



**HAL**  
open science

# Simple tangential families and perestroikas of their envelopes

Gianmarco Capitanio

► **To cite this version:**

Gianmarco Capitanio. Simple tangential families and perestroikas of their envelopes. 2004. hal-00002788

**HAL Id: hal-00002788**

**<https://hal.science/hal-00002788>**

Preprint submitted on 6 Sep 2004

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Simple tangential families and perestroikas of their envelopes

GIANMARCO CAPITANIO

## Abstract

Tangential families are 1-parameter families of rays emanating tangentially from smooth curves. We classify tangential family germs up to Left-Right equivalence: we prove that there are two infinite series and four sporadic simple singularities of tangential family germs (in addition to two stable singularities). We give their normal forms and miniversal tangential deformations (i.e., deformations among tangential families), and we describe the corresponding envelope perestroikas of small codimension. We also discuss envelope singularities of non simple tangential families.

KEYWORDS : Envelope theory, Left-Right equivalence, Tangential families.

2000 MSC : 14B05, 14H15, 58K25, 58K40, 58K50.

## 1 Introduction

A tangential family is a system of rays emanating tangentially from a smooth curve. Tangential families naturally arise in the Geometry of Caustics (see e.g. [3]) and in Differential Geometry. For instance, every smooth curve in a Riemannian surface defines the tangential family of its tangent geodesics. Tangential family theory is a generalization of Envelope theory. Indeed, every 1-parameter family of plane curves is tangential, with respect to every generic point of any (geometric) branch of its envelope.

In [10] we classified stable singularities of tangential family germs (under small tangential deformations, i.e. deformations among tangential families), and we proved that their envelopes are smooth or have a second order self-tangency. In [7] and [8] we considered tangential families with singular support (the singularity being a semicubic cusp).

In this paper we classify simple singularities of tangential family germs. We prove that there are two infinite series and four sporadic simple tangential family singularities (in addition to the two stable singularities). We give their normal forms and miniversal tangential deformations, and we describe perestroikas of small codimension occurring to envelopes of simple tangential family germs under small tangential deformations. We also study envelopes of non simple tangential family germs.

Our study of tangential families is related to Envelope Theory and Singularity Theory, namely to the theories of Thom and Arnold on envelopes of families of plane curves (see [1], [2] and [15]), and to several branches of Projection Theory (see [5]), namely the classifications of simple projections of surfaces and of projections of generic surfaces with boundary (due to J.W. Bruce, P.J. Giblin and V.V. Goryunov, see [12], [6] and [13]), in our case the boundary being fixed to be contained in the projection critical set.

## 2 Simple tangential family germs

Unless otherwise specified, all the objects considered below are supposed to be of class  $\mathcal{C}^\infty$ ; by plane curve we mean an embedded 1-submanifold of  $\mathbb{R}^2$ .

Let us consider a map  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of the plane  $\mathbb{R}^2$ , whose coordinates are denoted by  $\xi$  and  $t$ . If  $\partial_t f$  vanishes nowhere, then  $f_\xi := f(\xi, \cdot)$  parameterizes an immersed smooth curve  $\Gamma_\xi$ . Hence,  $f$  defines the 1-parameter family of curves  $\{\Gamma_\xi : \xi \in \mathbb{R}\}$ .

**Definition.** The family parameterized by  $f$  is a *tangential family* whenever (1) the curve parameterized by  $f(\cdot, 0)$ , called the *support* of the family, is smooth and (2) for every  $\xi \in \mathbb{R}$ , the family curve  $\Gamma_\xi$  is tangent to the support at  $f(\xi, 0)$ .

In other terms, a tangential family is a fibration, whose base is the support and whose fibers are the curves  $\Gamma_\xi$ . A *p-parameter tangential deformation* of a tangential family  $f$  is a map  $F : \mathbb{R}^2 \times \mathbb{R}^p \rightarrow \mathbb{R}^2$ , such that  $F_\lambda := F(\cdot; \lambda)$  is a tangential family for every  $\lambda \in \mathbb{R}^p$  and  $F_0 = f$ . For instance, the tangent geodesics to a curve in a Riemannian surface form a tangential family. A perturbation of the metric induces a tangential deformation on this family.

The *graph* of a tangential family  $\{\Gamma_\xi : \xi \in \mathbb{R}\}$  is the surface  $\Phi := \cup_{\xi \in \mathbb{R}} \{(\xi, \Gamma_\xi)\} \subset \mathbb{R} \times \mathbb{R}^2$ . Let us denote by  $\pi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the projection  $(\xi, P) \mapsto P$ . The *criminant set* of the tangential family is the critical set of  $\pi|_\Phi$ . The *envelope* is the apparent contour of the graph in the plane (i.e. the critical value set of  $\pi|_\Phi$ ). By the very definition, the support of the family belongs to its envelope.

Below we will study tangential family germs, so we will consider their local parameterizations as map germs  $(\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ . Graphs of tangential family germs are smooth.

