



HAL
open science

1D action and partition function for the 2D Ising model with boundary magnetic field

Maxime Clusel, Jean-Yves Fortin

► **To cite this version:**

Maxime Clusel, Jean-Yves Fortin. 1D action and partition function for the 2D Ising model with boundary magnetic field. 2004. hal-00002720v1

HAL Id: hal-00002720

<https://hal.science/hal-00002720v1>

Preprint submitted on 30 Aug 2004 (v1), last revised 11 Jan 2005 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1D action and partition function for the 2D Ising model with a boundary magnetic field

Maxime Clusel[†] and Jean-Yves Fortin[‡]

[†] Laboratoire de Physique, École normale supérieure de Lyon, 46 allée d'Italie, 69364 Lyon cedex 07, France

[‡] Laboratoire de Physique Théorique, Université Louis Pasteur, 3 rue de l'Université, 67084 Strasbourg cedex, France

E-mail: maxime.clusel@ens-lyon.fr, fortin@lpt1.u-strasbg.fr

Abstract. In this article we obtain some exact results for the 2D Ising model with a general boundary magnetic field and for a finite size system, by an alternative method to that developed by B. McCoy and T.T. Wu. This method is a generalization of ideas from V.N. Plechko presented for the 2D Ising model in zero field, based on the representation of the Ising model using a Grassmann algebra. In this way, a Gaussian 1D action is obtained for a general configuration of the boundary magnetic field. In the special case where the magnetic field is homogeneous, we check that our results are in agreement with McCoy and Wu's previous work, and we also compute the two point correlation functions on the boundary. We use this correlation function to obtain the exact partition function and the free energy in the special case of an inhomogeneous boundary magnetic field.

PACS numbers: 02.30.Ik ; 05.50.+q ; 05.70.Fh

Submitted to: *J. Phys. A: Math. Gen.*

1. Introduction

Studied for the first time in 1925 (1), the Ising model is one of the most important models of statistical physics. The two dimensional case, solved exactly for the first time in 1944 by Onsager (2), is the prototype of systems with second order phase transition and non Gaussian critical exponents. It has therefore been studied extensively by various exact and approximate methods. It also provides fruitful links with 1D Quantum Field Theory. In particular, a way to express the Ising Hamiltonian as a Gaussian Grassmannian action was established long time ago (3; 4), and used by Plechko to compute the partition function of the 2D Ising model in zero field, on a large class of lattices (5; 6).

The 2D Ising model has also been used to study some boundary problems, for example the effect of a magnetic field on the boundary magnetisation and also the analogy to wetting transitions. It is quite remarkable that exact results can be obtained in this case since the problem with general uniform field has not yet been solved except for some exact results at the critical temperature (7; 8). In a series of papers, McCoy and Wu computed the partition function and the correlation functions for the model, using dimer statistics (9; 10; 11; 12; 13; 14). Another solution was proposed by Chatterjee, using boundary Quantum Field Theory (15).

Here we propose an alternative derivation of the exact partition function including a boundary magnetic field, generalizing the method introduced by Plechko. This method appears to be simpler than the McCoy and Wu's derivation, and allows exact calculations for more complicated cases. By this method we demonstrate that the partition function can be expressed as a Grassmann path integral of a 1D Gaussian action with general or random magnetic fields after integrating over the bulk degrees of freedom. In the special case of an homogeneous boundary magnetic field, we compute the free energy on the lattice, and the boundary spin-spin correlation functions. We check that our formula is equivalent to the one obtained by McCoy and Wu (9; 13). Moreover, we can also compute the exact partition function and the free energy when there is an interface between two opposite fields H and $-H$ on one boundary, as the simplest example of inhomogeneous boundary magnetic field.

The article is organized as follow: In section 2, we introduce the notations used throughout the article. In section 3, following Plechko, we obtain a Grassmannian path integral representation of the partition function, and in section 4, we present the fermionization of the boundary, and obtain the fermionic action. In section 5, as a preliminary and useful exercise, we solve in the same way the 1D Ising model with an homogeneous magnetic field, in order to introduce the method to the reader. Section 6 is dedicated to the explicit calculation of the partition function for the two dimensional Ising model ; We then give the corresponding 1D action. At this stage, we compare our results with McCoy and Wu, taking the thermodynamic limit. In the section 7, we compute the two point correlation function on the boundary, which is used, in section 8, to obtain the expressions of the partition function and the free energy in the case of an interface on the boundary.

2. General notation

In the following we consider the Ising model on a square lattice of size L with spins $\sigma_{mn} = \pm 1$. For sake of simplicity, we limit ourself here to the case where the coupling constant J is the same in both directions. The method works however if there are two different coupling constants along vertical and horizontal bounds. Until section 4 we consider inhomogeneous magnetic fields h_n , placed on the sites of the first column $m = 1$ (see Figure 1). Periodic boundary conditions for the spins are imposed in the direction parallel to the magnetic field line, $\sigma_{m1} = \sigma_{mL+1}$, and free boundary conditions in the transverse direction, formally equivalent to $\sigma_{0n} = \sigma_{L+1n} = 0$.

The Hamiltonian is then given by

$$\mathcal{H} = -J \sum_{m,n=1}^L (\sigma_{mn}\sigma_{m+1n} + \sigma_{mn}\sigma_{mn+1}) - \sum_{n=1}^L h_n \sigma_{1n}. \quad (1)$$

The partition function (PF) \mathcal{Z} is defined as

$$\mathcal{Z} = \sum_{\{\sigma_{mn}\}} \exp(-\beta\mathcal{H}),$$

where $\beta = 1/k_B T$ and the sum is over all possible spin configurations. We can write, using $\sigma_{mn}^2 = 1$,

$$e^{-\beta\mathcal{H}} = [\cosh(\beta J)]^{2L^2} \prod_{n=1}^L \cosh(\beta h_n) (1 + u \sigma_{1n}) \cdot \prod_{m,n=1}^L (1 + t \sigma_{mn} \sigma_{m+1n}) (1 + t \sigma_{mn} \sigma_{mn+1}),$$

with $u_n = \tanh(\beta h_n)$ and $t = \tanh(\beta J)$. We then define \mathcal{Q} as

$$\mathcal{Q}[h] = \text{Tr}_{\sigma_{mn}} \left[\prod_{m,n=1}^L (1 + t \sigma_{mn} \sigma_{m+1n}) (1 + t \sigma_{mn} \sigma_{mn+1}) \cdot \prod_{n=1}^L (1 + u_n \sigma_{1n}) \right], \quad (2)$$

where Tr_{σ} is the normalized sum $\frac{1}{2} \sum_{\sigma=\pm 1}$. This PF has already been calculated by McCoy and Wu (9) for a uniform field. This was done in two steps. First they proved

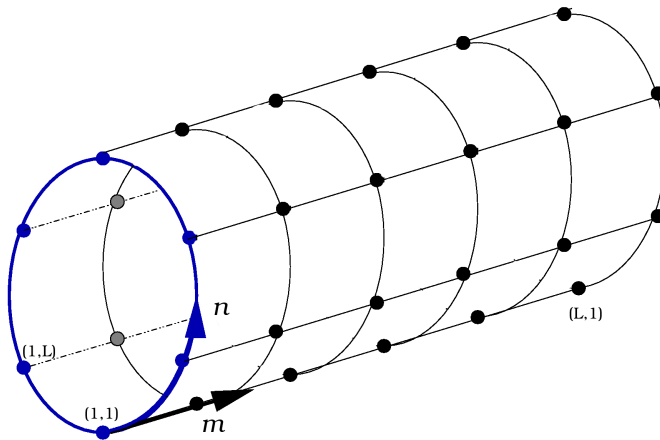


Figure 1. Description of the model on a periodical lattice with fixed conditions in one direction.

that this PF is the Pfaffian of a matrix, using dimer statistics, they then performed the direct calculation of this Pfaffian. Here we treat the problem in a different way. Our idea is to generalize the elegant method introduced by Plechko (5; 6) for the 2D Ising model without magnetic field since this method appears to be simple, and provides a direct link with Quantum Field Theory. This is because it is straightforward to obtain the expression of the quadratic fermionic action. Here we show that we can derive such a quadratic action in presence of a general boundary magnetic field.

