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# On generating series of complementary planar trees

Roland Bacher

*Abstract*<sup>1</sup>: We generalize and reprove an identity of Parker and Loday. It states that certain pairs of generating series associated to pairs of labelled rooted planar trees are mutually inverse under composition.

## 1 Introduction

In [1] Carlitz, Scoville and Vaughan consider finite words (in a finite alphabet  $\mathcal{A}$ ) such that all pairs of consecutive letters belong to a fixed subset  $L \subset \mathcal{A} \times \mathcal{A}$ . They show (Theorems 6.8 and 7.3 of [1]) that suitably defined pairs of signed generating series counting such words associated to  $L \subset \mathcal{A} \times \mathcal{A}$  and to its complementary set  $\bar{L} = \mathcal{A} \times \mathcal{A} \setminus L$  are each others inverse. Their result was generalized in the first part of Parker's thesis [5] who showed an analogous result for suitable classes of finite trees having labelled vertices. Loday in [3], motivated by questions concerning combinatorial realisations of operads, rediscovered Parkers result together with a different proof, based on homological arguments.

This paper presents a further generalisation of Parkers and Loday's result.

A typical example of our identity can be described as follows: Associate to two complex matrices

$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

the following two systems of algebraic equations

$$\begin{cases} g_1 &= (-X + ag_1 + bg_2)(-X + \alpha g_1 + \beta g_2) \\ g_2 &= (-X + cg_1 + dg_2)(-X + \gamma g_1 + \delta g_2) \\ g &= -X + g_1 + g_2 \end{cases}$$

and

$$\begin{cases} \tilde{g}_1 &= (X - (1-a)\tilde{g}_1 - (1-b)\tilde{g}_2)(X - (1-\alpha)\tilde{g}_1 - (1-\beta)\tilde{g}_2) \\ \tilde{g}_2 &= (X - (1-c)\tilde{g}_1 - (1-d)\tilde{g}_2)(X - (1-\gamma)\tilde{g}_1 - (1-\delta)\tilde{g}_2) \\ \tilde{g} &= -X + \tilde{g}_1 + \tilde{g}_2 \end{cases} .$$

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Choosing continuous determinations satisfying  $g = g_1 = g_2 = \tilde{g} = \tilde{g}_1 = \tilde{g}_2 = 0$  at  $X = 0$  we get holomorphic functions  $g = g(X)$ ,  $\tilde{g} = \tilde{g}(X)$  for  $X$  in an open neighbourhood of  $0 \in \mathbb{C}$ . We have now

$$g(\tilde{g}(X)) = X$$

for all  $X$  in a small open disc centered at 0.

This result holds of course formally for the corresponding generating series and can be verified by computing for instance a minimal polynomial  $P(u, v) = \sum_{i,j} p_{i,j} u^i v^j$  for  $g$  (i.e. satisfying in particular  $P(g(X), X) = 0$ ) and checking that we have  $P(X, \tilde{g}(X)) = 0$ . Since this identity is algebraic, the field  $\mathbb{C}$  can be replaced by an arbitrary commutative ring.

The sequel of this paper is organized as follows: The next section states the main result in purely algebraic terms over a not necessarily commutative ring. Parkers and Lodays result corresponds to the special case of matrices with coefficients in  $\{0, 1\}$ . Our main result removes the restriction on the coefficients. It is also somewhat easier to state (at least in a commutative setting) since it avoids combinatorial descriptions. It follows from our formulation that all involved generating functions are algebraic in a commutative setting and over a finite alphabet. Section 3 fixes notations concerning trees. Section 4 describes spin models on trees and recasts our main result using partition functions of spin models. Section 5 proves the main result using a spin model on grafted trees (called “graftings” in [3]). The proof avoids homological arguments and is thus in some sense more elementary (although perhaps not simpler) than the proof of [3]. Section 6 describes briefly a further generalisation involving arbitrary (not necessarily regular) finite trees which appears already in Parkers work. Section 7 is a digression generalizing the notion of grafted trees. Section 8 contains the computations for example (i) of [3]. We display the defining polynomial of the relevant (algebraic) generating function and discuss briefly its asymptotics.

## 2 Main result

Consider a (not necessarily finite) alphabet  $\mathcal{A}$  and a (not necessarily commutative) associative ring  $R$  having a unit 1. We denote by  $Y_{\mathcal{A}}$  a set of (non-commutative) variables indexed by elements  $\alpha \in \mathcal{A}$  and by  $X$  a supplementary (non-commutative) variable. We denote by  $\tilde{R}[[X]] = \mathbf{Z}[[R, Y_{\mathcal{A}}, R]] \otimes_{\mathbf{Z}} \mathbf{Z}$  the ring of formal power series. An element of  $\tilde{R}[[X]]$  is a (generally infinite) sum of monomials of the form

$$\pm r_1 Z_1 r_2 Z_2 \cdots r_l Z_l r_{l+1}$$

with  $r_i \in R$  and  $Z_i \in \{X\} \cup Y_{\mathcal{A}}$ . Let  $k \geq 2$  be a natural integer and let  $M_1, \dots, M_k$  be a set of matrices with rows and columns indexed by  $\mathcal{A}$  and

coefficients  $M_j(\alpha, \beta) \in R$  for  $\alpha, \beta \in \mathcal{A}$ . For  $\alpha \in \mathcal{A}$ , let  $g_\alpha \in \tilde{R}[[X]]$  be the power series  $g_\alpha = Y_\alpha X^k +$  (terms of higher order in  $X$ ) which satisfies

$$g_\alpha = Y_\alpha(X - (M_1 V)_\alpha)(X - (M_2 V)_\alpha) \cdots (X - (M_k V)_\alpha)$$

where  $V$  is the column vector with coordinates  $g_\beta$ ,  $\beta \in \mathcal{A}$  and where  $(M_j V)_\alpha = \sum_{\beta \in \mathcal{A}} M_j(\alpha, \beta) g_\beta$  denotes the  $\alpha$ -th coordinate of the matrix-product  $M_j V$ . Set  $g = -X + \sum_{\alpha \in \mathcal{A}} g_\alpha$ .

Remark that our notation is slightly misleading:  $g, g_1, \dots, g_l$  are power series in  $X$  defining “functions” of  $X$ . The letter  $X$  stands of course for the power series at  $X = 0$  of the identity function  $X \mapsto X$ . Remark also that the requirements  $k \geq 2$  and  $g_\alpha = Y_\alpha X^k +$  terms of higher order in  $X$  ensure that  $g_\alpha$  is well-defined: Its  $n$ -th coefficient involving  $X$   $n$  times depends only on coefficients with indices  $\leq n - k + 1$  of  $g_\beta$ ,  $\beta \in \mathcal{A}$  involving at most  $n - k + 1$  occurrences of  $X$ .

Define “complementary” matrices  $\tilde{M}_1, \dots, \tilde{M}_k$  with coefficients  $\tilde{M}_j(\alpha, \beta) = 1 - M_j(\alpha, \beta)$  by setting  $\tilde{M}_j = J - M_j$  where  $J$  is the all 1 matrix with rows and columns indexed by  $\mathcal{A}$ . For  $\alpha \in \mathcal{A}$  we introduce “complementary” functions  $\tilde{g}_\alpha = Y_\alpha(-X)^k +$  terms of higher order in  $X \in \tilde{R}[[X]]$  satisfying

$$\tilde{g}_\alpha = Y_\alpha(-X + (\tilde{M}_1 \tilde{V})_\alpha) \cdots (-X + (\tilde{M}_k \tilde{V})_\alpha)$$

where  $\tilde{V}$  is the column vector with coordinates  $\tilde{g}_\beta$ ,  $\beta \in \mathcal{A}$  and where  $(\tilde{M}_j \tilde{V})_\alpha = \sum_{\beta \in \mathcal{A}} \tilde{M}_j(\alpha, \beta) \tilde{g}_\beta$ . We define now  $\tilde{g} = -X + \sum_{j=1}^l \tilde{g}_j$ .

Given two formal power series  $f, h \in \tilde{R}[[X]]$  such that every monomial of  $h$  is at least of degree 1 in  $X$ , the composition  $f \circ_X h$  of  $f$  with  $h$  is defined as the formal power series obtained by replacing every occurrence of  $X^k$ ,  $k = 1, 2, 3, \dots$  in every monomial of  $f$  by the series  $h^k$ .

The main result of this paper can now be stated as follows:

**Theorem 2.1** *We have (formally)*

$$g \circ_X \tilde{g} = \tilde{g} \circ_X g = X .$$

**Remark 2.2** (i) *In a commutative setting with  $R = \mathbf{C}$  the field of complex numbers, a finite alphabet  $\mathcal{A}$ , and  $Y_\alpha \in \mathbf{C}$  for  $\alpha \in \mathcal{A}$ , the functions  $g_\alpha, \tilde{g}_\alpha$  and thus also  $g$  and  $\tilde{g}$  are holomorphic determinations of algebraic functions in an open neighbourhood of 0.*

(ii) *The above definitions of  $g_\alpha$  and  $\tilde{g}_\alpha$  are well-suited for iterative computations of the power-series  $g_\alpha, \tilde{g}_\alpha$  by “bootstrapping” (see e.g. section 5.4 of [4]). Indeed, given  $g_{\alpha, n}$ ,  $\alpha \in \mathcal{A}$  such that  $g_\alpha - g_{\alpha, n} = O(X^{1+(n+1)(k-1)})$  we have*

$$g_\alpha - Y_\alpha(t - (M_1 V_n)_\alpha) \cdots (t - (M_k V_n)_\alpha) = O(X^{1+(n+2)(k-1)})$$

where  $V_n$  is the column vector with coordinates  $g_{\alpha, n}$ ,  $\alpha \in \mathcal{A}$ . An analogous result holds of course for  $\tilde{g}_1, \dots, \tilde{g}_l$ .

(iii) Changing the grading and considering also all variables  $Y_\alpha$ ,  $\alpha \in \mathcal{A}$  as being of degree 1, Theorem 2.1 holds also for  $k = 1$  (as a special case, one obtains the main result of [1]). In this case,  $g \circ_X \tilde{g}$  boils down to a simple product in  $\tilde{R}$  since  $g$  and  $\tilde{g}$  are of the form  $cX$  with  $c \in R[[Y_\alpha]_{\alpha \in \mathcal{A}}]$ .

(iv) The choice of an integer  $k \geq 2$  (or  $k = 1$ ) corresponds to the case of  $k$ -regular trees. This restriction can be removed: Section 6 describes a further generalization (already contained in [5]), corresponding to arbitrary finite trees.