Denote by  $\text{Diff}(\mathbb{R}^2, 0)$  the group of diffeomorphism germs of the plane keeping fixed the origin and by  $\mathfrak{m}_{\xi,t}$  the ring of function germs  $(\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$  in the variables  $\xi$  and  $t$ . Then the group  $\mathcal{A} := \text{Diff}(\mathbb{R}^2, 0) \times \text{Diff}(\mathbb{R}^2, 0)$  acts on the space  $(\mathfrak{m}_{\xi,t})^2$  by  $(\varphi, \psi) \cdot f := \psi \circ f \circ \varphi^{-1}$ , where  $(\varphi, \psi) \in \mathcal{A}$  and  $f \in (\mathfrak{m}_{\xi,t})^2$ . Two elements of  $(\mathfrak{m}_{\xi,t})^2$  are said to be *Left-Right equivalent*, or  *$\mathcal{A}$ -equivalent*, if they belong to the same  $\mathcal{A}$ -orbit. Thus, envelopes of  $\mathcal{A}$ -equivalent tangential families are diffeomorphic.

The standard definition of versality of a deformation can be translated in the setting of tangential deformations. A tangential deformation  $F$  of a tangential family germ  $f$  is said to be *versal* if any tangential deformation of  $f$  is  $\mathcal{A}$ -equivalent to a tangential deformation induced from  $F$ . Such a deformation is said to be *miniversal* whenever the base dimension  $p$  is minimal. This minimal number  $p$ , depending only on the  $\mathcal{A}$ -orbit of the family, is called the *tangential codimension* of the singularity. A tangential family germ is *stable* if every  $\mathcal{A}$ -versal tangential deformation is trivial; it is *simple* if by arbitrary sufficiently small tangential deformations of it, we can obtain only a finite number of singularities.

Let us recall that a map germ  $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$  defines, by the formula  $f^*g := g \circ f$ , a homomorphism from the ring  $\mathcal{E}_{x,y}$  of the function germs in the target to the ring  $\mathcal{E}_{\xi,t}$  of the function germs in the source. Hence, we can consider every  $\mathcal{E}_{\xi,t}$ -module as an  $\mathcal{E}_{x,y}$ -module via this homomorphism. The *tangent space* at  $f$  is the  $\mathcal{E}_{x,y}$ -module defined by

$$Tf := \langle \partial_\xi f, \partial_t f \rangle_{\mathcal{E}_{\xi,t}} + f^*(\mathcal{E}_{x,y}) \cdot \mathbb{R}^2 .$$

The *codimension* of  $f$  is the dimension of the real vector space  $\mathcal{E}_{\xi,t}^2 / Tf$ .

We state now our main result, proven in Section 6.

**Theorem 1.** *Every simple tangential family germ is  $\mathcal{A}$ -equivalent to a tangential family germ  $(\xi + t, \varphi)$ , of support  $y = 0$ , where  $\varphi$  is one of the function germs listed in the table below, in which  $c$  and  $\tau$  are the codimension and the tangential codimension of the singularity. A miniversal tangential deformation of the normal form is obtained adding  $\sum_i \lambda_i e_i$  to  $\varphi$  according to the table.*

Singularity	$\varphi$	$c$	$\{e_i\}$	$\tau$
I	$t^2$	0	—	0
II	$t^2\xi$	1	—	0
$S_{1,n}$	$t^2(t + \xi) + t^4 + t^{2n+3}$	$n + 1$	$t^3, t^5, \dots, t^{2n+1}$	$n$
$T_n$	$t^3 + t^2(t + \xi)^{n+1}$	$2n + 1$	$t^2, t^2\xi, \dots, t^2\xi^{n-1}$	$n$
$S_{2,2}$	$t^2(t + \xi) + t^5 + t^6$	3	$t^3, t^4$	2
$S_{2,3}^\pm$	$t^2(t + \xi) + t^5 \pm t^9$	4	$t^3, t^4, t^6$	3
$S_{2,4}$	$t^2(t + \xi) + t^5$	5	$t^3, t^4, t^6, t^9$	4

**Remarks.** (1) This table is a part of the more general classification of the simple projections of surfaces in the plane, due to V. V. Goryunov (see [12]).

(2) In [10] we proved that singularities I and II are all the stable singularities; the corresponding envelopes are respectively smooth and have an order 2 self-tangency.

A curve with a singularity of order  $b/a$  is *non symmetric* with respect to a smooth curve tangent to it at the singular point if it is contained into one of the two domains cut off near the singular point by the smooth curve.

**Corollary.** *The envelope of every simple tangential family germ, having not a singularity I, has two tangent branches, one of which is its support. More precisely:*

- (i) *if the germ has a singularity  $S_{1,n}$ , the envelope second branch has a  $(2n + 3)/2$ -cusp at the tangency point, this cusp being non symmetric to the support;*
- (ii) *if the germ has a singularity  $S_{2,2}$ ,  $S_{2,3}^\pm$  or  $S_{2,4}$ , then the envelope second branch has a singularity of order  $5/3$  at its tangency point with the support;*
- (iii) *if the germ has a singularity  $T_n$ , the envelope second branch is smooth and it has a tangency of order  $3n + 2$  with the support.*

Envelope singularities of simple tangential family germs are shown in Figure 1.

