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Contributions of order $\mathcal{O}(m^2_{\rm quark})$ to $K_{\ell 3}$ form factors and unitarity of the CKM matrix *

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Abstract

The form factors for the $K_{\ell 3}$ semileptonic decay are computed to order $O(p^4)$ in generalized chiral perturbation theory. The main difference with the standard $O(p^4)$ expressions consists in contributions quadratic in quark masses, which are described by a single divergence-free low-energy constant, A_3. A new simultaneous analysis is presented for the CKM matrix element V_{us}, the ratio F_K/F_π, $K_{\ell 3}$ decay rates and the scalar form factor slope λ_0. This framework easily accommodates the precise value for V_{ud} deduced from superallowed nuclear β-decays.

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1 Introduction

Together with low-energy π–π scattering [1, 2], the semi-leptonic $K_{\ell 3}$ [3] and K_{e4} [4, 5] decays have been among the first applications of standard chiral perturbation theory (SχPT) [6, 2, 7] to be studied at the one-loop level. At this order, the mesonic form factors that describe these decays do not contain the poorly known low energy constants L_8 and L_6, and consequently they may be expected to be less sensitive to the size of the chiral condensate $\langle \bar{q}q \rangle$ than, e.g., the $\pi - \pi$ s-wave [8, 9]. In particular, the $K_{\ell 3}$ form factors at $\mathcal{O}(\mathrm{p}^4)$ merely involve, besides the masses and decay constants of the pseudoscalar states, the low energy constant L_9, whose value can be obtained [3] from the experimentally measured charge radius of the pion [10]. This fortunate circumstance has been used in the past to extract the CKM matrix element V_{us} from the K_{e3} decay rates [11], and presently this extraction is still considered by the Particle Data Group (PDG) compilation [12] to remain the most accurate and least model dependent. Yet this determination of V_{us} relies on a model-dependent estimate of $\mathcal{O}(m^2_{\mathrm{quark}})$ contributions to the form factor $f_+(0)$ (the notation will be given below), which, in SχPT, arise as contributions of order $\mathcal{O}(\mathrm{p}^6)$. With the new careful and accurate determinations of V_{ud} from superallowed nuclear β-decays [13, 14, 15], the size of these corrections is required to be comparable to the (parameter-free) genuine $\mathcal{O}(\mathrm{p}^4)$ contribution in order to preserve the unitarity of the CKM matrix. It is then legitimate to ask whether similarly important $\mathcal{O}(m^2_{\mathrm{quark}})$ contributions would not affect the parameter-free SχPT prediction for the slope λ_0 of the $K_{\mu 3}$ scalar form factor at order $\mathcal{O}(\mathrm{p}^4)$. In principle, these questions can be answered by performing the full two-loop SχPT calculation of the $K_{\ell 3}$ form factors, provided the several new $\mathcal{O}(\mathrm{p}^6)$ counterterms that will contribute can be measured independently or estimated in a reliable way [1]. The present status of this enterprise is limited [2] to the evaluation of the so-called chiral double-logs [19], which, although only part of the full two-loop corrections, do not seem to point towards huge effects coming from the chiral loops themselves (see in particular the numerical results in Table 1 of Ref. [19] and the comments preceding it).

The purpose of the present work is to address some of these issues from the point of view of generalized chiral perturbation theory (GχPT) [8, 20, 21]. The expressions for the form factors of semileptonic decay of kaons to $\mathcal{O}(\mathrm{p}^4)$ in GχPT have never been published (a discussion of the $K_{\ell 4}$ form factors at order $\mathcal{O}(\mathrm{p}^2)$ in GχPT can be found in Ref. [22]), although

[1] The structure of the order $\mathcal{O}(\mathrm{p}^6)$ effective lagrangian of SχPT has been discussed in Refs. [16, 17].

[2] Unfortunately, the $\mathcal{O}(\mathrm{p}^6)$ calculation of Ref. [18] only considers a very specific combination of the *slopes* of the $f_+^{K\pi}$ and of the mesonic electromagnetic form factors.

they have existed for some time and have been partially reported on various occasions. Keeping in mind the well-known and important exception of the phase of the K_{e4}^+ decay amplitude, the $K_{\ell 3}$ and K_{e4} form factors are indeed found to be to a large extent independent of the size of the condensate $\langle \bar{q}q \rangle$. The main consequence of the modified chiral counting ($m_q \sim \mathcal{O}(\mathrm{p})$) is that in G$\chi$PT the form factors receive a contribution quadratic in quark masses already at order $\mathcal{O}(\mathrm{p}^4)$, in addition to the standard $\mathcal{O}(\mathrm{p}^4)$ expressions. Furthermore, these new $\mathcal{O}(m_{\mathrm{quark}}^2)$ contributions, which within SχPT would count as $\mathcal{O}(\mathrm{p}^6)$, **are all related**. They all stem from a single term of the $\mathcal{L}_{(2,2)}$ component of the effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$ and they are described by a single divergence-free low-energy constant A_3 (see Ref. [9] and Appendix A for notation). (This statement is exact in the case of $K_{\ell 3}$ form factors f_+ and f_-, whereas in the case of K_{e4} it remains true for the dominant $\mathcal{O}(m_s^2)$ terms.) It thus appears that the $\mathcal{O}(\mathrm{p}^4)$ GχPT offers a predictive description of the terms quadratic in quark masses which is non-trivial compared to the one-loop SχPT (no quadratic terms) and yet much simpler than the standard $\mathcal{O}(\mathrm{p}^6)$ order, in which other unknown constants should contribute in addition to the $\mathcal{O}(\mathrm{p}^4)$ GχPT terms driven by the constant A_3. This framework, which is as systematic as the standard expansion, suggests a new simultaneous analysis of V_{us}, $K_{\ell 3}$ decay rates, F_K/F_π, and of the scalar slope λ_0 which may be of interest in connection with the constraint of unitarity of the CKM matrix and with the forthcoming new $K_{\ell 3}$ data. A closely related application, namely the determination of the $\mathcal{O}(\mathrm{p}^4)$ constant L_3 from the K_{e4}^0 decay rate, will be presented elsewhere [23], together with a full discussion of the $K_{\ell 4}$ form factors at order $\mathcal{O}(\mathrm{p}^4)$ in GχPT.

The present stage of our analysis involves one additional limitation, to the extent that no electromagnetic corrections are included, unless explicitly stated. For this reason we postpone a detailed analysis of the isospin asymmetry in the K_{e3}^0 and K_{e3}^+ decay rates that is due to the mass difference $m_d - m_u$. We just check that this asymmetry is consistent with the GχPT treatment of $\pi^0 - \eta$ mixing within errors.

The paper is organized as follows: Section 2 provides the necessary expressions of the $K_{\ell 3}$ form factors and of the $\pi^0 - \eta$ mixing angles in GχPT. Implications for the determinations of V_{us} and of the ratio F_K/F_π of pseudoscalar decay constants are discussed in Section 3. The slopes of the $K_{\ell 3}$ form factors are considered in Section 4. Concluding remarks are presented in Section 5. Details on the $\mathcal{O}(\mathrm{p}^4)$ structure of the GχPT effective Lagrangian and on its renormalization are presented in Appendix A. Useful expressions for the pseudoscalar masses and decay constants have been gathered in Appendix B.

2 $K_{\ell 3}$ form factors in GχPT to $\mathcal{O}(\mathrm{p}^4)$

We consider the two semileptonic decay channels

$$\begin{aligned} K^+(p) &\to \pi^0(p')\ell^+(p_\ell)\nu_\ell(p_\nu) \qquad [K^+_{\ell 3}] \\ K^0(p) &\to \pi^-(p')\ell^+(p_\ell)\nu_\ell(p_\nu) \qquad [K^0_{\ell 3}]. \end{aligned} \tag{2.1}$$

The symbol ℓ stands for μ or e. As stated before, we do not consider electromagnetic corrections. The processes (2.1) are then described by four form factors, $f_\pm^{K^+\pi^0}(t)$ and $f_\pm^{K^0\pi^-}(t)$, which depend on $t = (p'-p)^2 = (p_l + p_\nu)^2$, the square of the four momentum transfer to the leptons, and which are defined in terms of the hadronic matrix elements of the charged strangeness changing QCD vector current as follows:

$$\begin{aligned} < \pi^0(p')|(\overline{s}\gamma_\mu u)(0)|K^+(p) > &= \frac{1}{\sqrt{2}}[(p+p')_\mu f_+^{K^+\pi^0} + (p-p')_\mu f_-^{K^+\pi^0}] \\ < \pi^-(p')|(\overline{s}\gamma_\mu u)(0)|K^0(p) > &= (p+p')_\mu f_+^{K^0\pi^-} + (p-p')_\mu f_-^{K^0\pi^-}. \end{aligned} \tag{2.2}$$

2.1 GχPT expressions of the $K_{\ell 3}$ form factors

The one-loop GχPT expressions (using the notation of reference [3]) for the form factors are summarized below. We start with the two form factors $f_+^{K^0\pi^-}(t)$ and $f_+^{K^+\pi^0}(t)$, which are in practice sufficient for the description of the electron decay modes K^+_{e3} and K^0_{e3}, and keeping isospin breaking contributions due to the quark mass difference $m_d - m_u$. For the $K^0_{\ell 3}$ channel, which is somewhat simpler, since $\pi^0 - \eta$ mixing only enters the loop contributions, we obtain

$$\begin{aligned} f_+^{K^0\pi^-}(t) &= 1 \;+\; H_{\pi^+K^0}(t) \;+\; \frac{1}{2}\,H_{\pi^0K^+}(t) \;+\; \frac{3}{2}\,H_{\eta K^+}(t) \\ &+ \sqrt{3}\;\overset{o}{\varepsilon}\;\Big[H_{\pi K}(t) \;-\; H_{\eta K}(t)\Big] \\ &+ \Big[\frac{1}{8}\widehat{m}^2\,\xi^2 \;+\; \frac{1}{2}\widehat{m}^2 A_3\Big]\,(r-1)^2\,(1+\frac{1}{R})\;, \end{aligned} \tag{2.3}$$

with (the definitions of the loop functions $M^{\mathrm{r}}_{PQ}(t)$, $K_{PQ}(t)$ and $L_{PQ}(t)$ that we use below can be found in Ref. [7])

$$H_{PQ}(t) \;=\; \frac{1}{F_\pi^2}\Big[t\,M^{\mathrm{r}}_{PQ}(t) \;-\; L_{PQ}(t) \;+\; \frac{2}{3}\,L_9^{\mathrm{r}}\,t\Big]\;. \tag{2.4}$$

Here $\overset{o}{\varepsilon}$ denotes the leading order $\pi^0 - \eta$ mixing angle,

$$\overset{o}{\varepsilon} = \frac{\sqrt{3}}{4R}\Big[1 - \frac{\Delta_{GMO}}{M_\eta^2 - M_\pi^2} + \frac{2}{3}\,(r_2 - r)\,\frac{r-1}{r+1}\,\frac{M_\pi^2}{M_\eta^2 - M_\pi^2}\Big]\;, \tag{2.5}$$

where

$$
\begin{aligned}
r &\equiv m_s/\widehat{m} \\
r_2 &\equiv 2\frac{M_K^2}{M_\pi^2} - 1 \\
\Delta_{GMO} &\equiv 3M_\eta^2 - 4M_K^2 + M_\pi^2 \ .
\end{aligned}
\tag{2.6}
$$

We work only at first order in the quark mass difference $(m_d - m_u)$, *i.e.* we consider only terms that are at most of order $\mathcal{O}(1/R)$ [3], where

$$
R = \frac{m_s - \widehat{m}}{m_d - m_u}. \tag{2.7}
$$

The lowest order SχPT value for the $\pi^0 - \eta$ mixing angle, $\overset{o}{\varepsilon}_{\rm st} = \frac{\sqrt{3}}{4R}$, is recovered by dropping, in Eq. (2.5), the last two terms, which are counted as order $\mathcal{O}(\mathrm{p}^4)$ in SχPT. The ξ^2 and A_3 terms in Eq. (2.3) are $\mathcal{O}(\mathrm{p}^4)$ contributions in GχPT (see the detailed formulas for the effective Lagrangian in Appendix A), but are absent at this order in SχPT.