3. Transformation of the PF using a Grassmann representation

Following Plechko, we introduce pairs of Grassmann variables in order to remove the local interaction between spins. We briefly define some useful tools using Grassmann algebra. For more details, we refer the reader to the book by Nakahara (16).

A Grassmann algebra \mathcal{A} of size N is a set of N anti-commuting objects $\{a_i\}_{i=1,N}$ satisfying:

$$\forall 1 \leq i, j \leq N, \quad a_i a_j = -a_j a_i,$$

which implies $a_i^2 = 0$. Functions defined on such an algebra are particularly simple, they are polynomials. It is possible to define a notion of integration (16) with the following rules:

$$\int da \, a = 1, \quad \int da \, 1 = 0, \quad (3)$$

and for any function $f(a)$,

$$\int da \, f(a) = \frac{\partial f(a)}{\partial a}. \quad (4)$$

With these definitions, Gaussian integrals are expressed by

$$\int \prod_{i=1}^N da_i^* da_i \exp \left(\sum_{i,j=1}^N a_i M_{ij} a_j^* \right) = \det M. \quad (5)$$

We also define a trace operator over an algebra $\mathcal{A} = \{a, a^*\}$ as

$$\mathfrak{Tr}_{a,a^*} [f(a, a^*)] \equiv \int da^* da \, f(a, a^*) e^{aa^*}, \quad (6)$$

with the simple rules

$$\mathfrak{Tr}_{a,a^*} [1] = 1, \quad \mathfrak{Tr}_{a,a^*} [aa^*] = 1.$$

This operator will be useful in the following, and the subscripts may be omitted when the trace is performed on the Grassmann variables that are present in the expressions inside the brackets. Grassmann variables are introduced in the PF in order to decouple the spin variables. Terms containing the same spin are then put together and the sum over the spin configurations is performed. We use the fact that

$$1 + t\sigma\sigma' = \int da^* da (1 + a\sigma)(1 + ta^*\sigma') e^{aa^*} = \mathfrak{Tr}[(1 + a\sigma)(1 + ta^*\sigma')] \quad (7)$$

and follow closely the notation in reference (5). We consider the following link variables

$$\begin{aligned}\psi_{mn}^{(1)} &= 1 + t\sigma_{mn}\sigma_{m+1n}, \\ \psi_{mn}^{(2)} &= 1 + t\sigma_{mn}\sigma_{mn+1}.\end{aligned}\tag{8}$$

In order to decouple the products of two spins, we can express each object as a trace operator over a product of two Grassmann polynomials using equation (7)

$$\begin{aligned}\psi_{mn}^{(1)} &= \mathfrak{Tr} [A_{mn}A_{m+1n}^*], \\ \psi_{mn}^{(2)} &= \mathfrak{Tr} [B_{mn}B_{mn+1}^*],\end{aligned}\tag{9}$$

where

$$\begin{aligned}A_{mn} &= 1 + a_{mn}\sigma_{mn}, A_{mn}^* = 1 + ta_{m-1n}^*\sigma_{mn}, \\ B_{mn} &= 1 + b_{mn}\sigma_{mn}, B_{mn}^* = 1 + tb_{mn-1}^*\sigma_{mn}.\end{aligned}\tag{10}$$

The next step is to gather together the different terms corresponding to the same spin. For thus, we have to use the mirror symmetry introduced by Plechko (6). If we consider an ensemble of objects $\{\mathcal{O}_i; \mathcal{O}_i^*\}_{i=1..3}$ such as a pair of this objects $(\mathcal{O}_i^*\mathcal{O}_k)$ commutes with all overs \mathcal{O} or \mathcal{O}^* (for example Grassmann variables), but with no further particular commutation properties, we obtain the following mirror symmetry:

$$(\mathcal{O}_1^*\mathcal{O}_1)(\mathcal{O}_2^*\mathcal{O}_2)(\mathcal{O}_3^*\mathcal{O}_3) = \mathcal{O}_1^*\mathcal{O}_2^*\mathcal{O}_3^*\mathcal{O}_3\mathcal{O}_2\mathcal{O}_1 = \mathcal{O}_3^*\mathcal{O}_2^*\mathcal{O}_1^*\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3.\tag{11}$$

We want to apply this mirror ordering to objects (10), but the association $A_{mn}A_{m+1n}^*$ or $B_{mn}B_{mn+1}^*$ does not commute with all A , A^* , B and B^* . However, as we will finally trace over the Grassmann variables, we can weaken the condition on commutativity of the pairs, and only require that $\mathfrak{Tr}[(\mathcal{O}_i^*\mathcal{O}_k)\mathcal{O}_l] = \mathfrak{Tr}[\mathcal{O}_l(\mathcal{O}_i^*\mathcal{O}_k)]$. We then obtain the mirror ordering (11) in the sense of the trace which can be applied to the $A_{mn}A_{m+1n}^*$ and $B_{mn}B_{mn+1}^*$.

It is also important to treat the spin boundary conditions separately from the bulk quantities in order to obtain an expression valid not only in the thermodynamic limit, but for all finite values of N . Indeed, in the direction parallel to the line of magnetic fields where $\sigma_{mL+1} = \sigma_{m1}$, the corresponding link element can be expressed as

$$\psi_{mL}^{(2)} = \mathfrak{Tr} [B_{mL}B_{mL+1}^*] = \mathfrak{Tr} [B_{m1}^*B_{mL}],\tag{12}$$

where $B_{m1}^* = 1 + tb_{m0}^*\sigma_{m1}$. The equality (12) associated with definitions (10) imposes $b_{m0}^* = -b_{mL}^*$. The periodic conditions on spins therefore lead to anti-periodic conditions on Grassmann variables. In the transverse direction, we have $\sigma_{0n} = \sigma_{L+1n} = 0$, corresponding to free boundary conditions. This implies the boundary conditions on Grassmann variables $a_{0n}^* = 0$ and therefore $A_{1n}^* = 1$. The PF $\mathcal{Q}[h]$ (2) can be written in terms of the $\psi_{mn}^{(k)}$ as

$$\mathcal{Q}[h] = \text{Tr}_{\sigma_{mn}} \left[\prod_{m,n=1}^L \psi_{mn}^{(1)} \psi_{mn}^{(2)} \cdot \prod_{n=1}^L (1 + u_n \sigma_{mn}) \right].\tag{13}$$

Using the mirror symmetry (11), the boundary terms can be written as

$$\begin{aligned} \prod_{n=1}^L \psi_{Ln}^{(1)} &= \mathfrak{Tr} \left[\overrightarrow{\prod_{n=1}^L A_{Ln}} \right], \\ \prod_{m=1}^L \psi_{mL}^{(2)} &= \mathfrak{Tr} \left[\overrightarrow{\prod_{m=1}^L B_{m1}^*} \cdot \overleftarrow{\prod_{m=1}^L B_{mL}} \right]. \end{aligned} \quad (14)$$

These products can be reorganized as follow (6):

$$\prod_{n=1}^L \psi_{Ln}^{(1)} \cdot \prod_{m=1}^L \psi_{mL}^{(2)} = \mathfrak{Tr} \left[\overrightarrow{\prod_{m=1}^L B_{m1}^*} \cdot \overrightarrow{\prod_{n=1}^L A_{Ln}} \cdot \overleftarrow{\prod_{m=1}^L B_{mL}} \right]. \quad (15)$$

For the bulk elements, we obtain the following arrangement

$$\prod_{m=1}^L \psi_{mn}^{(2)} = \mathfrak{Tr} \left[\overleftarrow{\prod_{m=1}^L B_{mn}} \cdot \overrightarrow{\prod_{m=1}^L B_{mn+1}^*} \right], \quad (16)$$

$$\prod_{n=1}^{L-1} \prod_{m=1}^L \psi_{mn}^{(2)} \cdot \prod_{m=1}^L \psi_{mL}^{(2)} = \mathfrak{Tr} \left[\overrightarrow{\prod_{m=1}^L B_{m1}^*} \cdot \prod_{n=1}^{L-1} \prod_{m=1}^L \psi_{mn}^{(2)} \cdot \overleftarrow{\prod_{m=1}^L B_{mL}} \right], \quad (17)$$

$$\prod_{n=1}^L \prod_{m=1}^L \psi_{mn}^{(2)} \cdot \prod_{n=1}^L \psi_{Ln}^{(1)} = \mathfrak{Tr} \left[\overrightarrow{\prod_{n=1}^L \left(\overrightarrow{\prod_{m=1}^L B_{mn}^*} \cdot A_{Ln} \cdot \overleftarrow{\prod_{m=1}^L B_{mn}} \right)} \right], \quad (18)$$

where we use the fact that $\psi_{Ln}^{(1)}$ are commuting objects as well as the associativity of the product:

$$\mathcal{O}_1^*(\mathcal{O}_1\mathcal{O}_2^*)(\mathcal{O}_2\mathcal{O}_3^*)\mathcal{O}_3 = (\mathcal{O}_1^*\mathcal{O}_1)(\mathcal{O}_2^*\mathcal{O}_2)(\mathcal{O}_3^*\mathcal{O}_3). \quad (19)$$