### 3 Trees

A *tree* is a connected graph without cycles. A *rooted tree* contains a marked vertex  $r$ , called the *root*. In particular, a rooted tree is non-empty. A rooted tree  $T = \{r\}$  reduced to its root is *trivial*. The edges of a rooted tree are canonically oriented by requiring the root to be the unique source of the directed tree obtained by orienting all edges away from the root. We write  $e = (\alpha(e), \omega(e))$  for the edge  $e$  oriented from  $\alpha(e)$  to  $\omega(e)$ . Given an edge  $e = (\alpha(e), \omega(e))$  we call  $\omega(e)$  a *son* of  $\alpha(e)$  and  $\alpha(e)$  the *father* of  $\omega(e)$ . Each vertex  $v \neq r \in T$  except the root has a unique father. The set of vertices sharing a common father is a *brotherhood*. A *leaf* is a vertex without sons. A vertex having at least one son is *interior*. The *level* of a vertex is its (combinatorial) distance to the root. A rooted tree is *planar* if every brotherhood is totally ordered. Every brotherhood of a *locally finite* rooted tree is finite. A rooted tree is thus locally finite if and only if the set of vertices of given level  $N$  is finite for all  $N \in \mathbf{N}$ . A rooted tree is  *$k$ -regular* if every non-empty brotherhood of strictly positive level contains exactly  $k$  vertices. A rooted tree is finite if and only if the set of vertices of level  $N$  is finite for all  $N$  and empty for  $N$  large enough. Given a vertex  $v \in T$ , we denote by  $T(v)$  the subtree of  $T$  rooted at  $v$  which is defined by considering all vertices  $w \in T$  such that  $v$  belongs to the unique geodesic path joining the root  $r \in T$  to  $w$  (the vertices of  $T(v)$  are thus  $v$  and its descendants). We call such a subtree *maximal*. A *principal* subtree of a non-trivial tree  $T$  is a maximal subtree  $T(s_i)$  rooted at a son  $s_i$  of the root  $r \in T$ . A non-trivial  $k$ -regular tree has thus exactly  $k$  principal subtrees.

Unless otherwise stated, a ( $k$ -regular) tree will henceforth always denote a ( $k$ -regular) finite rooted planar tree.

Every tree  $T$  can be represented by a plane tree of the halfplane  $\mathbf{R} \times \mathbf{R}_{\geq 0}$  such that a totally ordered brotherhood  $s_1 < s_2 < \dots$  of level  $n \geq 0$  corresponds to vertices  $(x_1, n), (x_2, n), \dots$ ,  $x_1 < x_2 < \dots$ .

A vertex  $v \in T$  of level  $n$  has a unique recursively defined *address*

$$a(v) = a_0 a_1 \dots a_{n-1} \in \{1, 2, \dots\}^n$$

where  $a_{n-1}$  is the number of elements  $\leq v$  in the brotherhood of  $v$  and where

$a_0 \dots a_{n-2}$  is the address of the (unique) father of  $v$ . The address of the root  $r \in T$  is empty. The lexicographic order on addresses orders the set of vertices of a planar rooted tree completely.

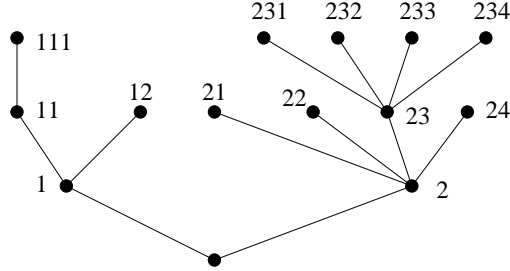


Figure 1: A (finite rooted planar) tree with addressed vertices.

Figure 1 illustrates the notions of this section. It displays a tree with 14 vertices

$$\emptyset < 1 < 11 < 111 < 12 < 2 < 21 < 22 < 23 < 231 < 232 < 233 < 234 < 24 ,$$

completely ordered by their addresses. The root corresponds to the lowest vertex having an empty address. The oriented edge  $e$  with vertices 23 and 232 starts at  $\alpha(e) = (23)$  and ends at the leaf  $\omega(e) = 232$ . The vertex 23 is thus the father of 232. The brotherhood of 232 are the four vertices 231, 232, 233 and 234 at level 3 having  $\alpha(e) = 23$  as their common father. This tree is of course not regular since the interior vertices 1 and 2 (for instance) have respectively 2 and 4 sons. The tree of Figure 1 has two principal subtrees. Two vertices with a non-empty address belong to a common principal subtree if and only if the first letter of their addresses coincides.

## 4 Non-commutative spin models on rooted planar trees

Let  $T = (E, V)$  be a (rooted planar) tree with edges  $E$  and vertices  $V$ . We denote by  $V^\circ$  the set of interior vertices (having at least one son) of  $T$  and by  $E' \subset E$  the subset of leafless edges (i.e.  $e \in E'$  if  $\omega(e) \in V^\circ$ ). We define a *spin-model*  $(T, w)$  on  $T$  by considering a (not necessarily) finite set or alphabet  $\mathcal{A}$  of *spins* and a *weight-function*

$$w : E' \times \mathcal{A} \times \mathcal{A} \longrightarrow R$$

with values in a (not necessarily commutative) ring  $R$  containing 1. Setting  $w_e(\alpha, \beta) = w(e, \alpha, \beta)$ , a weight function can be identified with an application

$$E' \longrightarrow M_{\mathcal{A}}(R)$$

where  $M_{\mathcal{A}}(R)$  denotes the set of matrices with coefficients in  $R$  and rows and columns indexed by  $\mathcal{A}$ . We call  $w_e \in M_{\mathcal{A}}(R)$  the *weight-matrix* of  $e \in E'$ .

We introduce furthermore (non-commutative) variables  $Y_\alpha$  for  $\alpha \in \mathcal{A}$  and a supplementary (non-commutative) variable  $X$ .

Given a spin-model  $(T, w)$ , its *complementary spin-model* is defined as  $(T, \tilde{w})$  where  $\tilde{w}_e(\alpha, \beta) = 1 - w_e(\alpha, \beta)$  for  $e \in E'$  and  $\alpha, \beta \in \mathcal{A}$ .

A colouring

$$\varphi : V^\circ \longrightarrow \mathcal{A}$$

of all interior vertices in  $T$  by elements of  $\mathcal{A}$  is a *state*. Its *energy*  $f_T(\varphi) \in R[X, (Y_\alpha)_{\alpha \in \mathcal{A}}]$  is recursively defined as follows: If  $T = \{r\}$  is trivial then  $f_T(\varphi) = X$ . Otherwise, we set

$$f_T(\varphi) = Y_{\varphi(r)} (w(1, \varphi) f_{T_1}(\varphi_1)) (w(2, \varphi) f_{T_2}(\varphi_2)) \cdots$$

where  $T_1, T_2, \dots$  are the principal subtrees associated to the linearly ordered sons  $s_1 < s_2 < \dots$  of the root  $r \in T$ , where

$$w(i, \varphi) = \begin{cases} w((r, s_i), \varphi(r), \varphi(s_i)) & \text{if } s_i \in V^\circ \\ 1 & \text{otherwise} \end{cases}$$

and where  $f_{T_i}(\varphi)$  is the energy of the spinmodel on the principal subtree  $T_i$  defined by the  $i$ -th son  $s_i$  of the root  $r \in T$  with weights and state obtained by restriction.

In a commutative setting, this boils down to

$$f(\varphi) = X^{\sharp(V \setminus V^\circ)} \prod_{v \in V^\circ} Y_{\varphi(v)} \prod_{e \in E'} w_e(\varphi(\alpha(e)), \varphi(\omega(e)))$$

which is the familiar definition used in statistical physics.

The total sum

$$Z = Z(T) = \sum_{\varphi \in \mathcal{A}^{V^\circ}} f(\varphi)$$

of energies over all states is the *partition function*. We denote by

$$Z_\alpha = Z_\alpha(T) = \sum_{\varphi \in \mathcal{A}^{V^\circ}, \varphi(r)=\alpha} f(\varphi)$$

the *restricted* partition function obtained by computing the total energy of all states with prescribed colour  $\varphi(r) = \alpha$  on the root. We have obviously

$$Z = \sum_{\alpha \in \mathcal{A}} Z_\alpha$$

if  $T$  is non-trivial. For  $T = \{r\}$  trivial, we have  $Z = X$  and  $Z_\alpha = 0$  for  $\alpha \in \mathcal{A}$ .

The partition function of a tree  $T$  can be computed as follows: We have  $Z(T) = X$  for  $T = \{r\}$  the trivial tree. Otherwise, consider the principal subtrees  $T_1, T_2, \dots$  associated to the  $k$  linearly ordered sons  $s_1 <$

$s_2 < \dots < s_k$  of  $r$ . For  $\alpha \in \mathcal{A}$  denote by  $Z_\alpha(T_j)$  the obvious restricted partition functions of the principal subtree  $T_j$  with colour  $\alpha$  on its root  $s_j$ . We denote by  $e_j = \{r = \alpha(e_j), s_j = \omega(e_j)\}$  the oriented edge joining the root  $r$  of  $T$  to its  $j$ -th son  $s_j = \omega(e_j)$ .

The definition of the restricted partition function implies then easily the following result:

**Proposition 4.1** *If  $T \neq \{r\}$  is non-trivial, we have*

$$Z_\alpha(T) = Y_\alpha C_1(\alpha) C_2(\alpha) \cdots C_k(\alpha)$$

where  $T_1, \dots, T_k$  are the principal subtrees associated to the  $k$  linearly ordered sons  $s_1 < s_2 < \dots < s_k$  of the root  $r \in T$  and where

$$C_k(\alpha) = \begin{cases} \sum_{\beta \in \mathcal{A}} w((r, s_i), \alpha, \beta) Z_\beta(T_i) & \text{if } T_i \neq \{s_i\} \\ X & \text{otherwise .} \end{cases}$$

An edge  $e \in E$  of a  $k$ -regular tree is of type  $j$  if the extremity  $\omega(e)$  (having level  $m + 1$ ) of  $e$  has an address

$$a_0 a_1 \dots a_m = a_0 a_1 \dots a_{m-1} j$$

ending with  $1 \leq j = a_m \leq k$ . The endpoint  $\omega(e)$  of a type  $j$  edge is thus the  $j$ -th element in its (totally ordered) brotherhood.

Given a (finite)  $k$ -regular tree  $T$  and  $k$  weight-matrices  $M_1, \dots, M_k$  (with indices in  $\mathcal{A} \times \mathcal{A}$ ) we consider the spin model with spins  $\mathcal{A}$  and weight function  $w_e(\alpha, \beta) = M_j(\alpha, \beta)$  on leafless edges of type  $j$ .

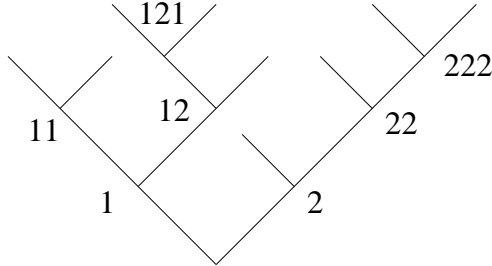


Figure 2: A 2-regular tree.

