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Modular representations of cyclotomic Hecke algebras of type $G(r, p, n)$

Gwenaëlle Genet*and Nicolas Jacon†

Abstract

We give a classification of the simple modules for the cyclotomic Hecke algebras over $\mathbb{C}$ in the modular case. We use the unitriangular shape of the decomposition matrices of Ariki-Koike algebras and Clifford theory.

1 Introduction

Let r, p and n be integers such that p divides r. The complex reflection group of type $G(r, p, n)$ is defined to be the groups of $n \times n$ permutation matrices such that the entries are either 0 or r^{th} roots of unity and the product of the nonzero entries is a $\frac{r}{p}$-root of unity.

In [2] and [5], Ariki and Broué-Malle have defined a Hecke algebra associated to each complex reflection group $G(r, p, n)$. It can be seen as a deformation of the algebra of the group $G(r, p, n)$. According to a conjecture of Broué and Malle, such algebras, called cyclotomic Hecke algebras, should occur as endomorphism algebras of the Lusztig induced character. The representation theory of cyclotomic Hecke algebras of type $G(r, 1, n)$ (also known as Ariki-Koike algebras) is beginning to be well understood. They are cellular algebras, the simple modules have been classified in both semi-simple and modular cases and the decomposition matrices are known in characteristic 0 (see [21] for a survey of these results).

In [2], Ariki has shown that a Hecke algebra of type $G(r, p, n)$ can be considered as the 0-component of a graded system for a Hecke algebra of type $G(r, 1, n)$ with a special choice of parameters. As a consequence, in the semi-simple case, he has given a complete set of non isomorphic simple modules by using Clifford theory. In the modular case, partial results have been obtained by Hu in [17] (see also [16] and [18] where the case $r = p = 2$ which corresponds to Hecke algebras of type D_n is studied). The main problem is that the restrictions of the simple modules for non semi-simple Ariki-Koike algebras are much more complicated to describe than for semi-simple ones.

The purpose of this paper is precisely to give a parametrization of the simple modules for the non semi-simple cyclotomic Hecke algebras of type $G(r, p, n)$

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over $\mathbb{C}$. It comes from the "canonical basic set" introduced by Geck and Rouquier in [13] and extended by the second author in [20]. This set induces a parametrization of the simple modules of non semi-simple Ariki-Koike algebras by some FLOTW multipartitions (this kind of multipartitions has been defined by Foda, Leclerc, Okado, Thibon and Welsh in [9]). We proceed in an analogous way than in [14, Theorem 2.1], using the unitriangular shape of the decomposition matrices of Ariki-Koike algebras and Clifford theory.

2 Cyclotomic Hecke algebras, Clifford theory

2.A Cyclotomic Hecke algebras

Let r, p and n be integers such that p divides r. We denote $d := \dfrac{r}{p}$. Let R be an integral ring, q a unit of R and a sequence $x = (x_1, \ldots, x_d)$ of elements in R. Denote $\mathfrak{H}^{q,x}_{r,p,n}(R)$, the Hecke R-algebra associated to the complex reflection group $G(r,p,n)$, for the parameters q, x, defined by the following presentation:

$$\begin{array}{ll}
\textit{generators}: & a_0, \ldots, a_n, \\
\textit{relations}: & (a_0 - x_1)\ldots(a_0 - x_d) = 0 \\
& (a_i - q)(a_i + 1) = 0,\ 1 \leq i \leq n, \\
& a_1 a_3 a_1 = a_3 a_1 a_3, \\
& a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1},\ 2 \leq i \leq n-1, \\
& (a_1 a_2 a_3)^2 = (a_3 a_1 a_2)^2, \\
& a_1 a_i = a_i a_1,\ 4 \leq i \leq n, \\
& a_i a_j = a_j a_i,\ 2 \leq i < j \leq n,\ j \geq i+2, \\
& a_0 a_1 a_2 = (q^{-1} a_1 a_2)^{2-r} a_2 a_0 a_1 + (q-1) \displaystyle\sum_{i=1}^{r-2} (q^{-1} a_1 a_2)^{1-k} a_0 a_1, \\
& \qquad\qquad = a_1 a_2 a_0, \\
& a_0 a_i = a_i a_0,\ 3 \leq i \leq n.
\end{array}$$

We have the following special cases:

- if $r = p = 1$, $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is the Hecke algebra of type A_{n-1},
- if $r = 2$ and $p = 1$, $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is the Hecke algebra of type B_n,
- if $r = p = 2$, $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is the Hecke algebra of type D_n,
- if $p = 1$, $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is the Ariki-Koike algebra defined in [4].

Suppose the ring R contains a pth root of unity η_p and a pth root y_i of x_i, for each $i \in [1,d]$. We define a sequence $Q = (Q_1, \ldots, Q_r)$ of elements in R from the sequence x: for $j \in [1,r]$, $j = sp + t$ with $s \in [0, d-1]$ and $t \in [1,p]$, let $Q_j := y_{s+1} \eta_p^{t-1}$. Then, $\mathfrak{H}^{q,Q}_{r,1,n}(R)$ is the R-algebra defined by

$$\begin{array}{ll}
\textit{generators}: & T_1, \ldots, T_n, \\
\textit{relations}: & (T_1^p - x_1)\ldots(T_1^p - x_d) = 0, \\
& (T_i - q)(T_i + 1) = 0,\ 2 \leq i \leq n, \\
& T_1 T_2 T_1 T_2 = T_2 T_1 T_2 T_1, \\
& T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},\ 2 \leq i \leq n-1, \\
& T_i T_j = T_j T_i,\ 1 \leq i < j \leq n,\ j \geq i+2.
\end{array}$$

We use [2, Proposition 1.6] and identify a_0 with T_1^p, a_1 with $T_1^{-1}T_2T_1$ and a_i with T_i, for $i \in [2,n]$, such that $\mathfrak{H}^{q,x}_{r,p,n}(R)$ is considered as a subalgebra of $\mathfrak{H}^{q,Q}_{r,1,n}(R)$. Then by [15, §4.1], the Ariki-Koike algebra $\mathfrak{H}^{q,Q}_{r,1,n}(R)$ is graded over the cyclotomic Hecke algebra $\mathfrak{H}^{q,x}_{r,p,n}(R)$ by a cyclic group of order p:

$$\mathfrak{H}^{q,Q}_{r,1,n}(R) = \bigoplus_{i=0}^{p-1} T_1^i\, \mathfrak{H}^{q,x}_{r,p,n}(R).$$

Put $\mathfrak{H} := \mathfrak{H}^{q,Q}_{r,1,n}(R)$ and $\mathfrak{H}' := \mathfrak{H}^{q,x}_{r,p,n}(R)$. We can define an automorphism $g^{\mathfrak{H}}_{\mathfrak{H}'}$ of $\mathfrak{H}'$ by

$$g^{\mathfrak{H}}_{\mathfrak{H}'}(h) := T_1^{-1}hT_1, \text{ for all } h \in \mathfrak{H}'.$$

We also have an automorphism $f^{\mathfrak{H}}_{\mathfrak{H}'}$ of $\mathfrak{H}$ which is defined on the grading by

$$f^{\mathfrak{H}}_{\mathfrak{H}'}({T_1}^j h) = \eta_p^j\, T_1^j h, \text{ for } j \in [0, p-1],\ h \in \mathfrak{H}'.$$

The next theorem of Dipper and Mathas will be very useful in the following of this paper.