We now insert the product over the remaining $\psi_{mn}^{(1)}$ inside the previous expression

$$\mathfrak{Tr} \left[\prod_{m=1}^{L-1} \psi_{mn}^{(1)} \cdot \overrightarrow{\prod_{m=1}^L B_{mn}^*} \cdot A_{Ln} \dots \right] = \mathfrak{Tr} \left[B_{1n}^* A_{1n} \cdot \overrightarrow{\prod_{m=2}^L A_{mn}^* B_{mn}^* A_{mn}} \dots \right], \quad (20)$$

and finally obtain

$$\prod_{m,n=1}^L \psi_{mn}^{(1)} \psi_{mn}^{(2)} = \mathfrak{Tr} \left[\overrightarrow{\prod_{n=1}^L B_{1n}^* A_{1n}} \left(\overrightarrow{\prod_{m=2}^L A_{mn}^* B_{mn}^* A_{mn}} \cdot \overleftarrow{\prod_{m=2}^L B_{mn}} \right) B_{1n} \right]. \quad (21)$$

The PF is rewritten in a way that the sum on each spin can be performed by iteration

$$\mathcal{Q}[h] = \text{Tr}_{\sigma_{mn}} \mathfrak{Tr} \left[\overrightarrow{\prod_{n=1}^L B_{1n}^* A_{1n}} (1 + u_n \sigma_{1n}) \left(\overrightarrow{\prod_{m=2}^L A_{mn}^* B_{mn}^* A_{mn}} \cdot \overleftarrow{\prod_{m=2}^L B_{mn}} \right) B_{1n} \right]. \quad (22)$$

In fact, as we do not yet fermionize $(1 + u_n \sigma_{1n})$, we reproduce here the Plechko's derivation in the special case of free-periodic boundary conditions (5). This expression is the basis of the rest of this paper. The sum over spins outside the magnetic field region will lead to a quadratic action over Grassmann variables which therefore commutes with the rest of the elements belonging to the first column $(1, n)$.

4. Grassmannian representation of the action on the lattice

The key point in equation (22) is that the trace over spin configurations is performed in an iterative way. For example, the first summation is done on spins σ_{Ln} in the products $A_{Ln}^* B_{Ln}^* A_{Ln} B_{Ln}$. This operation leads to a quantity which is quadratic in Grassmann variables, which can be put outside the general product (22). The same operation is then performed on spins σ_{L-1n} and so on. This makes Plechko's method efficient for the 2D Ising model in zero field. With a uniform magnetic field in the bulk, the spin trace over the product of the four previous operators would lead to a quantity which is linear and quadratic in Grassmann variables and does not commute with the other products. The boundary magnetic field affects only the last products depending on spins σ_{1n} and this makes the problem very *similar to a 1D Ising model in a uniform magnetic field*: it is therefore solvable.

4.1. Trace over spins inside the bulk

For spins σ_{mn} inside the bulk, $1 \leq n \leq L$ and $2 \leq m \leq L$, we have to evaluate successively

$$\text{Tr}_{\sigma_{mn}} [A_{mn}^* B_{mn}^* A_{mn} B_{mn}] = \exp(\mathbb{Q}_{mn}),$$

with

$$\mathbb{Q}_{mn} = a_{mn} b_{mn} + t^2 a_{m-1n}^* b_{mn-1}^* + t(a_{m-1n}^* + b_{mn-1}^*)(a_{mn} + b_{mn}). \quad (23)$$

These terms commute with all Grassmannian terms and can be pulled out of the remaining products. We obtain

$$\text{Tr}_{\{\sigma_{mn}\}_{m=2..L}} \left[\prod_{m=2}^{\overrightarrow{L}} A_{mn}^* B_{mn}^* A_{mn} \cdot \prod_{m=2}^{\overleftarrow{L}} B_{mn} \right] = \exp \left(\sum_{m=2}^L \mathbb{Q}_{mn} \right). \quad (24)$$

The PF can now be written as

$$\mathcal{Q}[h] = \mathfrak{T} \left[\exp \left(\sum_{n=1}^L \sum_{m=2}^L \mathbb{Q}_{mn} \right) \cdot \prod_{n=1}^{\overrightarrow{L}} \underbrace{\text{Tr}_{\sigma_{1n}} ((1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n})}_{\text{boundary spins } \sigma_{1n}} \right]. \quad (25)$$

4.2. Trace over the boundary spins

In the expression (25), we can evaluate separately the trace over spins σ_{1n} , leading to

$$\text{Tr}_{\sigma_{1n}} [1 + u_n \sigma_{1n}] B_{1n}^* A_{1n} B_{1n} = 1 + a_{1n} b_{1n} + t b_{1n-1}^* (a_{1n} + b_{1n}) + u_n \mathbb{L}_n, \quad (26)$$

$$\mathbb{L}_n = a_{1n} + b_{1n} + t b_{1n-1}^*. \quad (27)$$

The presence of a magnetic field on a site introduces a linear Grassmann term. This term no longer commutes with the others, and we need to compute the product in (25) carefully.

We would like to change artificially the fixed boundary conditions to periodic ones in order to simplify the subsequent calculations based on Fourier transformation. The quadratic part of equation (26) is equal to \mathbb{Q}_{1n} with fixed boundary conditions: We can write

$$\begin{aligned}\mathbb{Q}_{1n} &= \mathbb{Q}_{1n}^{\text{p}} - ta_{Ln}^* \mathbb{L}_n, \\ \mathbb{Q}_{1n}^{\text{p}} &= a_{1n} b_{1n} + t^2 a_{0n}^* b_{1n-1}^* + t(a_{0n}^* + b_{1n-1}^*)(a_{1n} + b_{1n}),\end{aligned}\quad (28)$$

where we introduce the boundary quantities $a_{0n}^* = a_{Ln}^*$. $\mathbb{Q}_{1n}^{\text{p}}$ corresponds to periodic boundary conditions for the Grassmann variables (or anti-periodic conditions for the spins). We obtain

$$\text{Tr}_{\sigma_{1n}} [(1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n}] = \exp(\mathbb{Q}_{1n}^{\text{p}} + u_n \mathbb{L}_n - ta_{Ln}^* \mathbb{L}_n). \quad (29)$$

The correction to periodic conditions due to the free boundary condition for the spins is included in $-ta_{Ln}^* \mathbb{L}_n$.

