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# Indecomposable representations of $\mathfrak{osp}(3, 2)$ , $D(2, 1; \alpha)$ and $G(3)$

Jérôme GERMONI

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*Abstract.* We classify the finite-dimensional representations of the Lie superalgebras  $\mathfrak{osp}(3, 2)$ ,  $D(2, 1; \alpha)$  (the one-parameter family of deformations of  $\mathfrak{osp}(4, 2)$ ) and  $G(3)$ . In short, indecomposable representations in the non-trivial blocks are, up to isomorphism and duality, naturally parametrized by positive roots of an infinite Dynkin diagram of type  $D_\infty$  or  $A_\infty^\infty$ . From this result we can deduce the representation type of all basic classical Lie superalgebras but  $F(4)$ .

## Introduction.

Let  $\mathfrak{g}$  be one of the complex basic classical Lie superalgebras  $\mathfrak{osp}(3, 2)$ ,  $D(2, 1; \alpha)$  for some  $\alpha \in \mathbb{C} \setminus \{0, -1\}$ , or  $G(3)$ . We want to classify all indecomposable finite-dimensional  $\mathfrak{g}$ -modules.

Let  $X^+$  be the set of isomorphism classes of simple finite-dimensional  $\mathfrak{g}$ -modules. For  $\lambda \in X^+$  we choose a simple module  $S(\lambda)$  in  $\lambda$ . We set an edge between two elements of  $X^+$  if they have a non-split extension. Blocks are connected components of the resulting graph. All the simple components of an indecomposable module belong to the same block ; we say that the indecomposable module itself belongs to this block.

If a block is reduced to one element, the corresponding simple module is typical in the sense of Kac [6] and splits in any finite-dimensional module in which it is a subquotient. The other modules and blocks are called atypical. For  $M$  a  $\mathfrak{g}$ -module in a block  $\Gamma$ , we set  $[M] = \sum_{\lambda \in \Gamma} [M : S(\lambda)] \cdot \lambda \in \mathbb{N}\Gamma$ .

The main result is the following :

**Theorem 1** *Let  $\Gamma$  be an atypical block of  $\mathfrak{osp}(3, 2)$ ,  $D(2, 1; \alpha)$  or  $G(3)$ . Then :*

- (a) *As a graph,  $\Gamma$  is an infinite Dynkin diagram of type  $D_\infty$  or  $A_\infty^\infty$ . (Recall its vertices are isomorphism classes of simple modules.)*
- (b) *Up to isomorphism, the indecomposable projective modules in  $\Gamma$  are naturally parametrized by the set of simple roots of  $\Gamma$  considered as a root system.*
- (c) *Up to isomorphism and to duality, the indecomposable non-projective modules in  $\Gamma$  are naturally parametrized by the set of positive roots of  $\Gamma$  : if  $\Lambda \in \mathbb{N}\Gamma$  is a positive root, there are two indecomposable representations  $M$  and  $M^*$  in  $\Gamma$ , dual of each other, such that  $[M] = [M^*] = \Lambda$ .*

More precise versions of this theorem will be given in theorems 2.1.1, 3.1.1 and 4.1.1.

Say that two blocks (of possibly different superalgebras) are equivalent if the categories of modules belonging to these two blocks are equivalent. According to the results announced by V. Serganova in [11], it turns out that there is only one equivalence class of atypical blocks for all the algebras  $\mathfrak{osp}(3, 2n)$  ( $n \geq 1$ ) and  $\mathfrak{osp}(2p + 1, 2)$  ( $p \geq 1$ ), and there are two of them for the algebras  $\mathfrak{osp}(2p, 2)$  ( $p \geq 2$ ). The representation type of special linear superalgebras and related algebras was determined in [2]. Hence we get :

**Corollary 2** *The following Lie superalgebras have tame representation type :  $\mathfrak{sl}(m, 1)$  ( $m \geq 1$ ),  $\mathfrak{osp}(m, 2p)$  ( $m \leq 3$  or  $p = 1$ ),  $D(2, 1; \alpha)$ ,  $G(3)$ .*

On the other hand, notice that the arguments used in [2] to prove that  $\mathfrak{sl}(m, n)$  has wild representation type for  $m, n \geq 2$  can be adapted to show that  $\mathfrak{osp}(m, 2p)$  has wild representation type if  $m \geq 4$  and  $p \geq 2$ . The only remaining case in the class of basic classical Lie superalgebras is  $F(4)$ , which is most likely to have tame representation type.

As simple Lie algebras, basic classical Lie superalgebras have a root system, which, modulo the choice of a basis, enables us to endow  $X^+$  with a partial order. We can define highest weight modules  $\mathcal{L}_0(\lambda)$  (for  $\lambda \in X^+$ ) which, to some extent, play the role of Verma modules in the Bernstein-Gelfand-Gelfand category  $\mathcal{O}$  : namely,  $\mathcal{L}_0(\lambda)$  is projective in the category of finite-dimensional modules with all subquotients lower than  $\lambda$  (see section 1.2).

For superalgebras of type II (*i.e.*  $\mathfrak{osp}(m, 2p)$  with  $m \geq 3$  and  $p \geq 1$ ,  $D(2, 1; \alpha)$  with  $\alpha \in \mathbb{C} \setminus \{0, -1\}$ ,  $G(3)$  or  $F(4)$ , see [6]), the modules  $\mathcal{L}_0(\lambda)$  are constructed by homological induction, and no longer by usual induction as highest-weight modules in  $\mathcal{O}$  or as the so-called Kac modules of  $\mathfrak{sl}(m, n)$  (see [10] or [2]). The situation is now more complicated : first, projective modules do not have a filtration by highest weight modules in general ; next, the character and the composition series of  $\mathcal{L}_0(\lambda)$  are not known.

Nevertheless, for  $\mathfrak{osp}(3, 2)$ ,  $D(2, 1; \alpha)$  and  $G(3)$ , everything can be computed by *ad hoc* methods : it simplifies a lot that the Casimir separates atypical blocks.

CONVENTIONS : The ground field is  $\mathbb{C}$ . Elements of  $\mathbb{Z}/2\mathbb{Z}$  will be denoted by  $\underline{0}$  and  $\underline{1}$ , while 0 and 1 lie in  $\mathbb{Z}$  : thus  $\mathfrak{g}_{\underline{0}}$  and  $\mathfrak{g}_{\underline{1}}$  denote different algebras. The symbols  $\mathbb{S}$  and  $\mathbb{\wedge}$  denote classical symmetric and exterior algebras.

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## 1 Homological induction

### 1.1 Preliminaries

In this section we denote by  $\mathfrak{g} = \mathfrak{g}_{\underline{0}} \oplus \mathfrak{g}_{\underline{1}}$  a complex basic classical Lie superalgebra (see [6] or [9]). We assume that  $\mathfrak{g}$  is either  $\mathfrak{osp}(m, 2n)$  with  $m \geq 3$  and  $n \geq 1$  or  $D(2, 1; \alpha)$  for some  $\alpha \in \mathbb{C} \setminus \{0, -1\}$  or  $G(3)$  or  $F(4)$ . We exclude the cases of  $\mathfrak{sl}(m, n)$  and  $\mathfrak{osp}(2, 2n)$  because for these algebras, we can define highest weight modules (denoted by  $V_\lambda$  in [10] and  $K(\lambda)$  in [2]) by usual induction, so that the considerations developed here are less useful. It follows from our assumption that  $\mathfrak{g}_{\underline{0}}$  is semi-simple.

We fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}_{\underline{0}}$ , and we denote by  $W$  the Weyl group of  $\mathfrak{g}_{\underline{0}}$ . Let  $\Delta_0$  (resp.  $\Delta_1$ ) the set of even (resp. odd) roots, *i.e.* the set of weights of  $\mathfrak{h}$  in  $\mathfrak{g}_{\underline{0}}$  (resp.  $\mathfrak{g}_{\underline{1}}$ ). We put  $\Delta = \Delta_0 \cup \Delta_1$ .

We choose a system of positive roots  $\Delta^+$  in  $\Delta$ . Let  $\leq$  be the partial order on  $\mathfrak{h}$  defined by :  $\mu \leq \lambda$  if  $\lambda - \mu \in \mathbb{N}\Delta^+$ . We denote by  $\rho_0$  (resp.  $\rho_1$ ) the half-sum of positive even (resp. odd) roots, and we set  $\rho = \rho_0 - \rho_1$ . The shifted action of  $W$  is defined, for  $w \in W$  and  $\lambda \in \mathfrak{h}^*$ , by :  $w.\lambda = w(\lambda + \rho) - \rho$ . We denote by  $\ell$  the length function on  $W$ .

The superalgebra  $\mathfrak{g}$  admits an even non degenerate bilinear invariant form, which allows to define a Casimir element in the center of the enveloping algebra, and induces on  $\mathfrak{h}^*$  a non degenerate

bilinear form denoted  $(.,.)$ . For a non-isotropic root  $\beta \in \Delta$ , we set  $\beta^\vee = 2\beta/(\beta, \beta)$ . If  $M$  is a module with a highest weight  $\lambda$ , the eigenvalue of the Casimir element on  $M$  is :  $\text{Cas}(\lambda) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho)$ .

We denote by  $\mathfrak{b}$  the Borel subalgebra corresponding to  $\Delta^+$ , and by  $\mathbb{C}_\lambda$  the one-dimensional  $\mathfrak{b}$ -module with weight  $\lambda \in \mathfrak{h}^*$ . When  $\lambda \in \mathfrak{h}^*$  is dominant for  $\mathfrak{g}_0$  with respect to  $\Delta_0^+$ , the simple  $\mathfrak{g}_0$ -module with highest weight  $\lambda$  will be denoted by  $L(\mathfrak{g}_0, \lambda)$ . We denote its character by  $\chi(e^\lambda)$  and we extend  $\chi$  to finite sums of symbols  $e^\lambda$  by linearity. Thus :

$$\chi(e^\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho_0)}}{\prod_{\beta \in \Delta_0^+} (e^{\beta/2} - e^{-\beta/2})}.$$

We denote by  $\mathfrak{g}\text{-mod}$  the category of finite-dimensional  $\mathfrak{g}$ -modules with even morphisms of representations. For every  $\lambda \in X^+$  we choose a module  $S(\lambda)$  in  $\lambda$ . Once a system of positive roots is chosen, a simple finite-dimensional module  $S$  has a unique maximal weight with respect to  $\leq$ , called the highest weight of  $S$ , which determines  $S$  up to isomorphism and parity. Thus  $X^+$  can be identified with a subset of the set of dominant weights for  $\mathfrak{g}_0$ , determined in proposition 2.3 of [6] for a distinguished choice of  $\Delta^+$ . Elements in  $X^+$  will be called dominant weights (for  $\mathfrak{g}$ , with respect to  $\Delta^+$ ). A dominant weight is typical if and only if for every  $\beta \in \Delta_1^+$  such that  $(\beta, \beta) = 0$ , the scalar product  $(\lambda + \rho, \beta)$  is not zero.

- Lemma 1.1.1** (i) *The category  $\mathfrak{g}\text{-mod}$  contains enough projective modules.*  
(ii) *Projective and injective modules in  $\mathfrak{g}\text{-mod}$  coincide.*  
(iii) *For any  $\lambda, \mu \in X^+$ , we have :  $\text{Ext}_{\mathfrak{g}}^1(S(\lambda), S(\mu)) \simeq \text{Ext}_{\mathfrak{g}}^1(S(\mu), S(\lambda))$ .*

PROOF : Since  $\mathfrak{g}_0$  is semi-simple, any finite-dimensional  $\mathfrak{g}_0$ -module is projective and injective. Besides induction takes projectives to projectives, so that for any finite-dimensional  $\mathfrak{g}$ -module  $M$ ,  $\text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \text{Res}_{\mathfrak{g}_0} M$  is projective in  $\mathfrak{g}\text{-mod}$ . Since it maps onto  $M$ , this proves (i).

By proposition 2.1.1 in [2], induced modules from  $\mathfrak{g}_0$  to  $\mathfrak{g}$  are also coinduced, so that projective and injective modules coincide, whence (ii).