**Example.** We consider the commutative spin model on the 2-regular tree  $T$  of Figure 2 with  $\mathcal{A} = \{1, 2\}$  and

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad M_2 = (M_1)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

For the sake of simplicity, we set  $X = Y_1 = Y_2 = 1$ . Given an interior vertex  $v$  of  $T$ , we denote by  $Z_*(v)$  the vector  $\begin{pmatrix} Z_1(T(v)) \\ Z_2(T(v)) \end{pmatrix}$  where  $T(v)$  denotes the



maximal subtree of  $T$  with root  $v$  (obtained by considering the connected component of  $v$  in  $T \setminus a(v)$  where  $a(v)$  is the father of  $v \neq r$ ). Since all sons of 11, 121, 222 are leaves, we have

$$Z_*(11) = Z_*(121) = Z_*(222) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where we denote a vertex by its address. We get now

$$Z_*(12) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} Z_*(121) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

$$Z_*(22) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Z_*(222) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$Z_*(2) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Z_*(22) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Denoting by  $U * W$  the Hadamard product (“student’s vector product”)  $\begin{pmatrix} \alpha\alpha' \\ \beta\beta' \end{pmatrix}$  of two vectors  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$  we have then

$$\begin{aligned} Z_*(1) &= \left( \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} Z_*(11) \right) * \left( \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Z_*(12) \right) \\ &= \left( \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) * \left( \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) \\ &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} * \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} Z_*(\emptyset) &= \left( \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} Z_*(1) \right) * \left( \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Z_*(2) \right) \\ &= \left( \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) * \left( \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 5 \\ 8 \end{pmatrix} * \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 25 \\ -24 \end{pmatrix}. \end{aligned}$$

This yields the partition function

$$Z(T) = Z_1(T) + Z_2(T) = 25 - 24 = 1$$

for the tree of Figure 2 with respect to the weights  $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$  and  $M_2 = M_1^{-1}$  associated to leafless edges indicating  $NW$  (type 1), respectively  $NE$  (type 2).

Given a spinset (alphabet)  $\mathcal{A}$  and pairs of complementary matrices  $M_1, \tilde{M}_1 = J - M_1, \dots, M_k, \tilde{M}_k = J - M_k$  (where  $J$  denotes the all 1 matrix with row and column-indices in  $\mathcal{A}$ ), we consider the associated complementary spin models on  $k$ -regular trees with partition function  $Z(T), \tilde{Z}(T)$  as above.

Denoting by  $\mathcal{T}_k$  the set of all (finite, rooted, planar)  $k$ -regular trees, we introduce the signed formal generating series

$$Z(\mathcal{T}_k) = - \sum_{T \in \mathcal{T}_k} (-1)^{d^\circ(T)} Z(T)$$

and

$$\tilde{Z}(\mathcal{T}_k) = \sum_{T \in \mathcal{T}_k} (-1)^{d(T)} \tilde{Z}(T)$$

where  $d^\circ(T) = \sharp(V^\circ)$  denotes the number of interior vertices and  $d(T) = \sharp(V \setminus V^\circ)$  the number of leaves of a  $k$ -regular tree  $T \in \mathcal{T}_k$ .

Let us moreover consider the restricted generating series

$$Z_\alpha(\mathcal{T}'_k) = - \sum_{T \in \mathcal{T}'_k} (-1)^{d^\circ(T)} Z_\alpha(T)$$

and

$$\tilde{Z}_\alpha(\mathcal{T}'_k) = \sum_{T \in \mathcal{T}'_k} (-1)^{d(T)} \tilde{Z}_\alpha(T)$$

where  $\mathcal{T}'_k = \mathcal{T}_k \setminus \{r\}$  denotes the set of all  $k$ -regular trees which are non-trivial. We have obviously

$$Z(\mathcal{T}_k) = -X + \sum_{\alpha \in \mathcal{A}} Z_\alpha(\mathcal{T}'_k) \quad \text{and} \quad \tilde{Z}(\mathcal{T}_k) = -X + \sum_{\alpha \in \mathcal{A}} \tilde{Z}_\alpha(\mathcal{T}'_k).$$

**Proposition 4.2** *We have*

$$Z_\alpha(\mathcal{T}'_k) = Y_\alpha (X - (M_1 W)_\alpha) \cdots (X - (M_k W)_\alpha)$$

and

$$\tilde{Z}_\alpha(\mathcal{T}'_k) = Y_\alpha \left( -X + (\tilde{M}_1 \tilde{W})_\alpha \right) \cdots \left( -X + (\tilde{M}_k \tilde{W})_\alpha \right)$$

where  $W$  (respectively  $\tilde{W}$ ) is the column vector with coordinates  $Z_\alpha(\mathcal{T}'_k)$  (respectively  $\tilde{Z}_\alpha(\mathcal{T}'_k)$ ) indexed by  $\alpha \in \mathcal{A}$ .

For a fixed natural integer  $k \geq 1$  and  $k$  (weight-)matrices  $M_1, \dots, M_k$  we have now:

**Corollary 4.3** *We have  $g_\alpha = Z_\alpha(\mathcal{T}'_k)$ ,  $\tilde{g}_\alpha = \tilde{Z}_\alpha(\mathcal{T}'_k)$  for all  $\alpha \in \mathcal{A}$  and  $g = Z(\mathcal{T}_k)$ ,  $\tilde{g} = \tilde{Z}(\mathcal{T}_k)$  where  $g_\alpha, \tilde{g}_\alpha, g, \tilde{g}$  are as in Theorem 2.1 and where  $Z_\alpha(\mathcal{T}'_k), \tilde{Z}_\alpha(\mathcal{T}'_k), Z(\mathcal{T}_k), \tilde{Z}(\mathcal{T}_k)$  are as above.*

**Proof of Proposition 4.2** Given a non-trivial  $k$ -regular tree  $T \neq \{r\}$  with root  $r$ , we denote by  $T_1, \dots, T_k$  the principal subtrees associated to the  $k$  linearly ordered sons  $s_1 < s_2 < \dots < s_k$  of  $r$ .

The generating function

$$Z_\alpha(\mathcal{T}'_k) = - \sum_{T \in \mathcal{T}'_k} (-1)^{d^\circ(T)} Z_\alpha(T) = \sum_{T \in \mathcal{T}'_k} (-1)^{\sum_j d^\circ(T_j)} Z_\alpha(T)$$

(with the last equality following from  $d^\circ(T) = 1 + \sum_{j=1}^k d^\circ(T_j)$ ) factorizes now as

$$\begin{aligned} Z_\alpha(\mathcal{T}'_k) &= Y_\alpha \left( X + \sum_{\beta \in \mathcal{A}} M_1(\alpha, \beta) \sum_{T \in \mathcal{T}'_k} (-1)^{d^\circ(T)} Z_\beta(T) \right) \cdot \\ &\quad \cdot \left( X + \sum_{\beta \in \mathcal{A}} M_2(\alpha, \beta) \sum_{T \in \mathcal{T}'_k} (-1)^{d^\circ(T)} Z_\beta(T) \right) \cdots \\ &\quad \cdots \left( X + \sum_{\beta \in \mathcal{A}} M_k(\alpha, \beta) \sum_{T \in \mathcal{T}'_k} (-1)^{d^\circ(T)} Z_\beta(T) \right) \\ &= Y_\alpha \left( X - \sum_{\beta \in \mathcal{A}} M_1(\alpha, \beta) Z_\beta(\mathcal{T}'_k) \right) \cdots \left( X - \sum_{\beta \in \mathcal{A}} M_k(\alpha, \beta) Z_\beta(\mathcal{T}'_k) \right). \end{aligned}$$

Indeed, neglecting the trivial term  $Y_\alpha$ , the partition function  $Z_\alpha(T)$  of a tree  $T \in \mathcal{T}'_k$  with prescribed spin  $\varphi(r) = \alpha$  on its root decomposes into  $k$  obvious factors corresponding to the  $k$  principal subtrees of  $T$ : A leafless edge  $e \in E'$  with  $\omega(e) \in T_j$  yields a contribution to the  $j$ -th factor of  $Z_\alpha(\mathcal{T}'_k)$ . Summing over all  $k$ -regular trees  $T \in \mathcal{T}'_k$  yields the above expression. The term  $X$  in the  $j$ -th factor corresponds to trees  $T \in \mathcal{T}'_k$  whose  $j$ -th principal subtree  $T_j$  is trivial. This proves the first equality of Proposition 4.2.

For

$$\tilde{Z}_\alpha(\mathcal{T}'_k) = \sum_{T \in \mathcal{T}'_k} (-t)^{d(T)} \tilde{Z}_\alpha(T) = \sum_{T \in \mathcal{T}'_k} (-t)^{\sum_j d(T_j)} \tilde{Z}_\alpha(T)$$

we get the analogous factorization

$$\begin{aligned} \tilde{Z}_\alpha(\mathcal{T}'_k) &= Y_\alpha \left( -X + \sum_{\beta \in \mathcal{A}} \tilde{M}_1(\alpha, \beta) \sum_{T \in \mathcal{T}'_k} (-1)^{d(T)} \tilde{Z}_\beta(T) \right) \cdot \\ &\quad \cdot \left( -X + \sum_{\beta \in \mathcal{A}} \tilde{M}_2(\alpha, \beta) \sum_{T \in \mathcal{T}'_k} (-1)^{d(T)} \tilde{Z}_\beta(T) \right) \cdots \\ &\quad \cdots \left( -X + \sum_{\beta \in \mathcal{A}} \tilde{M}_k(\alpha, \beta) \sum_{T \in \mathcal{T}'_k} (-1)^{d(T)} \tilde{Z}_\beta(T) \right) \\ &= Y_\alpha \left( -X + \sum_{\beta \in \mathcal{A}} \tilde{M}_1(\alpha, \beta) \tilde{Z}_\beta(\mathcal{T}'_k) \right) \cdots \left( -X + \sum_{\beta \in \mathcal{A}} \tilde{M}_k(\alpha, \beta) \tilde{Z}_\beta(\mathcal{T}'_k) \right) \end{aligned}$$

which proves the second equality of Proposition 4.2.  $\square$

The proof of Corollary 4.3 is immediate: corresponding series are recursively defined by the same formulae and initial data.

## 5 Proof of Theorem 2.1

By Corollary 4.3, the formal power series  $g = g(X) = -X + \sum_{\alpha \in \mathcal{A}} g_\alpha(X)$  and  $\tilde{g} = \tilde{g}(X) = -X + \sum_{\alpha \in \mathcal{A}} \tilde{g}_\alpha(X)$  involved in Theorem 2.1 are suitably

signed generating series for the partition functions of complementary spin-models defined on  $k$ -regular trees. In order to prove Theorem 2.1 we define spin-models on a set of combinatorial objects which we call grafted trees. A suitably signed generating series of the partition functions for these spin-models equals  $g \circ_X \tilde{g}$  and a direct computation establishes Theorem 2.1.