4.3. Grassmann variables associated with the magnetic field

Here we introduce a pair of Grassmann variables associated with the boundary magnetic field. In the rest of the article, we will refer to it as the fermionic magnetic field. We have

$$\exp(u_n \mathbb{L}_n) = 1 + u_n \mathbb{L}_n = \int dh_n^* dh_n (1 + u_n h_n)(1 + h_n^* \mathbb{L}_n) e^{h_n h_n^*}, \quad (30)$$

therefore

$$\text{Tr}_{\sigma_{1n}} [(1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n}] = \int_{h_n, h_n^*} \mathfrak{Tr} [\exp(\mathbb{Q}_{1n}^{\text{p}} + (h_n^* - ta_{Ln}^*) \mathbb{L}_n + u_n h_n)],$$

We now perform a translation in the fermionic magnetic field

$$H_n = h_n, \quad H_n^* = h_n^* - ta_{Ln}^*, \quad (31)$$

which leads to

$$\prod_{n=1}^{\overrightarrow{L}} \text{Tr}_{\sigma_{1n}} [(1 + u_n \sigma_{1n}) B_{1n}^* A_{1n} B_{1n}] = \int_{H_n, H_n^*} \mathfrak{Tr} \left[\exp \left(\sum_{n=1}^L \mathbb{Q}_{1n}^{\text{p}} + H_n^* \mathbb{L}_n + H_n ta_{Ln}^* \right) \cdot \prod_{n=1}^{\overrightarrow{L}} e^{u_n H_n} \right].$$

It is also useful to write the last L products as a non local action along the boundary line

$$\prod_{n=1}^{\overrightarrow{L}} e^{u_n H_n} = \exp \left(\sum_{n=1}^L u_n H_n + \sum_{m=1}^{L-1} \sum_{n=m+1}^L u_m u_n H_m H_n \right). \quad (32)$$

4.4. Fermionic action of the PF

Putting equations (32,32) into (25), we obtain the Grassmannian representation of the PF

$$\mathcal{Q}[h] = \int \mathcal{D}a^* \mathcal{D}a \mathcal{D}b^* \mathcal{D}b \mathcal{D}H^* \mathcal{D}H \exp \mathcal{S}[a, a^*, b, b^*, H, H^*], \quad (33)$$

with the action \mathcal{S} defined as

$$\begin{aligned} \mathcal{S} = & \sum_{mn=1}^L (\mathbb{Q}_{mn}^p + a_{mn}a_{mn}^* + b_{mn}b_{mn}^*) + \sum_{n=1}^L H_n^* \mathbb{L}_n + \sum_{n=1}^L H_n t a_{Ln}^* \\ & + \sum_{m<n} u_m u_n H_m H_n + \sum_{n=1}^L H_n H_n^*. \end{aligned} \quad (34)$$

This action can be separated into three terms

$$\mathcal{S} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}} + \mathcal{S}_{\text{field}}, \quad (35)$$

with

$$\begin{aligned} \mathcal{S}_{\text{bulk}} &= \sum_{mn=1}^L (\mathbb{Q}_{mn}^p + a_{mn}a_{mn}^* + b_{mn}b_{mn}^*), \\ \mathcal{S}_{\text{field}} &= \sum_{m<n} u_m u_n H_m H_n + \sum_{n=1}^L H_n H_n^*, \\ \mathcal{S}_{\text{int}} &= \sum_{n=1}^L H_n^* \mathbb{L}_n + \sum_{n=1}^L H_n t a_{Ln}^*. \end{aligned} \quad (36)$$

The PF written as (33) is just a Gaussian integral over the set of variables (a, a^*, b, b^*, H, H^*) . If we first integrate over the variables (a, a^*, b, b^*) corresponding to the action $\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}}$, this leads to a new Gaussian action depending only on fermionic magnetic field (H_n, H_n^*) . This new action is very similar to that for a one dimensional problem. Actually the way we integrate (33) is close to solving a 1D Ising model with a magnetic field and Grassmann variables. In the next section we present briefly this case since we will use similar tools later. Our method can then be checked using the transfer matrix technics.

5. 1D Ising model with a homogeneous magnetic field

The treatment is similar to the 2D Ising model, except that there is only one kind of link variables and no mirror symmetry involved. The exact solution in the case of a homogeneous magnetic field $u = \tanh(\beta h)$ using the transfer matrix method is simply

$$2^L \mathcal{Q}_{\text{1D}}(h) = \left(1 + t + \sqrt{(1-t)^2 + 4tu^2}\right)^L + \left(1 + t - \sqrt{(1-t)^2 + 4tu^2}\right)^L. \quad (37)$$

If we apply the Grassmann transformations as before, we can write an equation similar to (33):

$$\mathcal{Q}_{\text{1D}}(h) = \int \mathcal{D}a^* \mathcal{D}a \mathcal{D}H^* \mathcal{D}H \exp(\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}} + \mathcal{S}_{\text{field}}), \quad (38)$$

with

$$\begin{aligned} \mathcal{S}_{\text{bulk}} &= \sum_{n=1}^L (a_n a_n^* + t a_{n-1}^* a_n), \quad \mathcal{S}_{\text{int}} = \sum_n H_n^* \mathbb{L}_n, \\ \mathcal{S}_{\text{field}} &= \sum_n H_n H_n^* + u^2 \sum_{m<n} H_m H_n, \quad \mathbb{L}_n = a_n + t a_{n-1}^*. \end{aligned}$$

The Grassmann variables a_n and H_n are anti-periodic in space, and can be Fourier transformed using $a_n = \frac{1}{\sqrt{L}} \sum_q r_{q+\frac{1}{2}}^n a_{q+\frac{1}{2}}$ with $r_q = e^{2i\pi q/L}$. In the new basis, the actions are almost diagonalized

$$\begin{aligned} \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}} &= \sum_{q=0}^{L/2-1} (1 - tr_{q+\frac{1}{2}}) a_{q+\frac{1}{2}} a_{q+\frac{1}{2}}^* + (1 - t\bar{r}_{q+\frac{1}{2}}) a_{-q-\frac{1}{2}} a_{-q-\frac{1}{2}}^* \\ &+ \sum_{q=0}^{L/2-1} H_{q+\frac{1}{2}}^* (a_{q+\frac{1}{2}} + t\bar{r}_{q+\frac{1}{2}} a_{-q-\frac{1}{2}}^*) + \sum_{q=0}^{L/2-1} H_{-q-\frac{1}{2}}^* (a_{-q-\frac{1}{2}} + tr_{q+\frac{1}{2}} a_{q+\frac{1}{2}}^*), \end{aligned} \quad (39)$$

where the bar defines the complex conjugate. For L odd, we have to take care of the momenta on the diagonal of the Brillouin zone: it leads to additional terms in (39); However, these terms are irrelevant for large L . In the following, we restrict ourselves to L even.

The non local sum in $\mathcal{S}_{\text{field}}$ can be written in the Fourier modes as

$$\begin{aligned} \sum_{m<n} H_m H_n &= \sum_q \frac{1}{r_{q+\frac{1}{2}} - 1} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}}, \\ &= \sum_{q=0}^{L/2-1} \left(\frac{1}{r_{q+\frac{1}{2}} - 1} - \frac{1}{\bar{r}_{q+\frac{1}{2}} - 1} \right) H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}}. \end{aligned} \quad (40)$$

We can separate the previous sums (39) into independent blocks of four Grassmann variables $(a_{q+\frac{1}{2}}, a_{q+\frac{1}{2}}^*, a_{-q-\frac{1}{2}}, a_{-q-\frac{1}{2}}^*)$. To compute the individual block integrals, we use the general Gaussian formula:

$$\int da^* da db^* db \exp(\alpha a a^* + \bar{\alpha} b b^* + v a + v^* a^* + w b + w^* b^*) = \alpha \bar{\alpha} \exp\left(-\frac{w w^*}{\bar{\alpha}} - \frac{v v^*}{\alpha}\right),$$

where (v, v^*, w, w^*) are Grassmann variables and $(\alpha, \bar{\alpha})$ two independent complex numbers. We obtain

$$\begin{aligned} \mathcal{Q}_{1D}(h) &= \mathcal{Q}_{1D}(0) \int \mathcal{D}H^* \mathcal{D}H \exp\left(\sum_{q=0}^{L/2-1} H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* \right. \\ &\left. + \gamma_{q+\frac{1}{2}}^{\text{1D}} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* + u^2 \delta_{q+\frac{1}{2}} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}}\right), \end{aligned} \quad (41)$$

where

$$\mathcal{Q}_{1D}(0) = \prod_{q=0}^{L/2-1} |1 - tr_{q+\frac{1}{2}}|^2, \quad \gamma_{q+\frac{1}{2}}^{\text{1D}} = \frac{2it \sin \theta_{q+\frac{1}{2}}}{1 - 2t \cos \theta_{q+\frac{1}{2}} + t^2}, \quad (42)$$

$$\delta_{q+\frac{1}{2}} = -i \cot\left(\frac{1}{2}\theta_{q+\frac{1}{2}}\right), \quad \theta_{q+\frac{1}{2}} = \frac{2\pi}{L} \left(q + \frac{1}{2}\right). \quad (43)$$

$\mathcal{Q}_{1D}(0)$ is the PF in zero field, and is equal to $1 + t^L$ in this case.