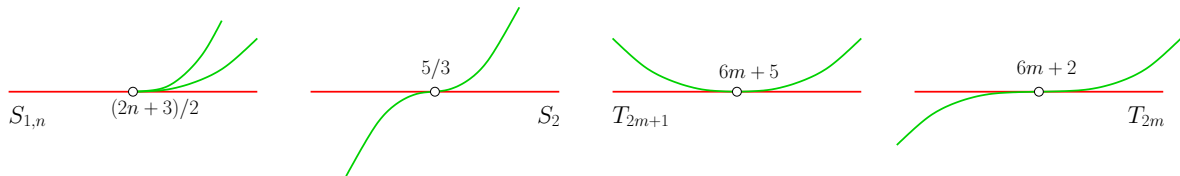


Figure 1: Envelopes of simple tangential family germs.



The discriminant of singularity  $S_{1,2}$  is represented in Figure 3. It has been found for me experimentally by Francesca Aicardi. Note that it contains the flag  $V_1 \supset V_0$  and a curve, corresponding to a self-tangency of the envelope second branch.

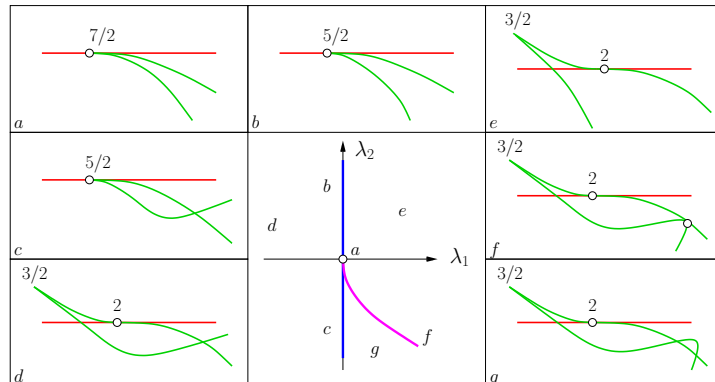


Figure 3: Bifurcation diagram of singularity  $S_{1,2}$ .

The  $S_{2,2}$ -discriminant in the plane  $\{\lambda_1, \lambda_2\}$ , shown in Figure 4, is the union of the  $\lambda_2$ -axis and four curve germs, tangent to it at the origin, the tangency order of all these curves being 1. To my knowledge, this bifurcation diagram did not appear earlier in Projection Theory.

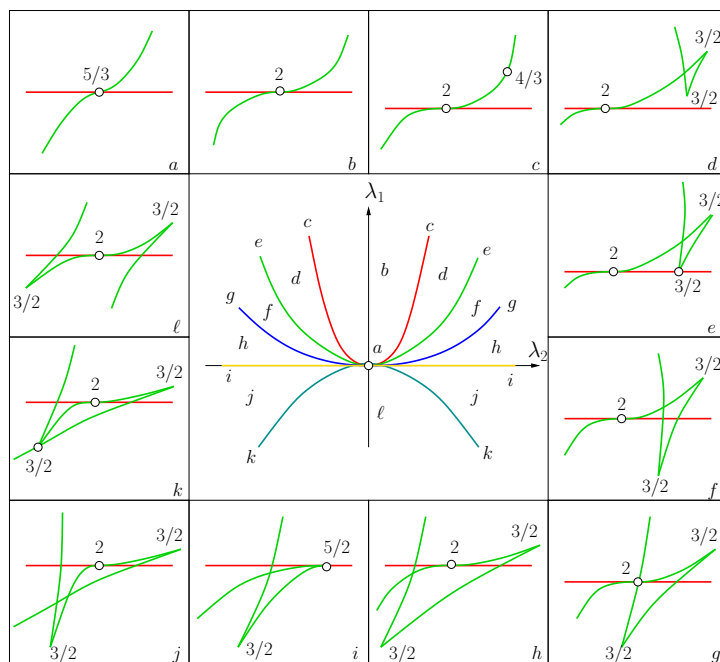


Figure 4: Bifurcation diagram of the singularity  $S_{2,2}$ .

Finally, we consider singularities  $T_n$ .

**Proposition 1.** *The discriminant of the singularity  $T_n$  is an  $n$ -dimensional swallowtail.*

*Proof.* The second branch of the envelope is the graph  $y = \frac{4}{27}Q_\lambda^3(x)$ , where  $Q_\lambda(x) := x^{n+1} + \lambda_n x^{n-1} + \dots + \lambda_1$ , is a miniversal deformation of  $x^{n+1}$  for the Left equivalence in the space

of function germs  $(\mathbb{R}, 0) \rightarrow \mathbb{R}$ . The  $T_n$ -discriminant is formed by the values  $\lambda$  for which  $P_\lambda$  has roots of multiplicity greater than 3, i.e., for which  $Q_\lambda$  has multiple roots. Thus, it is a swallowtail.  $\square$

Bifurcation diagrams of singularities  $T_n$  are depicted in Figure 5 for  $n = 1, 2, 3$ .