For the $K_{\ell 3}^+$ decay mode, we find

$$
\begin{aligned}
f_+^{K^+\pi^0}(t) &= f_+^{K^0\pi^-}(t) \times \Bigg\{ 1 \ + \ \frac{\sqrt{3}}{2}(\varepsilon_1 \ + \ \varepsilon_2) \\
&\quad - \ \frac{1}{\sqrt{3}}\widehat{m}^2\xi^2(r-1)^2 \ \overset{o}{\varepsilon} \ - \widehat{m}^2 C_2^P (r-1)^2 \left[\frac{1}{R} \ + \ \frac{2\,\overset{o}{\varepsilon}}{\sqrt{3}}\right] \Bigg\} \ ,
\end{aligned}
\tag{2.8}
$$

where the $\pi^0 - \eta$ mixing angles at order $\mathcal{O}(\mathrm{p}^4)$, ε_1, ε_2, are defined in section 2.2 below. Although the constant C_2^P corresponds to a counterterm of $\mathcal{L}_{(2,2)}$ (see Eq. (A.10) in Appendix A) that violates the Zweig rule, it is not expected to be suppressed, since this violation occurs in the 0^- channel.

At zero momentum transfer, Eq. (2.3) gives

$$
\begin{aligned}
f_+^{K^0\pi^-}(0) &= 1 \ - \ \frac{1}{256\pi^2 F_\pi^2} \Big\{ 2\,(M_{\pi^+}^2 + M_{K^0}^2)\, h_0(\frac{M_{\pi^+}^2}{M_{K^0}^2}) \\
&\quad + \ (M_{\pi^0}^2 + M_{K^+}^2)\, h_0(\frac{M_{\pi^0}^2}{M_{K^+}^2}) + 3\,(M_{K^+}^2 + M_\eta^2)\, h_0(\frac{M_{K^+}^2}{M_\eta^2}) \\
&\quad + \ 2\sqrt{3}\,\overset{o}{\varepsilon} \ \Big[(M_\pi^2 + M_K^2)\, h_0(\frac{M_\pi^2}{M_K^2}) \ - \ (M_K^2 + M_\eta^2)\, h_0(\frac{M_K^2}{M_\eta^2})\Big] \Big\} \\
&\quad + \ \left[\frac{1}{8}\widehat{m}^2\xi^2 + \frac{1}{2}\widehat{m}^2 A_3\right](r-1)^2 \ \left(1 + \frac{1}{R}\right) \ ,
\end{aligned}
\tag{2.9}
$$

[3] We have however kept the pion mass difference, which is mainly an electromagnetic effect [24, 25], in the loop contributions.

with

$$h_0(x) \;=\; 1 \;+\; \frac{2x}{1-x^2}\,\ln x \;. \tag{2.10}$$

To recover the expression at order $\mathcal{O}(\mathrm{p}^4)$ in SχPT [11, 3] for $f_+^{K^0\pi^-}(0)$, one replaces $\overset{o}{\varepsilon}$ by the leading order SχPT value $\overset{o}{\varepsilon}_{\rm st}$, and one simply drops the last line in Eq. (2.9). Notice that this last contribution is the only correction of order $\mathcal{O}(\mathrm{p}^4)$ that does not vanish in the large-N_c limit of QCD. The fact that the $\mathcal{O}(\mathrm{p}^4)$ corrections to $f_+^{K^0\pi^-}(0)$ in SχPT vanish altogether in the large-N_c limit might provide a natural explanation why contributions of the $\mathcal{O}(\mathrm{p}^6)$ counterterms could be comparatively sizeable.

The expressions of the form factors $f_-^{K^0\pi^-}(t)$ and $f_-^{K^+\pi^0}(t)$ are rather cumbersome if $\mathcal{O}(m_d - m_u)$ effects are included. Since we do not need the latter in this case, we give only the common expression of these two form factors in the isospin limit, which is

$$\begin{aligned}
f_-^{K\pi}(t) \;=\;& \frac{1}{2}\widehat{m}\xi(r-1) \;-\; \frac{1}{4}\widehat{m}^2\xi^2(r-1)(r+3) \;-\; \widehat{m}^2\xi\tilde{\xi}(r-1)(r+2) \\
&+\; \frac{1}{2}\widehat{m}^2(A_1+B_1)(r^2-1) \;+\; \frac{1}{2}\widehat{m}^2(A_2-2B_2)(r-1) \\
&+\; \widehat{m}^2 D^S(r-1)(r+2) \;+\; \frac{1}{4}[5\mu_\pi - 2\mu_K - 3\mu_\eta] \\
&-\; \frac{1}{4F_0^2}\left[5\,t - 2\,M_\pi^2 - 2\,M_K^2 + 16\widehat{m}^2(A_0 + 2\,Z_0^S)(r+1)\right] K_{\pi K}(t) \\
&-\; \frac{1}{4F_0^2}\left[3t - 2M_\pi^2 - 2M_K^2 + 4\widehat{m}^2(A_0 + 2Z_0^P)(r-1)(r+3)\right] K_{K\eta}(t) \\
&-\; \frac{3}{2F_0^2}\,(M_K^2 - M_\pi^2)\left\{\left[M_{K\pi}^{\rm r}(t) \;+\; \frac{2}{3}\,L_9^{\rm r}\right] \;+\; \left[M_{K\eta}^{\rm r}(t) \;+\; \frac{2}{3}\,L_9^{\rm r}\right]\right\} \;.
\end{aligned} \tag{2.11}$$

For the ease of comparison, we may rewrite this GχPT expression for the form factor $f_-(t)$ in terms of the corresponding SχPT expression:

$$\begin{aligned}
f_-^{K\pi;std}(t) \;=\;& \frac{F_K}{F_\pi} - 1 - \frac{2\,L_9^r\;(M_K^2 - M_\pi^2)}{F_\pi^2} \\
&+ \frac{(2\,M_K^2 + 2\,M_\pi^2 - 3\,t)}{4\,F_\pi^2}\,K_{K\eta}(t) \\
&+ \frac{(2\,M_K^2 + 2\,M_\pi^2 - 5\,t)}{4\,F_\pi^2}\,K_{\pi K}(t) \\
&- \frac{3\;(M_K^2 - M_\pi^2)}{2\,F_\pi^2}\;\left[M^r{}_{K\eta}(t) + M^r{}_{K\pi}(t)\right] \;,
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
f_-^{K\pi}(t) \;=\;& f_-^{K\pi;std}(t) \\
&- \frac{1}{4}\left(\frac{F_K}{F_\pi} - 1\right)^2 \left(\frac{F_K^2}{{F_\pi}^2} + 2\frac{F_K}{F_\pi} - 1\right)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}\, m^2\, A_3\, (r-1)(r+3) \\
& -\frac{\Delta_{GMO}\ (r+3)}{4\, F_\pi^2\ (r-1)}\, K_{K\eta}(t) - \frac{M_\pi^2\ (1+r)\ \hat{\epsilon}(r)}{F_\pi^2}\, K_{\pi K}(t)\ , \qquad (2.13)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\epsilon}(r) &\equiv \frac{4\,\widehat{m}^2\,(A_0 + 2\, Z_0^S)}{M_\pi^2} = 2\frac{r_2 - r}{r^2-1}(1+2\zeta) \\
\zeta &= Z_0^S / A_0\ , \qquad (2.14)
\end{aligned}
$$

and the $\mathcal{O}(\mathrm{p}^4)$ expression

$$
\begin{aligned}
\hat{m}\xi &= \frac{1}{r-1}\left(\frac{F_K^2}{F_\pi^2} - 1\right) \\
& -(r+1)\,\hat{m}^2\,(A_1 + B_1) - \hat{m}^2\,(A_2 - 2B_2) - (r+3)\,\hat{m}^2\, A_3 \\
& -2\,(r+2)\,\hat{m}^2 D^S \qquad (2.15) \\
& +2\,\hat{m}^2\,\xi^2 + 2\,(r+2)\,\hat{m}^2 \xi\tilde{\xi} \\
& -\frac{1}{2(r-1)}\,(5\mu_\pi - 2\mu_K - 3\mu_\eta)
\end{aligned}
$$

has been used. It is remarkable that, in Eq. (2.13), *all* $\mathcal{L}_{(2,2)}$ constants – *except* for A_3 – have been absorbed into the renormalization of the decay constants. Notice also that, in contrast to $f_+(t)$, $f_-(t)$ starts only at next-to-leading order in the chiral expansion, and that furthermore this contribution is not suppressed at $t = 0$ for $N_c \to \infty$.

2.2 Mixing angles

The expressions of the $K_{\ell 3}$ form factors given in the preceding subsection involve the $\pi^0 - \eta$ mixing angles $\overset{o}{\varepsilon}, \varepsilon_1, \varepsilon_2$. In defining them we ignore isospin breaking through electromagnetic effects, so that the only source of isospin violation is the quark mass difference $m_d - m_u$. If $m_d \neq m_u$, the isosinglet and isovector axial currents A_μ^8 and A_μ^3 have nonvanishing off-diagonal matrix elements between the vacuum and one-meson states. The *two* $\pi^0\eta$ mixing angles ε_1 and ε_2 are introduced such as to define combinations of the axial currents having vanishing off-diagonal matrix elements:

$$
\begin{aligned}
\langle\Omega|\ \cos\varepsilon_1\, A_\mu^8(0) - \sin\varepsilon_1\, A_\mu^3(0)\, |\pi^0(p)\rangle &= 0 \\
\langle\Omega|\ \cos\varepsilon_2\, A_\mu^3(0) + \sin\varepsilon_2\, A_\mu^8(0)\, |\eta(p)\rangle &= 0. \qquad (2.16)
\end{aligned}
$$

Both ε_1 and ε_2 are of order $\mathcal{O}(m_d - m_u)$. Note that we were not forced to define mixing angles in terms of matrix elements of the axial currents: we could have chosen to use the pseudoscalar densities, or any other operators with the appropriate quantum numbers; however, the corresponding expressions for the mixing angles would in general differ from those given below.