Theorem 2.1 ([8]) *With the previous notations, assume that we have a partition of Q:*

$$Q = Q^1 \coprod Q^2 \coprod \cdots \coprod Q^s$$

such that

$$f_\Gamma(q,Q) = \prod_{1\le\alpha<\beta\le s} \prod_{\substack{Q_i\in Q^\alpha \\ Q_j\in Q^\beta}} \prod_{-n<a<n} (q^a Q_i - Q_j)$$

is a unit of R. Then, $\mathfrak{H}^{q,Q}_{r,1,n}(R)$ is Morita-equivalent to the following algebra:

$$\bigoplus_{\substack{n_1,\ldots,n_s>0 \\ n_1+\ldots+n_s=n}} \mathfrak{H}^{q,Q^1}_{r_1,1,n_1}(R) \otimes \ldots \otimes \mathfrak{H}^{q,Q^s}_{r_s,1,n_s}(R)$$

where for $i \in [1,s]$, $r_i = |Q^i|$.

2.B Clifford theory

Let $\mathbf{y}_1$, $\mathbf{y}_2$,...,$\mathbf{y}_d$ and $\mathbf{v}$ be indeterminates over $\mathbb{C}$. For $i \in [1,d]$, put

$$\mathbf{x}_i := \mathbf{y}_i^p$$

and for $i \in [1,r]$, $i = sp+t$ with $s \in [0,d-1]$, $t \in [1,p]$, $\eta_p := \exp(\frac{2i\pi}{p})$,

$$\mathbf{Q}_i := \eta_p^{t-1}\mathbf{y}_{s+1}.$$

Form the sequences $\mathbf{x} := (\mathbf{x}_1,\ldots,\mathbf{x}_d)$ and $\mathbf{Q} := (\mathbf{Q}_1,\ldots,\mathbf{Q}_r)$.

Let $A := \mathbb{C}[\mathbf{x}_1, \mathbf{x}_1^{-1}, \ldots, \mathbf{x}_d, \mathbf{x}_d^{-1}, \mathbf{v}, \mathbf{v}^{-1}]$, $K := \mathbb{C}(\mathbf{x}_1, \ldots, \mathbf{x}_d, \mathbf{v})$ its field of fractions.

Let $\theta : A \to \mathbb{C}$ be a ring homomorphism such that $\mathbb{C}$ is the field of fractions of $\theta(A)$.
We put for $i \in [1, d]$,

$$x_i := \theta(\mathbf{x}_i),$$

for $i \in [1, r]$,

$$Q_i := \theta(\mathbf{Q}_i)$$

and

$$v := \theta(\mathbf{v}).$$

Note that for $i \in [1, r]$, $i = sp + t$ with $s \in [0, d-1]$, $t \in [1, p]$, then $Q_i := \eta_p^{t-1} y_{s+1}$, for a complex number y_{s+1} such that $y_{s+1}^p = x_{s+1}$.
We also form the sequences $x := (x_1, \ldots, x_d)$ and $Q := (Q_1, \ldots, Q_r)$.

We will note for short $\mathbf{H}$, resp. H, for the Ariki-Koike algebras $\mathfrak{H}_{r,1,n}^{\mathbf{v},\mathbf{Q}}(K)$, resp. $\mathfrak{H}_{r,1,n}^{v,Q}(\mathbb{C})$ and $\mathbf{H}'$, resp. H', for the cyclotomic Hecke algebras $\mathfrak{H}_{r,p,n}^{\mathbf{v},\mathbf{x}}(K)$, resp. $\mathfrak{H}_{r,p,n}^{v,x}(\mathbb{C})$.

Since $\mathbf{H}$, resp. H, is graded over $\mathbf{H}'$, resp. H', we will give some results which comes from Clifford theory.

Let $\mathfrak{H}$ be the Ariki-Koike algebra $\mathbf{H}$, resp. H, and $\mathfrak{H}'$ be the cyclotomic Hecke algebra $\mathbf{H}'$, resp. H'. In both cases, we will denote by Res the functor of restriction from the category of the $\mathfrak{H}$-modules to the category of the $\mathfrak{H}'$-modules and by Ind the functor $\mathfrak{H} \otimes_{\mathfrak{H}'} -$ from the category of the $\mathfrak{H}'$-modules to the category of the $\mathfrak{H}$-modules.

Let V be a $\mathfrak{H}$-simple module and U be a simple submodule of $\operatorname{Res} V$. Put

$$o_{f,V} := \min \{k \in \mathbb{N}_{>0} \mid {}^{f^k}V \simeq V\}$$

where $f := f_{\mathfrak{H}'}^{\mathfrak{H}}$ and for $i \in [0, o_{f,V} - 1]$, ${}^{f^i}V$ is the $\mathfrak{H}$-module with the same underlying space as V and the structure of module is given by composing the original action of $\mathfrak{H}$ with f^i.
In the same way, let

$$o_{g,U} := \min \{k \in \mathbb{N}_{>0} \mid {}^{g^k}U \simeq U\}$$

where $g := g_{\mathfrak{H}'}^{\mathfrak{H}}$ and for $i \in [0, o_{g,U} - 1]$, ${}^{g^i}U$ is the $\mathfrak{H}'$-module with the same underlying space as U with the action of $\mathfrak{H}'$ twisted by g^i.
Denote $[X : Y]$ the multiplicity of a simple module Y in the semi-simple module X.