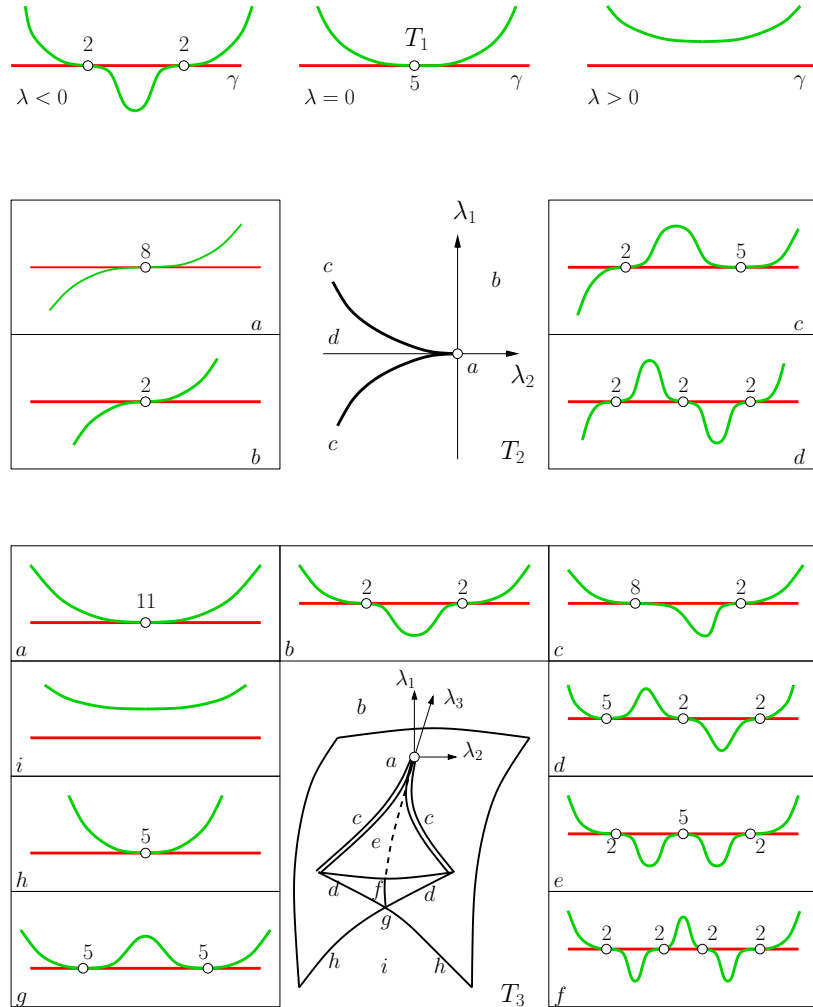


Figure 5: Bifurcation diagrams of singularities  $T_n$  for  $n = 1, 2, 3$ .

## 4 Classification of tangential family germs

The set of tangential family germs is naturally decomposed according to the configurations of their discriminant sets. We define now this decomposition.

Given a tangential family germ, the fiber  $\pi^{-1}(0, 0)$  defines a *vertical direction* in the tangent plane to its graph at the origin. The germ is said to be of *first type* if its discriminant set has only one branch, of *second type* if it has exactly two branches and these branches are smooth, non vertical and transversal each other. In [10] we proved that a tangential family germ has a singularity I (resp., II) if and only if it is of first type (resp., of second type).

**Definition.** Let  $n \in \mathbb{N} \cup \{\infty\}$ . A tangential family germ is said to be (1) of type  $S_n$  if its discriminant set has two branches, both smooth, one of which has an order  $n$  tangency with the vertical direction; (2) of type  $T_n$  if its discriminant set has two branches, both smooth, which have an order  $n$  tangency; (3) of type  $U$  if its discriminant set has more than two branches or at least a singular branch.

These singularity classes, in addition to stable singularities I and II, cover all the tangential family singularities.

**Definition.** A tangential family germ of type  $S_1$  is said to be of type  $S_{1,n}$  if its envelope has a cusp of order  $(2n + 3)/2$  at the origin.

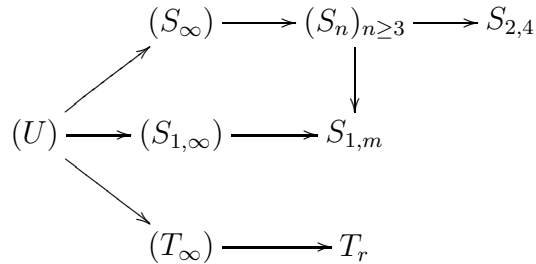
**Theorem 2.** For every  $n \in \mathbb{N}$ , a tangential family germ has a singularity  $S_{1,n}$  (resp.,  $T_n$ ) if and only if it is of type  $S_{1,n}$  (resp.,  $T_n$ ). A tangential family germ has a singularity  $S_{2,2}$ ,  $S_{2,3}^\pm$  or  $S_{2,4}$  if and only if it is of type  $S_2$ .

The next result describes envelopes of non simple tangential family germs of type  $S_{n \geq 3}$ .

**Theorem 3.** For  $n \in \mathbb{N}$ ,  $n \geq 3$ , the envelope of a  $S_n$ -type tangential family germ has, in addition to the support, a branch with a singularity of order  $(n + 3)/(n + 1)$ , tangent to the support. If  $n$  is even, this singularity is a cusp, non symmetric to the support.

