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► **To cite this version:**

Alain Bérard, Herve Mohrbach. Non Abelian Berry Phase in Noncommutative Quantum Mechanics. 2004. hal-00002832v1

HAL Id: hal-00002832

<https://hal.science/hal-00002832v1>

Preprint submitted on 14 Sep 2004 (v1), last revised 27 Mar 2006 (v4)

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Non Abelian Berry Phase in Noncommutative Quantum Mechanics

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(Dated: September 14, 2004)

We consider the adiabatic evolution of the Dirac equation in order to compute its Berry curvature in momentum space. It is found that the position operator acquires an anomalous contribution due to the non Abelian Berry gauge connection rendering the quantum mechanic algebra noncommutative. A generalization to any known spinning particles is possible by using the Bargmann-Wigner equation of motions. The noncommutativity of the coordinates is responsible of the topological spin transport of spinning particle similarly to the spin Hall effect in spintronic physics or the Magnus effect in optics. As an application we predict new dynamics for nonrelativistic particles in an electric field and for photons in a gravitational field.

PACS numbers:

Recently, Quantum Mechanics involving noncommutative space time coordinates has led to numerous works in high energy [1] and condensed matter physics[2]. In this framework an antisymmetric θ^{ij} parameter usually taken to be constant is introduced in the commutation relation of the coordinates in the space manifold $[x^i, x^j] = i\hbar\theta^{ij}$. In a recent paper [3] we generalized the quantum mechanics in noncommutative geometry by considering a quantum particle of mass m whose coordinates satisfy the deformed Heisenberg algebra $[x^i, x^j] = i\hbar\theta^{ij}(\mathbf{x}, \mathbf{p})$, $[x^i, p^j] = i\hbar\delta^{ij}$, and $[p^i, p^j] = 0$. From the Jacobi identity $[p^i, [x^j, x^k]] + [x^j, [x^k, p^i]] + [x^k, [p^i, x^j]] = 0$, we deduced the important property that the θ field is only momentum dependent. An important consequence of the noncommutativity between the coordinates is that neither the position operator does satisfy the usual law $[x^i, L^j] = i\hbar\varepsilon^{ijk}x_k$, nor the angular momentum satisfy the standard $so(3)$ algebra $[L^i, L^j] = i\hbar\varepsilon^{ijk}L_k$. Actually we have $[x^i, L^j] = i\hbar\varepsilon^{ijk}x_k + i\hbar\varepsilon^j_{kl}p^l\theta^{ik}(\mathbf{p})$, and $[L^i, L^j] = i\hbar\varepsilon^{ijk}L_k + i\hbar\varepsilon^i_{kl}\varepsilon^j_{mn}p^l p^n \theta^{km}(\mathbf{p})$. To remedy this absence of generators of rotations in the noncommutative geometry we had to introduce a generalized angular momentum $\mathbf{J} = \mathbf{r} \wedge \mathbf{p} + \lambda \frac{\mathbf{p}}{p}$, that satisfies the $so(3)$ algebra. The position operator then transforms as a vector under rotations i.e., $[x^i, J^j] = i\hbar\varepsilon^{ijk}J_k$. The presence of the dual Poincare momentum $\lambda\mathbf{p}/p$ leads to a dual Dirac monopole in momentum space for the position algebra

$$[x^i, x^j] = -i\hbar\lambda\varepsilon^{ijk}\frac{p^k}{p^3}. \quad (1)$$

This result immediately implies that the coordinates of spinless particles are commuting. Another consequence is the quantification of the helicity $\lambda = n\hbar/2$ that arises from the restoration of the translation group of momentum that is broken by the monopole [3][7]. Note also that other recent theoretical works concerning the anomalous Hall effect in two-dimensional ferromagnets predicted a topological singularity in the Brillouin zone [8]. In addition, in recent experiments a monopole in the crystal momentum space was discovered and interpreted in terms of an Abelian Berry curvature [9].