Lemma 2.2 *Denote for short $g := g_{\mathfrak{H}'}^{\mathfrak{H}}$ and $f := f_{\mathfrak{H}'}^{\mathfrak{H}}$. Every simple $\mathfrak{H}'$-module U appears as a direct summand of the restriction* $\operatorname{Res} V$ *of a simple $\mathfrak{H}$-module V with multiplicity-free. With such modules U and V, a conjugate ${}^{g^i}U$ of U, $i \in [0, o_{g,U} - 1]$, occurs in the restriction of the conjugate ${}^{f^j}V$ of V, for all*

$j \in [0, o_{f,V}-1]$ and the conjugates of V are the only simple $\mathfrak{H}$-modules whose restrictions contain such a ${}^{g^i}U$. Besides

$$\begin{aligned} \operatorname{Res} V &\simeq \bigoplus_{i=0}^{o_{f,V}-1} {}^{f^i}V, & (1)\\ \operatorname{Ind} U &\simeq \bigoplus_{i=0}^{o_{g,U}-1} {}^{g^i}U, & (2)\\ p &= o_{g,U}o_{f,V}. & (3) \end{aligned}$$

Proof: It is easy to see that

$$\operatorname{Res} V \simeq [\operatorname{Res} V : U] \bigoplus_{i=0}^{o_{g,U}-1} {}^{g^i}U, \tag{4}$$

(see [6, Theorem 11.1]). Using [14, Proposition 2.2], we get

$$\operatorname{Ind} U \simeq [\operatorname{Ind} U : V] \bigoplus_{i=0}^{o_{f,V}-1} {}^{f^i}V$$

as well as

$$\frac{p}{o_{f,V}o_{g,U}} = [\operatorname{Res} V : U][\operatorname{Ind} U : V]. \tag{5}$$

As $\operatorname{Res} V$ is g-stable, [6, Proposition 11.14] implies that $\operatorname{End}_{\mathfrak{H}}(\operatorname{Ind}\operatorname{Res} V)$ is graded over $\operatorname{End}_{\mathfrak{H}'}(\operatorname{Res} V)$ by a cyclic group of order p. By (4) and (5), the dimension of $\operatorname{End}_{\mathfrak{H}}(\operatorname{Ind}\operatorname{Res} V)$ over $\mathbb{C}$ is $\frac{p^2}{o_{f,V}}$ and by the grading over $\operatorname{End}_{\mathfrak{H}'}(\operatorname{Res} V)$, this dimension is equal to $p[\operatorname{Res} V : U]^2 o_{g,U}$. By comparison with (5), it follows that

$$[\operatorname{Res}\ V : U] = [\operatorname{Ind}\ U : V]. \tag{6}$$

Now, by definition of $o_{f,V}$, for short $\mathfrak{o}$, there exists an $\mathfrak{H}$-automorphism

$$t : V \to {}^{f^{\mathfrak{o}}}V.$$

So, $t^{\frac{p}{\mathfrak{o}}}$ is an $\mathfrak{H}$-isomorphism. By Schur's lemma, $t^{\frac{p}{\mathfrak{o}}}$ is a scalar. There exists a complex number c such that $(\frac{t}{c})^{\frac{p}{\mathfrak{o}}} = \operatorname{Id}_V$. Then

$$\operatorname{Res} V \simeq \bigoplus_{i=0}^{\frac{p}{\mathfrak{o}}-1} \operatorname{Ker}(\frac{t}{c} - \eta_p^{\mathfrak{o}i}\operatorname{Id}_V).$$

It is easy to see that for each $i \in [0, \frac{p}{\mathfrak{o}}-1]$, $\operatorname{Ker}(\frac{t}{c} - \eta_p^{\mathfrak{o}i}\operatorname{Id}_V)$ is an $\mathfrak{H}'$-module isomorphic to ${}^{g^i}\operatorname{Ker}(\frac{t}{c} - \operatorname{Id}_V)$. So the semi-simple module $\operatorname{Res} V$ decomposes into more than $\frac{p}{\mathfrak{o}}$ simple modules and by (5) by less than $\frac{p}{\mathfrak{o}}$ simple modules. Therefore $\operatorname{Res} V$ decomposes into exactly $\frac{p}{\mathfrak{o}}$ simple modules and $[\operatorname{Ind} U : V] = 1$ so $[\operatorname{Res} V : U] = 1$ by (6).
As the module induced from a simple $\mathfrak{H}'$-module to $\mathfrak{H}$ is semi-simple, the assertions of the Lemma are proved. ∎

2.C The semi-simple case

We keep the notations of the previous subsection.

We study the representation theory of the algebras $\mathbf{H} := \mathfrak{H}^{\mathbf{v},\mathbf{Q}}_{r,1,n}(K)$ and $\mathbf{H}' := \mathfrak{H}^{\mathbf{v},\mathbf{x}}_{r,p,n}(K)$. By [1, Main Theorem], $\mathbf{H}$ is a split semi-simple algebra. This implies that $\mathbf{H}'$ is also split semi-simple.

A complete set of non isomorphic modules for $\mathbf{H}$ has been given in [4] and [7]. Let Π^r_n be the set of r-partitions of n. We say that $\lambda = \big(\lambda(i)\big)_{i\in[1,r]}$ is a r-partition of size n if for every i, $\lambda(i)$ is a partition and $\Sigma^r_{i=1}|\lambda(i)| = n$.
For each r-partition $\lambda \in \Pi^r_n$, we can associate an $\mathfrak{H}^{\mathbf{v},\mathbf{Q}}_{r,1,n}(A)$-module $S^{\mathbf{v},\mathbf{Q}}(\lambda)$ called a "Specht module" that is free over A. Here, we use the definition of "classical" Specht module contrary to [7] where the dual Specht modules are defined. We have the following theorem.