Keeping only contributions which are at most linear in the quark mass difference $m_d - m_u$, the off-diagonal matrix elements of the flavor neutral axial currents read

$$\begin{aligned} \langle\Omega|A^8_\mu(0)|\pi^0(p)\rangle &= ip_\mu\, F_\pi\, \varepsilon_1 \\ \langle\Omega|A^3_\mu(0)|\eta(p)\rangle &= -ip_\mu\, F_\eta\, \varepsilon_2. \end{aligned} \tag{2.17}$$

The decay constants F_π and F_η define the diagonal matrix elements of the same currents,

$$\begin{aligned} \langle\Omega|A^3_\mu(0)|\pi^0(p)\rangle &= ip_\mu\, F_\pi \\ \langle\Omega|A^8_\mu(0)|\eta(p)\rangle &= ip_\mu\, F_\eta\ . \end{aligned} \tag{2.18}$$

Up to corrections of order $\mathcal{O}((m_u - m_d)^2)$ that we neglect, F_π can be identified with the charged pion decay constant, $F_\pi = 92.4$ MeV.

At order $\mathcal{O}(\mathrm{p}^2)$, the two mixing angles coincide and are equal to $\overset{o}{\varepsilon}$. At order $\mathcal{O}(\mathrm{p}^3)$, they both receive $\mathcal{O}(m_{\mathrm{quark}})$ corrections, and are no longer equal. Explicitly,

$$\begin{aligned} \frac{\varepsilon_1 \ + \ \varepsilon_2}{2} &= \overset{o}{\varepsilon} + \frac{1}{\sqrt{3}R(M_\eta^2 - M_\pi^2)} \Big\{ \widehat{m}^3 X_\rho(r) \\ &\quad - \Big(\frac{F_K^2}{F_\pi^2} - 1\Big) \Big[\frac{\Delta_{GMO}}{2} + \frac{1}{3}(r_2 - r)\Big(\frac{r+2}{r+1}\Big) M_\pi^2 \Big] \Big\}\ , \end{aligned} \tag{2.19}$$

and

$$\varepsilon_1 - \varepsilon_2 = \frac{1}{\sqrt{3}R(M_\eta^2 - M_\pi^2)} \left(\frac{F_K^2}{F_\pi^2} - 1\right) \left[\Delta_{GMO} - \frac{2}{3}(r_2 - r)\left(\frac{r-2}{r+1}\right) M_\pi^2\right], \tag{2.20}$$

where we have defined

$$X_\rho(r) = (r-1)^2 \left[(3\rho_1 + 4\rho_3)\,(r-1) - \rho_2\,(r+1)\right] \tag{2.21}$$

in terms of the $\mathcal{L}_{(0,3)}$ quantities $\rho_{1,2,3}$.

3 Decay rates, V_{us} and F_K/F_π revisited

3.1 The $K^0 \to \pi^- e^+ \nu$ rate

For the K_{e3} decay we only need to consider the form factor $f_+(t)$, and for $K^0 \to \pi^- e^+ \nu$ Eq. (2.3) is all we need. Ignoring the tiny term in Eq. (2.3) proportional to $\overset{o}{\varepsilon}$, we may write the decay rate in units of 10^{-15} MeV as

$$\begin{aligned}\Gamma[K^0 \to \pi^- e^+ \nu] \;=\; & V_{us}^2 \Big[105.056 \\ & + 203.031\,(r-1)^2 \left(\frac{\widehat{m}^2 A_3}{2} + \frac{\widehat{m}^2 \xi^2}{8} \right) \\ & + 98.267\,(r-1)^4 \left(\frac{\widehat{m}^2 A_3}{2} + \frac{\widehat{m}^2 \xi^2}{8} \right)^2 \Big].\end{aligned} \tag{3.1}$$

Strictly speaking, the last contribution on the right-hand side of this expression, although coming from the square of the $\mathcal{O}(\mathrm{p}^4)$ expression of $f_+^{K^0\pi^-}(t)$, represents an order $\mathcal{O}(\mathrm{p}^6)$ effect in the chiral expansion of the decay rate. However, as can be checked *e.g.* on Figs. 1 and 2 below, it does not affect our analysis in the range of values considered for the Cabibbo angle.

We express $\widehat{m}\xi$ in terms of V_{us} as follows: first, we use Eq. (2.15), but only at lowest order,

$$\widehat{m}\xi = \frac{1}{r-1}\left(\frac{F_K^2}{F_\pi^2} - 1 \right) , \tag{3.2}$$

and then use the formula for the ratio of the branching rates

$$\frac{\Gamma[K \to \mu\nu]}{\Gamma[\pi \to \mu\nu]} = \frac{|V_{us}|^2}{|V_{ud}|^2} \frac{F_K^2}{F_\pi^2} \frac{M_{K^+}}{M_{\pi^+}} \frac{\{1-(M_\mu/M_{K^+})^2\}^2}{\{1-(M_\mu/M_{\pi^+})^2\}^2} \cdot (1+\delta_K-\delta_\pi) , \tag{3.3}$$

where the radiative corrections $\delta_K = 0.0020 \pm 0.0002, \delta_\pi = 0.0017 \pm 0.0002$ nearly cancel [26].

Next, we use the experimental values [12] for the branching rates to fix [11] the combination of constants

$$\frac{F_K}{F_\pi}\frac{V_{us}}{V_{ud}} = 0.2758 \pm 0.0005, \tag{3.4}$$

together with the unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix

$$V_{ud}^2 + V_{us}^2 = 1. \tag{3.5}$$

We ignore $|V_{ub}|$, which has been obtained [27, 28] from a model-dependent analysis of data from B semileptonic decays to be $(3.25 \pm 0.6) \times 10^{-3}$; however, even if the central value of V_{ub}

turned out to be three times larger, this would not affect our results. After expanding Eq. (3.1) in V_{us} around $V_{us} = 0.220$, this implies

$$\begin{aligned}\Gamma[K^0 \to \pi^- e^+ \nu] &= 5.390 + 31.10\,(V_{us} - 0.220) \\ &\quad + (r-1)^2\,\widehat{m}^2 A_3\,[5.059 + 37.58\,(V_{us} - 0.220)] \\ &\quad + (r-1)^4\,\widehat{m}^4 A_3^2\,[1.189 + 10.81\,(V_{us} - 0.220)],\end{aligned} \tag{3.6}$$

to be compared with the experimental rate of 4.937 ± 0.052. Consequently, from the knowledge of V_{us} one may extract a value for the $\mathcal{L}_{(2,2)}$ quantity $\widehat{m}^2(r-1)^2 A_3 = (m_s - \widehat{m})^2 A_3$. For example, in Figure 1 we show the band in the $(V_{us}\,,\ A_3)$ plane indicated by experiment, together with lines of constant V_{us} corresponding to the Particle Data Group (PDG) [12] value $V_{us} = 0.2196 \pm 0.0023$. Accepting this constraint on V_{us} would imply $\widehat{m}^2(r-1)^2 A_3 = -0.089 \pm 0.024$. A naive dimensional analysis (NDA) [9] would give

$$|(r-1)^2\,\widehat{m}^2\,A_3|\Big|_{NDA} \approx \frac{m_s^2(\Lambda_H)}{\Lambda_H^2} \approx 0.04\ , \tag{3.7}$$

where we have taken $m_s(\Lambda_H) \approx 200$ MeV, and $\Lambda_H \approx 1$ GeV is a typical hadronic mass scale.

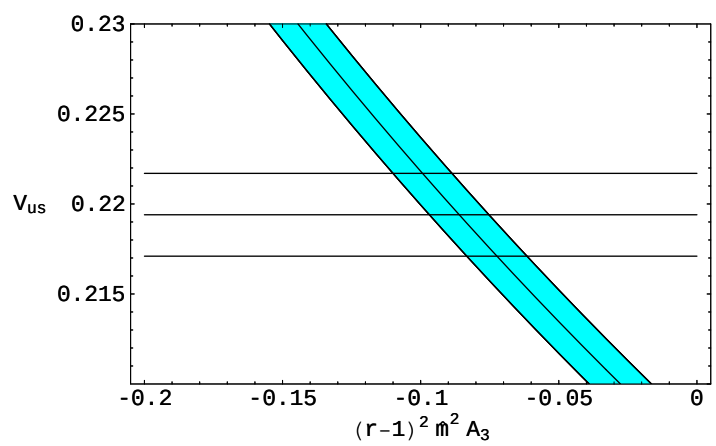

Figure 1: The dependence of V_{us} on A_3. Horizontal lines indicate the range of values for V_{us} given by PDG [12].

3.2 Determination of V_{ud} from nuclear beta decay

In its 1998 update, PDG [12] recommends for V_{us} only the value determined from K_{e3} decay, arguing that the value obtained from hyperon decays (see [29] for an early attempt along these

lines) suffers from theoretical uncertainties due to first-order SU(3) symmetry-breaking effects in the axial-vector couplings. However, within the context of the present GχPT analysis, we may not, without independent knowledge of A_3, use the values for V_{us} thereby extracted from K_{e3} decay. Moreover, F_K is determined from the $K_{\mu 2}$ decay together with the knowledge of V_{us}, and we note that F_K/F_π appears implicitly in our theoretical expressions for the $K_{\ell 3}$ form factors. The origin of this is, of course, our re-expression Eq. (3.2) of the $\mathcal{L}_{(0,3)}$ constant ξ in terms of F_K/F_π.

¿From the unitarity condition for the CKM matrix, it follows that $|V_{us}|$ may be fixed from the knowledge of the up-down quark-mixing matrix element of the CKM matrix, $|V_{ud}|$, alone. The value of V_{ud} can be determined from several independent sources: nuclear superallowed Fermi beta decays, free neutron decay, and pion beta decay.

Currently, superallowed Fermi $0^+ \to 0^+$ nuclear beta decays [13, 14, 15], together with the muon lifetime, provide the most accurate value,

$$V_{ud} = 0.9740 \pm 0.0005. \tag{3.8}$$

The precision is limited not by experimental error but by the estimated uncertainty in theoretical corrections [15]. In Figure 2 we show the band in the V_{ud} , A_3 plane indicated by experiment, together with lines of constant V_{ud} corresponding to these values.