To prove (iii), recall that  $\mathfrak{g}$  admits a ‘‘Chevalley automorphism’’  $\sigma$  that maps a root space  $\mathfrak{g}_\alpha$  to  $\mathfrak{g}_{-\alpha}$  and fixes  $\mathfrak{h}$  (see [9], proposition 1.2). Hence we can define a contravariant duality functor  $D$  by twisting the usual dual of a module by  $\sigma$ . This functor does not change the character of a module, so it fixes finite-dimensional simple modules. Interpreting the groups  $\text{Ext}_{\mathfrak{g}}^1$  as equivalence classes of short exact sequences, we get the isomorphism in (iii) by applying  $D$ .  $\square$

The lemma shows that  $\mathfrak{g}\text{-mod}$  is a nice category in the sense of [2], section 1. We will use the notions and notations introduced therein. Recall that the Ext-quiver of  $\mathfrak{g}\text{-mod}$  is the quiver  $Q$  with vertex set  $X^+$ , where the number of arrows from  $\lambda$  to  $\mu$  is  $\dim \text{Ext}_{\mathfrak{g}}^1(S(\lambda), S(\mu))$ . We know there exists a system of relations  $R$  on  $Q$  such that the category of representations of the quiver with relations  $Q/R$  is equivalent to  $\mathfrak{g}\text{-mod}$ . Blocks are connected components of  $Q$  (or, by abuse, parts of  $X^+$ ). The block containing the trivial representation  $\mathbb{C}$  is called principal and denoted by  $\Gamma_0$ . We denote by  $\mathfrak{g}\text{-mod}_\Gamma$  the category of finite dimensional  $\mathfrak{g}$ -modules all subquotients of which belong to a given block  $\Gamma$ . Any indecomposable  $\mathfrak{g}$ -module lies in a unique  $\mathfrak{g}\text{-mod}_\Gamma$ .

## 1.2 Harish-Chandra categories and Zuckerman functors

Given two Lie superalgebras  $\mathfrak{n} \subset \mathfrak{m}$ , we denote by  $\mathcal{HC}(\mathfrak{m}, \mathfrak{n})$  the category of  $\mathfrak{m}$ -modules that restrict to a (possibly infinite) direct sum of finite-dimensional simple  $\mathfrak{n}$ -modules.

Let  $\mathfrak{s}_0$  be a Levi subalgebra of  $\mathfrak{g}_0$  containing  $\mathfrak{h}$ . Notice that any finite-dimensional  $\mathfrak{g}$ -module lies in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$ , since by assumption  $\mathfrak{g}_0$  is semi-simple. The goal in the next two sections is to collect a few properties of certain functors  $\mathcal{L}_i^{\mathfrak{s}_0} : \mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0) \rightarrow \mathcal{HC}(\mathfrak{g}, \mathfrak{g}_0)$  introduced by Santos in [9], section 4. This section contains mainly abstract nonsense, but things become more specific in the next one.

Let  $S_0$  be a simply-connected connected complex Lie group with Lie algebra  $\mathfrak{s}_0$ . Let  $\hat{\mathcal{M}}(S_0)$  be the algebra defined as the inverse limit of  $\mathfrak{s}_0$ -semi-simple quotients of  $U(\mathfrak{s}_0)$  : as a  $\mathfrak{s}_0$ -module, it is the direct product of  $L^* \otimes L$ , where  $L$  runs over the set  $\hat{S}_0$  of isomorphism classes of finite-dimensional  $S_0$ -modules. This algebra contains a unique subalgebra  $\mathcal{M}(S_0)$  isomorphic as a  $\mathfrak{s}_0$ -module to the direct sum of  $L^* \otimes L$ ,  $L$  running over  $\hat{S}_0$ , i.e. to the matrix coefficient algebra of  $S_0$ .

We can define an approximately unital associative superalgebra  $R(\mathfrak{g}, S_0)$  in [8], (1.131), which is isomorphic to  $U(\mathfrak{g}) \otimes_{U(\mathfrak{s}_0)} \mathcal{M}(S_0)$  as a left  $\mathfrak{g}$ -module and a right  $\mathfrak{s}_0$ -module, and to  $\mathcal{M}(S_0) \otimes_{U(\mathfrak{s}_0)} U(\mathfrak{g})$  as a right  $\mathfrak{g}$ -module and a left  $\mathfrak{s}_0$ -module. For  $\mathfrak{s}_0 = \mathfrak{g}_0$ ,  $R(\mathfrak{g}, S_0)$  is the algebra denoted by  $\mathcal{M}(G)$  in [9], section 4. Extending [8], theorem 1.117, we can identify  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$  to the category of approximately unital  $R(\mathfrak{g}, S_0)$ -modules.

The following lemma describes some projectives in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$ .

- Lemma 1.2.1** (i) *If  $V$  lies in  $\mathcal{HC}(\mathfrak{s}_0, \mathfrak{s}_0)$  then  $\text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}} V$  is projective in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$ . Such a module is called a standard projective. Any module in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$  is a quotient of a standard projective module.*  
(ii) *If  $V$  lies in  $\mathcal{HC}(\mathfrak{s}_0, \mathfrak{s}_0)$ , then  $\text{Res}_{\mathfrak{g}_0} \text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}} V$  is projective in  $\mathcal{HC}(\mathfrak{g}_0, \mathfrak{s}_0)$ .*  
(iii) *If  $V$  lies in  $\mathcal{HC}(\mathfrak{h}, \mathfrak{h})$ , then  $\text{Res}_{\mathfrak{s}_0} \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V$  is projective in  $\mathcal{HC}(\mathfrak{s}_0, \mathfrak{h})$ .*

PROOF : (i) Let  $\mathfrak{a}_0$  (resp.  $\mathfrak{a}_1$ ) be the even (resp. odd) part of  $\mathfrak{g}/\mathfrak{s}_0$  : they are semi-simple  $\mathfrak{s}_0$ -modules. Extending [9], lemma 2.6,  $\text{Res}_{\mathfrak{s}_0} \text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}} V$  has an increasing filtration by submodules  $(W_j)_{j \geq -1}$  with  $W_{-1} = 0$  and

$$W_j/W_{j-1} \simeq \bigoplus_{k+l=j} S^k(\mathfrak{a}_0) \otimes \bigwedge^l(\mathfrak{a}_1) \otimes V.$$

Since  $\mathfrak{s}_0$  is reductive and since  $\mathfrak{g}/\mathfrak{s}_0$  and  $V$  are semi-simple modules,  $W_j/W_{j-1}$  splits in  $W_j$ . Thus, passing to the limit, we get that

$$\text{Res}_{\mathfrak{s}_0} \text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}} V \simeq S(\mathfrak{a}_0) \otimes \bigwedge(\mathfrak{a}_1) \otimes V.$$

This shows that  $\text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}} V$  does lie in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$ . Since  $\text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}}$  is left adjoint to the exact restriction functor, it maps projectives in  $\mathcal{HC}(\mathfrak{s}_0, \mathfrak{s}_0)$  (that is to say, any module !) to projectives in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$ . The last part of (i) follows since any module  $M$  in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$  is a quotient of  $\text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}} \text{Res}_{\mathfrak{s}_0} M$ .

(ii) and (iii) : The same filtration argument as above shows that  $\text{Res}_{\mathfrak{g}_0} \text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}} V$  (resp.  $\text{Res}_{\mathfrak{s}_0} \text{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V$ ) is isomorphic to  $\text{Ind}_{\mathfrak{s}_0}^{\mathfrak{g}_0}(\bigwedge(\mathfrak{g}_1) \otimes V)$  (resp.  $\text{Ind}_{\mathfrak{h}}^{\mathfrak{s}_0}(S(\mathfrak{a}_0) \otimes \bigwedge(\mathfrak{a}_1) \otimes V)$ ), which by (i) is projective in  $\mathcal{HC}(\mathfrak{g}_0, \mathfrak{s}_0)$  (resp.  $\mathcal{HC}(\mathfrak{s}_0, \mathfrak{h})$ ).  $\square$

Santos defined a right exact covariant functor

$$\mathcal{L}_0^{\mathfrak{s}_0} = R(\mathfrak{g}, G_0) \otimes_{U(\mathfrak{g})} ? : \mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0) \longrightarrow \mathcal{HC}(\mathfrak{g}, \mathfrak{g}_0).$$

Notice that since  $S_0$  is connected,  $\otimes_{U(\mathfrak{g})}$  is the same as  $\otimes_{R(\mathfrak{g}, S_0)}$  on  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$ . By lemma 1.2.1 (i), for  $i \geq 0$ , the  $i$ -th left derived functor  $\mathcal{L}_i^{\mathfrak{s}_0}$  of  $\mathcal{L}_0^{\mathfrak{s}_0}$  is well defined.

The functor  $\mathcal{L}_0^{\mathfrak{s}_0}$  has a purely classical counterpart, namely the Bernstein-Zuckerman right exact functor

$${}^0\mathcal{L}_0^{\mathfrak{s}_0} = \mathcal{M}(G_0) \otimes_{U(\mathfrak{g}_0)} ? : \mathcal{HC}(\mathfrak{g}_0, \mathfrak{s}_0) \longrightarrow \mathcal{HC}(\mathfrak{g}_0, \mathfrak{g}_0).$$

Its derived functors will be denoted by  ${}^0\mathcal{L}_i^{\mathfrak{g}_0}$ . They can be computed by a variant of the Borel-Weil-Bott theorem. The following lemma relates the super functor and the classical one : it means that the  $\mathfrak{g}_0$ -module structure of  $\mathcal{L}_i^{\mathfrak{g}_0}(M)$  only depends on that of  $M$ . It is also a consequence of proposition 4.4 in [9].

**Lemma 1.2.2** *There is a commutative diagram of functors, for all  $i \geq 0$  :*

$$\begin{array}{ccc} \mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0) & \xrightarrow{\mathcal{L}_i^{\mathfrak{g}_0}} & \mathcal{HC}(\mathfrak{g}, \mathfrak{g}_0) \\ \text{Res}_{\mathfrak{g}_0} \downarrow & & \downarrow \text{Res}_{\mathfrak{g}_0} \\ \mathcal{HC}(\mathfrak{g}_0, \mathfrak{s}_0) & \xrightarrow{{}^0\mathcal{L}_i^{\mathfrak{g}_0}} & \mathcal{HC}(\mathfrak{g}_0, \mathfrak{g}_0), \end{array}$$

where  $\text{Res}_{\mathfrak{g}_0}$  is the forgetful functor from  $\mathfrak{g}$ -modules to  $\mathfrak{g}_0$ -modules.

PROOF : First assume  $i = 0$ . As a left  $\mathfrak{g}_0$ -module,  $R(\mathfrak{g}, G_0) \simeq \mathcal{M}(G_0) \otimes_{U(\mathfrak{g}_0)} U(\mathfrak{g})$ , so that by transitivity of tensor product we have for any  $M$  in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$  :

$$\text{Res}_{\mathfrak{g}_0} \mathcal{L}_0^{\mathfrak{g}_0}(M) \simeq \mathcal{M}(G_0) \otimes_{U(\mathfrak{g}_0)} U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M \simeq \mathcal{M}(G_0) \otimes_{U(\mathfrak{g}_0)} M = {}^0\mathcal{L}_0^{\mathfrak{g}_0} \text{Res}_{\mathfrak{g}_0} M.$$

Now, the functor  $\text{Res}_{\mathfrak{g}_0}$  is exact and, by lemma 1.2.1 (ii), it maps projectives in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$  to projectives in  $\mathcal{HC}(\mathfrak{g}_0, \mathfrak{s}_0)$ . Hence the assertion for all  $i$  follows by deriving the assertion for  $i = 0$ .  $\square$

Now we want to relate the three categories  $\mathcal{HC}(\mathfrak{g}, \mathfrak{g}_0)$ ,  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$  and  $\mathcal{HC}(\mathfrak{g}, \mathfrak{h})$ . First, mimicking [9], section 4, or in [8], proposition 2.9, we define a right exact functor

$$\mathcal{L}'_0 = R(\mathfrak{g}, S_0) \otimes_{U(\mathfrak{g})} ? : \mathcal{HC}(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0).$$

Notice that since  $H$  is connected,  $\otimes_{U(\mathfrak{g})}$  is the same as  $\otimes_{R(\mathfrak{g}, H)}$ . This functor has a purely classical counterpart, namely :

$${}^0\mathcal{L}'_0 = \mathcal{M}(S_0) \otimes_{U(\mathfrak{s}_0)} ? : \mathcal{HC}(\mathfrak{s}_0, \mathfrak{h}) \longrightarrow \mathcal{HC}(\mathfrak{s}_0, \mathfrak{s}_0).$$

The functor  ${}^0\mathcal{L}'_0$  and its derived functors, denoted by  ${}^0\mathcal{L}'_i$  in the sequel, can be computed by a variant of the Borel-Weil-Bott theorem.