## 5.1 A spin model on grafted trees

A *grafted tree* is given by

$$(A; B_1, B_2, \dots, B_{d(A)})$$

where  $A$  is a tree with  $d(A)$  leaves and where  $B_1, \dots, B_{d(A)}$  is a sequence of  $d(A)$  trees. A grafted tree is  *$k$ -regular* if it involves only  $k$ -regular trees.

The *skeleton* of a grafted tree  $(A; B_1, \dots, B_{d(A)})$  is the tree obtained by grafting (gluing) the root of  $B_1$  to the smallest (leftmost) leaf of  $A$ , by grafting the root of  $B_2$  to the second-smallest leaf of  $A$  etc. The skeleton of  $(A; B_1, \dots, B_{d(A)})$  has thus  $\sum_{j=1}^{d(A)} d(B_j)$  leaves.

The data of a  $k$ -regular grafted tree  $(A; B_1, \dots, B_{d(A)})$  and  $k$  weight-matrices  $M_1, \dots, M_k$  indexed by  $\mathcal{A} \times \mathcal{A}$  defines a ‘‘compositional’’ spin model as follows: Compute first the ordinary partition function  $Z(A)$  as defined previously. Compute also the partition functions  $\tilde{Z}(B_1), \dots, \tilde{Z}(B_{d(A)})$  with respect to the complementary weight matrices  $\tilde{M}_1 = J - M_1, \dots, \tilde{M}_k = J - M_k$  (where we denote by  $J$  the all 1 matrix indexed by  $\mathcal{A} \times \mathcal{A}$ ). Replace now the  $i$ -th (occurrence of the) letter  $X$  in every monomial of  $Z(A)$  by  $\tilde{Z}(B_i)$ . The resulting formal power series (or non-commutative polynomial for a finite alphabet  $\mathcal{A}$ ) is by definition  $Z(A; B_1, \dots, B_{d(A)})$ . It is of degree  $\sum_{i=1}^{d(A)} d(B_i)$  in  $X$ .

In particular, we have

$$Z(A; B_1, \dots, B_{d(A)}) = X^{-d(V_A)} Z(A) \prod_{j=1}^{d(A)} \tilde{Z}(B_j)$$

(with  $d(V_A)$  denoting the number of leaves in  $A$ ) in a commutative setting.

Let  $g(X) = -X + \sum_{\alpha \in \mathcal{A}} g_\alpha(X)$ ,  $\tilde{g}(X) = -X + \sum_{\alpha \in \mathcal{A}} \tilde{g}_\alpha(X)$  be the generating series involved in Theorem 2.1 associated to  $k$  matrices  $M_1, \dots, M_k$ .

**Proposition 5.1** *We have*

$$g \circ_X \tilde{g}(X) = - \sum_{(A; B_1, \dots, B_{d(A)}) \in \mathcal{G}_k} (-1)^{d^\circ(A) + \sum_{j=1}^{d(A)} d(B_j)} Z(A; B_1, \dots, B_{d(A)})$$

where  $\mathcal{G}_k$  denotes the set of all  $k$ -regular grafted trees.

**Proof.** This follows at once from Corollary 4.3 and the definition of spin-models on grafted trees.  $\square$

Given a  $k$ -regular tree  $T \in \mathcal{T}_k$  we denote by  $\mathcal{S}(T)$  the set of all  $k$ -regular grafted trees with skeleton  $T$ . Elements of  $\mathcal{S}(T)$  are in bijection with  $k$ -regular rooted subtrees of  $T$  containing the root  $r \in T$ : The subtree  $A \subset T$  of a  $k$ -regular grafted tree  $(A; B_1, \dots, B_{d(A)}) \in \mathcal{S}(T)$  with skeleton  $T$  clearly contains  $r$ . Since the subtree  $B_j \subset T$  is the maximal subtree of  $T$  rooted in the  $j$ -th leaf  $v_j$  of  $A$ , the pair of trees  $A \subset T$  defines the grafted tree  $(A; B_1, \dots, B_{d(A)})$  completely.

We denote by  $\mathcal{S}'(T) = \mathcal{S} \setminus \{(r); T\}$  the set of all  $k$ -regular grafted trees  $(A; B_1, \dots, B_{d(A)})$  with skeleton  $T$  and non-trivial  $A \neq \{r\}$ .

For  $\alpha \in \mathcal{A}$  and  $(A; B_1, \dots, B_{d(A)}) \in \mathcal{S}'(T)$  we define  $Z_\alpha(A; B_1, \dots, B_{d(A)})$  in the obvious way by replacing the  $i$ -th occurrence of the letter  $X$  in  $Z_\alpha(A)$  with  $\tilde{Z}(B_i)$  for  $i = 1, \dots, d(A)$ .

**Proposition 5.2** *We have for a non-trivial  $k$ -regular tree  $T \in \mathcal{T}_k'$  and for all  $\alpha \in \mathcal{A}$*

$$\sum_{(A; B_1, \dots, B_{d(A)}) \in \mathcal{S}'(T)} (-1)^{d^\circ(A)} Z_\alpha(A; B_1, \dots, B_{d(A)}) = -\tilde{Z}_\alpha(T) .$$

**Proof.** The proof is by induction on the number  $d(T)$  of leaves in  $T$ . If all  $k$  principal subtrees associated to the linearly ordered sons  $s_1 < \dots < s_k$  of the root  $r \in T$  are trivial, we have  $\mathcal{S}'(T) = (T; \{s_1\}, \dots, \{s_k\})$  and

$$\begin{aligned} & \sum_{(A; B_1, \dots, B_{d(A)}) \in \mathcal{S}'(T)} (-1)^{d^\circ(A)} Z_\alpha(A; B_1, \dots, B_{d(A)}) \\ &= (-1)^{d^\circ(T)} Z_\alpha(T; \{s_1\}, \dots, \{s_k\}) = -Y_\alpha . \end{aligned}$$

Since we have  $\tilde{Z}_\alpha(T) = Y_\alpha$ , Proposition 5.2 follows in this case.

Otherwise, we have

$$\sum_{(A; B_1, \dots, B_{d(A)}) \in \mathcal{S}'(T)} (-1)^{d^\circ(A)} Z_\alpha(A; B_1, \dots, B_{d(A)}) = -Y_\alpha C_1 C_2 \cdots C_k$$

where  $C_i = X$  if the principal subtree  $T_i = \{s_i\}$  is trivial and

$$C_i = \tilde{Z}(T_i) + \sum_{\beta \in \mathcal{A}} M_i(\alpha, \beta) \sum_{(A; B_1, \dots, B_{d(A)}) \in \mathcal{S}'(T_i)} Z_\beta(A; B_1, \dots, B_{d(A)})$$

if  $T_i \neq \{s_i\}$ . The contribution of a non-trivial principal subtree  $T_i \neq \{s_i\}$  is thus by induction

$$\begin{aligned} C_i &= \tilde{Z}(T_i) - \sum_{\beta \in \mathcal{A}} M_i(\alpha, \beta) \tilde{Z}_\beta(T_i) \\ &= \sum_{\beta \in \mathcal{A}} \left( \tilde{Z}_\beta(T_i) - M_i(\alpha, \beta) \tilde{Z}_\beta(T_i) \right) \\ &= \sum_{\beta \in \mathcal{A}} (1 - M_i(\alpha, \beta)) \tilde{Z}_\beta(T_i) \\ &= \sum_{\beta \in \mathcal{A}} \tilde{M}_i(\alpha, \beta) \tilde{Z}_\beta(T_i) \end{aligned}$$

and

$$-Y_\alpha C_1 C_2 \cdots C_k = -\tilde{Z}_\alpha(T) . \quad \square$$

**Corollary 5.3** *We have for a non-trivial  $k$ -regular tree  $T \in \mathcal{T}'_k$*

$$\sum_{(A; B_1, \dots, B_{d(A)}) \in \mathcal{S}(T)} (-1)^{d^\circ(A)} Z(A; B_1, \dots, B_{d(A)}) = 0 .$$

**Proof.** Sum the equality of Proposition 5.2 over  $\alpha \in \mathcal{A}$  and use

$$Z(\{r\}; T) = \tilde{Z}(T) = \sum_{\alpha \in \mathcal{A}} \tilde{Z}_\alpha(T) . \quad \square$$

**Proof of Theorem 2.1.** By Proposition 5.1 we have

$$g \circ_X \tilde{g} = - \sum_{T \in \mathcal{T}'_k} (-1)^{d(T)} \sum_{(A; B_1, \dots, B_{d(A)}) \in \mathcal{S}(T)} (-1)^{d^\circ(A)} Z(A; B_1, \dots, B_{d(A)}) .$$

It follows from Corollary 5.3 that only the trivial tree  $T = \{r\}$  contributes to the right hand side. The contribution of  $T = \{r\}$  amounts to

$$-(-1)^{d(\{r\})} Z(\{r\}; \{r\}) = -(-1) \cdot X = X$$

which proves Theorem 2.1.  $\square$

## 6 Further generalizations

We give first a generalization of Theorem 2.1 associated to not necessarily regular trees. The special case of this generalization where all weight-matrices have coefficients in  $\{0, 1\}$  appears already in [5].

We state then an even more general version involving trees with vertices labelled by a set  $\mathcal{T}$  of types and formal power-series in  $R[[X_\tau]_{\tau \in \mathcal{T}}, (Y_\alpha)_{\alpha \in \mathcal{A}}]$ .

### 6.1 Trees with vertices of arbitrary degrees

Consider a subset  $\mathcal{K} \subset \mathbb{N}_{>0}$  of strictly positive integers and a partition  $\mathcal{A} = \cup_{k \in \mathcal{K}} \mathcal{A}_k$  of the alphabet  $\mathcal{A}$  into non-empty parts indexed by  $\mathcal{K}$ . For each  $k \in \mathcal{K}$  choose  $k$  matrices  $M_1^k, M_2^k, \dots, M_k^k$  with rows indexed by  $\mathcal{A}_k$  and columns indexed by  $\mathcal{A}$ . For  $i = 1, \dots, k$ , set  $\tilde{M}_i^k = J - M_i^k$  where  $J$  is the all 1 matrix with indices in  $\mathcal{A}_k \times \mathcal{A}$ .