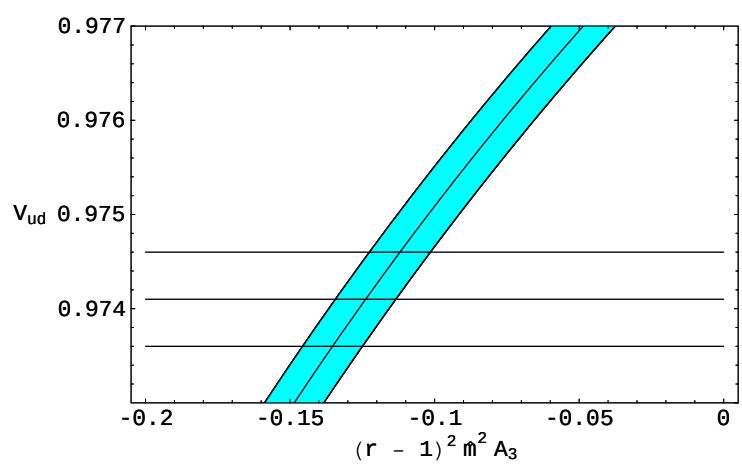

Figure 2: The dependence of V_{ud} on A_3. Horizontal lines indicate the range of values for V_{ud} determined from superallowed nuclear beta decay.

The determinations of V_{ud} from free neutron decay data are approximately a factor of four poorer in precision (see the discussion in [15] and references therein),

$$V_{ud} = 0.9759 \pm 0.0021, \tag{3.9}$$

due to the difficulty in separating the vector from the axial piece, but planned experiments aiming at an accurate measurement of the electron emission asymmetry [30] could change this situation qualitatively, since the error in this case is primarily of experimental origin.

Finally, the theoretical corrections in the nuclear Fermi transitions are absent in the case of the pion beta decay. The present status of this type of experiments results in the value [15]

$$V_{ud} = 0.9670 \pm 0.0161. \tag{3.10}$$

Here also, the situation might improve in the future [31, 30].

Combining the above results, one obtains

$$V_{ud} = 0.9741 \pm 0.0005. \tag{3.11}$$

Accepting this value for V_{ud} would imply $\widehat{m}^2(r-1)^2 A_3 = -0.124 \pm 0.022$, somewhat larger than the NDA estimate Eq. (3.7), but still acceptable.

Finally, we note that recent (model-dependent) analyses of hyperon semileptonic decays give [32]

$$V_{ud} = 0.9750 \pm 0.0004, \tag{3.12}$$

and [33]

$$V_{ud} = 0.9743 \pm 0.0009. \tag{3.13}$$

Taking the value Eq. (3.11) for V_{ud} (*i.e.*, excluding results from hyperon decays), the unitarity relation Eq. (3.5) gives

$$V_{us} = 0.2261 \pm 0.0023. \tag{3.14}$$

Incorporating the above values for the CKM matrix elements into Eq. (3.4) then implies

$$\frac{F_K}{F_\pi} = 1.189 \pm 0.012 \quad [V_{ud} \text{ from nuclear beta decay}], \tag{3.15}$$

which may be compared with the corresponding result using the PDG values for V_{us},

$$\frac{F_K}{F_\pi} = 1.226 \pm 0.014 \quad [V_{us} \text{ from PDG}]. \tag{3.16}$$

Using Eq. (3.4), which relates F_K/F_π to V_{us} and V_{ud}, Eq. (3.5) which relates V_{us} and V_{ud}, and Eq. (3.6) which relates V_{us} and $\widehat{m}^2(r-1)^2 A_3$, we may directly relate F_K/F_π and $\widehat{m}^2(r-1)^2 A_3$, see Figure 3.

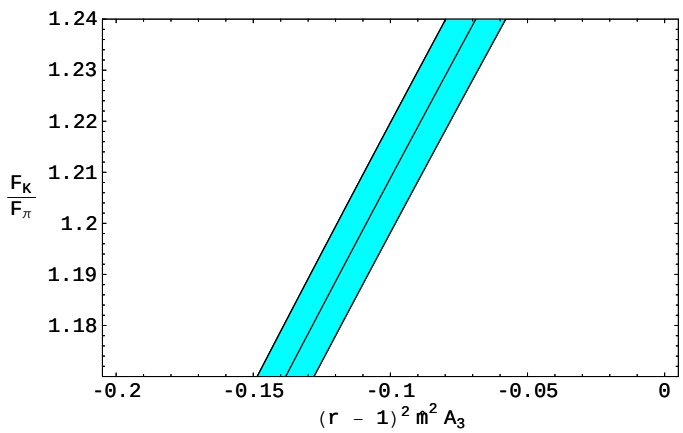

Figure 3: The dependence of F_K/F_π on A_3.

3.3 $m_u \neq m_d$ effects on the $K^+ \to \pi^0 e^+ \nu$ rate

We may treat the decay of the K^+ similarly; the result is most conveniently expressed in terms of the ratio

$$\begin{aligned}\frac{\Gamma[K^+ \to \pi^0 e^+ \nu]}{\Gamma[K^0 \to \pi^- e^+ \nu]} &= (R^{+0})^2 \left[0.49615 - 0.00240\,(r-1)^2 \left(\frac{\widehat{m}^2 A_3}{2} + \frac{\widehat{m}^2 \xi^2}{8}\right)\right. \\ &\quad \left. + 0.00234\,(r-1)^4 \left(\frac{\widehat{m}^2 A_3}{2} + \frac{\widehat{m}^2 \xi^2}{8}\right)^2\right],\end{aligned} \tag{3.17}$$

where R^{+0} is the ratio

$$R^{+0} \equiv \frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)}\,. \tag{3.18}$$

Note that the last two terms in the brackets in Eq. (3.17) arise from the small differences in phase space which are due to mass differences between K^0 and K^+, and between π^+ and π^0.

Proceeding as we did for the K^0 rate above, *i.e.*, using Eqs. (3.2) and (3.4), we obtain

$$\begin{aligned}\frac{\Gamma[K^+ \to \pi^0 e^+ \nu]}{\Gamma[K^0 \to \pi^- e^+ \nu]} &= (R^{+0})^2 \left\{0.4961 + 0.0040\,(V_{us} - 0.22)\right. \\ &\quad -(r-1)^2\,\widehat{m}^2\,A_3\,[0.0011 + 0.0041\,(V_{us} - 0.22)] \\ &\quad \left. +0.0005\,(r-1)^4\,\widehat{m}^4\,A_3^2\right\}\,.\end{aligned} \tag{3.19}$$

The correction terms in the curly brackets are completely negligible for A_3 and V_{us} in the range determined above, since they are at most of the order of 0.0001, compared to the leading term 0.4961.

The measured rates may be deduced from the data given by the Particle Data Group [12]:

$$\begin{aligned}\Gamma[K^+ \to \pi^0 e^+ \nu]_{exp} &= 2.561 \pm 0.032 \\ \Gamma[K^0 \to \pi^- e^+ \nu]_{exp} &= 4.937 \pm 0.052,\end{aligned} \tag{3.20}$$

where, as above, we express all rates in units of 10^{-15} MeV; consequently,

$$\frac{\Gamma[K^+ \to \pi^0 e^+ \nu]_{exp}}{\Gamma[K^0 \to \pi^- e^+ \nu]_{exp}} = 0.519 \pm 0.016. \tag{3.21}$$

Consequently, the experimental value for R^{+0} is 1.023 ± 0.016.

Up to and including terms of order $\mathcal{O}(\mathrm{p}^4)$, the theoretical expression for R^{+0} is given by Eq. (2.8); the terms proportional to $\widehat{m}^2\xi^2(r-1)^2$ and to $\widehat{m}^2 C_2^P$ are of order $\mathcal{O}(\mathrm{p}^4)$. We first estimate the $\mathcal{O}(\mathrm{p}^3)$ correction terms in Eq. (2.19) by NDA. For the first one, we obtain

$$\left|\frac{\widehat{m}^3 X_\rho(r)}{M_\pi^2}\right|_{NDA} \approx \left|\frac{8\, r^3 \widehat{m}^3 \rho}{M_\pi^2}\right|_{NDA} \approx \frac{8\, m_s^3(\Lambda_H)}{M_\pi^2 \Lambda_H} \approx 3.2, \tag{3.22}$$

where ρ characterizes the size of the $\mathcal{L}_{(0,3)}$ low-energy constants ρ_α, $\alpha = 1, 2, 3$. This implies (using $R = 43.5$, from reference [24])

$$\frac{\widehat{m}^3 X_\rho(r)}{R(M_\eta^2 - M_\pi^2)} \approx \frac{0.22}{R} \approx 0.005. \tag{3.23}$$

Similarly, the last term in Eq. (2.19) is

$$\begin{aligned}\frac{1}{\sqrt{3}R(M_\eta^2 - M_\pi^2)}\left(\frac{F_K^2}{F_\pi^2} - 1\right)\left[\frac{\Delta_{GMO}}{2} + \frac{1}{3}(r_2 - r)\left(\frac{r+2}{r+1}\right)M_\pi^2\right] \\ \approx \frac{-0.033}{R} + \frac{0.006(r_2 - r)}{R}\left(\frac{r+2}{r+1}\right) \\ \approx -0.00075 + 0.00014\,(r_2 - r)\left(\frac{r+2}{r+1}\right),\end{aligned} \tag{3.24}$$

which is quite negligible.

Finally, we turn to the $\mathcal{O}(\mathrm{p}^4)$ corrections to R^{+0}. From

$$\widehat{m}^2\xi^2(r-1)^2 \approx \left(\frac{F_K^2}{F_\pi^2} - 1\right)^2 \approx 0.24, \tag{3.25}$$

we obtain

$$\frac{\widehat{m}^2\xi^2(r-1)^2\, \overset{o}{\varepsilon}}{\sqrt{3}} \approx 0.14\, \overset{o}{\varepsilon}\ . \tag{3.26}$$

The NDA estimate for the C_2^P term gives

$$\widehat{m}^2 \left|C_2^P\right| (r-1)^2 \left[\frac{1}{R} \; + \; \frac{2\,\overset{o}{\varepsilon}}{\sqrt{3}}\right] \approx 0.04 \cdot \left[\frac{1}{R} \; + \; \frac{2\,\overset{o}{\varepsilon}}{\sqrt{3}}\right], \tag{3.27}$$

where we note that no assumption has been made here of any suppression of C_2^P due to Zweig rule violation. We have verified that $\mathcal{O}(\mathrm{p}^4)$ corrections to $\varepsilon_1 \; + \; \varepsilon_2$ are negligibly small. Using the numerical form of Eq. (2.5),

$$\overset{o}{\varepsilon} = \frac{1}{R}\left[0.533 + 0.502\,\frac{r-1}{r+1}\left(1-\frac{r}{r_2}\right)\right], \tag{3.28}$$

the upshot is that R^{+0} may be written (recall Eq. (2.8))

$$R^{+0} \;\; = \;\; 1 + \sqrt{3}\;\overset{o}{\varepsilon} - 0.14\;\overset{o}{\varepsilon} \pm \frac{0.22}{R}\;, \tag{3.29}$$

where the term $-0.14\;\overset{o}{\varepsilon}$ comes from Eq. (3.26); the indicated uncertainty, coming from other higher-order corrections, is dominated by the estimation of the contributions from the $\mathcal{L}_{(0,3)}$ terms (see Eq. (3.23) above). Numerically, this implies

$$R^{+0} \; = \; 1 + \frac{1}{R}\left[0.85 + 0.80\,\frac{r-1}{r+1}\left(1-\frac{r}{r_2}\right)\right] \pm \frac{0.22}{R}, \tag{3.30}$$

and taking, for example, the commonly accepted value [24] $R = 43.5 \pm 2.2$,

$$R^{+0} = 1.020 \pm 0.005 \; + \; 0.018\,\frac{r-1}{r+1}\left(1-\frac{r}{r_2}\right), \tag{3.31}$$

which is consistent, for any permissible value of r, with the experimental value 1.023 ± 0.016 which we deduced above. In view of this experimental uncertainty, we will leave for a later time the careful investigation of the $\mathcal{O}(\mathrm{p}^4)$ effects mentioned above, together with the analysis of electromagnetic corrections (the radiative corrections to R^{+0} in the standard case have been investigated in Ref. [34]). Note that some (but not all) of the existing experimental data has been published with radiative corrections, but often without mention of how these corrections have been implemented [35]. More precise knowledge of R^{+0} would be useful in constraining the relationship displayed in Eq. (3.29) between the two quark-mass ratios R and r, thereby testing the relevance of GχPT.