The remaining integrals over (H, H^*) are easy to evaluate, if we use

$$\int da^* da db^* db \exp(aa^* + bb^* + \alpha a^* b^* + \beta ab) = 1 - \alpha\beta. \quad (44)$$

After some algebra and simplifications, we finally obtain

$$\mathcal{Q}_{1D}(h) = \prod_{q=0}^{L/2-1} \left(1 + t^2 - 2t \cos \theta_{q+\frac{1}{2}} + 4tu^2 \cos^2\left(\frac{1}{2}\theta_{q+\frac{1}{2}}\right) \right). \quad (45)$$

Results (37) and (45) are equivalent when L is even but are written in a different way. This has been tested numerically for finite size systems, and analytically in the thermodynamic limit.

Using Grassmann variables to express the PF in terms of products over Fourier modes is of course less efficient in this case. However in 2D, the method is very similar and leads to a final expression which is close to (45) and (41) as seen in the next section.

6. Explicit calculation of the PF

In this section, we perform the Gaussian integration of the Grassmannian form of \mathcal{Q} (33) : we start by reducing the number of Grassmann variables per site, then we integrate over the variables in the bulk, in order to obtain the *1D action*, expressed in terms of the fermionic magnetic field. Finally, this last integral is evaluated, leading to the PF on the lattice. In order to validate our method, we check that our result is identical that obtained by McCoy and Wu, in the thermodynamic limit.

6.1. Reduction of number of Grassmann variables per site

In the 2D case, we can easily integrate half of the Grassmann variables, for example (a_{mn}, b_{mn}) , by using the identity

$$\int db da e^{ab+aL+b\bar{L}} = e^{\bar{L}L}. \quad (46)$$

Since the original measure is $db^* db da^* da$, moving db to the right of da^* implies a minus sign. After integrating over (a, b) we define $c_{mn} = a_{mn}^*$ and $c_{mn}^* = -b_{mn}^*$, which removes the minus signs from the new measure $dc^* dc$. We thus obtain the following actions

$$\begin{aligned} \mathcal{S}_{\text{bulk}} &= \sum_{m,n=1}^L c_{mn} c_{mn}^* + t(c_{mn}^* + c_{mn})(c_{m-1n} - c_{mn-1}^*) - t^2 c_{m-1n} c_{mn-1}^*, \\ \mathcal{S}_{\text{int}} &= \sum_{n=1}^L t H_n c_{0n} + (t c_{1n-1}^* + c_{1n}^* + c_{1n}) H_n^*, \\ \mathcal{S}_{\text{field}} &= \sum_{m < n} u_m u_n H_m H_n + \sum_{n=1}^L H_n H_n^*. \end{aligned} \quad (47)$$

Taking into account the different boundary conditions, we Fourier transform these variables as in the 1D case

$$c_{mn} = \frac{1}{L} \sum_{p,q=0}^{L-1} r_p^m r_{q+\frac{1}{2}}^n c_{pq+\frac{1}{2}}, \quad c_{mn}^* = \frac{1}{L} \sum_{p,q=0}^{L-1} \bar{r}_p^m \bar{r}_{q+\frac{1}{2}}^n c_{pq+\frac{1}{2}}^*, \quad (48)$$

and compute the bulk action

$$\begin{aligned} \mathcal{S}_{\text{bulk}} = & \sum_{p,q=0}^{L-1} \left(1 - t\bar{r}_p - tr_{q+\frac{1}{2}} - t^2\bar{r}_p r_{q+\frac{1}{2}} \right) c_{pq+\frac{1}{2}} c_{pq+\frac{1}{2}}^* + \\ & - t\bar{r}_p c_{pq+\frac{1}{2}} c_{L-pL-q-\frac{1}{2}} + tr_{q+\frac{1}{2}} c_{pq+\frac{1}{2}}^* c_{L-pL-q-\frac{1}{2}}^*. \end{aligned} \quad (49)$$

The result implies a coupling between modes $(p, q + \frac{1}{2})$ and $(L - p, L - q - \frac{1}{2}) \sim (-p, -q - \frac{1}{2})$. As in 1D, the sum can be expressed as a sum over independent blocks containing the variables $c_{pq+\frac{1}{2}}, c_{-p-q-\frac{1}{2}}, c_{pq+\frac{1}{2}}^*$ and $c_{-p-q-\frac{1}{2}}^*$. These different blocks are indeed independent if we restrict to momenta (p, q) corresponding to the white points (set S_1) of the Brillouin zone in figure 2. In this case, the modes (p, q) belonging to S_1 and the modes $(L - p, L - q)$ (corresponding to S_2) completely fill the Brillouin zone. This avoids counting the Grassmann variables twice. We also need to symmetrize the action, which can be written as

$$\begin{aligned} \mathcal{S}_{\text{bulk}} = & \sum_{p,q \in S_1} \left(\alpha_{pq+\frac{1}{2}} c_{pq+\frac{1}{2}} c_{pq+\frac{1}{2}}^* + \bar{\alpha}_{pq+\frac{1}{2}} c_{-p-q-\frac{1}{2}} c_{-p-q-\frac{1}{2}}^* + \beta_p c_{pq+\frac{1}{2}} c_{-p-q-\frac{1}{2}} \right. \\ & \left. + \beta_{q+\frac{1}{2}} c_{pq+\frac{1}{2}}^* c_{-p-q-\frac{1}{2}}^* \right), \end{aligned} \quad (50)$$

$$\alpha_{pq+\frac{1}{2}} = 1 - t\bar{r}_p - tr_{q+\frac{1}{2}} - t^2\bar{r}_p r_{q+\frac{1}{2}}, \quad \beta_p = t(r_p - \bar{r}_p), \quad (51)$$

and for the interacting part

$$\begin{aligned} \mathcal{S}_{\text{int}} = & \frac{1}{L} \sum_{p,q \in S_1} c_{pq+\frac{1}{2}} (r_p H_{q+\frac{1}{2}}^* - t H_{-q-\frac{1}{2}}) + c_{pq+\frac{1}{2}}^* \bar{r}_p (1 + tr_{q+\frac{1}{2}}) H_{-q-\frac{1}{2}}^* + \\ & c_{-p-q-\frac{1}{2}} (\bar{r}_p H_{-q-\frac{1}{2}}^* - t H_{q+\frac{1}{2}}) + c_{-p-q-\frac{1}{2}}^* r_p (1 + t\bar{r}_{q+\frac{1}{2}}) H_{q+\frac{1}{2}}^*. \end{aligned} \quad (52)$$

The last action, $\mathcal{S}_{\text{field}}$, is the same as in 1D.

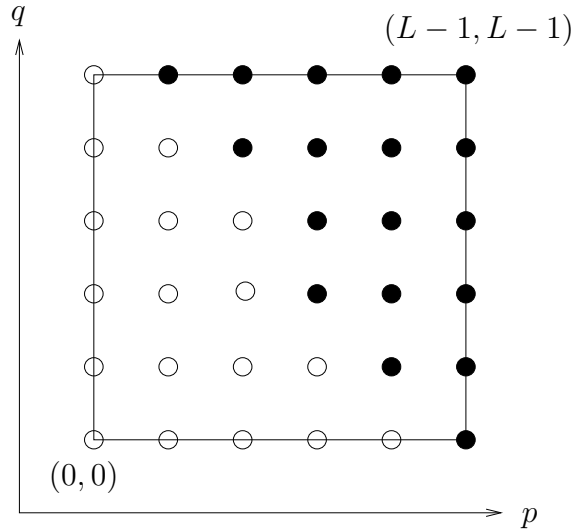


Figure 2. Integration domain for the Fourier modes in the case where L is even ($L=6$). The set S_1 of modes corresponds to the white points. The other modes (set S_2) are obtained by the symmetry $(p, q) \rightarrow (L - p, L - q)$.