For  $\alpha \in \mathcal{A}_k$  consider the uniquely defined (non-commutative) formal power series  $g_\alpha, \tilde{g}_\alpha$  consisting only of monomials of degree at least 1 in  $X$ , involving at least one variable of  $(Y_\beta)_{\beta \in \mathcal{A}}$  and such that

$$g_\alpha = Y_\alpha (X - (M_1^k V)_\alpha) (X - (M_2^k V)_\alpha) \cdots (X - (M_k^k V)_\alpha)$$

and

$$\tilde{g}_\alpha = Y_\alpha(-X + (\tilde{M}_1^k \tilde{V})_\alpha) \cdots (-X + (\tilde{M}_k^k \tilde{V})_\alpha)$$

where  $V$ , respectively  $\tilde{V}$ , is the column vector with coordinates  $g_\beta$ , respectively  $\tilde{g}_\beta$ , for  $\beta \in \mathcal{A}$ .

$$\text{Set } g = -X + \sum_{\alpha \in \mathcal{A}} g_\alpha \text{ and } \tilde{g} = -X + \sum_{\alpha \in \mathcal{A}} \tilde{g}_\alpha.$$

**Theorem 6.1** *We have*

$$g \circ_X \tilde{g} = \tilde{g} \circ_X g = X .$$

The proof is an adaption of the proof of Theorem 2.1: Consider trees whose internal vertices have degrees in  $\mathcal{K}$  and colour internal vertices of degree  $k \in \mathcal{K}$  by elements in  $\mathcal{A}_k$ .

**Example (Inversion of power series).** In this example, we work over a commutative ring  $R$  and set  $Y_\alpha = 1$ . Choose an integer  $l \geq 1$  and set  $\mathcal{K} = \{2, 3, 4, \dots\}$  (this ensures existence of all relevant power-series). We consider trees with internal vertices of degree  $\geq 2$  having spins in a finite alphabet  $\mathcal{A}$  containing  $l$  elements.

For  $1 \leq j \leq k$ ,  $2 \leq k$  choose elements  $\beta_{k,j} \in R$  and consider the diagonal weight function

$$w_e(\varphi(\alpha(e)), \varphi(\omega(e))) = \beta_{k,j} \delta_{\varphi(\alpha(e)), \varphi(\omega(e))} = \begin{cases} \beta_{k,j} & \text{if } \varphi(\alpha(e)) = \varphi(\omega(e)) \\ 0 & \text{otherwise} \end{cases}$$

where the leafless edge  $e$  joins a vertex  $\alpha(e)$  of degree  $k$  to its  $j$ -th son  $\omega(e)$ .

Introduce the formal power series  $g_*$ ,  $\tilde{g}_* \in R[[X]]$  without constant term which satisfy

$$\begin{aligned} g_* &= \sum_{k=2}^{\infty} \prod_{j=1}^k (X - \beta_{k,j} g_*) \\ \tilde{g}_* &= \sum_{k=2}^{\infty} \prod_{j=1}^k (-X + (l - \beta_{k,j}) \tilde{g}_*) . \end{aligned}$$

Set  $g(X) = -X + l g_*$  and  $\tilde{g}(X) = -X + l \tilde{g}_*$ . Theorem 6.1, applied to the present situation shows that we have

$$g(\tilde{g}(X)) = \tilde{g}(g(X)) = X .$$

Since this formula holds for any  $l \in \mathbb{N}$ , it extends to an arbitrary value of  $l \in R$  which can thus be considered as a parameter.

This formula provides a mean (other than the celebrated Lagrange inversion formula) for computing the compositional inverse of a formal power series  $h(X) = \sum_{k=1}^{\infty} \gamma_k X^k$  with  $\gamma_1 \in R^*$  invertible: Set  $f = h(-X/\gamma_1) = -X + \sum_{k=2}^{\infty} \beta_k t^k$ . If  $\beta_2 \neq 0$  set  $l = \beta_2$  and choose constants  $\beta_{k,j}$  such that  $g(X) = f(X)$  with  $g(X)$  defined as above.

Each such choice can be used to compute the compositional inverse  $\tilde{g} = f^{-1}$  of  $f$ . The compositional inverse of the initial series  $h(X)$  is then given by  $-\gamma_1 \tilde{g}(X)$ . The case  $\beta_2 = 0$  can be handled similarly by allowing the trees to have internal vertices of degree 1.

## 6.2 Vertices of different types

This still more general version of Theorem 2.1 is perhaps easier to formulate in a combinatorial way:

Consider a set  $\mathcal{T}$  of *vertex-types* indexing the non-empty parts of a partition  $\mathcal{A} = \cup_{\tau \in \mathcal{T}} \mathcal{A}_\tau$ . Consider also a set  $\mathcal{E}$  of *edge-types* together with weight-functions

$$w_\epsilon : \mathcal{A} \times \mathcal{A} \longrightarrow R$$

indexed by edge-types  $\epsilon \in \mathcal{E}$ .

A *labelled tree* is a finite rooted planar tree  $L$  with leaves labelled by elements of  $\mathcal{T}$ , internal vertices labelled by pairs  $(\tau, \alpha \in \mathcal{A}_\tau) \in \mathcal{T} \times \mathcal{A}$  and edges labelled by elements of  $\mathcal{E}$ . A vertex  $v \in L$  is of type  $\tau \in \mathcal{T}$  if it is either a leaf labelled  $\tau$  or an internal vertex labelled  $(\tau, \alpha \in \mathcal{A}_\tau)$ . A labelled tree  $L$  is of type  $\tau \in \mathcal{T}$  if its root vertex is of type  $\tau$ .

The *energy*  $f(L) \in R[(X_\tau)_{\tau \in \mathcal{T}}, (Y_\alpha)_{\alpha \in \mathcal{A}}]$  of a labelled tree  $L$  of type  $\tau \in \mathcal{T}$  is defined in the obvious way:  $f(L) = X_\tau$  if  $L$  is trivial (reduced to its labelled root) and

$$f(L) = Y_\alpha C_1 \cdots C_k$$

otherwise where  $(\tau, \alpha \in \mathcal{A}_\tau)$  is the label of the root  $r \in L$  and where  $C_i \in R[(X_\tau)_{\tau \in \mathcal{T}}, (Y_\alpha)_{\alpha \in \mathcal{A}}]$  is associated to the  $i$ -th labelled principal subtree  $L(s_i)$  of  $L$  as follows:  $C_i = X_{\tau_i}$  if  $L(s_i)$  is trivial of type  $\tau_i$  and

$$C_i = w_{\epsilon_i}(\alpha, \alpha_i) f(L_i)$$

where  $\epsilon_i$  is the label of the edge joining the root  $r$  to its  $i$ -th son  $s_i$  which is labelled  $(\tau_i, \alpha_i \in \mathcal{A}_{\tau_i})$ .

Define complementary weight-functions by  $\tilde{w}_\epsilon(\alpha, \beta) = 1 - w_\epsilon(\alpha, \beta)$  and compute the *complementary energy*  $\tilde{f}(L)$  of  $L$  as above using the complementary weight-functions.

Call two labelled trees  $L_1, L_2$  *edge-equivalent* if they differ only on their edge-labels (but share the same underlying tree-structure and vertex-labels).

Let  $\mathcal{L}$  be a subset of the set of all labelled (finite rooted planar) trees. We denote by  $\mathcal{L}_\tau \subset \mathcal{L}$  the subset of all labelled trees of type  $\tau$  in  $\mathcal{L}$  and assume that the set  $\mathcal{L}$  satisfies the following three conditions:

(i) If  $L \in \mathcal{L}$  and  $v$  is a leaf of  $L$ , then  $L(v) \in \mathcal{L}$  where  $L(v)$  denotes the labelled subtree defined in the obvious way by considering the maximal subtree of  $L$  rooted at  $v$ .

(ii) If  $L' \in \mathcal{L}_\tau$  and  $v$  is a leaf of type  $\tau$  in  $L \in \mathcal{L}$  then the labelled tree obtained in the obvious way by gluing  $L'$  onto the leaf  $v \in L$  is again in  $\mathcal{L}$ .

(iii) All equivalence classes of the edge-equivalence relation are finite.

Otherwise stated, all maximal labelled subtrees of an element in  $\mathcal{L}$  are in  $\mathcal{L}$  by (i) and  $\mathcal{L}$  is “closed under composition” by (ii). Condition (iii) is a finiteness condition (which can perhaps be slightly weakened or replaced by



a similar statement) ensuring the existence of the generating series  $Z_\tau$  and  $\tilde{Z}_\tau$  defined below.

For such a set  $\mathcal{L}$  we define

$$Z_\tau = - \sum_{L \in \mathcal{L}} (-1)^{d^\circ(L)} f(L)$$

and

$$\tilde{Z}_\tau = \sum_{L \in \mathcal{L}} (-1)^{d(L)} \tilde{f}(L)$$

where  $d^\circ(L)$  denotes the number of internal vertices of a labelled tree  $L$  and where  $d(L)$  denotes the number of leaves in  $L$ .

**Theorem 6.2** *We have*

$$Z_\tau \circ_{X_\mathcal{L}} (\tilde{Z}_\sigma)_{\sigma \in \mathcal{T}} = \tilde{Z}_\tau \circ_{X_\mathcal{L}} (Z_\sigma)_{\sigma \in \mathcal{T}} = X_\tau$$

where the notation  $Z_\tau \circ_{X_\mathcal{L}} (\tilde{Z}_\sigma)_{\sigma \in \mathcal{T}}$  means that every occurrence of  $X_\tau$  in  $Z_\tau$  is replaced by the generating series  $\tilde{Z}_\tau$ .

This result can be used for the formal inversion of power-series in several variables.

**Sketch of Proof.** Define grafted labelled trees in the obvious way and check that Proposition 5.2 remains valid in the present context.

## 7 Morphisms of rooted trees into posets

This section is a digression discussing a generalization of grafted trees.

The vertex set  $V$  of a rooted (not necessarily finite or planar) tree  $T$  with oriented edges  $E$  can be considered as a poset (partially ordered set) by considering the order relation induced by  $\omega(e) > \alpha(e)$  for  $e \in E$ .

**Remark 7.1** *Considering a rooted tree as a poset, one might wonder how many unordered pairs of comparable vertices are contained in ( $k$ -regular) rooted planar trees having  $n$  vertices. Let  $a_n$  (respectively  $a_{n,k}$ ) denote this number. We have then*

$$\sum_{n=1}^{\infty} a_n t^n = \frac{\partial y}{\partial u}(t, 1)$$

where  $y(t, u) = t + (\text{terms of higher order in } t)$  satisfies the functional equation

$$y(t, u) = t + t \frac{y(tu, u)}{1 - y(tu, u)} .$$

The corresponding sequence  $a_2, a_3, a_4, \dots$  starts as

1, 5, 22, 93, 386, 1586, 6476, 26333, 106762, 431910,  $\dots$  cf. A346 of [2].