Theorem 2.3 ([4],[7]) *The following set is a complete set of non isomorphic absolutely irreducible $\mathfrak{H}^{\mathbf{v},\mathbf{Q}}_{r,1,n}(K)$-modules:*

$$\left\{S^{\mathbf{v},\mathbf{Q}}_K(\lambda) := K \otimes_A S^{\mathbf{v},\mathbf{Q}}(\lambda) \mid \lambda \in \Pi^r_n\right\}.$$

For each $\lambda \in \Pi^r_n$, the set of standard tableaux of shape λ is a basis of the underlying $\mathbb{C}$-vector space of $S^{\mathbf{v},\mathbf{Q}}_K(\lambda)$ (see [4]). Using [4, Propositions 3.16, 3.17] and some combinatorial properties, it is easy to get the

Proposition 2.4 *Let $\lambda \in \Pi^r_n$. The map*

$$h^{\mathbf{v},\mathbf{Q}}_\lambda : {}^{f^{\mathbf{H}}_{\mathbf{H}'}}S^{\mathbf{v},\mathbf{Q}}_K(\lambda) \to S^{\mathbf{v},\mathbf{Q}}_K\big(\varpi(\lambda)\big) \tag{7}$$

that sends a standard λ-tableau $\mathbb{T} = (\mathbb{T}_i)_{i\in[1,r]}$ to the standard $\varpi(\lambda)$-tableau $\varpi(\mathbb{T}) := (\mathbb{T}_{\varpi^{-1}(i)})_{i\in[1,r]}$ is an $\mathbf{H}$-isomorphism where ϖ is the permutation of $\mathfrak{S}_r$ defined by $\mathbf{Q}_{\varpi(i)} := \eta_p\mathbf{Q}_i$, for all $i \in [1,r]$ and $\varpi(\lambda) = (\lambda(\varpi^{-1}(1)), \ldots, \lambda(\varpi^{-1}(r)))$.

Define

$$o_\lambda := \min\{k \in \mathbb{N}_{>0} \mid \varpi^k(\lambda) = \lambda\}.$$

It is clear that $o_\lambda = o_{f,S^{\mathbf{v},\mathbf{Q}}_K(\lambda)}$ and

$$\operatorname{Res} S^{\mathbf{v},\mathbf{Q}}_K(\lambda) \simeq \bigoplus_{i=0}^{\frac{p}{o_\lambda}-1} \operatorname{Ker}(h^{o_\lambda}_\lambda - \eta_p^{o_\lambda i}\operatorname{Id}_{S^{\mathbf{v},\mathbf{Q}}_K(\lambda)}),$$

where $h^{o_\lambda}_\lambda : {}^{f^{o_\lambda}}S^{\mathbf{v},\mathbf{Q}}_K(\lambda) \to S^{\mathbf{v},\mathbf{Q}}_K\big(\lambda\big)$ sends a standard λ-tableau $\mathbb{T}$ to the standard λ-tableau $\varpi^{o_\lambda}(\mathbb{T})$. For $i \in [0, \frac{p}{o_\lambda} - 1]$, put

$$S^{\mathbf{v},\mathbf{Q}}_K(\lambda, i) := \operatorname{Ker}(h^{o_\lambda}_\lambda - \eta_p^{o_\lambda i}\operatorname{Id}_{S^{\mathbf{v},\mathbf{Q}}_K(\lambda)}). \tag{8}$$

Let $\mathcal{L}$ be a set of representatives of r-partitions for the action of the cyclic group generated by ϖ over Π_n^r. It follows from Lemma 2.2, the

Proposition 2.5 *The set*

$$\{S_K^{\mathbf{v},\mathbf{Q}}(\lambda,i)\ ;\ i\in[0,\frac{p}{o_\lambda}-1],\ \lambda\in\mathcal{L}\}$$

is a complete set of non isomorphic simple $\mathbf{H}'$*-modules.*

If $\mathbb{T}$ is a standard λ-tableau, then for $j\in[0,\frac{p}{o_\lambda}-1]$, the element

$$(\mathbb{T},j):=\sum_{i=0}^{\frac{p}{o_\lambda}-1}\eta_p^{-o_\lambda ij}\varpi^{o_\lambda i}(\mathbb{T})$$

belongs to the eigenspace $\mathrm{Ker}(h_\lambda^{o_\lambda}-\eta_p^{o_\lambda j}\mathrm{Id}_{S_K^{\mathbf{v},\mathbf{Q}}(\lambda)})$. It comes from the fact that $\mathbb{T}=\dfrac{o_\lambda}{p}\displaystyle\sum_{j=0}^{\frac{p}{o_\lambda}-1}(\mathbb{T},j)$, that for every $j\in[0,\frac{p}{o_\lambda}-1]$, the eigenspace $\mathrm{Ker}(h_\lambda^{o_\lambda}-\eta_p^{o_\lambda j}\mathrm{Id}_{S_K^{\mathbf{v},\mathbf{Q}}(\lambda)})$ is generated by the $(\mathbb{T},j)$, $\mathbb{T}$ a standard λ-tableau. So we have just described the simple $\mathbf{H}'$-modules, see also [2].

The following part is now concerned with the non semi-simple case: simple $\mathfrak{H}_{r,1,n}^{q,Q}(\mathbb{C})$-modules and $\mathfrak{H}_{r,p,n}^{q,x}(\mathbb{C})$-modules.

2.D The non semi-simple case

We keep the notations of §2.B.

2.D.1 Decomposition maps

The aim of this part is to define the decomposition maps for $H:=\mathfrak{H}_{r,1,n}^{q,Q}(\mathbb{C})$ and $H':=\mathfrak{H}_{r,p,n}^{q,x}(\mathbb{C})$. Recall that $A:=\mathbb{C}[\mathbf{x}_1,\mathbf{x}_1^{-1},\dots,\mathbf{x}_d,\mathbf{x}_d^{-1},\mathbf{v},\mathbf{v}^{-1}]$, $K:=\mathbb{C}(\mathbf{x}_1,\dots,\mathbf{x}_d,\mathbf{v})$ and that $\theta:A\to\mathbb{C}$ is a ring homomorphism.

Using [10], there exists a discrete valuation ring $\mathcal{O}$ with maximal ideal $I(\mathcal{O})$ such that $A\subset\mathcal{O}$ and $I(\mathcal{O})\cap A=\ker(\theta)$. Let $k_0:=\mathcal{O}/I(\mathcal{O})$ be the residue field of $\mathcal{O}$. Denote by $\bar{\ }:\mathcal{O}\to k_0$ the canonical map. We obtain the algebras $\mathfrak{H}_{r,1,n}^{\overline{\mathbf{q}},\overline{\mathbf{Q}}}(k_0)$ and $\mathfrak{H}_{r,p,n}^{\overline{\mathbf{q}},\overline{\mathbf{x}}}(k_0)$. The field k_0 can be considered as an extension of $\mathbb{C}$ that is the quotient field of $\theta(A)$. By [12, Lemma 7.3.4], there is an isomorphism between the Grothendieck groups of finitely generated H-modules and $\mathfrak{H}_{r,1,n}^{\overline{\mathbf{q}},\overline{\mathbf{Q}}}(k_0)$-modules (resp. H'-modules and $\mathfrak{H}_{r,p,n}^{\overline{\mathbf{q}},\overline{\mathbf{x}}}(k_0)$-modules):

$$R_0(H)\simeq R_0(\mathfrak{H}_{r,1,n}^{\overline{\mathbf{q}},\overline{\mathbf{Q}}}(k_0))\qquad\text{and}\qquad R_0(H')\simeq R_0(\mathfrak{H}_{r,p,n}^{\overline{\mathbf{q}},\overline{\mathbf{x}}}(k_0)).$$