4 Form factor slopes

Analyses of $K_{\ell 3}$ data frequently assume a linear dependence

$$f_{+,0}^{K\pi}(t) = f_+^{K\pi}(0)\left[1 + \lambda_{+,0}\frac{t}{M_{\pi^+}^2}\right] , \tag{4.1}$$

where, as usual, the scalar form factor is defined as

$$f_0^{K\pi}(t) = f_+^{K\pi}(t) + \frac{t}{M_K^2 - M_\pi^2} f_-^{K\pi}(t) . \tag{4.2}$$

The parameter $\lambda_+^{K\pi}$ is identical to that of SχPT,

$$\begin{aligned} \frac{\lambda_+^{K\pi}}{M_{\pi^+}^2} &= \frac{1}{6}\langle\, \mathrm{r}^2 \,\rangle_{\mathrm{V}}^{\pi} \\ &+ \frac{1}{64\pi^2 F_0^2}\left[1 - \frac{1}{2}h_1(\frac{M_\pi^2}{M_K^2}) - \frac{1}{2}h_1(\frac{M_K^2}{M_\eta^2}) + \frac{5}{12}\ln(\frac{M_\pi^2}{M_K^2}) + \frac{1}{4}\ln(\frac{M_K^2}{M_\eta^2})\right] , \end{aligned} \tag{4.3}$$

with

$$h_1(x) + \frac{1}{2}\frac{(x^3 - 3x^2 - 3x + 1)}{(x-1)^3}\ln x + \frac{1}{2}\left(\frac{x+1}{x-1}\right)^2 - \frac{1}{3} , \tag{4.4}$$

and

$$F_\pi^2 \langle\, \mathrm{r}^2 \,\rangle_{\mathrm{V}}^{\pi} = 12\, L_9^r - \frac{1}{32\pi^2}\left\{2\ln\frac{M_\pi^2}{\mu^2} + \ln\frac{M_K^2}{\mu^2} + 3\right\} . \tag{4.5}$$

On the other hand, the GχPT expression for $\lambda_0^{K\pi}$,

$$\begin{aligned} \frac{\lambda_0^{K\pi}}{M_{\pi^+}^2} &= \frac{1}{2}\Big[\left(\frac{F_K^2}{F_\pi^2} - 1\right) - \frac{1}{2}\left(\frac{F_K^2}{F_\pi^2} - 1\right)^2 - \widehat{m}^2\,(r-1)^2\, A_3\,\frac{r+3}{r-1}\Big]\frac{1}{M_K^2 - M_\pi^2} \\ &- \frac{1}{384\pi^2 F_0^2}\left[5 - \frac{4M_\pi^2\,(r+1)\,\hat{\epsilon}(r)}{M_K^2 + M_\pi^2}\right] h_2(\frac{M_\pi^2}{M_K^2}) \\ &- \frac{1}{384\pi^2 F_0^2}\Big[3 + \\ &\quad 2\left(\frac{M_\eta^2 - M_K^2}{M_\eta^2 + M_K^2}\right)\left(\frac{M_K^2 + M_\pi^2}{M_K^2 - M_\pi^2}\right)\left(1 - \frac{\Delta_{GMO}\,(r+3)}{2\,(M_K^2 + M_\pi^2)\,(r-1)}\right)\Big] h_2(\frac{M_K^2}{M_\eta^2}) \\ &+ \frac{1}{64\pi^2 F_0^2} , \end{aligned} \tag{4.6}$$

where

$$h_2(x) = \frac{3}{2}\left(\frac{1+x}{1-x}\right)^2 + \frac{3x(1+x)}{(1-x)^3}\ln x , \tag{4.7}$$

differs from the SχPT result. This expression for λ_0 explicitly displays the dependence on $\widehat{m}^2(r-1)^2 A_3$ and on r, while its dependence on V_{us} is implicit in the F_K/F_π terms. However, as we saw above in Figure 3, F_K/F_π and $\widehat{m}^2(r-1)^2 A_3$ are correlated once we know the rate for $K^0 \to \pi^- e^+ \nu$. Therefore, for any choice of the quark mass ratio r, Eq. (4.6) gives a range of values for λ_0 corresponding to a given region in the $(F_K/F_\pi,\ \widehat{m}^2(r-1)^2 A_3)$ plane. For example, Eq. (3.15) defines one region in Figure 3, while Eq. (3.16) defines another. We display in Figure 4 the corresponding regions in the (λ_0, r) plane.

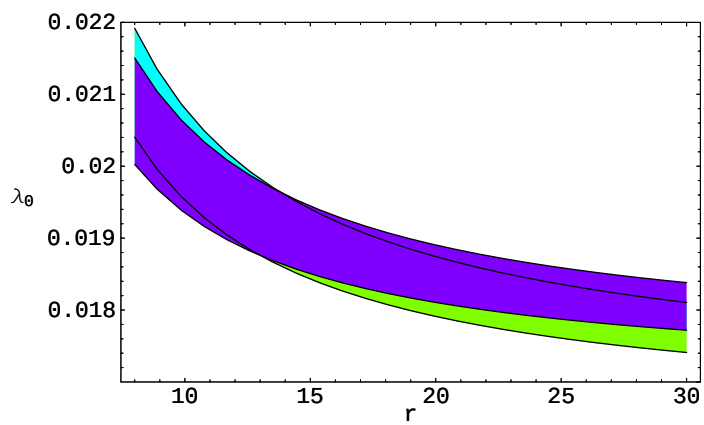

Figure 4: The dependence of λ_0 on r. The darker shaded region indicates the range of values Eq. (3.15) for λ_0 determined by using V_{ud} from superallowed nuclear beta decay, while the lighter shaded region shows the corresponding range Eq. (3.16) determined by using the PDG values for V_{us}. See text for details.

This analysis was done assuming that the Zweig-violating parameter Z_0^S=0. The whole dependence on Z_0^S is contained (through $\zeta = Z_0^S/A_0$) in the parameter $\hat{\epsilon}(r)$. As pointed out in [21], the vacuum stability requirement $B_0 \geq 0$ implies a upper bound on ζ, which yields

$$\hat{\epsilon}(r) - \hat{\epsilon}(r)|_{\zeta=0} \ < \ (1 - \hat{\epsilon}(r)|_{\zeta=0}) \ \frac{2}{r+2} \tag{4.8}$$

We find that taking $\zeta = 0$ or its maximal allowed value makes a difference of less than 0.0004 in λ_0, for any choice of r.

In SχPT, including one-loop corrections, the result [3] is

$$\lambda_0 = 0.017 \pm 0.004, \tag{4.9}$$

where the error is an estimate of the uncertainties due to higher-order contributions. The experimental situation remains unclear, in view of the inconsistency between some more recent

data [12] and the result ($\lambda_0 = 0.019 \pm 0.004$) of the high-statistics experiment of Donaldson *et al* [36].

5 Summary and discussion

In the preceding pages, we have studied the $K_{\ell 3}$ form factors at $\mathcal{O}(\mathrm{p}^4)$ precision within the generalized framework of chiral perturbation theory. In this connection, three (related) issues have been discussed: the extraction of the CKM matrix element V_{us} from the experimental value of the $K_{\ell 3}$ branching ratios, the determination of the ratio F_K/F_π of pseudoscalar decay constants, and the prediction for the slope λ_0 of the scalar form factor.

In the Leutwyler-Roos analysis [11], the expression of $f_+^{K^0\pi^-}(0)$ is written as

$$f_+^{K^0\pi^-}(0) = 1 + f_1 + f_2 + \cdots , \tag{5.1}$$

where the contributions f_1 and f_2 are of order m_{quark} and m^2_{quark} respectively. The one-loop SχPT contribution f_1 arises from Goldstone boson loops only, a counterterm contribution at this order of the standard counting being forbidden by the Ademollo-Gatto theorem. This leads to a parameter free prediction $f_1 = -0.023$. Furthermore, f_2, which would arise at chiral order $\mathcal{O}(\mathrm{p}^6)$ in SχPT, has been estimated in [11] to be -0.016 ± 0.008 by using a model for pion and kaon wave functions to compute matrix elements in the infinite momentum frame. This leads to a value $V_{us} = 0.2196 \pm 0.0023$. Assuming unitarity of the CKM matrix in a three generation standard model, and using the existing estimates of $|V_{ub}|$, the determination $V_{ud} = 0.9741 \pm 0.0005$ from superallowed nuclear β-decays leads instead to $V_{us} = 0.2261 \pm 0.0022$. Therefore, unitarity of the CKM matrix can only be restored at the expense of having the SχPT two-loop correction f_2 at least as large as the one-loop contribution f_1. This is far from signaling a failure of the chiral expansion in the present case, since f_1 might be anomalously small, being, for instance, suppressed, even compared to f_2, in the large-N_c limit (another consequence of the Ademollo-Gatto theorem). In GχPT, the corresponding expansion of $f_+^{K^0\pi^-}(0)$ reads

$$f_+^{K^0\pi^-}(0) = 1 + \tilde{f}_1 + \tilde{f}_2 + \cdots , \tag{5.2}$$

where $\tilde{f}_1$ collects all $\mathcal{O}(\mathrm{p}^4)$ contributions in the generalized chiral counting, and contains $\mathcal{O}(m^2_{\mathrm{quark}})$ terms. The difference between $\tilde{f}_1$ and f_1

$$\tilde{f}_1 - f_1 = \left[\frac{1}{8}\widehat{m}^2\,\xi^2 \,+\, \frac{1}{2}\widehat{m}^2 A_3\right](r-1)^2\,(1+\frac{1}{R}) \,, \tag{5.3}$$

involves a single low-energy constant from $\mathcal{L}_{(2,2)}$, A_3, which would appear only at order $\mathcal{O}(\mathrm{p}^6)$ in SχPT. The contribution of the $\mathcal{L}_{(2,1)}$ low-energy constant ξ is determined by the value of the ratio F_K/F_π. The value of V_{ud} from the nuclear β-decay data can be accommodated by

$$(r-1)^2\widehat{m}^2 A_3 = -0.124 \pm 0.022\,, \quad \frac{F_K}{F_\pi} = 1.189 \pm 0.012\,. \tag{5.4}$$

The corresponding difference is $\tilde{f}_1 - f_1 = -0.04 \pm 0.01$, and it represents the value which would be required in the Leutwyler-Roos analysis for the two-loop contribution f_2 (instead of the estimate $f_2 = -0.016 \pm 0.008$ of [11]) in order to maintain the unitarity of the CKM matrix. Of course, a confirmation, with a comparable accuracy, from other sources (neutron decay, pion β-decay) of the value of V_{ud} obtained from nuclear β-decays can only be welcome.