Theorems 2 and 3 are proven in Sections 5 and 6.

We end this section with the description of the hierarchy of non simple singularities. We denote by  $(L)$  and  $(K)$  some classes of non simple singularities. The arrow  $(L) \rightarrow (K)$  means that there exists two singularities  $L' \subset (L)$  and  $K' \subset (K)$  such that  $L' \rightarrow K'$ . The main adjacencies of non simple singularities are as follows.



## 5 Preliminary results

Here we introduce some technical results we will use in next section. We start describing standard parameterizations of tangential family germs.

Let us fix a tangential family germ. One easily verifies (see [10]) that in a coordinate system in which the family support is the germ of  $y = 0$ , the tangential family is parameterized by a map germ  $(\xi + t, \varphi)$ , where  $\varphi$  is of the form  $\alpha t^3 + t^2 \sum_{i=0}^{\infty} k_i \xi^i + t^3 \cdot \delta(0) + R(\xi)$ , where  $\alpha, k_i \in \mathbb{R}$ ,  $\delta(n)$  denotes any function of two variables with vanishing  $n$ -jet at the origin and  $R$  is flat (i.e., its Taylor expansion is zero). Such a parameterization is called *prenormal form*. The prenormal form of a tangential family germ is not unique.

In [10] we proved that a tangential family germ is of first type if and only if  $k_0 \neq 0$ ; it is of second type if and only if  $k_0 = 0$  and  $k_1 \neq 0, \alpha$ . Therefore, for any tangential family germ neither of first nor of second type, we have  $k_0 = 0$  and  $k_1(k_1 - \alpha) = 0$ .



**Lemma 1.** Let  $(\xi + t, \varphi)$  be a prenormal form of a tangential family germ, where  $\varphi$  is as above. The germ is of type  $(S_n)_{n \in \mathbb{N}}$ ,  $(T_n)_{n \in \mathbb{N}}$  or  $U$  if and only if  $\alpha = k_1 \neq 0$ ,  $\alpha \neq k_1 = 0$  or  $\alpha = k_1 = 0$  respectively. More precisely,

- (i) the family is of type  $S_n$  if and only if  $\varphi(0, t) = \alpha t^3 + O(t^{2n+3})$ ;
- (ii) the family is of type  $T_n$  if and only if  $k_0 = \dots = k_n = 0$  and  $\alpha, k_{n+1} \neq 0$ ,
- (iii) the family is of type  $U$  if and only if  $\varphi(\xi, t) = t^2 \cdot \delta(1)$ .

Theorem 3 follows from Lemma 1 by explicit computation of the envelope.

**Remark 1.** Let  $(\xi + t, \varphi)$  the prenormal form of a tangential family germ. Then every  $p$ -parameter tangential deformation of it is equivalent to a tangential deformation of the form  $(\xi + t, \varphi + t^2\psi)$ , where  $\psi = \psi(\xi, t; \lambda)$  is a function germ  $(\mathbb{R}^2 \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^2, 0)$ .

Let us consider now the space  $\mathbb{R}[[\xi, t]]$ , endowed with a quasihomogeneous filtration defined by weights  $\deg(\xi) = a$  and  $\deg(t) = b$ , where  $a$  and  $b$  are coprime natural numbers. We denote by  $\tilde{\mathfrak{m}}_{\xi, t}^p$  the  $\mathcal{E}_{\xi, t}$ -ideal generated by the monomials of weighted degree  $p$ , and by  $\tilde{\delta}(p)$  any function germ with zero weighted  $p$ -jet.

The *reduced tangent space* at  $f$  is the  $\mathcal{E}_{x, y}$ -module defined by  $T_r f := \mathfrak{g}_+ + \mathcal{M}$ , where  $\mathfrak{g}_+$  is the space of the vector field germs having positive weighted order,  $\mathcal{M}$  is the  $\mathcal{E}_{x, y}$ -module  $f^*(\mathfrak{m}_{x, y}^2 \oplus \mathbb{R} \cdot y) \times f^*(\mathfrak{m}_{x, y}^2 \oplus \mathbb{R} \cdot x)$ , and  $\mathfrak{m}_{x, y}^2$  is the second power of the maximal ideal  $\mathfrak{m}_{x, y}$ .

**Remark 2.** Let  $R$  be an element of  $\tilde{\mathfrak{m}}^p \times \tilde{\mathfrak{m}}^q$ , such that  $R \in T_r f / (\tilde{\mathfrak{m}}^{p+1} \times \tilde{\mathfrak{m}}^{q+1})$ . Then the weighted  $(p, q)$ -jet of  $f$  is  $\mathcal{A}$ -equivalent to that of  $f + R$ .

## 6 Proofs

Here we prove Theorems 1 and 2. For the computations we skip, we refer to [9]. A *vector monomial* is by definition an element of  $(\mathfrak{m}_{\xi, t})^2$  having a component which is a monomial, the other component being zero.