As a consequence, we obtain well-defined decomposition maps by choosing $\mathcal{O}$-forms for the simple $\mathfrak{H}_{r,1,n}^{\mathbf{q},\mathbf{Q}}(K)$-modules (resp. $\mathfrak{H}_{r,p,n}^{\mathbf{q},\mathbf{x}}(K)$-modules) and reducing them modulo $I(\mathcal{O})$:

$$d^{r,1,n}:R_0(\mathbf{H})\to R_0(H),$$

$$d^{r,p,n} : R_0(\mathbf{H}') \to R_0(H').$$

For a simple $\mathbf{H}$-module V and W a simple $\mathbf{H}'$-module, there exist non negative integers $d^{(1)}_{V,M}$ with M a simple H-module and $d^{(p)}_{W,N}$ with N a simple H'-module such that:

$$d^{r,1,n}([V]) = \sum_{M \in \mathrm{Irr}(H)} d^{(1)}_{V,M}[M]$$

$$\text{and} \qquad d^{r,p,n}([W]) = \sum_{N \in \mathrm{Irr}(H')} d^{(p)}_{W,N}[N].$$

By using the same argument as in [11, Lemma 5.2] (see also [12, Theorem 7.4.3.c]), we obtain the following result.

Proposition 2.6 *The following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbf{H}) & \xrightarrow{\mathrm{Res}} & R_0(\mathbf{H}') \\ {\scriptstyle d^{r,1,n}}\downarrow & & \downarrow{\scriptstyle d^{r,p,n}} \\ R_0(H) & \xrightarrow{\mathrm{Res}} & R_0(H') \end{array}$$

Moreover, for any simple $\mathbf{H}$*-module* V *and any simple* H*-module* M*, we have*

$$d^{(1)}_{V,M} = d^{(1)}_{f^{\mathbf{H}}_{\mathbf{H}'}V, f^{H}_{H'}M},$$

and for any simple $\mathbf{H}'$*-module* W *and any simple* H'*-module* N*, we have:*

$$d^{(p)}_{W,N} = d^{(p)}_{g^{\mathbf{H}}_{\mathbf{H}'}W, g^{H}_{H'}N}.$$

In the following, we will see that under some additionnal hypotheses, the above property leads to some interesting results about the decomposition map of cyclotomic Hecke algebras of type $G(r,p,n)$ and about the simple H'-modules.

2.D.2 Simple modules of non semi-simple Ariki-Koike algebras

We will work under the following hypothesis. We assume that v is a primitive root of unity,

$$v := \eta_e = \exp(\frac{2i\pi}{e}),$$

for a positive integer $e > 1$. Let

$$f := \gcd(e,p),\ e = fe',\ p = fp',$$

then $\eta_f = \eta_p^{p'} = \eta_e^{e'}$.
By the definition of Q and Theorem 2.1, without lost of generality, we can suppose that there exist integers $v_1 \leq \ldots \leq v_\delta$ that belong to $[0, e'-1]$ such that Q consists of the complex numbers $\eta_e^{v_i}\eta_p^j$, where i ranges over $[1,\delta]$ and j ranges over $[0, p-1]$. Then $r = \delta f p'$ and we can split Q into p' sets:

$$Q = Q^1 \coprod \ldots \coprod Q^{p'},$$

with Q^j formed by the $\eta_e^{v_i}\eta_p^{lp'+j}$, $i \in [1,\delta]$ and $l \in [0, f-1]$. We consider Q^j as an ordered sequence

$$Q^j = (\eta_p^{j-1}\eta_e^{v_1}, \ldots, \eta_p^{j-1}\eta_e^{v_\delta}, \eta_p^{p'+j-1}\eta_e^{v_1}, \ldots, \eta_p^{p'+j-1}\eta_e^{v_\delta}, \ldots \\ \eta_p^{(f-1)p'+j-1}\eta_e^{v_1}, \ldots, \eta_p^{(f-1)p'+j-1}\eta_e^{v_\delta}). \tag{9}$$

Remark: First note that, for $j \in [1,p']$, $Q^j = \eta_p^{j-1}Q^1$ and

$$Q^1 = (\eta_e^{v_1}, \ldots, \eta_e^{v_\delta}, \eta_e^{v_1+e'}, \ldots, \eta_e^{v_\delta+e'}, \ldots, \eta_e^{v_1+(f-1)e'}, \ldots, \eta_e^{v_\delta+(f-1)e'}),$$

with $v_1 \leq \ldots \leq v_\delta \leq v_1 + e' \leq \ldots \leq v_\delta + (f-1)e'$.

Note also that the quotient of two elements that belong to Q^j, for $j \in [1,p']$, is a power of $v = \eta_e$ while the quotient of an element of Q^j by an element of Q^l, $l \in [1,p']$, $l \neq j$ is not such a power, by definition of f.

If we order the elements of Q by the ordered elements of Q^1, then the ordered elements of Q^2, etc, Theorem 2.1 implies that the Ariki-Koike algebra H is Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1,\ldots,n_{p'}\geq 0\\ n_1+\ldots+n_{p'}=n}} \mathfrak{H}^{q,Q^1}_{f\delta,1,n_1}(\mathbb{C}) \otimes \ldots \otimes \mathfrak{H}^{q,Q^{p'}}_{f\delta,1,n_{p'}}(\mathbb{C}).$$

So it is clear that H is also Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1,\ldots,n_{p'}\geq 0\\ n_1+\ldots+n_{p'}=n}} \mathfrak{H}^{q,Q^1}_{f\delta,1,n_1}(\mathbb{C}) \otimes \ldots \otimes \mathfrak{H}^{q,Q^1}_{f\delta,1,n_{p'}}(\mathbb{C}).$$