In quantum mechanics this construction may look formal because it is always possible to introduce commuting coordinates with the transformation $\mathbf{R} = \mathbf{r} - \mathbf{p} \wedge \mathbf{S}/p^2$. The angular momentum is then $\mathbf{J} = \mathbf{R} \wedge \mathbf{p} + \mathbf{S}$ which satisfies the usual $so(3)$ algebra, whereas the potential energy term in the Hamiltonian now contains spin-orbit interactions $V(\mathbf{R} + \mathbf{p} \wedge \mathbf{S}/p^2)$. In fact, the inverse procedure is usually more efficient: considering an Hamiltonian with a particular spin-orbit interaction one can try to obtain a trivial Hamiltonian with a dynamics due to the noncommutative coordinates algebra. This procedure has been apply with success to the study of adiabatic transport in semiconductor with spin-orbit couplings [10] where the particular dynamics of the charges is due the commutation relation (1). The important point is to figure out which one of the two position operators \mathbf{r} or \mathbf{R} gives rise to the real mean trajectory of the particle. In fact it is well known that \mathbf{R} has not the genuine property of a position operator for a relativistic particle. As we shall see this crucial remark implies a new prediction concerning the non-relativistic limit of a Dirac particle.

In particle physics it is by now well known that the noncommutativity of the coordinates of massless particles is a fundamental property because the position operator does not transform like a vector unless it satisfies equation (1) and that $\theta^{ij}(\mathbf{p})$ is the Berry curvature for a massless particle with a given helicity λ [11].

In the present paper we present another point of view of the origin of the monopole in high energy and condensed matter physics by considering the adiabatic evolution of relativistic massive spinning particles. In particular the computation of the Berry curvature of Dirac particles gives rise to a noncommutative position operator that was already postulated by Bacry [12] some times ago. A generalization to any spin is possible by using the Bargmann-Wigner [13] equations of motion. By doing that construction, we are brought to make a generalization of noncommutative algebra by considering a θ field which is momentum as well as spin dependent. The associated connection is then non Abelian but becomes Abelian in the limit of vanishing mass leading to a monopole configuration for the Berry curvature. In this

respect our approach is different from [11] because the description of the photons is obtained by taking the zero mass limit of the massive representation of a spin one particle.

The Dirac's Hamiltonian for a relativistic particle of mass m has the form

$$\hat{H} = \alpha \cdot \mathbf{p} + \beta m + \hat{V}(\mathbf{R}),$$

where \hat{V} is an operator that acts only on the orbital degrees of freedom. Using the Foldy-Wouthuysen unitary transformation

$$U(\mathbf{p}) = \frac{E_p + mc^2 + c\beta\alpha \cdot \mathbf{p}}{\sqrt{2E_p(E_p + mc^2)}},$$

with $E_p = \sqrt{p^2c^2 + m^2c^4}$, we get the following transformed Hamiltonian

$$U(\mathbf{p})\hat{H}U(\mathbf{p})^+ = E_p\beta + U(\mathbf{p})\hat{V}(i\hbar\partial_{\mathbf{p}})U(\mathbf{p})^+.$$

The kinetic energy is now diagonal whereas the potential term becomes $\hat{V}(\mathbf{D})$ with the covariant derivative defined by $\mathbf{D} = i\hbar\partial_{\mathbf{p}} + \mathbf{A}$, and with the gauge potential $\mathbf{A} = i\hbar U(\mathbf{p})\partial_{\mathbf{p}}U(\mathbf{p})^+$, which reads

$$\mathbf{A} = \frac{\hbar c (ic^2\mathbf{p}(\alpha \cdot \mathbf{p})\beta + i\beta (E_p + mc^2) E_p\alpha - cE_p \boldsymbol{\Sigma} \wedge \mathbf{p})}{2E_p^2 (E_p + mc^2)}, \quad (2)$$

where $\boldsymbol{\Sigma} = 1 \otimes \boldsymbol{\sigma}$, is a (4×4) matrix. We consider adiabatic approximation by identifying the momentum degree of freedom as slow and the spin degree of freedom as fast, similar to the nuclear configuration in adiabatic treatment of molecular problems, which allows to neglect the interband transition. We then keep only the bloc diagonal matrix element in the gauge potential and project on the subspace of positive energy. This projection cancels the zitterbewegung which corresponds to an oscillatory motion around the mean position of the particle that mixes the positive and negative energies. In this way we obtain a non trivial gauge connection allowing us to define a new position operator \mathbf{r} for this particle

$$\mathbf{r} = i\hbar\partial_{\mathbf{p}} + \frac{c^2\hbar(\mathbf{p} \wedge \boldsymbol{\sigma})}{2E_p(E_p + mc^2)}, \quad (3)$$

which is a (2×2) matrix. The position operator (3) is not new, as it was postulated by H. Bacry [12]. In fact considering the irreducible representation of the Poincare group, this author proposed to adopt a general position operator for free massive or massless particles with any spin. In our approach which is easily generalizable to any known spin (see formula (6)) the anomalous part of the position operator arises from an adiabatic process of an interacting system and as we will now see is related to the Berry connection. For a different work with operator valued position connected to the spin-degree of freedom see [14]. Zitterbewegung-free non-commutative coordinates were also introduced for massless particle with rigidity and in the context of anyons [15].