**$S_1$ -type tangential families.** Set  $\deg(t) = 1$ ,  $\deg(\xi) = 2$ . Consider a tangential family germ of type  $S_1$ ; by Lemma 1, its prenormal form is  $\mathcal{A}$ -equivalent to  $(\xi, t^4 + t^2\xi + t^2 \cdot \tilde{\delta}(2))$ . It follows from the geometry of the Newton Diagram of  $t^4 + t^2\xi$  that the preceding map germ is formally  $\mathcal{A}$ -equivalent to  $(\xi, t^4 + t^2\xi) + \sum_{i=1}^{\infty} b_i(0, t^{2i+3})$ , for some  $b_i \in \mathbb{R}$ . Such a germ is of type  $S_{1, n}$  if and only if  $b_i = 0$  for  $i < n$  and  $b_n \neq 0$ . In this case we may assume  $b_n = 1$  after rescaling.

We set  $f_n(\xi, t) := (\xi, t^4 + t^2\xi + t^{2n+3})$ . Since  $T_r f_n$  contains  $\tilde{\mathfrak{m}}_{\xi, t}^{2n+2} \times \tilde{\mathfrak{m}}_{\xi, t}^{2n+4}$  and the vector monomials  $(0, t^{2\ell+4})$  for  $\ell \in \mathbb{N}$ , Remark 2 implies that every  $S_{1, n}$  type tangential family germ is  $\mathcal{A}$ -equivalent to  $f_n$  ( $n \in \mathbb{N}$ ), and hence to the  $S_{1, n}$ -normal form  $f'_n$  listed in Theorem 1.

The  $\mathcal{E}_{\xi, t}$ -ideal  $\langle f_n \rangle$  is generated by  $\xi$  and  $t^4$ . By the Preparation Theorem of Malgrange and Mather (see e.g. [14]),  $\{1, t, t^2, t^3\}$  is a generator system of the  $\mathcal{E}_{x, y}$ -module  $\mathcal{E}_{\xi, t}$ . Now,  $T f_n$  contains  $\tilde{\mathfrak{m}}_{\xi, t}^{2n} \times \tilde{\mathfrak{m}}_{\xi, t}^{2n+2}$  and the vector monomials  $(0, t^{2\ell})$  for  $\ell \in \mathbb{N}$ . Hence, the vectors  $(0, t^{2i+1})$ ,  $i = 0, \dots, n$ , form a basis of the real vector space  $\mathcal{E}_{\xi, t}^2 / T f_n$ . Therefore, the codimension of  $f_n$  is  $n + 1$  and  $f_n + \sum_{i=0}^n (0, \lambda_i t^{2i+1})$  is an  $\mathcal{A}$ -miniversal deformation of  $f_n$ . As a consequence,

$f'_n + \sum_{i=0}^n (0, \lambda_i t^{2i+1})$  is an  $\mathcal{A}$ -miniversal deformation of the normal form  $f'_n$ ; moreover, the singularities of class  $S_{1,\infty}$  have infinite codimension.

Let us consider a tangential deformation  $F = F(\xi, t; \alpha)$  of the tangential family germ  $f'_n$ . By the very definition of versality, there exist function germs  $\lambda_i(\alpha)$  such that  $F$  is equivalent to  $f'_n + \sum_{i=0}^n (0, \lambda_i(\alpha) t^{2i+1})$ . By Remark 1, such a deformation is tangential if and only if  $\lambda_0 \equiv 0$ . Thus,  $f'_n + \sum_{i=1}^n (0, \lambda_i t^{2i+1})$  is a miniversal tangential deformation of  $f'_n$ .

**$S_2$ -type tangential families.** Set  $\deg(\xi) = 3$ ,  $\deg(t) = 1$ . By Lemma 1, every tangential family germ of type  $S_2$  is  $\mathcal{A}$ -equivalent to a map germ  $g + (0, \tilde{\delta}(5))$ , where  $g(\xi, t) := (\xi, t^5 + t^2\xi)$ . It follows from the inclusion  $\tilde{\mathbf{m}}_{\xi,t}^8 \times \tilde{\mathbf{m}}_{\xi,t}^{10} \subset T_r g$  and a well known Theorem of Gaffney (see [11], [4]) that  $g$  is 9- $\mathcal{A}$ -determined. One checks that there are exactly four  $\mathcal{A}$ -orbits in  $(\mathbf{m}_{\xi,t})^2$  over the weighted 5-jet  $g$ , represented by the map germs  $g_2 := g + (0, t^6)$ ,  $g_3^\pm := g \pm (0, t^9)$  and  $g_4 := g$  (the forthcoming computations being identical for  $g_3^+$  and for  $g_3^-$ , we omit the sign  $\pm$ ). These germs are  $\mathcal{A}$ -equivalent to the normal forms  $g'_i$  given in Theorem 1.