**Lemma 1.2.3** (i) *There is an isomorphism of functors :  $\mathcal{L}_0^{\mathfrak{h}} \simeq \mathcal{L}_0^{\mathfrak{g}_0} \circ \mathcal{L}'_0$ .*

(ii) *For any  $i \geq 0$ , there is a commutative diagram of functors :*

$$\begin{array}{ccc} \mathcal{HC}(\mathfrak{g}, \mathfrak{h}) & \xrightarrow{\mathcal{L}'_i} & \mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0) \\ \text{Res}_{\mathfrak{s}_0} \downarrow & & \downarrow \text{Res}_{\mathfrak{s}_0} \\ \mathcal{HC}(\mathfrak{s}_0, \mathfrak{h}) & \xrightarrow{{}^0\mathcal{L}'_i} & \mathcal{HC}(\mathfrak{s}_0, \mathfrak{s}_0), \end{array}$$

PROOF : As in [8], proposition 2.18 (a), we see that if  $N$  is in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$ , then  $N \otimes_{U(\mathfrak{g})} R(\mathfrak{g}, S_0) \simeq N$ . In particular, for  $N = R(\mathfrak{g}, G_0)$ , we get for any  $M$  in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{h})$  that :

$$\mathcal{L}_0^{\mathfrak{h}}(M) = R(\mathfrak{g}, G_0) \otimes_{U(\mathfrak{g})} M = R(\mathfrak{g}, G_0) \otimes_{U(\mathfrak{g})} R(\mathfrak{g}, S_0) \otimes_{U(\mathfrak{g})} M = \mathcal{L}_0^{\mathfrak{g}_0} \circ \mathcal{L}'_0(M).$$

This proves (i). Assertion (ii) is proved by the same method as lemma 1.2.2 : associativity of tensor product implies the claim for  $i = 0$  ;  $\text{Res}_{\mathfrak{s}_0}$  is exact and, by lemma 1.2.1 (iii), maps projectives to projectives, therefore we can derive the claim for all  $i$  from the case  $i = 0$ .  $\square$

### 1.3 Zuckerman functors and generalized Verma modules

Let  $\mathfrak{p}$  be a parabolic subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{b}$ , with nilpotent radical  $\mathfrak{u}$  and Levi component  $\mathfrak{s} = \mathfrak{p}/\mathfrak{u} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$  (see [9], section 2). Then  $\mathfrak{s}_0$  is a Levi subalgebra of  $\mathfrak{g}_0$ . In this section, we use the functor  $\mathcal{L}_0^{\mathfrak{h}}$  to define nice highest weight modules (lemma 1.3.2(i)). But when we want to compute these modules, it will be convenient to use a second construction with  $\mathfrak{s}_0 = \mathfrak{g}_0$  (see lemma 1.3.4(ii)) and the character formula in lemma 1.3.3(ii).

**Lemma 1.3.1** (i) *Let  $E$  be a finite-dimensional  $\mathfrak{p}$ -module and  $M = \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} E$ . Then  $\mathcal{L}_0^{\mathfrak{s}_0} M$  is the biggest finite-dimensional quotient of  $M$ ; all the modules  $\mathcal{L}_i^{\mathfrak{s}_0} M$  are finite-dimensional.*

(ii) *The annihilator in  $U(\mathfrak{g})$  of any module  $M$  in  $\mathcal{HC}(\mathfrak{g}, \mathfrak{s}_0)$  is contained in the annihilator of  $\mathcal{L}_i^{\mathfrak{s}_0} M$  for any  $i$ : in particular all these modules have the same central character.*

PROOF : (i) is proposition 4.10 and (ii) is proposition 4.5 in [9].  $\square$

**Lemma 1.3.2** (i) *Let  $\lambda \in X^+$ . Then  $\mathcal{L}_0^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$  is projective in the category  $\mathfrak{g}\text{-mod}_{\leq \lambda}$  of finite-dimensional  $\mathfrak{g}$ -modules all simple subquotients of which have a highest weight lower than or equal to  $\lambda$ .*

(ii) *For every  $\lambda \in X^+$ , we have :  $\text{Ext}_{\mathfrak{g}}^1(S(\lambda), S(\lambda)) = 0$ .*

PROOF : (i) As a quotient of  $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$ , the module  $\mathcal{L}_0^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$  is generated by a vector of weight  $\lambda$ . Assume we are given a surjective map  $\pi : M \rightarrow \mathcal{L}_0^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$  in the category  $\mathfrak{g}\text{-mod}_{\leq \lambda}$ . Let  $v \in M$  be a vector of weight  $\lambda$  that maps to a generator of  $\mathcal{L}_0^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$  under  $\pi$ . Since  $\mathfrak{g}_0$  is semi-simple, the  $\mathfrak{h}$ -module  $M$  is semi-simple, hence, by maximality of  $\lambda$ , there is a map of  $\mathfrak{b}$ -modules  $\mathbb{C}_{\lambda} \rightarrow M$ ,  $1 \mapsto v$ , which induces a map of  $\mathfrak{g}$ -modules  $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda} \rightarrow M$ . As  $M$  is finite-dimensional, this map factors through  $\mathcal{L}_0^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$ , thereby producing a section of  $\pi$ . This proves assertion (i).

Assertion (ii) follows since any extension of  $S(\lambda)$  by itself is in  $\mathfrak{g}\text{-mod}_{\leq \lambda}$  and since the multiplicity of  $S(\lambda)$  in  $\mathcal{L}_0^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$  is 1.  $\square$

For  $\mathfrak{p}$ ,  $\mathfrak{s}_0$  as above, let  $E$  be a finite-dimensional  $\mathfrak{p}$ -module. We denote by  $\Pi(\mathfrak{p}, E)$  the multiset of highest weights of the  $\mathfrak{s}_0$ -module  $\bigwedge(\mathfrak{g}/(\mathfrak{g}_0 + \mathfrak{p})) \otimes E$  (the weights appear with multiplicity, and the exterior algebra is a classical, not super exterior algebra).

For  $i \in \mathbb{N}$ , we denote by  $\Pi_i(\mathfrak{p}, E)$  the set of elements  $\mu \in \Pi(\mathfrak{p}, E)$  such that there is an element  $w \in W$  of length  $i$  such that  $w(\mu + \rho_0) - \rho_0$  is dominant for  $\mathfrak{g}_0$ . (If  $\mu + \rho_0$  is singular,  $\mu$  does not belong to any  $\Pi_i(\mathfrak{p}, E)$ .) Notice that these definitions involve only the semi-simple Lie algebra  $\mathfrak{g}_0$ .

The character formulas in the following lemma were first proved in [9], proposition 5.2.

**Lemma 1.3.3** (Notations as above.)

(i) *The set of highest weights of  $\mathfrak{g}_0$ -modules occurring in  $\mathcal{L}_i^{\mathfrak{s}_0} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} E$  is included in  $\Pi_i(\mathfrak{p}, E)$ .*

(ii) 
$$\sum_{i \geq 0} (-1)^i \text{ch } \mathcal{L}_i^{\mathfrak{s}_0} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} E = \sum_{i \geq 0} (-1)^i \sum_{\mu \in \Pi_i(\mathfrak{p}, E)} \chi(e^{\mu}).$$

(iii) *In particular, if  $\mathfrak{p} = \mathfrak{b}$  and  $E = \mathbb{C}_{\lambda}$ , then :  $\sum_{i \geq 0} (-1)^i \text{ch } \mathcal{L}_i^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda} = \chi\left(\prod_{\beta \in \Delta_1^+} (1 + e^{-\beta}) e^{\lambda}\right)$ . This*

*expression is anti-invariant under the shifted action of  $W$ .*

REMARK. Theorem 6.8 in [9] states that when  $\lambda$  is dominant and typical, the module  $\mathcal{L}_i^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$  is simple if  $i = 0$  and zero if  $i > 0$ . Thus one could expect that in general the right handside of (iii) is the character of the highest weight module  $\mathcal{L}_0^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$ . It is often the case (see lemma 1.3.5) but not always (e.g.,  $\lambda = \lambda_0$  for  $\mathfrak{g} = \text{osp}(3, 2)$ ).

PROOF : By lemma 2.6 in [9], the  $\mathfrak{g}_0$ -module  $\text{Res}_{\mathfrak{g}_0} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} E$  admits a finite filtration by  $\mathfrak{g}_0$ -submodules  $V_r'$  such that the successive subquotients are  $\text{Ind}_{\mathfrak{p}_0}^{\mathfrak{g}_0} \bigwedge^r (\mathfrak{g}/(\mathfrak{g}_0 + \mathfrak{p})) \otimes E$ . This filtration can be refined into a filtration  $(V_p)_{0 \leq p \leq \sharp \Pi(\mathfrak{p}, E)}$  with subquotients  $V_p/V_{p-1} \simeq \text{Ind}_{\mathfrak{p}_0}^{\mathfrak{g}_0} L(\mathfrak{p}_0, \mu_p)$ , where  $p \mapsto \mu_p$  is an indexation of  $\Pi(\mathfrak{p}, E)$  and  $L(\mathfrak{p}_0, \mu_p)$  is the simple  $\mathfrak{s}_0$ -module with highest weight  $\mu_p$  on which  $\mathfrak{u}_0$  acts trivially. Thus we can write down a convergent spectral sequence of  $\mathfrak{g}_0$ -modules with  $E_1$ -term given by :

$$E_1^{p,q} = {}^0\mathcal{L}_{p+q}^{\mathfrak{s}_0} \text{Ind}_{\mathfrak{p}_0}^{\mathfrak{g}_0} L(\mathfrak{p}_0, \mu_p) \Rightarrow \text{Res}_{\mathfrak{g}_0} \mathcal{L}_{p+q}^{\mathfrak{s}_0} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} E.$$

By a variant of the Borel-Weil-Bott theorem,  ${}^0\mathcal{L}_i^{\mathfrak{s}_0} \text{Ind}_{\mathfrak{p}_0}^{\mathfrak{g}_0} L(\mathfrak{p}_0, \mu_p)$  is  $L(\mathfrak{g}_0, w(\mu_p + \rho_0) - \rho_0)$  if there exists an element  $w \in W$  of length  $i$  such that  $w(\mu_p + \rho_0) - \rho_0$  is dominant for  $\mathfrak{g}_0$ , and it is zero otherwise. This implies (i). Assertion (ii) follows from the Euler-Poincaré principle, and the formula in (iii) is a particular case. This formula can also be written as :

$$\begin{aligned} X(\lambda) &:= \chi \left( \prod_{\beta \in \Delta_1^+} (1 + e^{-\beta}) e^{\lambda} \right) = \frac{\sum_{w \in W} (-1)^{\ell(w)} w \left( \prod_{\beta \in \Delta_1^+} (e^{\beta/2} + e^{-\beta/2}) \cdot e^{\lambda + \rho_0 - \rho_1} \right)}{\prod_{\beta \in \Delta_0^+} (e^{\beta/2} - e^{-\beta/2})} \\ &= \frac{\prod_{\beta \in \Delta_1^+} (e^{\beta/2} + e^{-\beta/2})}{\prod_{\beta \in \Delta_0^+} (e^{\beta/2} - e^{-\beta/2})} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}. \end{aligned}$$

The anti-invariance under the shifted action of  $u \in W$  is now obvious :  $X(u.\lambda) = (-1)^{\ell(u)} X(\lambda)$ .  $\square$

Now we want to compare  $\mathcal{L}_i^{\mathfrak{s}_0} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\mathfrak{p}, \lambda)$  and  $\mathcal{L}_i^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$  in special situations : the goal here is part (ii) of lemma 1.3.4. Once the abstract nonsense is removed, the proof boils down to compute Zuckerman functors in a classical setting, using the Borel-Weil-Bott theorem.