We order $\mathbf{Q}$ and we split it into p' sets following what we have done for Q:

$$\mathbf{Q} = \mathbf{Q}^1 \coprod \ldots \coprod \mathbf{Q}^{p'}.$$

2.D.3 Parametrization of simple H-modules

We can now give a parametrization for the simple H-modules. By [3, Theorem 2.5], the simple modules of the Ariki-Koike algebra $\mathfrak{H}^{q,Q_1}_{f\delta,1,n_i}(\mathbb{C})$, $i \in [1,p']$, are the quotient modules $D^{v,Q^1}_{\mathbb{C}}(\lambda) := S^{\mathbf{v},\mathbf{Q^1}}_{\mathbb{C}}(\lambda)/J(S^{\mathbf{v},\mathbf{Q^1}}_{\mathbb{C}}(\lambda))$ of the specialisations of the Specht modules $S^{\mathbf{v},\mathbf{Q^1}}_{\mathbb{C}}(\lambda) := \mathbb{C} \otimes S^{\mathbf{v},\mathbf{Q^1}}(\lambda)$ by their radical of Jacobson, with λ a Kleshchev $f\delta$-partition (for the parameters Q^1) of size n_i. So the simple modules of H are labelled by the p'-tuples of Kleshchev $f\delta$-partitions (associated to Q^1) $(\lambda^1, \ldots, \lambda^{p'})$ with $\sum_{i=1}^{p'} |\lambda^i| = n$. We will denote this set by Λ^0 and for $\lambda \in \Lambda^0$, $D^{v,Q}_{\mathbb{C}}(\lambda)$ the corresponding simple H-module.

In [20], another parametrization for the simple $\mathfrak{H}^{q,Q_1}_{f\delta,1,n_i}(\mathbb{C})$-modules has been found by using Lusztig's a-function (see Theorem 2.7 below). Following this paper, we will associate to each r-partition an a-value. To do this, we first describe the a-value of a $f\delta$-partition $\mu = (\mu(1), \mu(2), ..., \mu(f\delta))$.

For $k = 1, ..., \delta$, $s = 1, ..., f$ and $j = (s-1)\delta + k$ we put $w_j := v_k + (s-1)e'$ so that w_j is the power of the jth-component of Q^1.

Then, for $j := 1, ..., f\delta$, we define:

$$m^{(j)} := w_j - \frac{je}{f\delta} + e$$

and for $s = 1, ..., n$, we put:

$$B'^{(j)}_s := \mu(j)_s - s + n + m^{(j)}$$

where we use the convention that $\mu(j)_p := 0$ if p is greater than the height of $\mu(j)$. For $j = 1, ..., f\delta$, let $B'^{(j)} = (B'^{(j)}_1, ..., B'^{(j)}_n)$. Then, we define:

$$a^{(1)}_{f\delta}(\mu) := \sum_{\substack{0 \le i \le j < f\delta \\ (a,b) \in B'^{(i)} \times B'^{(j)} \\ a > b \text{ if } i = j}} \min\{a, b\} - \sum_{\substack{1 \le i,j \le f\delta \\ a \in B'^{(i)} \\ 1 \le k \le a}} \min\{k, m^{(j)}\} + g(n).$$

where $g(n)$ is a rational number which only depends on the $m^{(j)}$ and on n (the expression of g is given in [20]).

Let $\lambda \in \Pi^r_n$, write $\lambda = (\lambda(1), \ldots, \lambda(r))$ as $\lambda = (\lambda^1, \ldots, \lambda^{p'})$ where, for $i \in [1, p']$, $\lambda^i = \big(\lambda(f\delta(i-1)+1), \ldots, \lambda(f\delta(i-1)+f\delta)\big)$.

Finally, we define

$$a^{(1)}_r(\lambda) := \sum_{i=1}^{p'} a^{(1)}_{f\delta}(\lambda^i).$$

To each Specht module $S^{\mathbf{v},\mathbf{Q^1}}_K(\lambda)$ of $\mathfrak{H}^{\mathbf{q},\mathbf{Q_1}}_{f\delta,1,n'}(K)$, $n' \in [0, n]$ and λ an $f\delta$-partition, we also associate an a-value

$$a^{(1)}_{f\delta}\big(S^{\mathbf{v},\mathbf{Q^1}}_K(\lambda)\big) := a^{(1)}_{f\delta}(\lambda).$$

To each simple $\mathfrak{H}^{q,Q_1}_{f\delta,1,n'}(\mathbb{C})$-module $D^{v,Q^1}_{\mathbb{C}}(\lambda)$, $n' \in [0, n]$ and λ a Kleshchev $f\delta$-partition of size n', we again associate an a-value

$$a^{(1)}_{f\delta}(D^{v,Q^1}_{\mathbb{C}}(\lambda)) := \min\{a^{(1)}_{f\delta}(\mu) \mid \mu \in \Pi^{f\delta}_{n'},\ d^{(1)}_{S^{\mathbf{v},\mathbf{Q^1}}_K(\mu), D^{v,Q^1}_{\mathbb{C}}(\lambda)} \neq 0\}.$$

Theorem 2.7 ([19, Theorem 2.3.8]) *Let λ be a Kleshchev $f\delta$-partition of n' for a non-negative integer n'. There exists a unique $\mu := \kappa(\lambda)$ such that $a^{(1)}_{f\delta}(D^{v,Q^1}_{\mathbb{C}}(\lambda)) = a^{(1)}_{f\delta}(\mu)$ and $d^{(1)}_{S^{\mathbf{v},\mathbf{Q^1}}_K(\mu), D^{v,Q^1}_{\mathbb{C}}(\lambda)} \neq 0$. The function κ is a bijection between the set of the Kleshchev $f\delta$-partitions of size n' and the set of the FLOTW $f\delta$-partitions of size n'.*

Recall what is a FLOTW $f\delta$-partition $\lambda = \big(\lambda(1), \ldots, \lambda(f\delta)\big)$ of size n' associated to the parameters v and Q^1, see [9]. Denote each partition $\lambda(i)$, $i \in [1, f\delta]$, as $(\lambda(i)_1, \lambda(i)_2, \ldots)$. The multipartition λ satisfies

1. for $i \in [1, f]$, $j \in [1, \delta - 1]$, k positive integer,
$$\lambda\big((i-1)\delta + j\big)_k \geq \lambda\big((i-1)m + j + 1\big)_{k+v_{j+1}-v_j},$$
2. for $i \in [1, f-1]$, k positive integer,
$$\lambda(i\delta)_k \geq \lambda(i\delta + 1)_{k+v_1+e'-v_\delta},$$
3. for k positive integer,
$$\lambda(f\delta)_k \geq \lambda(1)_{k+v_1+e'-v_\delta},$$
as well as
4. to each node of the diagram of λ which is located in the ath row and the bth column of the cth partition of λ (for two poditive integers a and b and $1 \leq c \leq f\delta$), we associate its residue $\eta_e^{b-a+v_c}$. Then, for any positive integer k, the cardinality of the set of the residues associated to the nodes in both the kth columns and the right rims of the Young diagrams of λ is not e.