Similarly, for the numbers  $a_{n,k}$  associated to  $k$ -regular trees, we have

$$\sum_{n=1}^{\infty} a_{n,k} t^n = \frac{\partial y}{\partial u}(t, 1)$$

with  $y(t, u) = t + (\text{ terms of higher order in } t )$  satisfying

$$y(t, u) = t + t (y(tu, u))^k .$$

For  $k = 2$  we get the sequence  $\frac{1}{2}(a_{3,2}, a_{5,2}, a_{7,2}, \dots)$  starting as

$$1, 6, 29, 130, 562, 2380, 9949, 41226, 169766, \dots, \text{ cf. A8549 of [2]}$$

and for  $k = 3$ , the sequence  $\frac{1}{3}(a_{4,3}, a_{7,3}, a_{10,3}, \dots)$  starting as

$$1, 9, 69, 502, 3564, 24960, 173325, 1196748, \dots, \text{ cf. A75045 of [2]}.$$

Given a second poset  $P$ , a *morphism* from  $T$  to  $P$  is an application  $\mu : V \rightarrow P$  such that  $\mu(\omega(e)) \geq \mu(\alpha(e))$  for every edge  $e$  of  $T$ .

Denote by  $\{1, 2\}$  (with  $1 < 2$ ) the obvious totally ordered poset. Call a morphism  $\mu : T \rightarrow \{1, 2\}$  *restricted* if  $\mu^{-1}(2)$  contains all leaves of  $T$ .

**Proposition 7.2** *If  $T$  is a rooted (planar) tree, then the set of grafted trees  $(A; B_1, \dots, B_{d(A)})$  with skeleton  $T$  corresponds bijectively to the set of restricted morphisms  $\mu : T \rightarrow \{1, 2\}$ .*

**Proof.** Given a restricted morphism  $\mu : T \rightarrow \{1, 2\}$  we set  $A = \{\mu^{-1}(1) \cup \text{ sons of } \mu^{-1}(1)\}$  where the root  $r \in T$  is by convention the son of the empty set. The rooted trees  $\{B_1, \dots, B_{d(A)}\}$  are the connected components of the maximal subforest with vertices  $\mu^{-1}(2)$ . It is obvious that  $(A; B_1, \dots, B_{d(A)})$  is a grafted tree with skeleton  $T$ . Reciprocally, given a grafted tree  $(A; B_1, \dots, B_{d(A)})$  with skeleton  $T$ , we get a morphism  $\mu : T \rightarrow \{1, 2\}$  by setting  $\mu(v) = 2$  if  $v \in B_j$  for some  $j = 1, \dots, d(A)$  and  $\mu(v) = 1$  otherwise. Since the forest  $\cup_{j=1}^{d(A)} B_j$  contains all leaves of  $T$ , the morphism  $\mu$  is restricted.  $\square$

Proposition 7.2 suggests to generalize the notion of grafted trees by considering sets of (suitable) morphisms from rooted trees into posets. We consider now a few special cases. Henceforth all rooted trees and posets will be finite. The number of morphisms of a rooted tree  $T$  having  $\sharp(V)$  vertices into a poset  $P$  with  $\sharp(P)$  elements is bounded by  $\sharp(P)^{\sharp(V)}$ .

## 7.1 Grafted trees

In this subsection we indicate how to count the number of grafted trees with given skeleton  $T$ . A slight modification counts all morphisms from  $T$  into

$\{1, 2\}$ . Further generalizations consist in counting all (suitable) morphisms from  $T$  into the totally ordered set  $\{1, \dots, n\}$ .

Given a (finite rooted planar) tree  $T$  we denote by  $\gamma(T)$  the number of grafted trees with skeleton  $T$ . We denote by  $\tilde{\gamma}(T) \geq \gamma(T)$  the number of all morphisms  $T \rightarrow \{1, 2\}$ . We have  $\gamma(\{r\}) = 1$ ,  $\tilde{\gamma}(\{r\}) = 2$  if  $T = \{r\}$  is the trivial tree reduced to its root. Remark that  $\tilde{\gamma}(\tilde{T}) = \gamma(T)$  where  $\tilde{T}$  is obtained by removing all leaves from the rooted tree  $\gamma$  (where  $\tilde{\gamma}(\emptyset) = 1$  by convention). This remark generalizes easily to the analogous numbers enumerating (restricted) morphisms  $T \rightarrow \{1, \dots, m\}$ .

**Proposition 7.3** *For  $T$  a non-trivial (finite rooted planar) tree we have*

$$\gamma(T) = 1 + \prod_{j=1}^k \gamma(T_j)$$

and

$$\tilde{\gamma}(T) = 1 + \prod_{j=1}^k \tilde{\gamma}(T_j)$$

where  $T_1, \dots, T_k$  are the principal subtrees defined by the  $k$  sons  $s_1, \dots, s_k$  of the root  $r \in T$ .

**Proof.** We count all (restricted) morphisms  $\mu$  from  $T$  into  $\{1, 2\}$ .

Consider a morphism  $\mu : V \rightarrow \{1, 2\}$ . If  $\mu(r) = 2$  we have  $\mu(v) = 2$  for all  $v \in V$  and there is exactly one such morphism yielding a contribution of 1 to  $\gamma(T)$  and  $\tilde{\gamma}(T)$ . If  $\mu(r) = 1$ , the restriction  $\mu_i$  of  $\mu$  to a principal subtrees  $T_i$  is an arbitrary (restricted) morphism from  $T_i$  into  $\{1, 2\}$  and the restrictions  $\mu_1, \dots, \mu_k$  can be arbitrary thus proving the formula.  $\square$

Proposition 7.3 leads to a fast algorithm for computing  $\gamma(T)$  illustrated by the following example.

**Example.** For the tree  $T$  of Figure 2 we have

$$\begin{aligned} \gamma(v_1) &= \gamma(v_3) = \gamma(v_8) = 2 \\ \gamma(v_4) &= 1 + \gamma(v_3) \cdot 1 = 3, \quad \gamma(v_7) = 1 + 1 \cdot \gamma(v_8) = 3 \\ \gamma(v_2) &= 1 + \gamma(v_1) \cdot \gamma(v_4) = 7, \quad \gamma(v_6) = 1 + 1 \cdot \gamma(v_7) = 4 \\ \gamma(v_5) &= 1 + \gamma(v_2) \cdot \gamma(v_6) = 29 \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma}(v_1) &= \tilde{\gamma}(v_3) = \tilde{\gamma}(v_8) = 1 + 2 \cdot 2 = 5 \\ \tilde{\gamma}(v_4) &= 1 + \tilde{\gamma}(v_3) \cdot 2 = 11, \quad \tilde{\gamma}(v_7) = 1 + 2 \cdot \tilde{\gamma}(v_8) = 11 \\ \tilde{\gamma}(v_2) &= 1 + \tilde{\gamma}(v_1) \cdot \tilde{\gamma}(v_4) = 56, \quad \tilde{\gamma}(v_6) = 1 + 2 \cdot \tilde{\gamma}(v_7) = 23 \\ \tilde{\gamma}(v_5) &= 1 + \tilde{\gamma}(v_2) \cdot \tilde{\gamma}(v_6) = 1289 \end{aligned}$$

which shows that the tree  $T$  of Figure 2 is the skeleton of  $\gamma(T) = \gamma(v_5) = 29$  different grafted trees (or equivalently, that  $T$  contains 29 different 2-regular

subtrees sharing the root  $r$  with  $T$ ) and has 1289 distinct morphisms into the totally ordered poset  $\{1, 2\}$ .

Proposition 7.3 shows also that the generating function

$$y(t) = \sum_{T \in \mathcal{T}_2} \gamma(T) t^{d(T)}$$

counting the number  $\gamma(k) = [t^k]y$  of 2-regular grafted trees whose skeletons have  $k$  leaves satisfies

$$\begin{aligned} y &= \frac{1-\sqrt{1-4t}}{2} + y^2 = \frac{1-\sqrt{-1+2\sqrt{1-4t}}}{2} \\ &= t + 2t^2 + 6t^3 + 21t^4 + 80t^5 + 322t^6 + 1348t^7 + 5814t^8 + \dots \end{aligned}$$

The similar generating function  $\tilde{y}(t) = \sum_{T \in \mathcal{T}_2} \tilde{\gamma}(T) t^{d(T)}$  (which counts all morphisms of binary rooted trees into  $\{1, 2\}$ ) is given by

$$\begin{aligned} \tilde{y} &= t + \frac{1-\sqrt{1-4t}}{2} + y^2 = \frac{1-\sqrt{-1-4t+2\sqrt{1-4t}}}{2} \\ &= 2t + 5t^2 + 22t^3 + 118t^4 + 706t^5 + 4530t^6 + \dots \end{aligned}$$

More generally, consider a fixed natural integer  $k \geq 2$ . We call a morphism  $\mu : T \rightarrow \{1, \dots, k\}$  *restricted* if  $\mu^{-1}$  contains all leaves of the rooted tree  $T$ . We denote by  $y_m(t)$  the generating function counting the number of (non-isomorphic) restricted morphisms from a  $k$ -regular tree having  $n$  leaves into the completely ordered finite set  $\{1, \dots, m\}$ . Similarly, we denote by  $\tilde{y}_m(t)$  the analogous generating function counting all morphisms without restriction on the leaves.

**Proposition 7.4** *We have*

$$\begin{aligned} y_1(t) &= \tilde{y}_1(t) = t + (y_1(t))^k \\ y_m(t) &= t + \sum_{h=1}^m (y_h(t))^k \\ \tilde{y}_m(t) &= mt + \sum_{h=1}^m (\tilde{y}_h(t))^k \end{aligned}$$

**Sketch of proof.** The trivial tree  $T$  yields a contribution of  $t$  to  $y_k$  and of  $mt$  to  $\tilde{y}(t)$ . For a morphism  $\mu$  from a nontrivial tree into  $\{1, \dots, m\}$  we consider the image  $\mu(r)$  of its root and the remaining possibilities of the induced restrictions  $\mu_1, \dots, \mu_k$  on the principal subtrees  $T_1, \dots, T_k$  of  $T$ .  $\square$

For arbitrary finite planar non-empty trees we consider the generating function counting the number of different restricted morphisms  $\mu$  from a tree having  $n$  vertices into the completely ordered finite set  $\{1, \dots, m\}$ . The generating function  $\tilde{y}_m(t)$  counts all such morphisms into  $\{1, \dots, m\}$ . One can then prove the following result.