From now on, we assume that  $\mathfrak{b}$  is distinguished, *i.e.* that only one simple root  $\beta_s$  is odd (see [6], proposition 1.5). We denote by  $\tilde{\theta}$  the unique simple root of  $\mathfrak{g}_0$  which is not a simple root for  $\mathfrak{g}$  as in [7], example 3.3. There is a  $\mathbb{Z}$ -grading on  $\mathfrak{g}$  such that  $\mathfrak{g}_{\alpha}$  has degree 0 for  $\alpha$  a simple even root and  $\mathfrak{g}_{\beta_s}$  has degree 1. This grading is concentrated in degrees  $-2, -1, 0, 1, 2$  and the homogenous spaces are eigenspaces for a central element  $z \in \mathfrak{g}_0$ . Let  $\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$  : it is a parabolic subalgebra of  $\mathfrak{g}$  with Levi component  $\mathfrak{s}_0 = \mathfrak{g}_0$ . Notice that  $\mathfrak{g}/(\mathfrak{g}_0 + \mathfrak{p}) \simeq \mathfrak{g}_{-1}$ . Given  $\lambda \in X^+$ , we denote by  $L(\mathfrak{p}, \lambda)$  the  $\mathfrak{g}_0$ -module with highest weight  $\lambda$  on which  $\mathfrak{g}_1 + \mathfrak{g}_2$  acts trivially.

**Lemma 1.3.4** *Assume  $\mathfrak{b}$  that is distinguished and that  $\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$ . Let  $\lambda \in \mathfrak{h}^*$  be dominant for  $\mathfrak{s}_0 = \mathfrak{g}_0$ . Then :*

- (i)  $\mathcal{L}'_0 \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda} \simeq \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\mathfrak{p}, \lambda)$  and  $\mathcal{L}'_i \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda} = 0$  if  $i > 0$ .
- (ii) For any  $i \in \mathbb{N}$ , we have :  $\mathcal{L}_i^{\mathfrak{s}_0} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\mathfrak{p}, \lambda) \simeq \mathcal{L}_i^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$ .

PROOF : Set  $I = \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda}$ . First, we want to show that (i) is true at the level of  $\mathfrak{s}_0$ -modules.

Extending [9], lemma 2.6,  $\text{Res}_{\mathfrak{s}_0} I$  has an increasing filtration by modules  $(V_j)_{j \geq -1}$  with  $V_{-1} = 0$  and

$$V_j/V_{j-1} \simeq \bigoplus_{k+l=j} \text{Ind}_{\mathfrak{b}_0 \cap \mathfrak{s}_0}^{\mathfrak{s}_0} S^k(\mathfrak{g}_{-2}) \otimes \bigwedge^l(\mathfrak{g}_{-1}) \otimes \mathbb{C}_{\lambda}.$$

By the Borel-Weil theorem, since  $\lambda$  is dominant for  $\mathfrak{s}_0$ , we get :

$${}^0\mathcal{L}'_0(V_j/V_{j-1}) \simeq \bigoplus_{k+l=j} S^k(\mathfrak{g}_{-2}) \otimes \bigwedge^l(\mathfrak{g}_{-1}) \otimes L(\mathfrak{p}, \lambda) \quad \text{and} \quad \mathcal{L}'_i(V_j/V_{j-1}) = 0 \quad (i \geq 1).$$



It follows by induction that

$${}^0\mathcal{L}'_0(V_j) \simeq \bigoplus_{k+l \leq j} S^k(\mathfrak{g}_{-2}) \otimes \bigwedge^l(\mathfrak{g}_{-1}) \otimes L(\mathfrak{p}, \lambda) \quad \text{and} \quad \mathcal{L}'_i(V_j) = 0 \quad (i \geq 1).$$

Since  $\lim_{\rightarrow} V_j = \text{Res}_{\mathfrak{s}_0} I$  and since the direct limit functor commutes with  ${}^0\mathcal{L}'_0 = \mathcal{M}(S_0) \otimes_{U(\mathfrak{s}_0)} ?$ , we get :

$$(*) \quad {}^0\mathcal{L}'_0 \text{Res}_{\mathfrak{s}_0} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda \simeq \bigoplus_{k,l \in \mathbb{N}} S^k(\mathfrak{g}_{-2}) \otimes \bigwedge^l(\mathfrak{g}_{-1}) \otimes L(\mathfrak{p}, \lambda).$$

Moreover the direct limit functor commutes with any tensor product, so that it takes standard projectives in  $\mathcal{HC}(\mathfrak{s}_0, \mathfrak{h})$  to standard projectives in  $\mathcal{HC}(\mathfrak{s}_0, \mathfrak{h})$ . It is moreover exact and commutes with  ${}^0\mathcal{L}'_0$ , hence :  $\lim_{\rightarrow} {}^0\mathcal{L}'_i = {}^0\mathcal{L}'_i \lim_{\rightarrow}$ . Consequently, we get :  ${}^0\mathcal{L}'_i I = 0$  for  $i \geq 1$ .

To prove (i), since by (\*) the  $\mathfrak{s}_0$ -modules  $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$  and  $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\mathfrak{p}, \lambda)$  are isomorphic, it is enough to show there is a epimorphism of  $\mathfrak{g}$ -modules  $\phi : \mathcal{L}'_0 I \rightarrow \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\mathfrak{p}, \lambda)$ . But (\*) implies there is a morphism of  $\mathfrak{p}$ -modules  $\mathcal{L}'_0 I \rightarrow L(\mathfrak{p}, \lambda)$ , because by weight considerations,  $\mathfrak{g}_1 \oplus \mathfrak{g}_2$  kills the summand corresponding to  $k = l = 0$ . The existence of  $\phi$  follows, hence (i) is proved.

(ii) Since we know that  $\mathcal{L}_0^{\mathfrak{h}} \simeq \mathcal{L}_0^{\mathfrak{s}_0} \circ \mathcal{L}'_0$ , we can write down a Grothendieck spectral sequence with  $E_2$ -term given by :  $\mathcal{L}_i^{\mathfrak{s}_0} \circ \mathcal{L}'_j(\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda)$ . By (i), this spectral sequence degenerates and converges to  $\mathcal{L}_{i+j}^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ , hence we get the isomorphism in (ii).  $\square$

To simplify notations, we set  $\mathcal{L}_i(\lambda) = \mathcal{L}_i^{\mathfrak{s}_0} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\mathfrak{p}, \lambda) = \mathcal{L}_i^{\mathfrak{h}} \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ , and  $\Pi(\lambda_i) = \Pi(\mathfrak{p}, L(\mathfrak{p}, \lambda_i))$ , and we use the same convention for  $\Pi_i$ .

**Lemma 1.3.5** *Assume  $\mathfrak{b}$  is distinguished and that  $\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$ . Let  $\lambda \in X^+$ . Then :*

- (i) *If  $(\lambda, \tilde{\theta}^\vee) > -2(\rho, \tilde{\theta}^\vee)$ , then :  $\mathcal{L}_i(\lambda) = 0$  for any  $i > 0$ .*
- (ii) *If  $\lambda$  is atypical, and if the modules  $\mathcal{L}_i(\lambda)$  are zero for  $i > 0$ , then the module  $\mathcal{L}_0(\lambda)$  is not simple.*

PROOF : To prove (i), we show that any  $\mu \in \Pi(\lambda)$  is dominant or singular whence  $\Pi_i(\lambda) = \emptyset$  for  $i > 0$ . We know that  $\mu$  is dominant for  $\mathfrak{g}_0$ , so it is enough to check that  $(\mu, \tilde{\theta}^\vee) \geq -1$ . For that purpose, we notice that  $\mu = \lambda - \sum_{\beta \in I} \beta$  for some  $I \subset \Delta_1^+$ . A case-by-case verification shows that for any  $\beta \in \Delta_1^+$ , we have :  $(\beta, \tilde{\theta}^\vee) \geq 0$ . Hence :

$$(\mu, \tilde{\theta}^\vee) = (\lambda - \sum_{\beta \in I} \beta, \tilde{\theta}^\vee) \geq (\lambda - 2\rho_1, \tilde{\theta}^\vee) = (\lambda, \tilde{\theta}^\vee) - 2(\rho_1, \tilde{\theta}^\vee) + 2(\rho_0, \tilde{\theta}^\vee) - 2 > -2.$$

Under the assumption of (ii), the character of the module  $\mathcal{L}_0(\lambda)$  is given by the right-hand side of lemma 1.3.3 (iii). By theorem 1 in [6], if it is simple, then  $\lambda$  is typical. This proves (ii).  $\square$

## 2 Representations of $\text{osp}(3, 2)$

### 2.1 Notations and result

Let  $\mathfrak{g} = \text{osp}(3, 2)$  (see [6]). The even part of  $\mathfrak{g}$  is  $\mathfrak{g}_0 \simeq \text{so}(3) \oplus \text{sp}(2) \simeq \text{sl}(2) \oplus \text{sl}(2)$ , hence  $\dim \mathfrak{h}^* = 2$ . We can choose a basis  $(\varepsilon, \delta)$  of  $\mathfrak{h}^*$  such that

$$\begin{aligned} \Delta_0 &= \{\pm\varepsilon; \pm 2\delta\}, & \Delta_1 &= \{\pm\varepsilon \pm \delta; \pm\delta\}, \\ \text{and} \quad (\varepsilon, \varepsilon) &= -(\delta, \delta) = 1, & (\varepsilon, \delta) &= 0. \end{aligned}$$

We choose the following distinguished set of positive roots :  $\Delta^+ = \{\varepsilon; 2\delta\} \cup \{\pm\varepsilon + \delta; \delta\}$  which corresponds to the simple roots  $-\varepsilon + \delta$  and  $\varepsilon$ . Here we have  $\tilde{\theta} = 2\delta$  and  $2\rho = \varepsilon - \delta$  so that  $-2(\rho, \tilde{\theta}^\vee) = 1$ . Besides,  $\mathfrak{g}_0$  is isomorphic to  $\mathfrak{sl}(2) \oplus \mathbb{C}$ . Using the basis  $(\varepsilon, \delta)$ , dominant weights for  $\mathfrak{g}_0$  (resp.  $\mathfrak{g}$ ) will be identified with  $\mathbb{N}/2 \times \mathbb{N}$  (resp.  $\mathbb{N}/2 \times \mathbb{C}$ ). By proposition 2.3 in [6], the set of dominant weights for  $\mathfrak{g}$  is :  $X^+ = \{(a, b) \in \mathbb{N}/2 \times \mathbb{N} \mid b = 0 \Rightarrow a = 0\}$ . The atypical dominant weights are  $\lambda_0 = (0, 0)$  and  $\lambda_l = (l - 1, l)$  (for  $l \in \mathbb{N}^*$ ). For example,  $S(\lambda_0)$  is the trivial module,  $S(\lambda_1)$  is the standard module. It can be checked that  $\text{Cas}(\lambda_l) = 0$  for any  $l \in \mathbb{N}$ .

**Theorem 2.1.1** *Let  $\mathfrak{g} = \text{osp}(3, 2)$ .*

- (a) *Up to a shift of parity, the principal block  $\Gamma_0 = \{\lambda_l \mid l \in \mathbb{N}\}$  is the unique atypical block.*  
(b) *The projective indecomposable modules have the following radical layer structure :*

$$\begin{array}{cccccccc} S(\lambda_0) & & S(\lambda_1) & & S(\lambda_2) & & S(\lambda_l) & \\ S(\lambda_2) & & S(\lambda_2) & & S(\lambda_0) & S(\lambda_1) & S(\lambda_3) & S(\lambda_{l-1}) & S(\lambda_{l+1}) & (l \geq 3) \\ S(\lambda_0) & & S(\lambda_1) & & S(\lambda_2) & & S(\lambda_l) & & & \end{array}$$

- (c) *Consider  $\Gamma_0$  as the basis of a root system of type  $D_\infty$  as follows :*

$$\begin{array}{ccccccc} \lambda_1 & - & \lambda_2 & - & \lambda_3 & - & \lambda_4 & - & \dots \\ & & | & & & & & & \\ & & \lambda_0 & & & & & & \end{array}$$

*Given a non-projective indecomposable module  $M$  in  $\Gamma_0$ , let  $[M] = \sum_{l \in \mathbb{N}} [M : S(\lambda_l)] \cdot \lambda_l \in \mathbb{N}\Gamma_0$ . The map  $M \mapsto [M]$  is onto the set of positive roots of  $\Gamma_0$ . Two indecomposable module are mapped to the same positive root if and only if they are isomorphic or dual to each other. The only self-dual non-projective indecomposable modules are the simple ones.*

This theorem follows from proposition 2.3.2 and section 2.4.