6.2. Integration over bulk variables : 1D action

The integration over the variables c and c^* is performed using the following identity, similar to formula (41):

$$\begin{aligned} & \int da^* da db^* db \exp(\alpha aa^* + \bar{\alpha} bb^* + \beta ab + \bar{\beta} a^* b^* + av + bw + a^* v^* + b^* w^*) \\ &= (\alpha \bar{\alpha} - \beta \bar{\beta}) \exp \left[\frac{1}{\alpha \bar{\alpha} - \beta \bar{\beta}} (\bar{\alpha} v^* v + \alpha w^* w + \bar{\beta} v w + \beta v^* w^*) \right]. \end{aligned} \quad (53)$$

We obtain

$$\begin{aligned} \int \mathcal{D}c^* \mathcal{D}c e^{\mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{int}}} &= \prod_{p,q \in S_1} \left(\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}} \right) \exp \left(\gamma_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* + \lambda_{pq+\frac{1}{2}} H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* \right. \\ & \left. + \bar{\lambda}_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* + \epsilon_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}} H_{q+\frac{1}{2}} \right), \end{aligned} \quad (54)$$

where we have identified the different coefficients of the Grassmannian fields forming a quadratic action:

$$\begin{aligned} \gamma_{pq+\frac{1}{2}} &= \frac{1}{L} \frac{1}{\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}}} \left(-\alpha_{pq+\frac{1}{2}} (1 + t \bar{r}_{q+\frac{1}{2}}) + \bar{\alpha}_{pq+\frac{1}{2}} (1 + t r_{q+\frac{1}{2}}) \right. \\ & \left. - \beta_p (1 + t r_{q+\frac{1}{2}}) (1 + t \bar{r}_{q+\frac{1}{2}}) + \bar{\beta}_{q+\frac{1}{2}} \right), \end{aligned} \quad (56)$$

and

$$\begin{aligned} \lambda_{pq+\frac{1}{2}} &= \frac{1}{L} \frac{1}{\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}}} \left(t r_p \beta_{q+\frac{1}{2}} + t r_p \alpha_{pq+\frac{1}{2}} (1 + t \bar{r}_{q+\frac{1}{2}}) \right), \\ \epsilon_{pq+\frac{1}{2}} &= \frac{1}{L} \frac{t^2 \beta_{q+\frac{1}{2}}}{\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}}}. \end{aligned} \quad (57)$$

Inside the product (54), the quantities in front of the exponentials can be simplified using cosine functions,

$$\alpha_{pq+\frac{1}{2}} \bar{\alpha}_{pq+\frac{1}{2}} - \beta_p \beta_{q+\frac{1}{2}} = (1 + t^2)^2 - 2t(1 - t^2) \left[\cos \theta_p + \cos \theta_{q+\frac{1}{2}} \right],$$

and are invariant under the transformation $(p, q + \frac{1}{2}) \rightarrow (L - p, L - q - \frac{1}{2})$. In (54), the product over S_1 of these coefficients define a bulk PF \mathcal{Q}_0 :

$$\mathcal{Q}_0^2 = \prod_{p,q=0}^{L-1} \left[(1 + t^2)^2 - 2t(1 - t^2) \left(\cos \theta_p + \cos \theta_{q+\frac{1}{2}} \right) \right]. \quad (58)$$

In the thermodynamic limit and in zero field, the free energy per site corresponding to \mathcal{Q}_0 is equal to the one corresponding to $\mathcal{Q}[0]$, since the boundary conditions do not play any role on the bulk properties. In this limit, the second order phase transition occurs at a temperature given by the solutions of the equation $(1 + t^2)^2 - 4t(1 - t^2) = 0$, or $t_c = \sqrt{2} - 1$, when the cosines, in the long-wave length limit $(p, q) \sim (0, 0)$, approach unity. In this case the product, hence the free energy, is singular.

The previous coefficients $\gamma_{pq+\frac{1}{2}}$ are not symmetrical in (p, q) since the model itself is not symmetrical in both directions. However they are antisymmetric $\gamma_{p,q+\frac{1}{2}} = -\gamma_{L-p,L-q-\frac{1}{2}}$.

This is useful in order to reduce the summation over the variables (p, q) in the action (54). This implies

$$\sum_{p, q \in \mathcal{S}_1} \gamma_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* = \sum_{q=0}^{L/2-1} \gamma_{q+\frac{1}{2}}^{2D} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^*,$$

with

$$\gamma_{q+\frac{1}{2}}^{2D} = \frac{1}{2} \sum_{p=0}^{L-1} \left(\gamma_{pq+\frac{1}{2}} - \gamma_{p-q-\frac{1}{2}} \right),$$

After simplification, we find that

$$\gamma_{pq+\frac{1}{2}} = \frac{1}{L} \frac{2it \sin \theta_{q+\frac{1}{2}}}{(1+t^2)^2 - 2t(1-t^2)(\cos \theta_p + \cos \theta_{q+\frac{1}{2}})}.$$

and

$$\gamma_{q+\frac{1}{2}}^{2D} = \frac{1}{L} \sum_{p=0}^{L-1} \frac{2it \sin \theta_{q+\frac{1}{2}}}{(1+t^2)^2 - 2t(1-t^2)(\cos \theta_p + \cos \theta_{q+\frac{1}{2}})}. \quad (59)$$

The factors γ^{2D} play the role of Fourier coefficients of an effective interaction between the boundary spins in the magnetic field. We can notice from (41) that coefficients γ^{1D} describe the nearest neighbor interaction of the 1D Ising model. Here the spins on the boundary can be mapped onto a 1D model.

The factors $\lambda_{pq+\frac{1}{2}}$ have a different symmetry, $\lambda_{pq+\frac{1}{2}} = \bar{\lambda}_{-p-q-\frac{1}{2}}$, which allows the same kind of manipulation as before. Hence, defining

$$\begin{aligned} \Lambda_{q+\frac{1}{2}} &= \frac{1}{2} \sum_{p=0}^{L-1} (\lambda_{pq+\frac{1}{2}} + \bar{\lambda}_{p-q-\frac{1}{2}}), \\ &= \frac{t}{L} \sum_{p=0}^{L-1} \frac{(1-t^2) \cos(\theta_p) - t(1+2t \cos \theta_{q+\frac{1}{2}} + t^2)}{(1+t^2)^2 - 2t(1-t^2)(\cos \theta_p + \cos \theta_{q+\frac{1}{2}})}, \end{aligned} \quad (60)$$

we can write

$$\sum_{p, q \in \mathcal{S}_1} \lambda_{pq+\frac{1}{2}} H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + \bar{\lambda}_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* = \sum_{q=0}^{L/2-1} \Lambda_{q+\frac{1}{2}} H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + \Lambda_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^*.$$

Moreover, further reduction of the terms containing $\epsilon_{pq+\frac{1}{2}}$ in the action (54) leads to the following simplification

$$\sum_{p, q \in \mathcal{S}_1} \epsilon_{pq+\frac{1}{2}} H_{-q-\frac{1}{2}} H_{q+\frac{1}{2}} = \sum_{q=0}^{L/2-1} t^2 \gamma_{q+\frac{1}{2}}^{2D} H_{-q-\frac{1}{2}} H_{q+\frac{1}{2}}.$$

Finally, the problem of the boundary field is reduced to a 1D Gaussian action in Grassmann variables which is given by:

$$\begin{aligned} \mathcal{S}_{1D} &= \sum_{q=0}^{L/2-1} \left[(1 + \Lambda_{q+\frac{1}{2}}) H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + (1 + \Lambda_{-q-\frac{1}{2}}) H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* + \gamma_{q+\frac{1}{2}}^{2D} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* \right. \\ &\quad \left. - t^2 \gamma_{q+\frac{1}{2}}^{2D} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}} \right] + \sum_{q, q'=0}^{L-1} \Delta_{q, q'}[u] H_{q+\frac{1}{2}} H_{q'+\frac{1}{2}}, \end{aligned} \quad (61)$$

with

$$\Delta_{q,q'}[u] = \frac{1}{L} \sum_{m < n} u_m u_n r_{q+\frac{1}{2}}^m r_{q'+\frac{1}{2}}^n.$$

This action is of the same form as that for the 1D problem (39), except for the additional terms Λ and $-t^2\gamma^{2D}$, which are not present in the 1D case. This is due to the fact that \mathcal{Q}_0 is not the true action in the zero field case : if we integrate (61) with respect to the Grassmann fields when $u = 0$, this will lead to a non zero corrective factor in front of \mathcal{Q}_0 , which is however irrelevant in the thermodynamic limit (the free energy corresponding to this factor is of order L instead of L^2). \mathcal{Q}_0 is therefore not the finite size zero field partition function for the periodic/free spin boundary case. The factor comes from the free boundary conditions that restore these conditions.