**Proposition 7.5** *We have*

$$y_1(t) = \tilde{y}_1(t) = t + \frac{ty_1}{1 - y_1}$$

$$y_m(t) = t + t \sum_{h=1}^m \frac{y_h}{1 - y_h}$$

$$\tilde{y}_m(t) = mt + t \sum_{h=1}^m \frac{\tilde{y}_h}{1 - \tilde{y}_h}$$

The first instances are  $y_1 = \tilde{y}_1 = \frac{1 - \sqrt{1 - 4t}}{2}$  (defining the Catalan numbers, cf. A108 of [2])

$$\begin{aligned} y_2 &= \frac{3 - 2t - \sqrt{1 - 4t} - \sqrt{2 - 16t + 4t^2 + (2 + 4t)\sqrt{1 - 4t}}}{4} \\ &= t + 2t^2 + 5t^3 + 15t^4 + 50t^5 + 178t^6 + 663t^7 + 2553t^8 + \dots \end{aligned}$$

which is sequence A7853 of [2] and

$$\begin{aligned} \tilde{y}_2 &= \frac{3 - \sqrt{1 - 4t} - \sqrt{2 - 20t + 2\sqrt{1 - 4t}}}{4} \\ &= 2t + 3t^2 + 9t^3 + 34t^4 + 145t^5 + 667t^6 + 3231t^7 + 16247t^8 + \dots \end{aligned}$$

Finally, let us mention the well-known fact that the function  $p(m)$  counting all morphisms from a fixed finite poset  $E$  into the totally ordered poset  $\{1, \dots, m\}$  is a polynomial of degree  $\sharp(E)$ . Indeed

$$p(m) = \sum_{k=1}^m \alpha_k \binom{m}{k}$$

where  $\alpha_k$  denotes the number of surjective morphisms of  $E$  into  $\{1, \dots, k\}$ . One can thus also consider generating functions associated to such polynomials. The number of restricted morphisms from a fixed rooted tree into  $\{1, \dots, m\}$  is of course also a polynomial function. Its degree in  $m$  is the number of interior leaves in  $T$ .

## 7.2 Surjective morphisms

Given a finite rooted tree  $T$  having  $n$  vertices, the number  $\sigma(T)$  of surjective morphisms from  $T$  into  $\{1, \dots, n\}$  can be recursively computed by remarking that

$$\sigma(T) = (n - 1)! \prod_{j=1}^k \frac{\sigma(T_j)}{n_j!}$$

where  $T_1, \dots, T_k$  are the principal (rooted) subtrees of  $T$  having  $n_1, \dots, n_k$  vertices. Denoting by  $\binom{n-1}{n_1, \dots, n_k} = \frac{(n-1)!}{n_1! \dots n_k!}$  the obvious multinomial coefficient (where  $n-1 = \sum n_i$ ) we get for our favorite (binary) tree  $T$  of Figure 2 the following numbers:  $\sigma(v) = 1$  if  $v$  is a leaf and

$$\begin{aligned}\sigma(v_1) = \sigma(v_3) = \sigma(v_8) &= \binom{2}{1, 1} = 2 \\ \sigma(v_4) = \sigma(v_7) &= \binom{4}{1, 3} = 2 \cdot 8 \\ \sigma(v_2) &= \binom{8}{3, 5} = 2 \cdot 8 = 896, \quad \sigma(v_6) = \binom{6}{1, 5} = 1 \cdot 8 = 48 \\ \sigma(T) = \sigma(v_5) &= \binom{16}{9, 7} = 896 \cdot 48 = 492011520\end{aligned}$$

The generating function

$$y(t) = \sum_{k=1}^{\infty} \alpha_k t^k$$

encoding the number  $\alpha_n$  of surjective morphisms into  $\{1, \dots, n\}$  from all rooted binary planar trees on  $n$  vertices satisfies

$$\alpha_n = (n-1)! \sum_{k=1}^{n-2} \frac{\alpha_k \alpha_{n-1-k}}{k! (n-1-k)!}$$

(cf. sequence A182 of [2]). Otherwise stated, the exponential generating function

$$z(t) = \sum_{k=1}^{\infty} \alpha_k \frac{t^k}{k!}$$

satisfies  $z' = z^2 - 1$  thus proving that  $z(t) = \tanh(t)$ .

Similarly, considering the exponential generating function

$$z(t) = \sum_{k=1}^{\infty} \beta_k \frac{t^k}{k!}$$

enumerating the number  $\beta_n$  of all surjective homomorphisms from rooted planar trees having  $n$  vertices into  $\{1, \dots, n\}$  we have

$$z' = \frac{z}{1-z} + 1 = \frac{1}{1-z}$$

which implies  $z(t) = \frac{1-\sqrt{1-4t}}{2}$  (cf. sequence A108 of [2]) and shows that  $\beta_n = n! \binom{2(n-1)}{n-1} / n = \frac{(2n-2)!}{(n-1)!}$ .

## 8 Loday's example (i)

The aim of this section is a partial analysis of example (i) in [3]. The framework is somewhat simpler as in the previous sections: We work over the commutative ground field  $\mathbf{C}$  of complex numbers. The alphabet  $\mathcal{A}$  consists of nine elements and equals

$$\mathcal{A} = \{\circ, N, NW, W, SW, S, SE, E, NE\}$$

suggesting the graphical notations of [3]. We set  $X = t$  in order to stick to [3].

Loday's example (i) corresponds to  $k = 2$ . The two  $9 \times 9$  matrices  $M_1, M_2$  are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Writing  $V = (g_\circ, g_N, g_{NW}, g_W, g_{SW}, g_S, g_{SE}, g_E, g_{NE})^t$  we get the equations

$$\begin{aligned} g_\circ &= (-t + g_\circ)(-t + g_N + g_{NW} + g_W) \\ g_N &= (-t + g_\circ + g_N + g_{NW})(-t + g_W) \\ g_{NW} &= (-t + g_\circ + g_N + g_{NW} + g_W)(-t) \\ g_W &= (-t + g_\circ + g_{NW} + g_W)(-t + g_N) \\ g_{SW} &= (-t + g_\circ + g_N + g_{NW} + g_W + g_{SW})(-t + g_S) \\ g_S &= (-t + g_\circ + g_N)(-t + g_{NW} + g_W + g_{SW} + g_S) \\ g_{SE} &= (-t + g_\circ + g_N + g_{NW} + g_W + g_S) \\ &\quad (-t + g_{SW} + g_{SE} + g_E + g_{NE}) \\ g_E &= (-t + g_\circ + g_N + g_W)(-t + g_{NW} + g_E + g_{NE}) \\ g_{NE} &= (-t + g_\circ + g_N + g_{NW} + g_W)(-t + g_E + g_{NE}) \end{aligned}$$

One sees easily that we have

$$g_\circ = g_N = g_W .$$

Eliminating  $g_N, g_W$  we get the simpler equations:

$$\begin{aligned} g_\circ &= (-t + g_\circ)(-t + 2g_\circ + g_{NW}) \\ g_{NW} &= (-t + 3g_\circ + g_{NW})(-t) \\ \hline g_{SW} &= (-t + 3g_\circ + g_{NW} + g_{SW})(-t + g_S) \\ g_S &= (-t + 2g_\circ)(-t + g_\circ + g_{NW} + g_S + g_{SW}) \\ \hline g_E &= (-t + 3g_\circ)(-t + g_{NW} + g_E + g_{NE}) \\ g_{NE} &= (-t + 3g_\circ + g_{NW})(-t + g_E + g_{NE}) \\ \hline g_{SE} &= (-t + 3g_\circ + g_{NW} + g_S)(-t + g_{SW} + g_{SE} + g_E + g_{NE}) \end{aligned}$$

where functions above a horizontal line are independent of functions below the line.

Computations (done with Maple) using Groebner-bases show that

$$y = g_o + g_N + g_{NW} + g_W + g_{SW} + g_S + g_{SE} + g_E + g_{NE}$$

satisfies the algebraic equation  $P(y(t), t) = 0$  where

$$P(y, t) = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4$$

with

$$\begin{aligned} c_0 = & t (288 t^{31} - 1008 t^{30} - 17696 t^{29} + 35124 t^{28} + 513042 t^{27} \\ & - 352654 t^{26} - 8834409 t^{25} - 2315100 t^{24} + 94293622 t^{23} \\ & + 92841847 t^{22} - 608228325 t^{21} - 1031578684 t^{20} \\ & + 2072381165 t^{19} + 5859780674 t^{18} - 1127775119 t^{17} \\ & - 16287829166 t^{16} - 15833938922 t^{15} + 9251292427 t^{14} \\ & + 38652814035 t^{13} + 44572754075 t^{12} + 10866248029 t^{11} \\ & - 40129564125 t^{10} - 59007425756 t^9 - 36829453004 t^8 \\ & - 10216139916 t^7 - 63849664 t^6 + 693364800 t^5 + 187804368 t^4 \\ & + 24111840 t^3 + 1694752 t^2 + 63488 t + 1024) \end{aligned}$$

$$\begin{aligned} c_1 = & -768 t^{31} + 1488 t^{30} + 44636 t^{29} - 49538 t^{28} - 1198111 t^{27} \\ & + 359773 t^{26} + 19127286 t^{25} + 7602856 t^{24} - 192894854 t^{23} \\ & - 193322898 t^{22} + 1208988418 t^{21} + 1968967542 t^{20} \\ & - 4212884427 t^{19} - 10893520130 t^{18} + 4099837581 t^{17} \\ & + 31343794098 t^{16} + 22508418249 t^{15} - 27437733598 t^{14} \\ & - 67449042813 t^{13} - 62529644946 t^{12} - 3629552721 t^{11} \\ & + 84243589625 t^{10} + 130637976096 t^9 + 104165077688 t^8 \\ & + 50704667612 t^7 + 15902199040 t^6 + 3290858704 t^5 \\ & + 451630576 t^4 + 40498432 t^3 + 2274080 t^2 + 72704 t + 1024 \end{aligned}$$

$$\begin{aligned} c_2 = & t (680 t^{29} - 932 t^{28} - 39270 t^{27} + 33926 t^{26} + 1020385 t^{25} \\ & - 335552 t^{24} - 15580483 t^{23} - 2999586 t^{22} + 151572589 t^{21} \\ & + 104425399 t^{20} - 945694543 t^{19} - 1131146667 t^{18} \\ & + 3558106593 t^{17} + 6544185368 t^{16} - 6226627151 t^{15} \\ & - 20759382401 t^{14} - 4197042728 t^{13} + 29133893401 t^{12} \\ & + 33005986439 t^{11} + 5921959164 t^{10} - 22398414511 t^9 \\ & - 37477032816 t^8 - 35648192872 t^7 - 21631096056 t^6 \\ & - 8298733544 t^5 - 1992896768 t^4 - 295849440 t^3 \\ & - 26179392 t^2 - 1255424 t - 24832) \end{aligned}$$



$$\begin{aligned}
c_3 = & -t^2 (t-2)(t+1) (2t^4 + 6t^3 - 11t^2 - 30t - 4) \\
& (124t^{21} - 398t^{20} - 5146t^{19} + 14694t^{18} + 92616t^{17} - 213234t^{16} \\
& - 966327t^{15} + 1518831t^{14} + 6391763t^{13} - 5003278t^{12} \\
& - 26227554t^{11} + 1248286t^{10} + 58532080t^9 + 36103178t^8 \\
& - 41699603t^7 - 64544195t^6 - 41818519t^5 - 21472740t^4 \\
& - 8578026t^3 - 1961960t^2 - 218928t - 9296)
\end{aligned}$$

and

$$\begin{aligned}
c_4 = & (t^2 - 2t - 2)(t^2 - 2t - 7)(t-2)^2(t+1)^2 \\
& (2t^4 + 6t^3 - 11t^2 - 30t - 4)^2 t^5 \\
& (8t^7 - 10t^6 - 171t^5 + 209t^4 + 948t^3 - 721t^2 - 1892t - 249)
\end{aligned}$$

The first coefficients of the series  $y(t)$  are

$$\begin{aligned}
& -t + 9t^2 - 49t^3 + 284t^4 - 1735t^5 + 10955t^6 - 70695t^7 + 463087t^8 \\
& - 3066450t^9 + 20471641t^{10} - 137540539t^{11} + 928791019t^{12} \mp \dots
\end{aligned}$$

The “complementary” function

$$\tilde{y} = \tilde{g}_o + \tilde{g}_N + \tilde{g}_{NW} + \tilde{g}_W + \tilde{g}_{SW} + \tilde{g}_S + \tilde{g}_{SE} + \tilde{g}_E + \tilde{g}_{NE}$$

is associated to the matrice  $J - M_1$  and  $J - M_2$  (where  $J$  is the square matrix of order 9 with all entries equal to 1) satisfies of course the same polynomial equation, after transposition of  $y$  and  $t$ .