## 2.2 Highest weight modules and simple modules

Part (iii) of the following lemma has been known for long (see *e.g.* [12] or [7]).

**Lemma 2.2.1** (i) *We have :  $\mathcal{L}_0(\lambda_0) \simeq \mathcal{L}_1(\lambda_1) \simeq S(\lambda_0)$  and  $\mathcal{L}_1(\lambda_0) \simeq \mathcal{L}_0(\lambda_1) \simeq S(\lambda_1)$ .*  
(ii) *For  $l \geq 2$ ,  $\mathcal{L}_i(\lambda_l)$  is zero if  $i > 0$  and  $\mathcal{L}_0(\lambda_l)$  is described by :*

$$\begin{array}{l} 0 \rightarrow S(\lambda_0) \oplus S(\lambda_1) \rightarrow \mathcal{L}_0(\lambda_2) \rightarrow S(\lambda_2) \rightarrow 0, \\ 0 \rightarrow S(\lambda_{l-1}) \rightarrow \mathcal{L}_0(\lambda_l) \rightarrow S(\lambda_l) \rightarrow 0 \quad \text{if } l \geq 3. \end{array}$$

- (iii) *The characters of the atypical simple modules of  $\text{osp}(3, 2)$  are :*

$$\text{ch } S(\lambda_l) = \begin{cases} \chi(e^{\lambda_0}) = 1/2 \cdot \chi\left(\prod_{\beta \neq -\varepsilon + \delta} (1 + e^{-\beta}) e^{\lambda_0}\right) & \text{if } l = 0, \\ \chi(e^{(0,1)} + e^{(1,0)}) = \chi\left(\prod_{\beta \neq \varepsilon + \delta} (1 + e^{-\beta}) e^{\lambda_1}\right) - \text{ch } S(\lambda_0) & \text{if } l = 1, \\ \chi\left(\prod_{\beta \neq \varepsilon + \delta} (1 + e^{-\beta}) e^{\lambda_l}\right) & \text{if } l \geq 2. \end{cases}$$

REMARK. If  $s_{\tilde{\theta}} \in W$  denotes the symmetry in  $\mathfrak{h}^*$  with respect to the root  $\tilde{\theta}$ , we notice that :  $s_{\tilde{\theta}}(\lambda_0 + \rho) - \rho = \lambda_1$ . The coefficient  $1/2$  in the character formula for  $S(\lambda_0)$  is the coefficient  $j_0^{-1}$  introduced in [7]. The module  $S(\lambda_1)$  is an “exceptional” representation (see [7], Example 3.3) : the sum  $\chi(\prod_{\beta \neq \epsilon + \delta} (1 + e^{-\beta}) e^{\lambda_1})$  is the character of the radical of  $\mathcal{L}_0(\lambda_2)$ , which “exceptionally” splits into two components.

PROOF : The highest weights of the submodules of the  $\mathfrak{g}_0$ -module  $\bigwedge \mathfrak{g}_{-1}$  are :  $(0, 0)$ ,  $(1, -1)$ ,  $(1, -2)$  and  $(0, -3)$ . We thus obtain :  $\Pi_0(\lambda_0) = \Pi_1(\lambda_1) = \{(0, 0)\}$  and  $\Pi_1(\lambda_0) = \Pi_0(\lambda_1) = \{(0, 1), (1, 0)\}$ . Assertion (i) now follows from lemma 1.3.3. We also obtain that  $\text{Ext}_{\mathfrak{g}}^1(S(\lambda_1), S(\lambda_0)) = 0$  (otherwise the extension would have appeared in  $\mathcal{L}_0(\lambda_1)$ , by lemma 1.3.2).

Let  $l \geq 2$ . Since  $(\lambda, \tilde{\theta}^\vee) = l > 1$ , lemma 1.3.5 implies that  $\mathcal{L}_i(\lambda_l) = 0$  for  $i > 0$ . Besides, we have :

$$\Pi(\lambda_l) = \{(l-1, l), (l-2, l-1), (l-1, l-1), (l, l-1), \\ (l-2, l-2), (l-1, l-2), (l, l-2), (l-1, l-3)\}.$$

We know that  $\mathcal{L}_0(\lambda_{l+1})$  is not simple. But the only weights in  $\Gamma_0 \cap \Pi(\lambda_{l+1})$  are  $\lambda_l$  and  $\lambda_{l+1}$  : hence  $\mathcal{L}_0(\lambda_{l+1})$  is a non-split extension of  $S(\lambda_{l+1})$  by  $S(\lambda_l)$ , and the highest weights of the  $\mathfrak{g}_0$ -modules in  $S(\lambda_l)$  belong to  $\Pi(\lambda_l) \cap \Pi(\lambda_{l+1})$ .

It is immediate to check that  $\Pi_0(\lambda_2)$  is the disjoint union of  $\Pi_0(\lambda_0)$ ,  $\Pi_0(\lambda_1)$  and  $\Pi(\lambda_2) \cap \Pi(\lambda_3)$ . It follows that the set of highest weights of the  $\mathfrak{g}_0$ -submodules of  $S(\lambda_2)$  is exactly  $\Pi(\lambda_2) \cap \Pi(\lambda_3)$ , that  $\mathcal{L}_0(\lambda_2)$  has three composition factors, namely  $S(\lambda_2)$ ,  $S(\lambda_0)$  and  $S(\lambda_1)$ , and (since  $S(\lambda_0)$  and  $S(\lambda_1)$  have no non-split extension) that the radical of  $\mathcal{L}_0(\lambda_2)$  is the direct sum  $S(\lambda_0) \oplus S(\lambda_1)$ .

To finish the proof of the lemma, we must compute  $\text{ch } S(\lambda_{l+1})$  for  $l \geq 2$ , which is inductively done using :  $\text{ch } S(\lambda_{l+1}) = \text{ch } \mathcal{L}_0(\lambda_{l+1}) - \text{ch } S(\lambda_l)$ .  $\square$

## 2.3 Projective modules

**Lemma 2.3.1** *The atypical projective modules of  $\text{osp}(3, 2)$  are described by assertion (b) of theorem 2.1.1.*

REMARK. By analogy with  $\text{sl}(m, n)$  (see [2], section 3.6), one could hope that the highest weight modules  $\mathcal{L}_0(\mu)$  play a part in a reciprocity principle with the modules  $P(\lambda)$  and  $S(\lambda)$ . But this is not the case, since the modules  $P(\lambda_0)$  and  $P(\lambda_1)$  do *not* have a filtration by modules  $\mathcal{L}_0(\mu)$ .

PROOF : By the proof of lemma 1.1.1, the projective cover  $P(\lambda_l)$  of  $S(\lambda_l)$  in  $\mathfrak{g}$ -mod is a direct summand in  $\tilde{Q}(\lambda_l) = \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L(\mathfrak{g}_0, \lambda_l)$ . As a  $\mathfrak{g}_0$ -module,  $\tilde{Q}(\lambda_l)$  is isomorphic to  $\bigwedge \mathfrak{g}_1 \otimes L(\mathfrak{g}_0, \lambda_l)$ . Now we know the characters of all the simple modules we can recover the composition series of  $\tilde{Q}(\lambda_l)$  by decomposing this tensor product of  $\mathfrak{g}_0$ -modules. If  $Q(\lambda_l)$  denotes the quotient of  $\tilde{Q}(\lambda_l)$  by its typical submodules (which are direct summands), we obtain after computations :

$$(\S) \quad \text{ch } Q(\lambda_l) = \begin{cases} 2 \text{ch } S(\lambda_l) + \text{ch } S(\lambda_2) & \text{if } l \in \{0, 1\}, \\ 2 \text{ch } S(\lambda_2) + \text{ch } S(\lambda_0) + \text{ch } S(\lambda_1) + \text{ch } S(\lambda_3) & \text{if } l = 2, \\ 2 \text{ch } S(\lambda_l) + \text{ch } S(\lambda_{l-1}) + \text{ch } S(\lambda_{l+1}) & \text{if } l \geq 3. \end{cases}$$

For all  $l \in \mathbb{N}$ , we have :  $P(\lambda_l)/\text{rad } P(\lambda_l) \simeq S(\lambda_l)$ . Moreover,  $\text{rad } P(\lambda_l)/\text{rad}^2 P(\lambda_l)$  is the direct sum of simple modules  $S(\mu)$  which have a non-split extension by  $S(\lambda_l)$ . By lemmas 1.1.1 and 2.2.1, we get :

$$(\S\S) \quad \text{rad } P(\lambda_l)/\text{rad}^2 P(\lambda_l) \simeq \begin{cases} S(\lambda_2) & \text{if } l \in \{0, 1\}, \\ S(\lambda_0) \oplus S(\lambda_1) \oplus S(\lambda_3) & \text{if } l = 2, \\ S(\lambda_{l-1}) \oplus S(\lambda_{l+1}) & \text{if } l \geq 3. \end{cases}$$



- Lemma 2.4.1** (i) *The modules  $\mathbf{p}(\lambda_l)$  are the projective and the injective modules in  $\Gamma_0/R\text{-mod}$ .*  
(ii) *Let  $\mathbf{m}$  be an indecomposable  $\Gamma_0/R$ -module. The following conditions are equivalent :*  
(a)  *$\mathbf{m}$  has no projective direct summand ;*  
(b)  *$d_l^- d_l^+ = 0$  for every  $l \in \mathbb{N}$ .*

PROOF : On any  $\Gamma_0/R$ -module, any composition of 3 or more arrows is zero, and the only possibly non-zero products of 2 arrows are  $d_l^+ d_l^-$  and  $d_l^- d_l^+$ . It follows easily that the maximal dimension of a  $\Gamma_0/R$ -module generated by a single vector  $v$  of degree  $\lambda_l$  is 3 (resp. 5, resp. 4) if  $l \in \{0, 1\}$  (resp.  $l = 2$ , resp.  $l \geq 3$ ).

Now, assume we are given a surjection  $\pi : \mathbf{m} \rightarrow \mathbf{p}(\lambda_l)$ . Let  $v \in \mathbf{m}$  be such that  $\pi(v)$  generates  $\mathbf{p}(\lambda_l)$ . The remark above shows that the submodule  $\mathbf{m}'$  generated by  $v$  has a lower dimension than  $\mathbf{p}(\lambda_l)$ . But  $\pi$  maps  $\mathbf{m}'$  onto  $\mathbf{p}(\lambda_l)$ , hence they are isomorphic,  $\pi$  has a section and  $\mathbf{p}(\lambda_l)$  is projective. Since it is self-dual, it is injective too.

To prove assertion (ii), assume there is a vector  $v \in \mathbf{m}$  such that  $d_l^- d_l^+ v \neq 0$ . Then  $v$  generates a submodule of  $\mathbf{m}$  that is isomorphic to  $\mathbf{p}(\lambda_l)$ . This latter module being injective, it splits in  $\mathbf{m}$ . Conversely, the product  $d_l^- d_l^+$  is not zero in  $\mathbf{p}(\lambda_l)$ .  $\square$

Let us introduce the following quiver with two connected components :

$$\begin{array}{ccccccc}
& & (\Gamma^+) & & & (\Gamma^-) & \\
\lambda_1^t \leftarrow & \lambda_2^s & \rightarrow & \lambda_3^t \leftarrow & \lambda_4^s & \rightarrow & \dots \\
& \downarrow & & & & \uparrow & \\
& \lambda_0^t & & & & \lambda_0^s & 
\end{array}$$

By Gabriel's theorem about representations of finite-type quivers (see [1], theorem 4.5.6), there is a bijection  $\mathbf{v}^+$  (resp.  $\mathbf{v}^-$ ) from the set of positive roots of the root system of type  $D_\infty$  to the set of isomorphism classes of finite-dimensional representations of  $\Gamma^+$  (resp.  $\Gamma^-$ ). Moreover, a representation of  $\Gamma^+$  or  $\Gamma^-$  can naturally be made into a representation of  $\Gamma_0/R$ .