It is straightforward to prove that the anomalous part of the position operator can be interpreted as a Berry connection in momentum space which, by definition is the (4×4) matrix $\mathbf{A}_{\alpha\beta}(\mathbf{p}) = i\hbar \langle \alpha \mathbf{p}+ | \partial_{\mathbf{p}} | \beta \mathbf{p}+ \rangle$ where $| \alpha \mathbf{p}+ \rangle$ is an eigenvector of the free Dirac equation of positive energy. The Berry connection can also be written

$$\mathbf{A}_{\alpha\beta}(\mathbf{p}) = i\hbar \langle \phi_{\alpha} | U\partial_{\mathbf{p}}U^+ | \phi_{\beta} \rangle,$$

in terms of the canonical base vectors $| \phi_{\alpha} \rangle = (1 \ 0 \ 0 \ 0)$ and $| \phi_{\beta} \rangle = (0 \ 1 \ 0 \ 0)$. The non zero element belonging only to the positive subspace, we can define the Berry connection by considering a 2×2 matrix $\mathbf{A}(\mathbf{p}) = i\hbar\mathcal{P}(U\partial_{\mathbf{p}}U^+)$, where \mathcal{P} is a projector on the positive energy subspace. In this context the θ field we postulated in [3] emerges naturally as a consequence of the adiabatic motion of a Dirac particle and corresponds to a non-Abelian gauge curvature satisfying the relation $\theta^{ij}(\mathbf{p}, \sigma) = \partial_{p^i}A^j - \partial_{p^j}A^i + [A^i, A^j]$. The commutation relations between the coordinates are then

$$[x^i, x^j] = i\hbar\theta^{ij}(\mathbf{p}, \sigma) = -i\hbar^2\varepsilon_{ijk}\frac{c^4}{2E_p^3} \left(m\sigma^k + \frac{p^k(\mathbf{p} \cdot \boldsymbol{\sigma})}{E_p + mc^2} \right), \quad (4)$$

relation which has very important consequences as it implies the nonlocalizability of the spinning particles. This is an intrinsic property and is not related to the creation of a pair during the measurement process (for a detailed discussion of this very important point see [12])

To generalize the construction of the position operator for a particle with unspecified $n/2$ ($n > 1$) spin, we start with the Bargmann-Wigner equations

$$(\gamma_{\mu}^{(i)}\partial_{\mu} + m + \hat{V})\psi_{(a_1 \dots a_n)} = 0 \quad (i = 1, 2, \dots, n),$$

where $\psi_{(a_1 \dots a_n)}$ is a Bargmann-Wigner amplitude and $\gamma^{(i)}$ are matrices acting on a_i . For each equation we have an Hamiltonian

$$\hat{H}^{(i)} = \alpha^{(i)} \cdot \mathbf{p} + \beta m + \hat{V},$$

then

$$\left(\prod_{j=1}^n U^{(j)}(\mathbf{p}) \right) \hat{H}^{(i)} \left(\prod_{j=1}^n U^{(j)}(\mathbf{p})^+ \right) = E_p\beta^{(i)} + \hat{V}(\mathbf{D}), \quad (5)$$

with $\mathbf{D} = i\hbar\partial_{\mathbf{p}} + \sum_{i=1}^n \mathbf{A}^{(i)}$, and $\mathbf{A}^{(i)} = i\hbar U^{(i)}(\mathbf{p})\partial_{\mathbf{p}}U^{(i)}(\mathbf{p})^+$. Again by considering the adiabatic approximation we deduce a general position operator \mathbf{r} for spinning particles

$$\mathbf{r} = i\hbar\partial_{\mathbf{p}} + \frac{c^2(\mathbf{p} \wedge \mathbf{S})}{E_p(E_p + mc^2)}, \quad (6)$$

with $\mathbf{S} = \hbar(\sigma^{(1)} + \dots + \sigma^{(n)})/2$. The generalization of (4) is then

$$[x^i, x^j] = i\hbar\theta^{ij}(\mathbf{p}, \mathbf{S}) = -i\hbar\varepsilon_{ijk}\frac{c^4}{E_p^3} \left(mS^k + \frac{p^k(\mathbf{p} \cdot \mathbf{S})}{E_p + mc^2} \right). \quad (7)$$