In all the four cases we deal with, the quotient space  $\mathcal{E}_{\xi,t}/\langle g_i \rangle$ , where  $i = 2, 3, 4$ , is generated by  $\{t, t^2, t^3, t^4\}$  (Preparation Theorem). Set  $e_0 := t$ ,  $e_1 := t^3$ ,  $e_2 := t^4$ ,  $e_3 := t^6$ ,  $e_4 := t^9$ . Using the inclusions  $\tilde{\mathbf{m}}_{\xi,t}^3 \times \tilde{\mathbf{m}}_{\xi,t}^5 \subset T_r g_2$ ,  $\tilde{\mathbf{m}}_{\xi,t}^5 \times \tilde{\mathbf{m}}_{\xi,t}^7 \subset T_r g_3$  and the above inclusion concerning  $g_4$ , one proves that  $\{(0, e_j) : j = 1, \dots, i\}$  is a basis of the real vector space  $\mathcal{E}_{\xi,t}^2/Tg_i$ , so the map germs  $g_i + \sum_{j=0, \dots, i} \lambda_j(0, e_j)$  are  $\mathcal{A}$ -miniversal deformations of the map germs  $g_i$ ; in particular, the codimensions of  $g_2$ ,  $g_3$  and  $g_4$  are 3, 4 and 5. Thus, the map germs  $g'_i + \sum_{j=0, \dots, i} \lambda_j(0, e_j)$  are  $\mathcal{A}$ -miniversal deformations of the normal forms  $g'_i$ . As for  $S_{1,n}$ -type tangential families, we see that these deformations are miniversal tangential deformations if and only if  $\lambda_0 \equiv 0$ .

**$T$ -type tangential families.** Set  $\deg(t) = n + 1$  and  $\deg(\xi) = 1$  and fix  $n \in \mathbb{N}$ . By Lemma 1, any  $T_n$ -type tangential family germ is  $\mathcal{A}$ -equivalent to  $h_n + (0, \tilde{\delta}(3n + 3))$ , where  $h_n(\xi, t) := (\xi, t^3 + t^2\xi^{n+1})$ . Since  $\tilde{\mathbf{m}}_{\xi,t}^2 \times \tilde{\mathbf{m}}_{\xi,t}^{3n+4} \subset T_r h_n$ , every such a germ is  $\mathcal{A}$ -equivalent to  $h_n$ , due to Remark 2, and then to the normal form  $h'_n$  listed in Theorem 1.

Now,  $\langle h_n \rangle$  is generated by  $\xi$  and  $t^3$ . Since  $\mathcal{E}_{\xi,t} \times \tilde{\mathbf{m}}_{\xi,t}^{3n+2} \subset Th_n$ , the Preparation Theorem implies that the vectors  $(0, t\xi^j)$ ,  $j = 0, \dots, n$  and  $(0, t^2\xi^i)$ ,  $i = 0, \dots, n - 1$  form a basis of the real vector space  $\mathcal{E}_{\xi,t}^2/Tf_n$ . Therefore, the codimension of  $h_n$  is  $2n + 1$  (so, the codimension of any  $T_\infty$  singularity is infinite). Actually, we get still a basis if we replace the vectors  $(0, t\xi^j)$  and  $(0, t^2\xi^i)$  by  $v_j := (0, t(\xi - t)^j)$  and  $w_i := (0, t^2(\xi - t)^i)$ , so  $h_n + \sum_{i=1}^n \lambda_i w_i + \sum_{j=0}^n \mu_j v_j$  is an  $\mathcal{A}$ -miniversal deformation of  $h_n$ . Thus,  $h'_n + \sum_{i=1}^n \lambda_i(0, t^2\xi^{i-1}) + \sum_{j=0}^n \mu_j(0, t\xi^j)$  is an  $\mathcal{A}$ -miniversal deformation of the normal form  $h'_n$ . Any tangential deformation of the  $T_n$ -normal form is equivalent to the above map germ, in which  $\lambda_j$  and  $\mu_i$  are function germs depending on the parameter of the tangential deformation. Due to Remark 1, all the function germs  $\mu_i$  have to be identically zero.

The above computations end the proof of Theorem 2. To complete the proof of Theorem 1, it remains to show that the singularity classes  $S_{1,\infty}$ ,  $T_\infty$ ,  $S_{n \geq 3}$  and  $U$  are non simple. For singularities of classes  $S_{1,\infty}$  and  $T_\infty$ , this follows from the adjacency of  $S_{1,\infty}$  (resp.,  $T_\infty$ ) to  $S_{1,n}$  (resp.,  $T_n$ ) for every  $n \in \mathbb{N}$ .

We prove now that the singularities of classes  $S_{n \geq 3}$  are non simple. Set  $\deg(\xi) = 4$ ,  $\deg(t) = 1$ . The germ  $f_a(\xi, t) := (\xi, t^6 + t^2\xi + t^7 + at^9)$ ,  $a \in \mathbb{R}$ , has a singularity of class  $S_3$ ,

due to Lemma 1. By a direct computation, one verifies that the weighted 9-jets  $f_a$  and  $f_b$  are not  $\mathcal{A}$ -equivalent if  $a \neq b$ . Thus, in the neighborhood of each such a map germ, there are infinitely many  $\mathcal{A}$ -orbits; that is, the germ is non simple. Let us consider a tangential family  $f$  of class  $S_3$ . By Lemma 1, its prenormal form is  $f(\xi, t) = (\xi, t^6 + t^2\xi + \tilde{\delta}(6))$ . It is easy to see that the weighted 9-jet of  $f$  is  $\mathcal{A}$ -equivalent to  $(\xi, t^6 + t^2\xi + At^7 + Bt^9)$  for some  $A, B \in \mathbb{R}$ . If  $A \neq 0$ , we normalize it to 1. Now the weighted 9-jet of  $f$  is of the form  $f_a$  for some  $a$ , so  $f$  is non simple. If  $A = 0$ , we consider the deformation  $F_\varepsilon(\xi, t) := f(\xi, t) + (0, \varepsilon t^7)$ . For every  $\varepsilon \neq 0$ ,  $F_\varepsilon$  is non simple, so  $f$  is also non simple. Finally, the singularities belonging to  $S_{n>3}$  and  $S_\infty$  are adjacent to  $S_3$ , so they are not simple.