**Lemma 2.4.2** *Any non-projective indecomposable representation of  $\Gamma_0/R$  is of the form  $\mathbf{v}^\pm(\beta)$ , where  $\beta$  is a positive root of  $D_\infty$ . The representations  $\mathbf{v}^\sigma(\beta)$  and  $\mathbf{v}^{\sigma'}(\beta')$  are isomorphic if and only if either  $\sigma = \sigma'$  and  $\beta = \beta'$ , or  $\sigma = -\sigma'$  and  $\beta = \beta'$  is a simple root.*

PROOF : If  $\mathbf{m} = \bigoplus_{\lambda_l \in \Gamma_0} \mathbf{m}_{\lambda_l}$  is a non-projective  $\Gamma_0/R$ -module, we define a representation  $\hat{\mathbf{m}}$  of  $\Gamma^+ \cup \Gamma^-$  by setting  $\hat{\mathbf{m}}_{\lambda_l^t}$  equal to the kernel of all the arrows starting from  $\lambda_l$ , and  $\hat{\mathbf{m}}_{\lambda_l^s} = \mathbf{m}_{\lambda_l} / \mathbf{m}_{\lambda_l^t}$ . It makes sense because by lemma 2.4.1 the product of any two arrows acting on  $\mathbf{m}$  is zero. If  $\mathbf{m}$  is indecomposable, then  $\hat{\mathbf{m}}$  is indecomposable too. The rest of the claim is trivial.  $\square$

### 3 Representations of $D(2, 1; \alpha)$

#### 3.1 Notations and result

Let  $\alpha \in \mathbb{C} \setminus \{0, -1\}$ , and let  $\mathfrak{g}$  be the basic classical Lie superalgebra  $D(2, 1; \alpha)$  (see [6]). It is denoted by  $\Gamma(-(1+\alpha)/2, 1/2, \alpha/2)$  in [13] and [14]. Recall from [13] that if  $\alpha = 1$ , it is isomorphic to  $\mathfrak{osp}(4, 2)$ , and that the algebras corresponding to  $\alpha$ ,  $-1 - \alpha^{-1}$  and  $\alpha^{-1}$  are isomorphic (hence if  $\alpha \in \mathbb{Q}$  we can assume  $\alpha > 0$ , or even  $\alpha > 1$ ).

The root system of  $\mathfrak{g}$  can be realized in the space  $\mathfrak{h}^* = \mathbb{C}^3$  endowed with a basis  $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$  and with the bilinear whose matrix is  $\text{diag}(-(1+\alpha)/2, 1/2, \alpha/2)$ , by  $\Delta_0 = \{\pm 2\varepsilon_i \mid 1 \leq i \leq 3\}$  and  $\Delta_1 = \{\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}$ . We choose as simple roots  $\beta_4 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3$ ,  $2\varepsilon_2$  and  $2\varepsilon_3$ . The last positive

even root is :  $\tilde{\theta} = 2\varepsilon_1$ . The non-simple positive odd roots are :  $\beta_1 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$ ,  $\beta_2 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3$  and  $\beta_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ . We have  $\rho = -\beta_4$  and  $-2(\rho, \tilde{\theta}^\vee) = 2$ .

The even part of  $\mathfrak{g}$  is :  $\mathfrak{g}_0 \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ , and the degree 0 part of  $\mathfrak{g}$  is :  $\mathfrak{g}_0 \simeq \mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ . We identify the set of dominant weights for  $\mathfrak{g}_0$  (resp.  $\mathfrak{g}_0$ ) with  $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$  (resp.  $\mathbb{C} \times \mathbb{N} \times \mathbb{N}$ ). By proposition 2.2 in [6], the set of dominant weights for  $\mathfrak{g}$  is :  $X^+ = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid a = 0 \Rightarrow b = c = 0 \text{ and } a = 1 \Rightarrow (b + 1) = \pm\alpha(c + 1)\}$ . Besides, the eigenvalue of the Casimir element is :

$$\text{Cas}(a, b, c) = -\frac{1 + \alpha}{2}(a - 1)^2 + \frac{1}{2}(b + 1)^2 + \frac{\alpha}{2}(c + 1)^2.$$

A dominant weight  $\lambda = (a, b, c)$  is atypical if and only if one of the following scalar products is zero :

$$\begin{aligned} 2(\lambda + \rho, \beta_1) &= -(1 + \alpha)(a - 1) + (b + 1) - \alpha(c + 1) \\ 2(\lambda + \rho, \beta_2) &= -(1 + \alpha)(a - 1) - (b + 1) + \alpha(c + 1) \\ 2(\lambda + \rho, \beta_3) &= -(1 + \alpha)(a - 1) + (b + 1) + \alpha(c + 1) \\ 2(\lambda + \rho, \beta_4) &= -(1 + \alpha)(a - 1) - (b + 1) - \alpha(c + 1) \end{aligned}$$

Notice that if  $\alpha \notin \mathbb{Q}$ , the only atypical dominant weights are  $\lambda_0 = (0, 0, 0)$  (corresponding to the trivial module) and  $\lambda_l = (l + 1, l - 1, l - 1)$  for  $l \geq 1$ . For example,  $\lambda_1$  is the highest root of  $\mathfrak{g}$ , *i.e.*  $S(\lambda_1)$  is the adjoint representation.

If  $\alpha \in \mathbb{Q}$ , we assume  $\alpha = p/q$ , with  $p$  and  $q$  relatively prime positive integers. For  $k \in \mathbb{N}$ , let  $\Gamma_k$  be the set of atypical simple modules  $\lambda \in X^+$  such that  $\text{Cas}(\lambda) = p(p + q)k^2/2$ . For  $l \in \mathbb{Z}$  we set moreover :

$$\lambda_{k,l} = \begin{cases} (-l + 2, -l - kp, -l + kq) & \text{if } l \leq -kp, \\ (-l + 1, l + kp - 1, -l + kq - 1) & \text{if } -kp + 1 \leq l \leq 0, \\ (l + 1, l + kp - 1, -l + kq - 1) & \text{if } 0 \leq l \leq kq - 1, \\ (l + 2, l + kp, l - kq) & \text{if } kq \leq l. \end{cases}$$

**Theorem 3.1.1** *Let  $\alpha \in \mathbb{C} \setminus \{-1, 0\}$  and let  $\mathfrak{g} = D(2, 1; \alpha)$ .*

- (i) *The principal block of  $\mathfrak{g}$  is :  $\Gamma_0 = \{\lambda_l \mid l \in \mathbb{N}\}$ . This block is equivalent to the principal block of  $\mathfrak{osp}(3, 2)$  : statements (b) and (c) in theorem 2.1.1 hold for the principal block of  $D(2, 1; \alpha)$ .*
- (ii) *If  $\alpha \notin \mathbb{Q}$ ,  $\Gamma_0$  is the unique atypical block.*
- (iii) *Let  $\alpha \in \mathbb{Q}$ ,  $\alpha > 0$ , written as  $\alpha = p/q$  with  $p$  and  $q$  relatively prime. Then any atypical block is one of the  $\Gamma_k$ . Besides, let  $k \geq 1$ .*

(a) *The map  $\mathbb{Z} \rightarrow \Gamma_k$ ,  $l \mapsto \lambda_{k,l}$  is a bijection ;*

(b) *The projective cover of  $S(\lambda_{k,l})$  ( $l \in \mathbb{Z}$ ) has the following radical layer structure :*

$$\begin{array}{c} S(\lambda_{k,l}) \\ S(\lambda_{k,l-1}) \quad S(\lambda_{k,l+1}) \\ S(\lambda_{k,l}) \end{array}$$

(c) *Consider the symbols  $\lambda_{k,l}$  ( $l \in \mathbb{Z}$ ) as a basis of a root system of type  $A_\infty^\infty$  as follows :*

$$\dots \text{---} \lambda_{k,-2} \text{---} \lambda_{k,-1} \text{---} \lambda_{k,0} \text{---} \lambda_{k,1} \text{---} \lambda_{k,2} \text{---} \dots$$

*Given a non-projective indecomposable module  $M$  in  $\Gamma_0$ , let  $[M] = \sum_{l \in \mathbb{N}} [M : S(\lambda_l)] \cdot \lambda_l \in \mathbb{N}\Gamma_0$ . The map  $M \mapsto [M]$  is onto the set of positive roots of  $\Gamma_0$ . Two indecomposable module are mapped to the same positive root if and only if they are isomorphic or dual to each other. The only self-dual non-projective indecomposable modules are the simple ones.*

Assertion (i) will be proved in section 3.2, the rest of the theorem in section 3.3. In [13], J. van der Jeugt described simple representations of  $D(2, 1; \alpha)$ , which gives some redundancy with some lemmas in this section. In [14], Yi Ming Zou gives a list of monogenous indecomposable modules for  $D(2, 1; \alpha)$ , but it seems to be incomplete. Indeed, for instance, Zou finds no non-simple representation admitting the trivial module as a simple component.

### 3.2 The principal block

**Lemma 3.2.1** (i) *We have :  $\mathcal{L}_0(\lambda_0) \simeq \mathcal{L}_1(\lambda_1) \simeq S(\lambda_0)$  and  $\mathcal{L}_1(\lambda_0) \simeq \mathcal{L}_0(\lambda_1) \simeq S(\lambda_1)$ .*  
(ii) *For  $l \geq 2$ ,  $\mathcal{L}_i(\lambda_l)$  is zero if  $i > 0$  and  $\mathcal{L}_0(\lambda_l)$  is described by :*

$$\begin{aligned} 0 &\rightarrow S(\lambda_0) \oplus S(\lambda_1) \longrightarrow \mathcal{L}_0(\lambda_2) \longrightarrow S(\lambda_2) \rightarrow 0, \\ 0 &\rightarrow S(\lambda_{l-1}) \longrightarrow \mathcal{L}_0(\lambda_l) \longrightarrow S(\lambda_l) \rightarrow 0 \quad \text{if } l \geq 3. \end{aligned}$$

(iii) *For  $l \geq 2$ , we have :  $\text{ch } S(\lambda_l) = \chi \left( \prod_{\beta \in \Delta_1^+ \setminus \{\beta_3\}} (1 + e^{-\beta}) e^{\lambda_l} \right)$ .*

PROOF : The highest weights of the  $\mathfrak{g}_0$ -module  $\bigwedge \mathfrak{g}_{-1}$  are :  $(0, 0, 0)$ ,  $(-1, 1, 1)$ ,  $(-2, 2, 0)$ ,  $(-2, 0, 2)$ ,  $(-3, 1, 1)$ ,  $(-4, 0, 0)$ . This implies :

$$\Pi_0(\lambda_0) = \Pi_1(\lambda_1) = \{(0, 0, 0)\} \text{ and } \Pi_1(\lambda_0) = \Pi_0(\lambda_1) = \{(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 1)\}.$$

Assertion (i) follows, and we also get :  $\text{Ext}_{\mathfrak{g}}^1(S(\lambda_1), S(\lambda_0)) = 0$ .

The end of the proof can be copied verbatim on that of lemma 2.2.1.  $\square$

**Lemma 3.2.2** *Assertion (i) of theorem 3.1.1 holds.*

PROOF : The proof of theorem 2.3.2 can be repeated verbatim : the only difference is that there are more  $\mathfrak{g}_0$ -submodules in the decomposition of  $\tilde{Q}(\lambda_l)$ , but the character formula (§) in the proof of lemma 2.3.1, from which everything else is deduced, still holds for the principal block of  $D(2, 1; \alpha)$ .  $\square$

### 3.3 Other blocks

**Lemma 3.3.1** *Let  $k \in \mathbb{N}$ , and let  $\Gamma_k$  be the set of atypical dominant weights  $\lambda \in X^+$  such that  $\text{Cas}(\lambda) = p(p+q)k^2/2$ . Then any atypical dominant weight belongs to a unique  $\Gamma_k$  (for some  $k \in \mathbb{N}$ ). If  $k \geq 1$ , we have :  $\Gamma_k = \{\lambda_{k,l} \mid l \in \mathbb{Z}\}$ .*

PROOF : It is straightforward to see that  $\lambda_{k,l}$  does belong to  $\Gamma_k$  and that all the  $\lambda_{k,l}$  are distinct. Thus it is enough to check that any atypical dominant weight  $\lambda = (a, b, c)$  is one of the  $\lambda_{k,l}$  or one of the  $\lambda_l$ .