For a massless particle we recover the relation $\mathbf{r} = i\hbar\partial_{\mathbf{p}} + \mathbf{p} \wedge \mathbf{S}/p^2$, with the commutation relation giving rise to the monopole $[x^i, x^j] = i\hbar\theta^{ij}(\mathbf{p}) = -i\hbar\varepsilon_{ijk}\lambda\frac{p^k}{p^3}$. The monopole in momentum introduced in [3] in order to construct genuine angular momenta has then a very simple physical interpretation. It corresponds to the Berry curvature resulting from an adiabatic process of massless particle with helicity λ . For $\lambda = \pm 1$ we have the position operator of the photon, whose noncommutativity property agrees with the weak localizability of the photon which is certainly an experimental fact. It is not surprising that a massless particle has a monopole Berry curvature as it is well known that the band touching point acts as a monopole in momentum space [16]. This is precisely the case for massless particles for which the positive and negative energy band are degenerate in $p = 0$. In our approach, the monopole appears as a limiting case of a more general Non Abelian Berry curvature arising from an adiabatic process of massive spinning particles.

The spin-orbit coupling term in (6) is a very small corrections to the usual operator in the particle physics context but it may be strongly enhanced and observable in solid state physics because the spin-orbit effect is more much important than in the vacuum. For instance in narrow gap semiconductors the equations of the bands theory are similar to the Dirac equation with the forbidden gap E_G between the valence and conduction bands instead of the Dirac gap $2mc^2$ [17]. The monopole in momentum space predicted and observed in semiconductors results from the limit of vanishing gap $E_G \rightarrow 0$ between the valence and conduction bands.

It is also interesting to consider the symmetry properties of the position operator with respect to the group of spatial rotations. In terms of commutative coordinates \mathbf{R} the angular momentum is by definition $\mathbf{J} = \mathbf{R} \wedge \mathbf{p} + \mathbf{S}$, whereas in terms of the noncommutative coordinates the angular momentum reads $\mathbf{J} = \mathbf{r} \wedge \mathbf{p} + \mathbf{M}$, where

$$\mathbf{M} = \mathbf{S} - \mathbf{A} \wedge \mathbf{p}. \quad (8)$$

One can explicitly check that in terms of the non commutative coordinates the relation $[x^i, J^j] = i\hbar\varepsilon^{ijk}x_k$ is satisfied, so \mathbf{r} like \mathbf{R} transforms as a vector under space rotations, but $d\mathbf{R}/dt = c\alpha$ which is physically unacceptable. For a massless particle (8) leads to the Poincaré momentum associated to the monopole in momentum space deduced in [3].

Let now look at some physical consequences of the non-commuting position operator on the dynamics of a quantum particle in an arbitrary potential. Due to the Berry phase in the definition of the position the equation of motion should be changed. But to compute commutator like $[x^k, V(x)]$ one resorts to the semiclassical approximation $[x^k, V(x)] = i\hbar\partial_l V(x)\theta^{kl} + O(\hbar^2)$ leading to new equations of motion

$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{E_p} + \dot{\mathbf{p}} \wedge \theta, \quad \text{and} \quad \dot{\mathbf{p}} = -\nabla V(\mathbf{r}) \quad (9)$$

with $\theta^i = \varepsilon^{ijk}\theta_{jk}$. Whereas the equation for the momentum is usual, the velocity one acquires a topological contribution due to the Berry phase, which is responsible of the relativistic topological spin transport as in the context of semi-conductor where similar non-relativistic equations lead to the spin Hall effect [10][18].