In the same way, to each Specht module $S_K^{\mathbf{v},\mathbf{Q}}(\lambda)$ of $\mathbf{H}$, $\lambda \in \Pi_n^r$, $\lambda = (\lambda^1, \ldots, \lambda^{p'})$, we can define $a_r^{(1)}(S_K^{\mathbf{v},\mathbf{Q}}(\lambda)) := a_r^{(1)}(\lambda)$ that is the sum $\displaystyle\sum_{i=1}^{p'} a_{f\delta}^{(1)}(\lambda^i)$. To each simple H-module $D_{\mathbb{C}}^{v,Q}(\lambda)$, $\lambda \in \Lambda^0$, we associate the a-value

$$a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)) := \min\{a_r^{(1)}(\mu) \mid \mu \in \Pi_n^r,\ d^{(1)}_{S_K^{\mathbf{v},\mathbf{Q}}(\mu), D_{\mathbb{C}}^{v,Q}(\lambda)} \neq 0\}. \tag{10}$$

With Theorem 2.1, it is clear that there exists a unique $\mu := \kappa'(\lambda)$ such that $a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)) = a_r^{(1)}(\mu)$ and $d^{(1)}_{S_K^{\mathbf{v},\mathbf{Q}}(\mu), D_{\mathbb{C}}^{v,Q}(\lambda)} \neq 0$. In fact,

$$\kappa'(\lambda) = (\kappa(\lambda^1), \ldots, \kappa(\lambda^{p'}))$$

if $\lambda = (\lambda^1, \ldots, \lambda^{p'})$.
The function κ' is a bijection between Λ^0 and Λ^1 the set of the p'-tuples of FLOTW $f\delta$-partitions $(\lambda^1, \ldots, \lambda^{p'})$ of size $\displaystyle\sum_{i=1}^{p'} |\lambda^i| = n$.

2.D.4 Preliminary results

In order to prove the main Theorem, we will need some preliminary results. First, recall that we have ordered $\mathbf{Q}$ by the ordered elements of $\mathbf{Q}^1$, then by the ordered elements of $\mathbf{Q}^2$ etc as in §2.D.2. Let ϖ be the permutation of $\mathfrak{S}_r$ defined in Proposition 2.4 and let

$$\lambda := (\lambda^1[1], ..., \lambda^1[f\delta], \lambda^2[1], ..., \lambda^2[f\delta], ..., \lambda^{p'}[1], ..., \lambda^{p'}[f\delta])$$

be an r-partition. Hence, following the notations of the previous paragraph, we have $\lambda^j[i] = \lambda(f\delta(j-1) + i)$. Then, it is easy to verify that

- for $i \in [1, (p'-1)f\delta]$, then $\varpi(i) = i + f\delta$,
- for $i \in [r - f\delta + 1, r - \delta]$, then $\varpi(i) = i - r + (f+1)\delta$,
- for $i \in [r - \delta + 1, r]$, then $\varpi(i) = i - r + \delta$.

Then, by Proposition 2.4, we have:

$$S_K^{\mathbf{v},\mathbf{Q}}(\lambda) \simeq S_K^{\mathbf{v},\mathbf{Q}}\big(\varpi(\lambda)\big)$$

where $\varpi(\lambda) = (\lambda^{p'}[f\delta - \delta + 1]\ , \ldots,\ \lambda^{p'}[f\delta], \lambda^{p'}[1]\ , \ldots,\ \lambda^{p'}[f\delta - \delta],\ \lambda^1[1]\ , \ldots,\ \lambda^1[f\delta], \ldots, \lambda^{p'-1}[1]\ , \ldots,\ \lambda^{p'-1}[f\delta])$.

Proposition 2.8 *Let $\lambda \in \Pi_n^r$, then $a_r^{(1)}(S_K^{\mathbf{v},\mathbf{Q}}(\lambda)) = a_r^{(1)}({}^{f_{\mathbf{H}'}^{\mathbf{H}}}S_K^{\mathbf{v},\mathbf{Q}}(\lambda))$. Besides, if $\lambda \in \Lambda^0$, then $a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)) = a_r^{(1)}({}^{f_{H'}^{H}}D_{\mathbb{C}}^{v,Q}(\lambda))$.*

Proof: By the definition of the a-value (see §2.D.3), we have:

$$a_r^{(1)}(S_K^{\mathbf{v},\mathbf{Q}}(\lambda)) := \sum_{i=1}^{p'} a_{f\delta}^{(1)}(\lambda^i[1], ..., \lambda^i[f\delta])$$

Thus, by using the above properties, it is sufficient to show that $a_{f\delta}^{(1)}$ $(\lambda^{p'}[1]\ , \ldots, \lambda^{p'}[f\delta])= a_{f\delta}^{(1)}(\lambda^{p'}[f\delta - \delta + 1], \ldots, \lambda^{p'}[f\delta], \lambda^{p'}[1], \ldots, \lambda^{p'}[f\delta - \delta])$.

Using the notations of §2.D.3, for $i = 1, ..., \delta$ and $j = 1, ..., f$ we write $m^{(i)}[j] := m^{((j-1)\delta+i)}$. Then, all we have to do is to prove that $m^{(i)}[j]$ doesn't depend on j. We have $m^{(i)}[j] = v_i - \dfrac{((j-1)\delta + i)e}{f\delta} + (j-1)e' = v_i - \dfrac{ie}{f}$ because $e'f = e$. Hence, we obtain the desired result.

With the definition (10), it is trivial that the previous result about Specht modules implies that, for any $\lambda \in \Lambda^0$, $a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)) = a_r^{(1)}({}^{f_{H'}^{H}}D_{\mathbb{C}}^{v,Q}(\lambda))$. ∎

With Proposition 2.8 and Lemma 2.2, we can associate an a-value to each simple $\mathbf{H}'$-module U. If U appears in the restriction $\operatorname{Res} S_K^{\mathbf{v},\mathbf{Q}}(\lambda)$, for $\lambda \in \Pi_n^r$, put

$$a_r^{(p)}(U) := a_r^{(1)}\big(S_K^{\mathbf{v},\mathbf{Q}}(\lambda)\big).$$

In the same way, we can associate an a-value to each simple H'-module W. If W appears in the restriction $\operatorname{Res} D_{\mathbb{C}}^{v,Q}(\lambda)$, for $\lambda \in \Lambda^0$, then

$$a_r^{(p)}(W) := a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda)).$$

Another useful proposition is the following one.