If  $(\lambda + \rho, \beta_1) = 0$ , then we have :  $q(b - a + 2) = p(a + c)$ . Since  $p$  and  $q$  are relatively prime, the ratio  $k = (a + c)/q$  is a positive integer. Set  $l = a - 1$ , we know by assumption that  $l \geq 0$  and  $l - kq + 1 = -c \leq 0$ , hence  $\lambda = \lambda_{k,l}$ . Similarly, if  $(\lambda + \rho, \beta_2) = 0$ , we set  $k = (a + b)/p$  and  $l = -a + 1$ , and again we get :  $\lambda = \lambda_{k,l}$ .

Finally, if  $(\lambda + \rho, \beta_3) = 0$ , then the number  $(b - a + 2)/p = (a - c - 2)/q$  is an integer. Let  $\sigma$  be its sign and  $k$  its absolute value. If  $k = 0$ , then  $\lambda = \lambda_{a-1} \in \Gamma_0$ . Otherwise, we set :  $l = \sigma(a - 2)$ . Again we have  $\lambda = \lambda_{k,l}$ .  $\square$

Clearly any  $\Gamma_k$  (for  $k \in \mathbb{N}^*$ ) is a union of blocks. In fact  $\Gamma_k$  is one block, as follows from the following lemma.

**Lemma 3.3.2** *Let  $k \in \mathbb{N}^*$ .*

(i) *We have :  $\mathcal{L}_0(\lambda_{k,0}) \simeq \mathcal{L}_1(\lambda_{k,0}) \simeq S(\lambda_{k,0})$ .*

(ii) *For  $l > 0$  (resp.  $l < 0$ ), the  $\mathfrak{g}$ -module  $\mathcal{L}_0(\lambda_{k,l})$  is a non-split extension of  $S(\lambda_{k,l})$  by  $S(\lambda_{k,l-1})$  (resp. by  $S(\lambda_{k,l+1})$ ). The modules  $\mathcal{L}_i(\lambda_{k,l})$  are zero for  $l \neq 0$  and  $i > 0$ .*

(iii) *We have :  $\text{ch } S(\lambda_{k,l}) = \chi \left( \prod_{\beta \in \Delta_1^+ \setminus \{\beta_{k,l}\}} (1 + e^{-\beta}) e^{\lambda_{k,l}} \right)$ , where  $\beta_{k,l} \in \Delta_1^+$  is the root orthogonal*

*to  $\lambda_{k,l} + \rho$ .*

REMARK. If  $l = 0$  the root  $\beta_{k,l}$  in (iii) can be either  $\beta_1$  or  $\beta_2$  but the character formula is the same. This follows from remark 3.2 in [7].

PROOF : Since  $\lambda_{k,0}$  is minimal in  $X^+$ , the module  $\mathcal{L}_0(\lambda_{k,0})$  is simple. Clearly  $\mathcal{L}_i(\lambda_{k,0})$  is zero for  $i \geq 2$ . We notice that  $s_{\bar{g}} \cdot \lambda_{k,0} = \lambda_{k,0}$ , hence by lemma 1.3.3, the sum  $\text{ch } \mathcal{L}_0(\lambda_{k,0}) - \text{ch } \mathcal{L}_1(\lambda_{k,0})$  is zero, and (i) follows. We could have seen this directly since  $\Pi_0(\lambda_{k,0})$  and  $\Pi_1(\lambda_{k,0})$  are equal to :

$$\{(1, kp - 1, kq - 1), (0, kp - 2, kq - 2), (0, kp, kq), (0, kp, kq - 2), (0, kp - 2, kq)\}.$$

(If  $kp$  or  $kq$  equals 1, a weight with a coordinate  $-1$  must be removed.)

If  $l = \pm 1$ , we have  $\Pi_1(\lambda_{k,l}) = \{(0, kp + l - 1, kq - l - 1)\}$  : it contains no dominant weight, so that  $\mathcal{L}_1(\lambda_{k,l}) = 0$ . Hence :  $\text{ch } \mathcal{L}_0(\lambda_{k,l}) = \sum_{\mu \in \Pi_0(\lambda_{k,l}) \setminus \Pi_1(\lambda_{k,l})} \chi(e^\mu)$ . Besides, if  $i \geq 2$  or if  $|l| \geq 2$  and  $i \geq 1$ , the set  $\Pi_i(\lambda_{k,l})$  is empty, so  $\mathcal{L}_i(\lambda_{k,l}) = 0$  too. Thus, by lemma 1.3.5, the module  $\mathcal{L}_0(\lambda_{k,l})$  is not simple for  $l \neq 0$ . But the only element in  $\Gamma_k \cap \Pi_0(\lambda_{k,l})$  is  $\lambda_{k,l-1}$  (resp.  $\lambda_{k,l+1}$ ) if  $l > 0$  (resp.  $l < 0$ ). Assertion (ii) follows.

Let  $l = \pm 1$  again. The set  $\Pi_0(\lambda_{k,l})$  is contained in :

$$\begin{aligned} &\{(2, kp + l - 1, kq - l - 1), \\ &(1, kp + l - 2, kq - l - 2), (1, kp + l, kq - l), (1, kp + l - 2, kq - l), (1, kp + l, kq - l - 2), \\ &(0, kp + l - 1, kq - l - 1), (0, kp + l - 1, kq - l - 1), (0, kp + l - 1, kq - l + 1), \\ &(0, kp + l + 1, kq - l - 1), (0, kp + l - 3, kq - l - 1), (0, kp + l - 1, kq - l - 3)\}. \end{aligned}$$

This inclusion is an equality except if  $kp$  or  $kq$  is 1 or 2 : in such a case, pairs of weights of the form  $\{(a, b, -2), (a, b, 0)\}$ ,  $\{(a, b, -3), (a, b, 1)\}$ ,  $\{(a, -2, c), (a, 0, c)\}$ ,  $\{(a, -3, c), (a, 1, c)\}$  should be removed.

All the highest weights of the  $\mathfrak{g}_0$ -modules occurring in  $S(\lambda_{k,0})$  must belong to :

$$A = \Pi_0(\lambda_{k,-1}) \cap \Pi(\lambda_{k,1}) = \{(1, kp - 1, kq - 1), (0, kp - 2, kq - 2), (0, kp, kq)\}.$$

Moreover,  $\Pi_0(\lambda_{k,l}) \setminus \Pi_1(\lambda_{k,l})$  is the disjoint union of  $A$  and  $B = \Pi_0(\lambda_{k,l}) \cap \Pi_0(\lambda_{k,2l})$ . Thus we obtain :  $\text{ch } S(\lambda_{k,0}) = \sum_{\mu \in A} \chi(e^\mu)$  and  $\text{ch } S(\lambda_{k,l}) = \sum_{\mu \in B} \chi(e^\mu)$ . The formula in (iii) is just another way to write down these formulas. The computation of  $\text{ch } S(\lambda_{k,l})$  for  $|l| \geq 2$  is now immediate by induction.  $\square$

**Proposition 3.3.3** *Let  $\alpha = p/q \in \mathbb{Q}_{>0}$  and  $k \in \mathbb{N}^*$ . The category of finite-dimensional  $D(2, 1; \alpha)$ -modules in  $\Gamma_k$  is equivalent to the category of representations of the following quiver :*

$$(\Gamma_k) \quad \cdots \bullet \xrightleftharpoons[d_2^-]{d_3^+} \lambda_{k,-2} \bullet \xrightleftharpoons[d_1^-]{d_2^+} \lambda_{k,-1} \bullet \xrightleftharpoons[d_0^-]{d_1^+} \lambda_{k,0} \bullet \xrightleftharpoons[d_1^-]{d_0^+} \lambda_{k,1} \bullet \xrightleftharpoons[d_2^-]{d_1^+} \lambda_{k,2} \bullet \xrightleftharpoons[d_3^-]{d_2^+} \bullet \cdots$$

*with relations  $(d^+)^2 = (d^-)^2 = (d^+ d^- + d^- d^+) = 0$ , where  $d^\pm = \sum_{l \in \mathbb{Z}} d_l^\pm$ .*



PROOF : We will determine only the structure of projective modules, *i.e.* prove assertion (iii-b) of theorem 2. We refer to section 5.2 in [2] for the end of the proof of this theorem, and to section 5.3 therein for the proof of assertion (iii-c) and a construction of indecomposable modules.

By the proof of the previous lemma, we see that  $\text{rad } P(\lambda_{k,l})/\text{rad}^2 P(\lambda_{k,l})$  is the direct sum  $S(\lambda_{k,l-1}) \oplus S(\lambda_{k,l+1})$ . Now, we proceed as before : we compute the composition series of  $\tilde{Q}(\lambda_{k,l}) = \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L(\mathfrak{g}_0, \lambda_{k,l})$  and we find that its atypical simple factors in  $\Gamma_k$  are :  $S(\lambda_{k,l})$  (with multiplicity 2),  $S(\lambda_{k,l+1})$  and  $S(\lambda_{k,l-1})$ . We can conclude as for the principal block of  $\text{osp}(3, 2)$ . This sketches the proof of assertion (iii-b) of theorem 2.  $\square$

## 4 Representations of $G(3)$

### 4.1 Roots and weights

Let  $\mathfrak{g}$  be the basic classical Lie superalgebra  $G(3)$  (see [6]). Its root system can be realized in the space  $\mathfrak{h}^* = \mathbb{C}^3$  endowed with a basis  $(\delta, \varepsilon_1, \varepsilon_2)$  and with the bilinear defined by :  $(\varepsilon_1, \varepsilon_1) = (\varepsilon_2, \varepsilon_2) = -2(\varepsilon_1, \varepsilon_2) = -(\delta, \delta) = 2$ . We take as simple roots  $\varepsilon_1, \varepsilon_2 - \varepsilon_1$  and  $-\varepsilon_1 - \varepsilon_2 + \delta$ . Then positive even roots are :  $\Delta_0^+ = \{\varepsilon_1, 2\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + 2\varepsilon_2, \varepsilon_2, \varepsilon_2 - \varepsilon_1, 2\delta\}$  and positive odd roots are :  $\Delta_1^+ = \{(\pm(u\varepsilon_1 + v\varepsilon_2) + \delta \mid (u, v) \in \{0, 1\}^2)\}$ . We have  $\rho = 2\varepsilon_1 + 3\varepsilon_2 - 5\delta/2$ ,  $\tilde{\theta} = 2\delta$  and  $-(\rho, \tilde{\theta}^\vee) = 5/2$ .

The even part of  $\mathfrak{g}$  is :  $\mathfrak{g}_0 \simeq \mathfrak{sl}_2 \oplus G_2$ , and the degree 0 part of  $\mathfrak{g}$  is :  $\mathfrak{g}_0 \simeq \mathbb{C} \oplus G_2$ . We identify the set of dominant weights for  $\mathfrak{g}_0$  (resp.  $\mathfrak{g}_0$ ) with  $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$  (resp.  $\mathbb{C} \times \mathbb{N} \times \mathbb{N}$ ) by means of the fundamental weights  $(\delta; \omega_1 = \varepsilon_1 + \varepsilon_2; \omega_2 = \varepsilon_1 + 2\varepsilon_2)$ . By proposition 2.2 in [6], the set of dominant weights for  $\mathfrak{g}$  is :  $X^+ = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid a = 0 \Rightarrow b = c = 0, a \neq 1 \text{ and } a = 2 \Rightarrow b = 0\}$ . Besides, the eigenvalue of the Casimir element is :

$$\text{Cas}(a, b, c) = 2b^2 + 6bc + 10b + 6c^2 + 18c - 2a^2 + 10a.$$

For  $k \in \mathbb{N}$ , we denote by  $\Gamma_k$  the set of dominant weights  $\lambda \in X^+$  such that  $\text{Cas}(\lambda) = 6k(k+1)$ . For  $l \in \mathbb{N}$ , we set :

$$\begin{aligned} \lambda_{0,0} &= (0, 0, 0), \\ \lambda_{0,1} &= (5, 0, 0), \\ \lambda_{k,0} &= (2, 0, k-1) \quad \text{if } k \geq 1, \\ \lambda_{k,1} &= (3, 0, k-1) \quad \text{if } k \geq 1, \\ \lambda_{k,l} &= \begin{cases} (l+2, 2l-2, k-l) & \text{if } 2 \leq l \leq k, \\ (l+3, 3k-l, l-k-1) & \text{if } k+1 \leq l \leq 3k, \\ (l+4, l-3k-1, 2k) & \text{if } 3k+1 \leq l. \end{cases} \end{aligned}$$

REMARK. For  $l \geq 2$ , the difference  $\lambda_{k,l} - \lambda_{k,l-1}$  is either the positive odd root  $\beta_{k,l}$  orthogonal to  $\lambda_{k,l} + \rho$ , or the sum  $\beta_{k,l} + \beta_{k,l-1}$ .