This determination of A_3 and the decrease in the value of the ratio of decay constants, as compared to the number $F_K/F_\pi = 1.22 \pm 0.01$ [11], is compatible with the present experimental information concerning the difference in the K^0_{e3} and K^+_{e3} decay rates, and induces only a mild modification in the prediction for the slope of the scalar form factor λ_0, which, as a function of the quark mass ratio $m_s/\widehat{m}$, varies between 0.0018 and 0.0022, well within the range set by the high-statistics K^0_L experiment of Donaldson *et al.* [36]. The higher values of λ_0 obtained by some of the more recent experiments [12] are therefore difficult to understand at the theoretical level, and cannot be ascribed, within GχPT, to the manifestation of a smaller value of the bilinear light quark condensate.

We thus conclude that the nuclear β-decay determination of $V_{ud} = 0.9741 \pm 0.0005$ need not be in contradiction with the present values of the K_{e3} decay rates and with chiral perturbation theory. One should then ask how the corresponding increase of $|V_{us}|$ by about 2.5 standard deviations would manifest itself in various observables. We have already mentioned that the present understanding of hyperon semi-leptonic decays is compatible with the suggested update of V_{us}. The effect on the K_{e4} decay rates should be analyzed separately [23]. Some effect on the extraction of the $\mathcal{O}(\mathrm{p}^4)$ low-energy constants L_1, L_2 and L_3 is to be expected *a priori*, but a precise statement requires a closer analysis. Finally, it is worth mentioning the possible effect on hadronic spectral functions which are extracted from the decays $\tau \to$ hadrons $+ \nu_\tau$ and used for a determination of fundamental QCD parameters [38, 39, 40]. While the non-strange ($\bar{u}d$) spectral functions should be only barely affected by an increase of $\sim 0.25\%$, the recently published [40] strange ($\bar{u}s$) spectral functions should be reduced by $\sim 4.6\%$. Consequently, we would expect no notable influence on the determination of $\alpha_S(M_\tau^2)$ [38], whereas the central value of the running strange quark mass $\mathrm{m_s}$ determined recently [39, 40] could increase by $\sim 15 - 20\%$. The issue certainly deserves a more detailed study.

Acknowledgments

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Appendices

A Expansion and renormalization of the effective Lagrangian

In this Appendix, we display the structure of the effective action of GχPT up to order $\mathcal{O}(\mathrm{p}^4)$. For a general discussion of the GχPT expansion framework, we refer the reader to the existing literature [8, 20, 21].

At leading order, the generalized expansion is described by $\tilde{\mathcal{L}}^{(2)}$, which was first given in Ref. [8]:

$$\begin{aligned}\tilde{\mathcal{L}}^{(2)} &= \frac{1}{4}F_0^2\Big\{\langle D_\mu U^+ D^\mu U\rangle + 2B_0\langle U^+\chi + \chi^+ U\rangle \\ &\quad + A_0\langle (U^+\chi)^2 + (\chi^+ U)^2\rangle + Z_0^S\langle U^+\chi + \chi^+ U\rangle^2 \\ &\quad + Z_0^P\langle U^+\chi - \chi^+ U\rangle^2 + H_0\langle \chi^+\chi\rangle\Big\} \ .\end{aligned} \tag{A.5}$$

The notation is as in Refs. [7, 3], except for the consistent removal of the factor $2B_0$ from χ, the parameter that collects the scalar and pseudoscalar sources,

$$\chi = s + ip = \mathcal{M} + \cdots \ , \ \mathcal{M} = \mathrm{diag}(m_u, m_d, m_s) \ . \tag{A.6}$$

In GχPT, the next-to-leading-order corrections are of order $\mathcal{O}(\mathrm{p}^3)$, and still occur at tree level only. They are embodied in $\tilde{\mathcal{L}}^{(3)} = \mathcal{L}_{(2,1)} + \mathcal{L}_{(0,3)}$, which reads [20, 37]

$$\begin{aligned}\tilde{\mathcal{L}}^{(3)} &= \frac{1}{4}F_0^2\Big\{\xi\langle D_\mu U^+ D^\mu U(\chi^+ U + U^+\chi)\rangle + \tilde{\xi}\langle D_\mu U^+ D^\mu U\rangle\langle \chi^+ U + U^+\chi\rangle \\ &\quad + \rho_1\langle (\chi^+ U)^3 + (U^+\chi)^3\rangle + \rho_2\langle (\chi^+ U + U^+\chi)\chi^+\chi\rangle \\ &\quad + \rho_3\langle \chi^+ U - U^+\chi\rangle\langle (\chi^+ U)^2 - (U^+\chi)^2\rangle \\ &\quad + \rho_4\langle (\chi^+ U)^2 + (U^+\chi)^2\rangle\langle \chi^+ U + U^+\chi\rangle \\ &\quad + \rho_5\langle \chi^+\chi\rangle\langle \chi^+ U + U^+\chi\rangle \\ &\quad + \rho_6\langle \chi^+ U - U^+\chi\rangle^2\langle \chi^+ U + U^+\chi\rangle + \rho_7\langle \chi^+ U + U^+\chi\rangle^3\Big\} \ .\end{aligned} \tag{A.7}$$

The tree-level contributions at order $\mathcal{O}(\mathrm{p}^4)$ are contained in [4]

$$\tilde{\mathcal{L}}^{(4)} = \mathcal{L}_{(4,0)} + \mathcal{L}_{(2,2)} + \mathcal{L}_{(0,4)} + B_0^2\mathcal{L}'_{(0,2)} + B_0\mathcal{L}'_{(2,1)} + B_0\mathcal{L}'_{(0,3)} \ . \tag{A.8}$$

[4] Contributions from the odd intrinsic parity sector are also present in the effective Lagrangian; at order $\mathcal{O}(\mathrm{p}^4)$ they are given by the Wess-Zumino term and are the same for SχPT and for GχPT, so that we do not display them here.

The part without explicit chiral symmetry breaking, $\mathcal{L}_{(4,0)}$, is described by the same low-energy constants L_1, L_2, L_3, L_9 and L_{10} as in SχPT [7]:

$$\begin{aligned}\mathcal{L}_{(4,0)} &= L_1\langle D_\mu U^+ D^\mu U\rangle\langle D_\nu U^+ D^\nu U\rangle + L_2\langle D_\mu U^+ D^\nu U\rangle\langle D_\mu U^+ D^\nu U\rangle \\ &\quad + L_3\langle D_\mu U^+ D^\mu U D_\nu U^+ D^\nu U\rangle - iL_9\langle F^R_{\mu\nu} D^\mu U D^\nu U^+ + F^L_{\mu\nu} D^\mu U^+ D^\nu U\rangle \\ &\quad + L_{10}\langle U^+ F^R_{\mu\nu} U F^{L\mu\nu}\rangle + H_1\langle F^R_{\mu\nu} F^{R\mu\nu} + F^L_{\mu\nu} F^{L\mu\nu}\rangle\ .\end{aligned} \tag{A.9}$$

The new term $\mathcal{L}_{(2,2)}$ would count as $\mathcal{O}(\mathrm{p}^6)$ in SχPT, and is given by:

$$\begin{aligned}\mathcal{L}_{(2,2)} &= \frac{1}{4}F_0^2\Big\{A_1\langle D_\mu U^+ D^\mu U(\chi^+\chi + U^+\chi\chi^+ U)\rangle \\ &\quad + A_2\langle D_\mu U^+ U\chi^+ D^\mu U U^+\chi\rangle \\ &\quad + A_3\langle D_\mu U^+ U(\chi^+ D^\mu\chi - D^\mu\chi^+\chi) + D_\mu U U^+(\chi D^\mu\chi^+ - D^\mu\chi\chi^+)\rangle \\ &\quad + A_4\langle D_\mu U^+ D^\mu U\rangle\langle\chi^+\chi\rangle \\ &\quad + B_1\langle D_\mu U^+ D^\mu U(\chi^+ U\chi^+ U + U^+\chi U^+\chi)\rangle \\ &\quad + B_2\langle D_\mu U^+\chi D^\mu U^+\chi + \chi^+ D_\mu U\chi^+ D^\mu U\rangle \\ &\quad + B_3\langle U^+ D_\mu\chi U^+ D^\mu\chi + D_\mu\chi^+ U D^\mu\chi^+ U\rangle \\ &\quad + B_4\langle D_\mu U^+ D^\mu U\rangle\langle\chi^+ U\chi^+ U + U^+\chi U^+\chi\rangle \\ &\quad + C_1^S\langle D_\mu U\chi^+ + \chi D^\mu U^+\rangle\langle D_\mu U\chi^+ + \chi D^\mu U^+\rangle \\ &\quad + C_2^S\langle D_\mu\chi^+ U + U^+ D^\mu\chi\rangle\langle D_\mu U^+\chi + \chi^+ D^\mu U\rangle \\ &\quad + C_3^S\langle D_\mu\chi^+ U + U^+ D^\mu\chi\rangle\langle D_\mu\chi^+ U + U^+ D^\mu\chi\rangle \\ &\quad + C_1^P\langle D_\mu U\chi^+ - \chi D^\mu U^+\rangle\langle D_\mu U\chi^+ - \chi D^\mu U^+\rangle \\ &\quad + C_2^P\langle D_\mu\chi^+ U - U^+ D^\mu\chi\rangle\langle D_\mu U^+\chi - \chi^+ D^\mu U\rangle \\ &\quad + C_3^P\langle D_\mu\chi^+ U - U^+ D^\mu\chi\rangle\langle D_\mu\chi^+ U - U^+ D^\mu\chi\rangle \\ &\quad + D^S\langle D_\mu U^+ D^\mu U(\chi^+ U + U^+\chi)\rangle\langle\chi^+ U + U^+\chi\rangle \\ &\quad + D^P\langle D_\mu U^+ D^\mu U(\chi^+ U - U^+\chi)\rangle\langle\chi^+ U - U^+\chi\rangle \\ &\quad + H_2\langle D_\mu\chi^+ D^\mu\chi\rangle\Big\}\ .\end{aligned} \tag{A.10}$$