Proposition 2.9 *Let $\lambda \in \Lambda^1$, then $\varpi(\lambda) \in \Lambda^1$, where ϖ is defined in the* Proposition 2.4.

Proof: Write $\lambda \in \Lambda^1$ as $\lambda = (\lambda^1, \dots, \lambda^{p'})$ where $\lambda^i = (\lambda^i[1], \dots, \lambda^i[f\delta])$ is a FLOTW $f\delta$-partition for Q^1. We will see that $\varpi(\lambda) \in \Lambda^1$ so the result will be proved. The r-partition $\varpi(\lambda)$ is equals to $(\lambda^{p'}[f\delta - \delta + 1]\ , \dots,\ \lambda^{p'}[f\delta], \lambda^{p'}[1]\ , \dots,\ \lambda^{p'}[f\delta - \delta],\ \lambda^1[1]\ , \dots,\ \lambda^1[f\delta],\ \dots\ \lambda^{p'-1}[1]\ , \dots,\ \lambda^{p'-1}[f\delta])$. It belongs to Λ^1 if and only if $(\lambda^{p'}[f\delta - \delta + 1]\ , \dots,\ \lambda^{p'}[f\delta],\ \lambda^{p'}[1]\ , \dots,\ \lambda^{p'}[f\delta - \delta])$ is a FLOTW $f\delta$-partition. It is easy to verify that the three first conditions hold. Now, the set of the residues associated to the nodes in both the kth columns and the right rims of the Young diagrams of this last multipartition consists of the residues of $\lambda^{p'}$ but multiplied by $\eta_e^{e'}$. So the fourth condition about FLOTW multipartitions hold as well as the Proposition. ∎

3 A parametrization of the simple H'-modules

We keep the above notations. The aim of this section is to describe a parametrization of the simple H'-modules, with $H' = \mathfrak{H}_{r,p,n}^{v,Q}(\mathbb{C})$. The proof is highly inspired by [14, Theorem 2.1].

Theorem 3.1 *For all simple H'-module N, there exists a simple $\mathbf{H}'$-module W_N such that*

$$d^{r,p,n}([W_N]) = [N] + \sum_{a_r^{(p)}(L) < a_r^{(p)}(N)} d_{W_N,L}[L],$$

and $a_r^{(p)}(N) = a_r^{(p)}(W_N)$.
Let $\Lambda^{1'} := \{W_N \mid N \in \mathrm{Irr}(H')\}$. Then $\Lambda^{1'}$ consists in the $S_K^{\mathbf{v},\mathbf{Q}}(\lambda, i)$, with $\lambda \in \Lambda^1 \cap \mathcal{L}$, $i \in [0, \frac{p}{o_\lambda} - 1]$ with the notations of the Proposition 2.5 and the map $N \in \mathrm{Irr}(H') \mapsto W_N \in \Lambda^{1'}$ is bijective. So $\Lambda^{1'}$ is a parametrization of the simple H'-modules.

Proof : Fix a simple H'-module N. By Lemma 2.2, there exists $\lambda \in \Lambda^0$ such that N appears in the restriction $\mathrm{Res} D_{\mathbb{C}}^{v,Q}(\lambda)$. We have

$$d^{r,1,n}([S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda))]) = [D_{\mathbb{C}}^{v,Q}(\lambda)] + \sum_{\substack{a_r^{(p)}(D_{\mathbb{C}}^{v,Q}(\mu)) < \\ a_r^{(p)}(D_{\mathbb{C}}^{v,Q}(\lambda))}} d^{(1)}_{S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda)), D_{\mathbb{C}}^{v,Q}(\mu)} [D_{\mathbb{C}}^{v,Q}(\mu)], \tag{11}$$

and $a_r^{(1)}\big(S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda))\big) = a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda))$. With the notations of (8) and Proposition 2.6, this equality implies that

$$\sum_{i=0}^{\frac{p}{o_{\kappa(\lambda)}}-1} d^{r,p,n}([S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda), i)]) = \sum_{i=0}^{\frac{p}{o_{f_{H'}^H, D_{\mathbb{C}}^{v,Q}(\lambda)}}-1} [{}^{(g_{H'}^H)^i}N] + [N'], \tag{12}$$

where N' is a sum of H'-modules, D, with a-value $a_r^{(p)}(D) < a_r^{(p)}(D_{\mathbb{C}}^{v,Q}(\lambda))$ and, for all i, j, $a_r^{(p)}(S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),i)) = a_r^{(p)}({}^{(g_{H'}^{H})^i}N)$.
Denote by $\tau(\lambda) \in \Lambda^0$, the multipartition defined by $D_{\mathbb{C}}^{v,Q}(\tau(\lambda)) := {}^{f_{H'}^{H}}D_{\mathbb{C}}^{v,Q}(\lambda)$. If we twist the action of H by $f_{H'}^{H}$, the equality (11) gives that $\kappa(\tau(\lambda)) = \varpi(\kappa(\lambda))$ by Proposition 2.9 and by definition of κ. As this map is bijective, it is clear that

$$o_{f_{\mathbf{H}'}^{\mathbf{H}},S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda))} = o_{f_{H'}^{H},D_{\mathbb{C}}^{v,Q}(\lambda)}$$

and therefore for all $i \in [0, \frac{p}{o_{\kappa(\lambda)}} - 1]$,

$$o_{g_{\mathbf{H}'}^{\mathbf{H}},S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),i)} = o_{g_{H'}^{H},N}$$

with the last equality of Lemma 2.2.
Recall that for $i \in [0, \frac{p}{o_{\kappa(\lambda)}} - 1]$, $S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),i)$ is conjugate to $S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),0)$ by $(g_{\mathbf{H}'}^{\mathbf{H}})^i$. The equality (12) implies that there exists an unique $i \in [0, \frac{p}{o_{\kappa(\lambda)}} - 1]$ such that

$$d^{r,p,n}([S_{\mathbb{C}}^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda),i)]) = [N] + [N''], \tag{13}$$

where N'' is a sum of H'-modules, D, with a-value $a_r^{(p)}(D) < a_r^{(p)}(N)$. This construction makes clear all the statements of the Theorem. ∎

Remark: The above theorem is only concerns with the case where v is a primitive e^{th}-root of unity. If v isn't a root of unity, it is readily checked that an analogue of this theorem holds by replacing Λ^1 by the set of Kleshchev multipartitions Λ^0 at $e = \infty$. The proof of this result can be easily obtained following the outline of [14].

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