**Theorem 4.1.1** *Let  $\mathfrak{g} = G(3)$ . Every atypical block of  $\mathfrak{g}$  is one of the  $\Gamma_k$ . For every  $k \in \mathbb{N}$ ,  $\Gamma_k$  is equivalent to the principal block of  $\text{osp}(3, 2)$ .*

*In other terms, as a graph,  $\Gamma_k$  is a Dynkin diagram of type  $D_\infty$  ; the map  $\mathbb{N} \rightarrow \Gamma_k, l \mapsto \lambda_{k,l}$  is bijective ; statements (b) and (c) of theorem 2.1.1 apply to  $\Gamma_k$ .*

## 4.2 Highest weight modules and simple modules

Easy calculations, in the spirit of lemma 3.3.1, show that  $\Gamma_k$  is the set  $\{\lambda_{k,l} \mid l \in \mathbb{N}\}$ , and that  $X^+$  is the union of all the  $\Gamma_k$ 's. Next lemma implies that  $\Gamma_k$  is only one block. It can be used to check the character formulas announced in [7].

**Lemma 4.2.1** *Assertions (i) and (ii) of lemma 2.2.1 hold for  $G(3)$ , with  $\lambda_{k,l}$  in place of  $\lambda_l$ .*

PROOF : We know that  $\lambda_{k,0}$  is minimal in  $\Gamma_k$ , hence  $\mathcal{L}_0(\lambda_{k,0})$  is simple. On the other hand,  $\lambda_{k,0}$  is the unique element in  $\Gamma_k$  lower than  $\lambda_{k,1}$ , but  $S(\lambda_{k,0})$  does not appear in  $\mathcal{L}_0(\lambda_{k,1})$  : indeed, for  $k = 0$ , the  $\mathfrak{g}_0$ -module  $L(\mathfrak{p}, \lambda_{0,1}) \otimes \bigwedge^5 \mathfrak{g}_{-1}$  does not contain  $L(\mathfrak{p}, \lambda_{0,0})$ , and for  $k \geq 1$ , the  $\mathfrak{g}_0$ -module  $L(\mathfrak{p}, \lambda_{k,1}) \otimes \mathfrak{g}_{-1}$  does not contain  $L(\mathfrak{p}, \lambda_{k,0})$ . Hence  $\mathcal{L}_0(\lambda_{k,1})$  is simple too.

If  $l \geq 3$ , the module  $\mathcal{L}_0(\lambda_{k,l})$  is not simple, hence is a non-split extension of  $S(\lambda_{k,l})$  by  $S(\lambda_{k,l-1})$  : indeed,  $\lambda_{k,l-1}$  is the only element in  $\Gamma_k$  of the form  $\lambda_{k,l} - \sum_{\beta \in I} \beta$ , for  $I \subset \Delta_1^+$ . If the radical of  $\mathcal{L}_0(\lambda_{k,2})$  did not contain both  $\lambda_{k,0}$  and  $\lambda_{k,1}$ , then either of these would be alone in their block, which is impossible since they are atypical.  $\square$

## 4.3 Projective modules

By the proof of theorem 2.1.1, the following lemma will imply theorem 4.1.1.

**Lemma 4.3.1** *Indecomposable projective modules in  $\Gamma_k$  are described by assertion (b) of theorem 2.1.1, with  $\lambda_{k,l}$  in place of  $\lambda_l$ .*

PROOF : By lemmas 4.2.1 and 1.1.1(iii), the first radical layer of indecomposable projective modules is  $S(\lambda_{k,2})$  if  $l = 0, 1$ , it is  $S(\lambda_{k,0}) \oplus S(\lambda_{k,1}) \oplus S(\lambda_{k,3})$  if  $l = 2$ , and  $S(\lambda_{k,l-1}) \oplus S(\lambda_{k,l+1})$  if  $l \geq 3$ .

We claim that  $\text{rad}^2 P(\lambda_{k,l}) / \text{rad}^3 P(\lambda_{k,l})$  cannot contain  $S(\lambda_{k,l \pm 2})$  if  $l \geq 3$ . Indeed, this would imply that  $P(\lambda_{k,l})$  has an indecomposable quotient with composition series  $S(\lambda_{k,l}), S(\lambda_{k,l \pm 1}), S(\lambda_{k,l \pm 2})$ , which is impossible by the previous lemma. This argument also excludes the presence of  $S(\lambda_{k,l+2})$  if  $l = 1, 2$  and of  $S(\lambda_{k,3})$  if  $l = 0$ . It follows that the second radical layer of  $P(\lambda_{k,l})$  contains only copies of  $S(\lambda_{k,l})$  (and, possibly, of  $S(\lambda_{k,1-l})$  if  $l = 0, 1$ ).

If the second radical layer of  $P(\lambda_{k,l})$  was not simple, then the third radical layer would not be empty : since  $P(\lambda_{k,l})$  is injective, it has an indecomposable socle. In this third radical layer, we could find only the same as in the first one, i.e.  $S(\lambda_{k,l \pm 1})$  (and  $S(\lambda_{k,0})$  if  $l = 2$ ). But assume  $S(\lambda_{k,l \pm 1})$  appears : then  $P(\lambda_{k,l})$  has an indecomposable quotient with composition series  $S(\lambda_{k,l}), S(\lambda_{k,l \pm 1}), S(\lambda_{k,l}), S(\lambda_{k,l \pm 1})$ . This is not compatible with the previous lemma.

It follows from these considerations that the modules  $P(\lambda_{k,l})$  have the structure described in theorem 2.1.1(b) for  $l \geq 2$ . For  $l \leq 1$ , the composition series of  $P(\lambda_{k,l})$  could be :  $S(\lambda_{k,l}), S(\lambda_{k,2}), S(\lambda_{k,1-l})$ .

To exclude this possibility, a direct argument works for  $k = 0$  : by the proof of lemma 1.1.1,  $P(\lambda_{0,0})$  is a direct summand in  $\bigwedge \mathfrak{g}_1$ , which does not contain the  $\mathfrak{g}_0$ -module with highest weight  $\lambda_{0,1}$ . Hence  $P(\lambda_{0,0})$  has the composition series we claim. The same is true for  $P(\lambda_{0,1})$  because projectives are injective, hence the same simple module (here,  $S(\lambda_{0,0})$ ) cannot appear as the socle of two different indecomposable projectives. For  $k \neq 0$ , proposition 4.4.1, which has an independent interest, allows to conclude.  $\square$

## 4.4 A translation functor

We introduce a little notation. Let  $\text{ad} = S(\lambda_{1,0})$  be the adjoint module : as a  $\mathfrak{g}_0$ -module, it is the sum of three modules with highest weights  $(2, 0, 0)$ ,  $(1, 1, 0)$  and  $(0, 0, 1)$ . For  $k \in \mathbb{N}$ , let  $\mathfrak{g}\text{-mod}_k$  be the category of  $\mathfrak{g}$ -modules that belong to  $\Gamma_k$ . Given any  $\mathfrak{g}$ -module  $M$ , we denote by  $p_k(M)$  the maximal summand of  $M$  that lies in  $\mathfrak{g}\text{-mod}_k$ , i.e. the generalised eigenspace of the Casimir with eigenvalue  $6k(k+1)$ . Given a  $\mathfrak{g}$ -module  $M$  in  $\mathfrak{g}\text{-mod}_k$ , we set :  $T_{k,k'}M = p_{k'}(M \otimes \text{ad})$ . It is standard to show that  $T_{k,k'}$  is an exact functor, left and right adjoint to  $T_{k',k}$  (see e.g. lemma II7.6 in [5]).

**Proposition 4.4.1** *The functors  $T_{k,k+1}$  and  $T_{k+1,k}$  are mutually inverse equivalences of categories between  $\mathfrak{g}\text{-mod}_k$  and  $\mathfrak{g}\text{-mod}_{k+1}$ .*

We start with a

**Lemma 4.4.2** *For every  $\mathfrak{g}_0$ -module  $L$ , we have :  $\mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L \otimes \text{ad}) \simeq \mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L) \otimes \text{ad}$ .*

PROOF : Let  $N$  be a finite-dimensional  $\mathfrak{g}$ -module.

$$\begin{aligned} \text{Hom}_{\mathfrak{g}}(\mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L \otimes \text{ad}), N) &\simeq \text{Hom}_{\mathfrak{g}}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L \otimes \text{ad}, N) \\ &\simeq \text{Hom}_{\mathfrak{g}}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L, \text{ad}^* \otimes N) \\ &\simeq \text{Hom}_{\mathfrak{g}}(\mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L), \text{ad}^* \otimes N) \\ &\simeq \text{Hom}_{\mathfrak{g}}(\mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L) \otimes \text{ad}, N). \square \end{aligned}$$

PROOF (OF THE PROPOSITION) : Since  $T_{k,k+1}$  and  $T_{k+1,k}$  are exact and adjoint of each other, it is enough to show that they do not annihilate simple modules.

Let  $L$  (resp.  $L'$ ) be the  $\mathfrak{g}_0$ -module with highest weight  $\lambda_{k,l}$  (resp.  $\lambda_{k+1,l}$ ). We claim that  $L \otimes \text{ad}$  is the direct sum of  $L'$  and of  $\mathfrak{g}_0$ -modules the highest weight of which is not in  $\Gamma_{k+1}$ . For any weight  $\beta$  of  $\text{ad}$  (i.e., any root of  $\mathfrak{g}$ ), the weight  $\lambda_{k,l} + \beta$  is dominant, singular or can be lifted to the dominant chamber (of  $\mathfrak{g}_0$ ) by an element of the Weyl group of length 1. By exercise 24.9 in [4], the highest weights of the  $\mathfrak{g}_0$ -modules that appear in  $L \otimes \text{ad}$  are of the form  $\lambda_{k,l} + \beta$ . To prove the claim, it is enough to check that  $(\lambda_{k,l} + \Delta) \cap \Gamma_{k+1} = \{\lambda_{k,l+1}\}$  and that no cancellation can happen.

Using the previous lemma, we get now :

$$\mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L) \otimes \text{ad} \simeq \mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L') \oplus M,$$

where  $M$  has a filtration by modules of the form  $\mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\mathfrak{p}, \mu))$ , with  $\mu \notin \Gamma_{k+1}$ . Hence :

$$(\nabla) \quad T_{k,k+1}(\mathcal{L}_0(\lambda_{k,l})) = p_{k+1}(\mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L) \otimes \text{ad}) \simeq \mathcal{L}_0(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L') = \mathcal{L}_0(\lambda_{k+1,l}).$$

Now, the radical  $R$  of  $\mathcal{L}_0(\lambda_{k,l})$  is a quotient of a sum of modules  $\mathcal{L}_0(\lambda_{k,m})$  with  $m < l$ . Using  $(\nabla)$  and the exactness of  $T_{k,k+1}$ , we obtain that  $T_{k,k+1}R$  is a quotient of a sum of modules  $\mathcal{L}_0(\lambda_{k+1,m})$ , with  $m < l$ , hence is properly embedded in  $\mathcal{L}_0(\lambda_{k+1,l})$ . It follows that  $T_{k,k+1}$  does not annihilate  $S(\lambda_{k,l})$ , hence it is faithful. Analogous considerations show that  $T_{k+1,k}$  is faithful as well.  $\square$

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