As an important particular application, consider the nonrelativistic limit of a charged spinning Dirac particle in a constant electric field $\hat{V}(\mathbf{r}) = e\mathbf{E}\cdot\mathbf{r}$, (the physical results are not affected if the electric field is time dependent). In this limit the Hamiltonian reads

$$\tilde{H}(\mathbf{R}, \mathbf{p}) \approx mc^2 + \frac{p^2}{2m} + \hat{V}(\mathbf{R}) - \frac{e\hbar}{4m^2c^2}\sigma\cdot(\mathbf{E} \wedge \mathbf{p}), \quad (10)$$

which is a Pauli Hamiltonian with a spin-orbit term. This expression has been already deduced in [19] by considering the Born-Oppenheimer approximation of the Dirac equation where the Berry phase in the nonrelativistic limit emerges naturally. The adiabaticity is satisfied as the non-diagonal terms of order λ_c/L can be made negligible in comparison to the diagonal one which is of order $\lambda_c^2/\lambda L$ when $\lambda \ll L$ (L is the length of variation of the potential and λ_c is the Compton length). From (10) we deduce

$$\frac{dX^i}{dt} = \frac{p^i}{m} - \frac{e\hbar}{4m^2c^2}\varepsilon^{ijk}\sigma_j E_k, \quad (11)$$

whereas the non relativistic limit of (9) leads to the following velocity

$$\frac{dx^i}{dt} = \frac{p^i}{m} - \frac{e\hbar}{2m^2c^2}\varepsilon^{ijk}\sigma_j E_k. \quad (12)$$

which shows an enhancement of the spin-orbit contribution to the velocity. This new fundamental prediction which could be experimentally verified is one of the two main result of this letter. This astonished result offers another manifestation beside the Thomas precession of the difference between the Galilean limit (11) and the non-relativistic limit (12).

Another example of topological spin transport that we consider now arises in the ultrarelativistic limit. Experimentally what we call a topological spin transport has been first observed in the case of the photon propagation in an inhomogeneous medium [20], where the right and left circular polarization propagate along different trajectories in a wave guide (the transverse shift is observable due to the multiple reflections), a phenomena interpreted quantum mechanically as arising from the interaction between the orbital momentum and the spin of the photon [20]. To interpret the experiments these authors introduced a complicate phenomenological Hamiltonian leading to generalized geometrical optic equation. Our approach provides a new satisfactory interpretation as this effect, also called optical Magnus effect, is now interpreted in terms of the non-commutative property of the position operator containing the Berry phase. Note that the adiabaticity conditions in this case are

given in [21]. To illustrate our purpose consider the simple photon Hamiltonian in the inhomogeneous medium $H = pc/n(r)$, from which we deduce the semi-classical equations of motion

$$\frac{dx^i}{dt} = \frac{c}{n} \left(\frac{p^i}{p} + \frac{\lambda \varepsilon^{ijk} p_k}{p^2} \frac{\partial \ln n}{\partial x^j} \right) \quad (13)$$

which are similar to those introduced phenomenologically in [20], but here are deduced rigorously from different physical consideration. This effect which is the manifestation of the Berry phase and can be seen as the ultra-relativistic spin-Hall effect implies that the velocity is no more equal to c/n . Note that similar equations are also given in [22] where the optical Magnus effect is also interpreted in terms of a monopole Berry curvature but in the context of geometrical optic. Our theory is easily generalizable to the photon propagation in a non isotropic medium, a situation which is mentioned in [20] but could not be studied with their phenomenological approach. As a typical non-isotropic medium consider the photon propagation in a static gravitational field whose metric $g^{ij}(x)$ is supposed to be time independent ($g^{0i} = 0$) and whose Hamiltonian is $H = c \left(-\frac{p_i g^{ij}(x) p_j}{g^{00}(x)} \right)^{1/2}$. In the semi-classical approximation the equation of motion are

$$\frac{dp_k}{dt} = \frac{c^2 p_i p_j}{2H} \partial_k \left(\frac{g^{ij}(x)}{g^{00}(x)} \right) \quad (14)$$

and

$$\frac{dx^k}{dt} = \frac{c \sqrt{g_{00}} g^{ki} p_i}{\sqrt{-g^{ij} p_i p_j}} + \frac{dp_l}{dt} \theta^{kl} \quad (15)$$

For a static gravitational field the velocity is then

$$v^i = \frac{c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0} = c \frac{g^{ij} p_j}{\sqrt{-g^{ij} p_i p_j}} + \frac{1}{\sqrt{g_{00}}} \frac{dp_l}{dt} \theta^{kl}$$

with $x^0 = ct$. Equation (14) and (15) are our new equations for the semiclassical propagation of light (or optical geometric equation) which take into account the noncommutative nature of the position operator, i.e the spin-orbit coupling of the photon. Then the spinning nature of photon introduces a quantum Berry phase, which affects at the semi-classical level the propagation of light in a static background gravitational field. This new fundamental prediction will be studied further in future paper, but we already point out that the Berry phase implies a velocity of light which is now different from the universal value c .