Using this 1D Gaussian action, the partition function for the 2D Ising model with an inhomogeneous or random boundary magnetic fields reads :

$$\mathcal{Q}[h] = \mathcal{Q}_0 \int dH^* dH \exp(\mathcal{S}_{1D}). \quad (62)$$

In the following, we will compute the remaining Gaussian integrals in two special cases where $\Delta_{qq'}[u]$ simplifies. The first one is the case of an homogeneous magnetic field, where $\Delta_{qq'}[u] = u^2\delta_{q+1/2}\delta(q+q')$ with $\delta(q-q')$ the Kronecker symbol, and $\delta_{q+1/2}$ defined in (43). The second case corresponds to the simplest case of inhomogeneous magnetic field, when half of the boundary spins is subject to $+H$ and the other half to $-H$.

6.3. Expression for the partition function and thermodynamic limit

In this section, we are interested in the special case of an homogeneous boundary magnetic field. The previous action (61) reduces to:

$$\begin{aligned} \mathcal{S}_{1D} = & \sum_{q=0}^{L/2-1} \left[(1 + \Lambda_{q+\frac{1}{2}}) H_{q+\frac{1}{2}} H_{q+\frac{1}{2}}^* + (1 + \Lambda_{-q-\frac{1}{2}}) H_{-q-\frac{1}{2}} H_{-q-\frac{1}{2}}^* + \right. \\ & \left. \gamma_{q+\frac{1}{2}}^{2D} H_{-q-\frac{1}{2}}^* H_{q+\frac{1}{2}}^* - t^2 \gamma_{q+\frac{1}{2}}^{2D} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}} \right] + u^2 \sum_{q=0}^{L-1} \delta_{q+\frac{1}{2}} H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}}, \end{aligned}$$

with $\delta_{q+\frac{1}{2}}$ defined in (43). The successive integrations over the blocks of Grassmann magnetic fields are easy to perform and we obtain:

$$\mathcal{Q}(h) = \mathcal{Q}_0 \prod_{q=0}^{L/2-1} Z_{q+\frac{1}{2}}(u),$$

with

$$Z_{q+\frac{1}{2}}(u) \equiv (1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) + \gamma_{q+\frac{1}{2}}^{2D} (u^2 \delta_{q+\frac{1}{2}} - t^2 \gamma_{q+\frac{1}{2}}^{2D}). \quad (63)$$

We factorize the previous expression, in order to distinguish between the boundary effect in zero field and the contribution due to the magnetic field:

$$\mathcal{Q}(h) = \mathcal{Q}_0 \prod_{q=0}^{L/2-1} \left[(1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) - t^2(\gamma_{q+\frac{1}{2}}^{2D})^2 \right] \left[1 + \frac{u^2 \delta_{q+\frac{1}{2}} \gamma_{q+\frac{1}{2}}^{2D}}{(1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) - t^2(\gamma_{q+\frac{1}{2}}^{2D})^2} \right].$$

The total free energy is therefore written as

$$F(h) = -Lk_B T \ln \cosh(\beta h) - k_B T \ln \mathcal{Q}_0 + Lf_b + Lf_{\text{field}}, \quad (64)$$

with f_b a corrective free energy that is needed to restore the free boundary conditions in the direction transverse to the field,

$$\beta f_b = -\frac{1}{L} \sum_{q=0}^{L/2-1} \ln \left[(1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) - t^2(\gamma_{q+\frac{1}{2}}^{2D})^2 \right], \quad (65)$$

and f_{field} the magnetic contribution to the free energy

$$\beta f_{\text{field}} = -\frac{1}{L} \sum_{q=0}^{L/2-1} \ln \left[1 + \frac{u^2 \delta_{q+\frac{1}{2}} \gamma_{q+\frac{1}{2}}^{2D}}{(1 + \Lambda_{q+\frac{1}{2}})(1 + \Lambda_{-q-\frac{1}{2}}) - t^2(\gamma_{q+\frac{1}{2}}^{2D})^2} \right]. \quad (66)$$

This decomposition is in agreement with McCoy and Wu's results(9; 13). Indeed in the thermodynamic limit, we can use the following identity ($a > b$),

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{1}{\sqrt{(a-b)(a+b)}},$$

to obtain

$$\int_0^{2\pi} \frac{d\theta_p}{2\pi} \frac{1}{(1+t^2)^2 - 2t(1-t^2)(\cos \theta_p + \cos \theta_{q+\frac{1}{2}})} = \frac{1}{\sqrt{R(\theta_{q+\frac{1}{2}})}},$$

with the function R defined by

$$R(\theta) = [(1+t^2)^2 + 2t(1-t^2)(1-\cos \theta)] [(1+t^2)^2 - 2t(1-t^2)(1+\cos \theta)].$$

Then the following coefficients can be evaluated in this limit:

$$\gamma_{q+\frac{1}{2}}^{2D} = \frac{2it \sin \theta_{q+\frac{1}{2}}}{\sqrt{R(\theta_{q+\frac{1}{2}})}}, \quad \Lambda_{q+\frac{1}{2}} = -\frac{1}{2} + \frac{1}{2} \frac{(1+t^2)(1-2t \cos \theta_{q+\frac{1}{2}} - t^2)}{\sqrt{R(\theta_{q+\frac{1}{2}})}}. \quad (67)$$

Using the previous results and after some algebra, we recover the result of McCoy and Wu, and the continuous form for the boundary free energy depending on the magnetic field is

$$f_{\text{field}} = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta \ln \left(1 + \frac{4u^2 t(1+\cos \theta)}{(1+t^2)(1-2t \cos \theta - t^2) + \sqrt{R(\theta)}} \right). \quad (68)$$

The expression (66) of the free energy allows the numerical computation of the specific heat in this case, even for L small. The results are presented on the figure 6.3 for various magnetic fields.

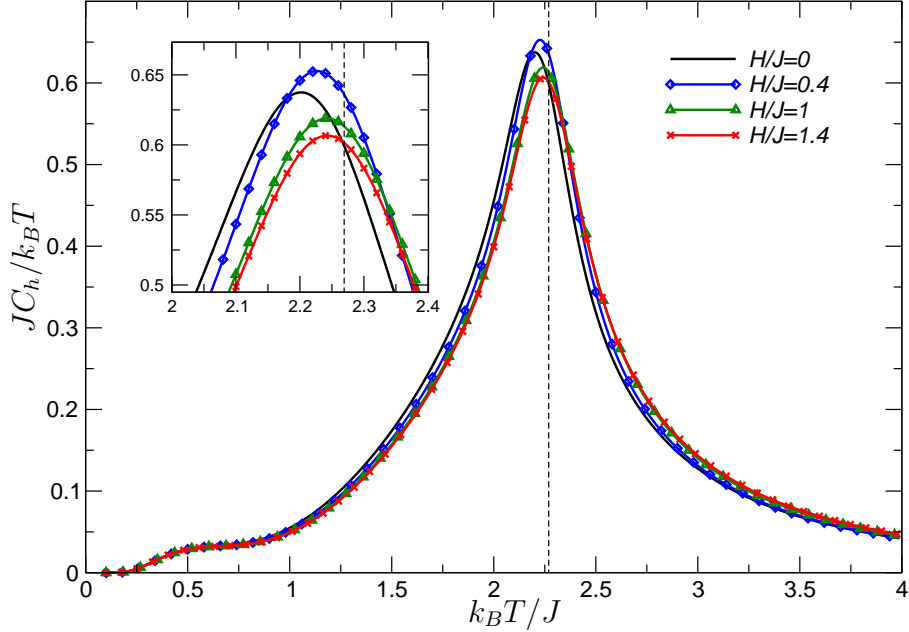


Figure 3. Specific heat for various values of homogeneous magnetic field on the boundary for $L=20$. The inset is a zoom in the region of the maximum. The vertical dashed line is the position of critical temperature for 2D Ising model in zero field, $k_B T_c/J \simeq 2.26$.