It remains to prove that the singularities of class  $U$  are non simple. Let us set  $f_a(\xi, t) := (\xi, t^4/4 + 2at^3\xi/3 + t^2\xi^2/2)$ . Due to Lemma 1,  $f_a$  is of type  $U$  for every  $a$ . The vertical direction at the origin and the three branches of the complex discriminant set define four points on the projective complex line, whose complex cross ratio is  $2a^2 - 1 + 2\sqrt{a^2 - 1}$ . This number is an invariant of tangential family germs, i.e.  $\mathcal{A}$ -equivalent tangential families have the same cross ratio. Therefore, two germs whose 4-jets are  $f_a$  and  $f_b$ , are not  $\mathcal{A}$ -equivalent whenever their cross ratios are different, i.e.  $|a| \neq |b|$ . Consider now the prenormal form  $f(\xi, t) = (\xi, t^3 \cdot \delta(1))$  of any  $U$ -type tangential family. Set  $F_\varepsilon(\xi, t) := f(\xi, t) + (0, \varepsilon(t^4 + t^2\xi^2))$ . For every  $\varepsilon$  small enough,  $F_\varepsilon$  is  $\mathcal{A}$ -equivalent to a non simple map germ, so  $f$  is non simple. This ends the proof of Theorem 1.

**Acknowledgments.** The results of this paper are contained in my PhD Thesis [9], written under the supervision of V. I. Arnold, who proposed to me this problem and carefully read this paper. I also like to thank F. Aicardi, M. Garay and V. V. Goryunov for useful discussions and comments.

## References

- [1] ARNOLD, V. I.: On the envelope theory, *Uspechi Math. Nauk.* **3**, **31** (1976), 248–249.
- [2] ARNOLD, V. I.: Wave front evolution and equivariant Morse lemma, *Comm. Pure Appl. Math.* **6**, **29** (1976), 557–582.
- [3] ARNOLD, V. I.: Astroidal geometry of hypocycloids and the Hessian topology of hyperbolic polynomials, *Russian Math. Surveys* **6**, **56** (2001), 1019–1083.
- [4] ARNOLD, V. I.; GORYUNOV, V. V.; LYASHKO, O. V.; VASIL'EV, V. A.: *Singularity theory. I*. Springer-Verlag, Berlin, 1998.
- [5] ARNOLD, V. I.; GORYUNOV, V. V.; LYASHKO, O. V.; VASIL'EV, V. A.: *Singularity theory. II*. Springer-Verlag, Berlin, 1998.
- [6] BRUCE, J. W.; GIBLIN, P. J.: Projections of surfaces with boundary. *Proc. London Math. Soc.* (3) **60** (1990), no. 2, 392–416.
- [7] CAPITANIO, G.: On the envelope of 1-parameter families of curves tangent to a semicubic cusp, *C. R. Math. Acad. Sci. Paris* **3**, **335** (2002), 249–254.

- [8] CAPITANIO, G.: Singularities of envelopes of curves tangent to a semicubic cusp. To appear in Proc. of Suzdal Int. Conf. 2002.
- [9] CAPITANIO, G.: Tangential families and minimax solutions to Hamilton–Jacobi equations. PhD Thesis, Université D. Diderot – Paris VII, 2004.
- [10] CAPITANIO, G.: Stable tangential families and singularities of their envelopes, submitted to Bull. Lond. Math. Soc., ArXiv: math.DG/0406584.
- [11] GAFFNEY, T.: A note on the order of determination of a finitely determined germ, *Invent. Math.*, **52** (1979), 127–130.
- [12] GORYUNOV, V. V.: Singularities of projections of complete intersections, *J. Soviet. Math.*, **27** (1984), 2785–2811.
- [13] GORYUNOV, V. V.: Projections of generic surfaces with boundaries. *Theory of singularities and its applications*, 157–200, Adv. Soviet Math., 1, Amer. Math. Soc., Providence, RI, 1990.
- [14] MARTINET, J.: *Singularities of smooth functions and maps*, L. M. S. Lecture Note Series, 58. Cambridge University Press, 1982. xiv+256 pp.
- [15] THOM, R.: Sur la théorie des enveloppes, *J. Math. Pures Appl.* 9, **41** (1962), 177–192.

GIANMARCO CAPITANIO

*Université D. Diderot – Paris VII*

*UFR de Mathématiques*

*Equipe de Géométrie et Dynamique*

*Case 7012 – 2, place Jussieu*

*75251 Paris Cedex 05*

*e-mail: Gianmarco.Capitanio@math.jussieu.fr*