In summary, we looked at the adiabatic evolution of the Dirac equation in order to clarify the relation between monopole and Berry curvature in momentum space. It was found that the position operator acquires naturally an anomalous contribution due to a non Abelian Berry gauge connection rendering noncommutative the quantum mechanic algebra. Using the Bargmann-Wigner equation of motions we generalized our formalism to all known spinning particles. The noncommutativity of the coordinate is responsible of the topological spin transport of spinning particle similarly to the spin Hall effect in spintronic physics or the optical Magnus effect in optics. In particular we predict two new effects. One is an unusual spin-orbit contribution of a non-relativistic particle which could be verified experimentally. The other one, concerns the effect of the Berry phase on the propagation of light in a static background gravitational field.

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- [1] H. Snyder, Phys. Rev. **71** (1947) 38; A. Connes, "Non-commutative Geometry", Academic press, San Diego (1994); N. Seiberg, E. Witten, JHEP. **09** (1999) 32; A.A Deriglazov, JHEP **03**, 03 (2003) 021; Phys. Lett. B **555** (2003) 83; M. Sheikh-Jabbari, Nucl. Phys. B **611** (2001) 383; M. Chaichan, M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett., **86** (2001) 2716; S. Ghosh, Phys. Rev. D **66** (2002) 045031; A. Konechny, A. Schwarz, Phys. Rep. **360** (2002) 353.
- [2] J. Belissard, Lecture Notes in Physics **257** (1986) 99; J. Belissard, A. Van Elst and H. Schulz-Baldes, cond-mat/9301005; J. Gamboa et al., Mod. Phys. Lett. A **16** (2001) 2075; C. Duval, P. A. Horvathy, J. Phys. A **34** (2001) 10097.
- [3] A. Bérard, H. Mohrbach, Phys. Rev. D **69** (2004) 045410.
- [4] A. Das, J. Frenkel, Phys. Rev. D **69** (2004) 065017.
- [5] M. Konsevich, Q-alg/970904.
- [6] J.E. Moyal, Proc. Cam. Phil. Soc. **45** (1949) 99.
- [7] R. Jackiw, Phys. Rev. Lett. **54** (1985) 159-162 and hep-th/0212058.
- [8] M. Onoda, N. Nagasoa, J. Phys. Soc. Jpn. **71** (1) (2002) 19.
- [9] Z. Fang et al., Science **302** (2003) 92-95.
- [10] S. Murakami, N. Nagaosa, S.C. Zhang, Science **301** (2003) 1348.
- [11] B.S. Skagerstam hep-th/9210054.
- [12] H. Bacry, "Localizability and Space in Quantum Physics", Lecture Notes in Physics, Vol **308**, Heidelberg, Springer-Verlag, (1988).
- [13] V. Bargmann, E.P. Wigner, Proc. Nat. Acad. Sci. **34** (1948) 211.
- [14] S. Ghosh, Phys. Lett. B **571** (2003) 97.
- [15] M.S. Plyushchay, Mod. Phys. Lett. A **4** (1989) 837; Phys. Lett. B **243** (1990) 383; J.L. Cortes, M.S. Plyushchay, Int. J. Mod. Phys. A **11** (1996) 3331; P.A. Horvathy, M.S. Plyushchay, JHEP **0206** (2002) 033; hep-th/0404137.
- [16] M.V. Berry, Proc. R. Soc. London A **392** (1984) 45.
- [17] E.I. Rashba, cond-mat/0309441.
- [18] J.E. Hirsch, Phys. Rev. Lett. **83** (1999) 1834.
- [19] H. Mathur, Phys. Rev. Lett. **67** (1991) 3325.
- [20] A.V. Dooghin et al., Phys. Rev. A **45** 11 (1992) 8204; V.S. Liberman, B.Y. Zeldovich, Phys. Rev. A **46** 8 (1992) 5199.
- [21] R.Y. Chiao, Y. Wu, Phys. Rev. Lett. **57** 8 (1986) 933.
- [22] K.Y. Bliokh, Y.P. Bliokh, physics/0402110.