**Remark.** For computations of huge terms in the series expansion of an algebraic function, one can use the following well-known trick: Any algebraic function  $y(t) = \sum a_n t^n$  of degree  $d$  satisfies a linear differential equation

$$\sum_{k=0}^d q_k(t) y^{(k)}(t) = 0$$

with polynomial coefficients  $q_0, \dots, q_d \in \mathbf{C}[t]$ . This allows a recursive computations of  $a_n$  with time and memory requirements linear in  $n$ . In our case, we get

$$q_0(t)y + q_1(t)y' + q_2(t)y'' + q_3(t)y^{(3)} + q_4(t)y^{(4)} = 0$$

where  $q_0, \dots, q_4 \in \mathbf{Z}[t]$  are polynomials of degrees respectively 150, 151, 155, 156 and 157.

## 8.1 Asymptotics

The asymptotic growth rate of the coefficients of  $y(t)$  is governed by the distance of the origin to the first ramification point of the corresponding sheet, see [4]. Ramifications are above the zeros of the discriminant  $D(t)$  of  $P(y, t)$  with respect to  $y$ . This discriminant is given by

$$D(t) = r_1^2 r_2 r_3^2 r_4^2 r_\infty$$

where

$$\begin{aligned} r_1 &= t^4 - 4t^3 + 6t^2 + 8t + 1 \\ r_2 &= t^{13} - 3t^{12} - 16t^{11} + 100t^{10} - 86t^9 - 222t^8 + 312t^7 - 544t^6 \\ &\quad + 4845t^5 + 10665t^4 + 9536t^3 + 4084t^2 + 528t + 16 \\ r_3 &= 20t^{16} + 50t^{15} - 849t^{14} - 937t^{13} + 11563t^{12} + 7833t^{11} \\ &\quad - 64177t^{10} - 58882t^9 + 141152t^8 + 259280t^7 + 82253t^6 \\ &\quad - 366913t^5 - 698955t^4 - 468324t^3 - 122700t^2 - 13720t - 552 \\ r_4 &= 5760t^{53} + 8864t^{52} - 425056t^{51} + \dots - 7200309248t - 76152832 \end{aligned}$$

( $r_4$  is not involved in coarse asymptotics of  $y(t)$ ). The roots of the polynomial

$$r_\infty = t^6 (t - 2)^2 (t + 1)^3 (2t^4 + 6t^3 - 11t^2 - 30t - 4)^2$$

are critical points for the critical value  $\infty$ .

Since the coefficients of  $y(t)$  have alternating signs, the “smallest” singularity of  $y(t)$  is on the negative real halfline. The following table resumes the relevant data for its computation. More precisely, the algebraic function defined by  $P$  has a ramification of order 3 with  $y = \infty$  above  $t = 0$ . The remaining sheet is unramified above  $t = 0$  and defines the generating function  $y(t)$  under consideration.

The table contains the following informations: The first column displays the argument  $t$  considered in the corresponding row. The second row indicates the factor of the discriminant  $D(t)$  if  $t$  is a root of  $D(t)$ . The remaining row displays information about the inverse images of  $t$  defined by the algebraic equation of  $y(t)$ .

We have

$t_0 = 0$	$r_\infty$	$y_1 = y_2 = y_3 = \infty, y_4 = 0$
$t_0 > t > t_1$		$y_1 < y_2 < y_3 < 0 < y_4$
$t_1 \sim -0.04355$	$r_2$	$y_1 \sim -2922 < y_2 = y_3 \sim -1.083 < 0 < y_4 \sim 0.066$
$t_1 > t > t_2$		$y_1 < 0 < y_4, y_2 = \overline{y_3} \in \mathbf{C} \setminus \mathbf{R}$
$t_2 \sim -0.1118$	$r_3$	$y_1 \sim -82.3 < y_2 = y_3 \sim 0.2194 < y_4 \sim 0.4499$
$t_2 > t > t_3$		$y_1 < 0 < y_4, y_2 = \overline{y_3} \in \mathbf{C} \setminus \mathbf{R}$
$t_3 \sim -0.14047$		$0 < y_4 \sim 4.113, y_1 = \infty, y_2 = \overline{y_3} \sim -2.98 \pm 8.481i$
$t_3 > t > t_4$		$0 < y_4 < y_1, y_2 = \overline{y_3} \in \mathbf{C} \setminus \mathbf{R}$
$t_4 \sim -0.14118$	$r_\infty$	$0 < y_4 \sim 8.692 < y_1 \sim 28.07, y_2 = y_3 = \infty$
$t_4 > t > t_5$		$0 < y_4 < y_1, y_2 = \overline{y_3} \in \mathbf{C} \setminus \mathbf{R}$
$t_5 \sim -0.14127$	$r_2$	$0 < y_4 = y_1 \sim 14.89, y_2 = \overline{y_3} \sim 23.95 \pm 59.72i$

The convergency radius of the series for  $y(t)$  is of course given by  $|t_5| = -t_5$  and the asymptotic growth of the coefficients of  $y(t)$  is roughly exponential with argument

$$1/t_5 \sim -7.07857458512410303820641252737538586816317182$$

A slightly more precise asymptotical behaviour of the coefficients of  $y(t)$  can be computed as follows:

At the root

$$\rho = t_5 \sim -.14127137998962933757540882196178714222253950575630$$

of  $r_2$ , the ramified sheet corresponds to the double root

$$y_\rho \sim 14.88738808602894055277970788094544394$$

of  $P(y, \rho) \in \mathbf{C}[y]$ . (Caution: when computing  $y_\rho$  as a root of  $P(y, \rho)$  one loses roughly half the digits since an error of order  $\epsilon$  on  $\rho$  induces an error of order  $\sqrt{\epsilon}$  on the corresponding two roots approximating the double root  $y_\rho$  of  $P(y, \rho)$ . A better strategy is of course to compute  $y_\rho$  as a (simple) root of the derivative  $\frac{d}{dy}P(y, \rho)$  of  $P(y, \rho)$ ).

In a open neighbourhood of  $\rho$  we get now a Puiseux series expansion

$$y(t) = h(t) + \sqrt{\rho - t} g(t)$$

with  $h(t)$ ,  $g(t)$  holomorphic in an open disc of radius

$$.141478015962983913779403501 > \rho \sim .141271379989629337575408821$$

containing no other singularities of  $y(t)$ . The asymptotics of the generating function  $y(t)$  are thus roughly given by

$$\begin{aligned} & \gamma_\rho \sqrt{\rho} \sqrt{1 - t/\rho} \\ &= \gamma_\rho \sqrt{\rho} \sum_{n=0}^{\infty} \binom{1/2}{n} \left(\frac{-t}{\rho}\right)^n \\ &= \gamma_\rho \sqrt{\rho} \left(1 - \sum_{n=1}^{\infty} \frac{1}{2} \frac{1/2 \cdot 3/2 \cdots (n-3/2)}{n!} \left(\frac{t}{\rho}\right)^n\right) \\ &= \gamma_\rho \sqrt{\rho} \left(1 - \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-3)}{2^n n!} \left(\frac{t}{\rho}\right)^n\right) \\ &= \gamma_\rho \sqrt{\rho} \left(1 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2n-2)!}{4^{n-1} n! (n-1)! \rho^n} t^n\right) \end{aligned}$$

where  $\gamma(\rho) = g(\rho)$ . Since

$$\frac{(2n-2)!}{n! (n-1)!} \sim \frac{1}{n} \frac{\sqrt{4\pi(n-1)} 4^{n-1} (n-1)^{2n-2} e^{-2n+2}}{2\pi (n-1) (n-1)^{2(n-1)} e^{-2(n-1)}} \sim \frac{4^{n-1}}{\sqrt{\pi} n^{3/2}}$$

we get the asymptotics

$$a_n \sim \frac{\gamma_\rho}{2 \sqrt{\pi} n^{3/2} \rho^{n-1/2}}$$

The constant  $\gamma_\rho$  can be computed by remarking that

$$\begin{aligned} 0 &= P(h(t) + \sqrt{\rho - t} g(t), t) \\ &= P(y_\rho + \gamma_\rho \sqrt{\rho - t} + O((\rho - t)), t) \\ &= \frac{\partial^2 P}{\partial y^2} \Big|_{(y_\rho, \rho)} \frac{\gamma_\rho^2 (\rho - t)}{2} + \frac{\partial P}{\partial t} \Big|_{(y_\rho, \rho)} (t - \rho) + O((\rho - t)^{3/2}) \end{aligned}$$

yielding

$$\gamma_\rho \sqrt{\rho} = \sqrt{2\rho \frac{\frac{\partial P}{\partial t} \Big|_{(y_\rho, \rho)}}{\frac{\partial^2 P}{\partial y^2} \Big|_{(y_\rho, \rho)}}} \sim 337.171657540870 .$$

We have thus asymptotically

$$a_n \sim 95.11436852604511894068836 \rho^{-n} n^{-3/2}$$

with  $\rho \sim -.1412713799896293375754088219617871422225395057563006418$ . Unfortunately, the right hand side is a fairly correct approximation of  $a_n$  only for very huge values of  $n$  (concretely,  $n \sim 10^4$  yields a only very few decimals). Indeed, the function  $y(t)$  ramifies again for

$$t \sim -.1414780159629839137794$$

which is extremely close to  $\rho$ .

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