Notice that the number of counterterms (17) involved in $\mathcal{L}_{(2,2)}$ agrees with Refs. [16, 17]. However, in both cases, different bases have been used. Finally, the tree-level contributions which behave as $\mathcal{O}(m^4_{\mathrm{quark}})$ in the chiral limit are contained in $\mathcal{L}_{(0,4)}$,

$$\begin{aligned}\mathcal{L}_{(0,4)} &= \frac{1}{4}F_0^2\Big\{E_1\langle(\chi^+ U)^4 + (U^+\chi)^4\rangle \\ &\quad + E_2\langle\chi^+\chi(\chi^+ U\chi^+ U + U^+\chi U^+\chi)\rangle \\ &\quad + E_3\langle\chi^+\chi U^+\chi\chi^+ U\rangle\end{aligned}$$

$$
\begin{aligned}
&+F_1^S\langle\chi^+U\chi^+U+U^+\chi U^+\chi\rangle^2\\
&+F_2^S\langle(\chi^+U)^3+(U^+\chi)^3\rangle\langle\chi^+U+U^+\chi\rangle\\
&+F_3^S\langle\chi^+\chi(\chi^+U+U^+\chi)\rangle\langle\chi^+U+U^+\chi\rangle\\
&+F_4^S\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+\chi\rangle\\
&+F_1^P\langle\chi^+U\chi^+U-U^+\chi U^+\chi\rangle^2\\
&+F_2^P\langle(\chi^+U)^3-(U^+\chi)^3\rangle\langle\chi^+U-U^+\chi\rangle\\
&+F_3^P\langle\chi^+\chi(\chi^+U-U^+\chi)\rangle\langle\chi^+U-U^+\chi\rangle\\
&+F_5^{SS}\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U+U^+\chi\rangle^2\\
&+F_6^{SS}\langle\chi^+\chi\rangle\langle\chi^+U+U^+\chi\rangle^2\\
&+F_5^{SP}\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U-U^+\chi\rangle^2\\
&+F_6^{SP}\langle\chi^+\chi\rangle\langle\chi^+U-U^+\chi\rangle^2\\
&+F_7^{SP}\langle(\chi^+U)^2-(U^+\chi)^2\rangle\langle\chi^+U-U^+\chi\rangle\langle\chi^+U+U^+\chi\rangle\\
&+H_3\langle\chi^+\chi\chi^+\chi\rangle+H_4\langle\chi^+\chi\rangle^2\Big\}\ .
\end{aligned}
\tag{A.11}
$$

Contributions from $\mathcal{L}_{(0,4)}$ would only appear at order $\mathcal{O}(\mathrm{p}^8)$ in SχPT, which, to the best of our knowledge, have not been discussed in the literature.

The $\mathcal{O}(\mathrm{p}^4)$ loop corrections to the processes studied here involve only graphs with one or two vertices from $\tilde{\mathcal{L}}^{(2)}$:

$$
\tilde{\mathcal{Z}}^{(4)}_{1\,\mathrm{loop}}=\tilde{\mathcal{Z}}^{(4)}_{\mathrm{tadpole}}+\tilde{\mathcal{Z}}^{(4)}_{\mathrm{unitarity}}+\cdots
\tag{A.12}
$$

The divergent parts of these one-loop graphs have been subtracted at a scale μ in the same dimensional renormalization scheme as described in [7]. Accordingly, the low energy constants of $\mathcal{L}_{(4,0)}$, $\mathcal{L}_{(2,2)}$, and $\mathcal{L}_{(0,4)}$ stand for the renormalized quantities, with an explicit logarithmic scale dependence ($X(\mu)$ denotes generically any of these renormalized low-energy constants)

$$
X(\mu)=X(\mu')+\frac{\Gamma_X}{(4\pi)^2}\cdot\ln(\mu'/\mu)\ .
\tag{A.13}
$$

At order $\mathcal{O}(\mathrm{p}^4)$, the low-energy constants of $\tilde{\mathcal{L}}^{(2)}$ and $\tilde{\mathcal{L}}^{(3)}$ also need to be renormalized. The corresponding counterterms, however, are of order $\mathcal{O}(B_0^2)$ and $\mathcal{O}(B_0)$, respectively, and they are gathered in the three last terms of Eq. (A.8): in GχPT, renormalization proceeds order by order in the expansion in powers of B_0 [20, 21]. Alternatively, one may think of Eqs. (A.5) and (A.7) as standing for the combinations $\tilde{\mathcal{L}}^{(2)}+B_0^2\mathcal{L}'_{(0,2)}$ and $\tilde{\mathcal{L}}^{(3)}+B_0\mathcal{L}'_{(2,1)}+B_0\mathcal{L}'_{(0,3)}$, respectively, with the corresponding low-energy constants representing the renormalized quantities. The full list of β-function coefficients Γ_X are tabulated below.

X	$F_0^2 \cdot \Gamma_X$
A_0	$5B_0^2/3$
Z_0^S	$11B_0^2/9$
Z_0^P	0
H_0	$10B_0^2/3$

Table 1: Subtraction scale dependences of the low energy constants of $\tilde{\mathcal{L}}^{(2)}$.

X	$F_0^2 \cdot \Gamma_X$
ξ	$3B_0$
$\tilde{\xi}$	B_0
ρ_1	$4B_0\left(\frac{1}{6}A_0 + Z_0^S + Z_0^P\right)$
ρ_2	$4B_0\left(\frac{1}{6}A_0 - Z_0^S + 3Z_0^P\right)$
ρ_3	$4B_0\left(\frac{1}{2}A_0 - \frac{2}{3}Z_0^S + \frac{5}{6}Z_0^P\right)$
ρ_4	$4B_0\left(\frac{11}{9}A_0 + \frac{5}{6}Z_0^S - \frac{2}{3}Z_0^P\right)$
ρ_5	$4B_0\left(A_0 + \frac{5}{3}Z_0^S - \frac{4}{3}Z_0^P\right)$
ρ_6	$4B_0\left(\frac{1}{9}Z_0^S + \frac{11}{18}Z_0^P\right)$
ρ_7	$4B_0\left(\frac{11}{18}Z_0^S + \frac{1}{9}Z_0^P\right)$

Table 2: Subtraction scale dependences of the low energy constants of $\tilde{\mathcal{L}}^{(3)}$.

X	Γ_X
L_1	$\frac{3}{32}$
L_2	$\frac{3}{16}$
L_3	0
L_9	$\frac{1}{4}$
L_{10}	$-\frac{1}{4}$
H_1	$-\frac{1}{8}$

Table 3: Subtraction scale dependences of the low energy constants of $\mathcal{L}_{(4,0)}$.

X	$F_0^2 \cdot \Gamma_X$
A_1	$-6(Z_0^S - Z_0^P)$
A_2	0
A_3	0
A_4	$2(Z_0^S - Z_0^P)$
B_1	$(3A_0 - 2Z_0^S - 2Z_0^P)$
B_2	$4(Z_0^S + Z_0^P)$
B_3	0
B_4	$(A_0 + Z_0^S + Z_0^P)$
C_1^S	$(A_0 - 2Z_0^P)$
C_2^S	0
C_3^S	0
C_1^P	$(A_0 - 2Z_0^S)$
C_2^P	0
C_3^P	0
D^S	$(A_0 + 7Z_0^S)$
D^P	$(A_0 + 7Z_0^P)$
H_2	0

Table 4: Subtraction scale dependences of the low energy constants of $\mathcal{L}_{(2,2)}$.

X	$F_0^2 \cdot \Gamma_X$
E_1	$-\frac{7}{3}A_0^2 + 8A_0(Z_0^S + Z_0^P) + \frac{34}{3}(Z_0^S + Z_0^P)^2$
E_2	$-16A_0(Z_0^S - Z_0^P) + 24(Z_0^S)^2 - 24(Z_0^P)^2$
E_3	$-\frac{14}{3}A_0^2 + 8A_0(Z_0^S + Z_0^P) + \frac{8}{9}\left[35(Z_0^S)^2 + 35(Z_0^P)^2 - 2Z_0^S Z_0^P\right]$
F_1^S	$\frac{26}{9}A_0^2 - \frac{25}{9}(Z_0^S + Z_0^P)^2$
F_2^S	$2A_0^2 + \frac{2}{3}A_0 Z_0^S - \frac{16}{3}A_0 Z_0^P - \frac{4}{9}\left[17(Z_0^S)^2 + 8(Z_0^P)^2 + 25Z_0^S Z_0^P\right]$
F_3^S	$2A_0^2 + \frac{2}{3}A_0 Z_0^S - \frac{16}{3}A_0 Z_0^P - \frac{4}{9}\left[71(Z_0^S)^2 + 8(Z_0^P)^2 - 29Z_0^S Z_0^P\right]$
F_4^S	$-20(Z_0^S)^2 + 20(Z_0^P)^2$
F_1^P	$A_0^2 - \frac{8}{9}(Z_0^S + Z_0^P)^2$
F_2^P	$2A_0^2 - \frac{16}{3}A_0 Z_0^S + \frac{2}{3}A_0 Z_0^P - \frac{4}{9}\left[8(Z_0^S)^2 + 17(Z_0^P)^2 + 25Z_0^S Z_0^P\right]$
F_3^P	$-2A_0^2 + \frac{16}{3}A_0 Z_0^S - \frac{2}{3}A_0 Z_0^P + \frac{4}{9}\left[8(Z_0^S)^2 + 71(Z_0^P)^2 - 29Z_0^S Z_0^P\right]$
F_5^{SS}	$\frac{44}{9}A_0 Z_0^S + \frac{8}{9}A_0 Z_0^P + \frac{1}{9}\left[85(Z_0^S)^2 + 4(Z_0^P)^2 + 26Z_0^S Z_0^P\right]$
F_6^{SS}	$4A_0 Z_0^S + \frac{2}{9}\left[77(Z_0^S)^2 - 4(Z_0^P)^2 - 26Z_0^S Z_0^P\right]$
F_5^{SP}	$\frac{8}{9}A_0 Z_0^S + \frac{44}{9}A_0 Z_0^P + \frac{1}{9}\left[4(Z_0^S)^2 + 85(Z_0^P)^2 + 26Z_0^S Z_0^P\right]$
F_6^{SP}	$-4A_0 Z_0^P + \frac{2}{9}\left[4(Z_0^S)^2 - 77(Z_0^P)^2 + 26Z_0^S Z_0^P\right]$
F_7^{SP}	$2A_0(Z_0^S + Z_0^P) - \frac{2}{9}\left[4(Z_0^S)^2 + 4(Z_0^P)^2 - 55Z_0^S Z_0^P\right]$
H_3	$8A_0(Z_0^S + Z_0^P) + \frac{4}{9}\left[35(Z_0^S)^2 + 35(Z_0^P)^2 - 2Z_0^S Z_0^P\right]$
H_4	$4A_0^2 - \frac{4}{9}\left[17(Z_0^S)^2 + 17(Z_0^P)^2 - 2Z_0^S Z_0^P\right]$

Table 5: Subtraction scale dependences of the low energy constants of $\mathcal{L}_{(0,4)}$.

B Masses and Decay Constants

For the reader's convenience and for later reference, we provide, in this Appendix, the expressions of the pseudoscalar decay constants and masses at order $\mathcal{O}(\mathrm{p}^4)$ [5]. Introducing the notation $(P = \pi, K, \eta)$

$$\mu_P \;=\; \frac{M_P^2}{32\pi^2 F_\pi^2}\,\ln\frac{M_P^2}{\mu^2}\;, \tag{B.1}$$

where μ is the renormalisation scale, we obtain the following formulae for the decay constants (in the limit $m_u = m_d$),

$$F_\pi^2 = F_0^2\left[1 + 2\widehat{m}\delta_{F,\pi}^{(2,1)} + 2\widehat{m}^2\delta_{F,\pi}^{(2,2)} - 4\mu_\pi - 2\mu_K\right]\;, \tag{B.2}$$

with

$$\begin{aligned}
\delta_{F,\pi}^{(2,1)} &= \xi + (2+r)\tilde{\xi} \\
\delta_{F,\pi}^{(2,2)} &= A_1 + \frac{1}{2}A_2 + 2A_3 + B_1 - B_2 \\
&\quad + \frac{1}{2}(2+r^2)(A_4 + 2B_4) + 2(2+r)D^S
\end{aligned} \tag{B.3}$$

$$F_K^2 = F_0^2\left[1 + 2\widehat{m}\delta_{F,K}^{(2,1)} + 2\widehat{m}^2\delta_{F,K}^{(2,2)} - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{3}{2}\mu_\eta\right]\;, \tag{B.4}$$

with

$$\begin{aligned}
\delta_{F,K}^{(2,1)} &= \frac{1}{2}(1+r)\xi + (2+r)\tilde{\xi} \\
\delta_{F,K}^{(2,2)} &= \frac{1}{2}(1+r^2)(A_1 + A_3 + B_1) + \frac{r}{2}(A_2 + 2A_3 - 2B_2) \\
&\quad + \frac{1}{2}(2+r^2)(A_4 + 2B_4) + (1+r)(2+r)D^S
\end{aligned} \tag{B.5}$$

$$F_\eta^2 = F_0^2\left[1 + 2\widehat{m}\delta_{F,\eta}^{(2,1)} + 2\widehat{m}^2\delta_{F,\eta}^{(2,2)} - 6\mu_K\right]\;, \tag{B.6}$$

[5] We also take this opportunity to correct a misprint in an earlier published expression [9] of the decay constant F_K.

with

$$
\begin{aligned}
\delta_{F,\eta}^{(2,1)} &= \frac{1}{3}(1+2r)\xi + (2+r)\tilde{\xi} \\
\delta_{F,\eta}^{(2,2)} &= \frac{1}{3}(1+2r^2)(A_1 + \frac{1}{2}A_2 + 2A_3 + B_1 - B_2) \\
&\quad + \frac{1}{2}(2+r^2)(A_4 + 2B_4) + \frac{2}{3}(1+2r)(2+r)D^S \\
&\quad - \frac{4}{3}(r-1)^2(C_1^P + C_2^P)
\end{aligned}
\tag{B.7}
$$

For the masses, we obtain:

$$
\begin{aligned}
F_\pi^2 M_\pi^2 &= F_0^2 \Big\{ 2\widehat{m}B_0 + 4\widehat{m}^2 A_0 + 4\widehat{m}^2(2+r)Z_0^S \\
&\quad + 2\widehat{m}^3 \delta_{M,\pi}^{(0,3)} + 2\widehat{m}^4 \delta_{M,\pi}^{(0,4)} + 4\widehat{m}^2 A_3 M_\pi^2 \\
&\quad - \mu_\pi \left[3\, M_\pi^2 + 5\, M_\pi^2\, \hat{\epsilon}(r)\right] \\
&\quad - 2\,\mu_K \left[M_\pi^2 + (1+r)\, M_\pi^2\, \hat{\epsilon}(r)\right] \\
&\quad - \frac{1}{3}\,\mu_\eta \Big[M_\pi^2 + (1+2r)\, M_\pi^2\, \hat{\epsilon}(r) - 2\,\frac{\Delta_{GMO}}{r-1} \Big] \Big\}
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
F_K^2 M_K^2 &= F_0^2 \Big\{ (1+r)\widehat{m}B_0 + (1+r)^2\widehat{m}^2 A_0 + 2(1+r)(2+r)\widehat{m}^2 Z_0^S \\
&\quad + (1+r)\widehat{m}^3 \delta_{M,K}^{(0,3)} + (1+r)\widehat{m}^4 \delta_{M,K}^{(0,4)} + (1+r)^2\widehat{m}^2 A_3 M_K^2 \\
&\quad - \frac{3}{2}\,\mu_\pi \left[M_K^2 + M_\pi^2\, \hat{\epsilon}(r)\right] \\
&\quad - 3\,\mu_K \left[M_K^2 + \frac{1}{2}\,(1+r)^2\, M_\pi^2\, \hat{\epsilon}(r)\right] \\
&\quad - \frac{1}{6}\,\mu_\eta \left[5\, M_K^2 + (r+1)\,(1+2r)\, M_\pi^2\, \hat{\epsilon}(r) + \frac{r+1}{r-1}\,\Delta_{GMO} \right] \Big\}
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
F_\eta^2 M_\eta^2 &= F_0^2 \Big\{ \frac{2}{3}(1+2r)\widehat{m}B_0 + \frac{4}{3}(1+2r^2)\widehat{m}^2 A_0 + \frac{4}{3}(1+2r)(2+r)\widehat{m}^2 Z_0^S + \frac{8}{3}(1-r)^2\widehat{m}^2 Z_0^P \\
&\quad + 2\widehat{m}^3 \delta_{M,\eta}^{(0,3)} + 2\widehat{m}^4 \delta_{M,\eta}^{(0,4)} + \frac{4}{3}(1+2r^2)\widehat{m}^2 A_3 M_\eta^2 - \frac{8}{3}(1-r)^2\widehat{m}^2 C_2^P M_\eta^2 \\
&\quad - \mu_\pi \left[M_\pi^2 + (2r+1)\, M_\pi^2\, \hat{\epsilon}(r) - 2\,\frac{\Delta_{GMO}}{r-1} \right] \\
&\quad - \frac{2}{3}\,\mu_K \left[8M_K^2 - 3M_\pi^2 + (r+1)(2r+1)\, M_\pi^2\, \hat{\epsilon}(r) + 2\,\frac{2r-1}{r-1}\,\Delta_{GMO} \right] \\
&\quad - \frac{1}{9}\,\mu_\eta \left[16M_K^2 - 7M_\pi^2 + 3\,(2r+1)^2\, M_\pi^2\, \hat{\epsilon}(r) + 4\,\frac{4r-1}{r-1}\,\Delta_{GMO} \right] \Big\}\,,
\end{aligned}
\tag{B.10}
$$

with

$$\delta_{M,\pi}^{(0,3)} = \frac{9}{2}\rho_1 + \frac{1}{2}\rho_2 + (10+4r+r^2)\rho_4 + \frac{1}{2}(2+r^2)\rho_5 + 6(2+r)^2\rho_7 \tag{B.11}$$

$$\delta_{M,K}^{(0,3)} = \frac{3}{2}(1+r+r^2)\rho_1 + \frac{1}{2}(1-r+r^2)\rho_2 + 3(2+2r+r^2)\rho_4 + \frac{1}{2}(2+r^2)\rho_5 + 6(2+r)^2\rho_7 \tag{B.12}$$

$$\begin{aligned}\delta_{M,\eta}^{(0,3)} = {} & \frac{3}{2}(1+2r^3)\rho_1 + \frac{1}{6}(1+2r^3)\rho_2 + \frac{8}{3}(1+r)(1-r)^2\rho_3 + \\ & \frac{1}{3}(10+8r+17r^2+10r^3)\rho_4 + \frac{1}{6}(1+2r)(2+r^2)\rho_5 + \\ & \frac{8}{3}(1-r)^2(2+r)\rho_6 + 2(1+2r)(2+r)^2\rho_7 \ ,\end{aligned} \tag{B.13}$$

while the $\mathcal{O}(\mathrm{p}^4)$ contributions are

$$\begin{aligned}\delta_{M,\pi}^{(0,4)} = {} & 8E_1 + 2E_2 + 4(1+r)(2+r^2)F_1^S + (20+9r+r^3)F_2^S \\ & +(4+r+r^3)F_3^S + 2(2+r^2)F_4^S + 4(2+r)(6+2r+r^2)F_5^{SS} \\ & +2(2+r)(2+r^2)F_6^{SS} \ ,\end{aligned} \tag{B.14}$$

$$\begin{aligned}\delta_{M,K}^{(0,4)} = {} & 2(1+r)(1+r^2)E_1 + (1+r^3)E_2 + \frac{1}{2}(1+r)(1-r)^2E_3 \\ & +4(1-r)(2+r^2)F_1^S + (8+9r+9r^2+4r^3)F_2^S \\ & +(4-r+r^2+2r^3)F_3^S + (1+r)(2+r^2)F_4^S \\ & +4(2+r)(4+3r+2r^2)F_5^{SS} + 2(2+r)(2+r^2)F_6^{SS} \ ,\end{aligned} \tag{B.15}$$

$$\begin{aligned}\delta_{M,\eta}^{(0,4)} = {} & \frac{8}{3}(1+2r^4)E_1 + \frac{2}{3}(1+2r^4)E_2 \\ & +\frac{8}{3}(1+2r^2)(2+r^2)F_1^S + \frac{1}{3}(20+13r+37r^3+20r^4)F_2^S \\ & +\frac{1}{3}(4+5r+5r^3+4r^4)F_3^S + \frac{2}{3}(1+2r^2)(2+r^2)F_4^S \\ & +\frac{16}{3}(1+r)^2(1-r)^2F_1^P + 4(1-r)^2(1+r+r^2)F_2^P \\ & +\frac{4}{3}(1-r)^2(1+r+r^2)F_3^P + 4(2+r)(2+2r+3r^2+2r^3)F_5^{SS} \\ & +\frac{2}{3}(2+r)(2+r^2)(1+2r)F_6^{SS} + \frac{8}{3}(1-r)^2(2+r^2)F_5^{SP} \\ & +\frac{4}{3}(1-r)^2(2+r^2)F_6^{SP} + \frac{16}{3}(1-r)^2(1+r)(2+r)F_7^{SP} \ .\end{aligned} \tag{B.16}$$

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