7. Boundary two point correlation function and magnetization

In this section, we compute the boundary spin-spin correlation functions along the boundary line between two different sites $(1, k)$ and $(1, l)$, $k < l$, in the case where the magnetic field is uniform. This is the easiest case since we can use simple properties of the Grassmann magnetic fields. Using (13), we have

$$\begin{aligned} \langle \sigma_{1k} \sigma_{1l} \rangle &\propto \text{Tr}_{\sigma} [\sigma_{1k} \sigma_{1l} e^{-\beta \mathcal{H}}], \\ &= \text{Tr}_{\sigma} \left[\prod_{m,n=1}^L \psi_{mn}^{(1)} \psi_{mn}^{(2)} \cdot \prod_{n=1}^L \sigma_{1k} \sigma_{1l} (1 + u \sigma_{mn}) \right]. \end{aligned} \quad (69)$$

We then write ($u \neq 0$)

$$\sigma_{1k} (1 + u \sigma_{1k}) = u \left(1 + \frac{1}{u} \sigma_{1k} \right), \quad (70)$$

and introduce local magnetic fields $u_n = u + (u^{-1} - u)(\delta_{kn} + \delta_{ln})$ so that

$$\langle \sigma_{1k} \sigma_{1l} \rangle \propto u^2 \text{Tr}_{\sigma} \left[\prod_{m,n=1}^L \psi_{mn}^{(1)} \psi_{mn}^{(2)} \cdot \prod_{n=1}^L (1 + u_n \sigma_{mn}) \right].$$

We remarks that this expression is the PF of the 2D Ising model in the particular case of an inhomogeneous boundary magnetic field (13). The integration over the bulk variables is not affected by this change. The difference appears *only* in the non local coupling between the fermionic magnetic fields H_n ,

$$\prod_{m=1}^{\overrightarrow{L}} e^{u_n H_n} = \exp \left[\sum_{n=1}^L u_n H_n + \sum_{m=1}^{L-1} \sum_{n=m+1}^L u_m u_n H_m H_n \right]. \quad (71)$$

Using the expression for u_n , we obtain

$$\sum_{m < n} u_m u_n H_m H_n = u^2 \sum_{m < n} H_m H_n + (1 - u^2)(H_k L_k + H_l L_l) + \frac{1 - u^2}{u^2} H_k H_l, \quad (72)$$

$$L_k = \sum_{n=k}^L H_n - \sum_{n=1}^{k-1} H_n,$$

and therefore,

$$\prod_{n=1}^{\overrightarrow{L}} e^{u_n H_n} = \prod_{n=1}^{\overrightarrow{L}} e^{u H_n} \cdot \left[1 + (1 - u^2)(H_k L_k + H_l L_l) + \frac{1 - u^2}{u^2} H_k H_l + (1 - u^2)^2 H_k L_k H_l L_l \right].$$

Then the two point correlation function can be simply expressed with correlation functions $\langle H_k H_l \rangle$, $\langle H_k L_k \rangle$, and $\langle H_k L_k H_l L_l \rangle$

$$\begin{aligned} \langle \sigma_{1k} \sigma_{1l} \rangle &= u^2 + u^2(1 - u^2) \langle H_k L_k + H_l L_l \rangle + (1 - u^2) \langle H_k H_l \rangle \\ &\quad + u^2(1 - u^2)^2 \langle H_k L_k H_l L_l \rangle. \end{aligned}$$

The correlation functions involving a product of four Grassmann fields can be written in term of products $\langle H_k H_l \rangle$ according to Wick's theorem:

$$\langle H_k L_k H_l L_l \rangle = \langle H_k L_k \rangle \langle H_l L_l \rangle - \langle H_k L_l \rangle \langle H_l L_k \rangle - \langle H_k H_l \rangle \langle L_k L_l \rangle. \quad (73)$$

Using a Fourier transformation, the two field correlation functions are expressed, using the previous definitions, as

$$\begin{aligned} \langle H_k H_l \rangle &= \frac{2i}{L} \sum_{q=0}^{L/2-1} \left\langle H_{q+\frac{1}{2}} H_{-q-\frac{1}{2}} \right\rangle \sin \left[\theta_{q+\frac{1}{2}}(k-l) \right], \quad (74) \\ &= \frac{2}{L} \sum_{q=0}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \sin \left[\theta_{q+\frac{1}{2}}(k-l) \right]. \end{aligned}$$

Each term on the right hand side of equation (73) can then be evaluated using the previous result:

$$\begin{aligned} \langle H_k L_k \rangle \langle H_l L_l \rangle &= \left(\frac{1}{L} \sum_{q=1}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \cot \frac{\theta_{q+\frac{1}{2}}}{2} \right)^2, \quad (75) \\ \langle H_k L_l \rangle &= -\frac{1}{L} \sum_{q=1}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \frac{\cos[\theta_{q+\frac{1}{2}}(k-l+1/2)]}{\sin(\theta_{q+\frac{1}{2}}/2)}, \\ \langle L_k L_l \rangle &= \frac{2}{L} \sum_{q=1}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \frac{\sin[\theta_{q+\frac{1}{2}}(k-l)]}{\sin(\theta_{q+\frac{1}{2}}/2)^2}. \end{aligned}$$

The magnetization can be computed the same way. Using the identity (70) we obtain

$$\langle \sigma_{1k} \rangle = u + u(1 - u^2) \langle H_k L_k \rangle. \quad (76)$$

The connected correlation function is then

$$\langle \sigma_{1k} \sigma_{1l} \rangle - \langle \sigma_{1k} \rangle \langle \sigma_{1l} \rangle = (1 - u^2) \langle H_k H_l \rangle - u^2(1 - u^2)^2 (\langle H_k L_l \rangle \langle H_l L_k \rangle + \langle H_k H_l \rangle \langle L_k L_l \rangle).$$

These correlation functions, and particularly $\langle H_k H_l \rangle$, can be extended for the study of more complex configurations of the boundary magnetic field. Without field ($u = 0$), the correlation function between two spins σ_{1k} and σ_{1l} are simply the correlation function between the two Grassmannian fields H_k and H_l and is given by equation (74). We can extract from this the dependence of the magnetization per spin m in the thermodynamic limit near the critical point T_c . Indeed, it is usual to define m^2 as the limit of the two point correlation function for large separation $r = |k - l|$:

$$\lim_{|k-l| \rightarrow \infty} \langle \sigma_{1k} \sigma_{1l} \rangle = m^2. \quad (77)$$

To obtain the main contribution from (74) near T_c in the thermodynamic limit, we use the expressions (67) to compute $Z_{q+\frac{1}{2}}(0)$ in (74) and then make an expansion around t_c in the low temperature limit ($t > t_c$) of the different quantities. We first write the correlation function as an integral

$$\begin{aligned} \langle \sigma_{1k} \sigma_{1l} \rangle &= \frac{2t}{\pi} \int_0^\pi d\theta \frac{\sin(\theta) \sqrt{R(\theta)}}{S(\theta)} \sin(\theta r), \\ S(\theta) &= \frac{1}{4} \left[\sqrt{R(\theta)} + (1 + t^2)(1 - 2t \cos \theta - t^2) \right]^2 + 4t^4 \sin^2 \theta, \end{aligned} \quad (78)$$

and expand R and S for $\Delta t = t - t_c$ and θ small, which is the region where the main contribution of the integral comes from. For R , we find the following expansion:

$$\begin{aligned} \sqrt{R(\theta)} &= \Delta t (1 + t_c^2)(1 + t_c + \sqrt{2}) \left[1 + \frac{t_c(1 - t_c^2)}{2(1 + t_c + \sqrt{2})} \tilde{\theta}^2 + \dots \right], \\ &= \Delta t (R_0 + R_2 \tilde{\theta}^2 + \dots) \end{aligned}$$

where we defined $\tilde{\theta} \Delta t = \theta$, $R_0 = (1 + t_c^2)(1 + t_c + \sqrt{2})$ and $R_2 = t_c(1 - t_c^4)/2$. For S , we obtain:

$$S(\theta) = \Delta t^2 \left(S_0 + S_2 \tilde{\theta}^2 + \dots \right),$$

where S_0 and S_2 are numerical coefficients evaluated at t_c : $S_0 = 16(3 - 2\sqrt{2}) \simeq 2.745$, $S_2 = 4t_c^4 \simeq .118$. After some algebra, we obtain the following behaviour near t_c of the two point correlation function:

$$\langle \sigma_{1k} \sigma_{1l} \rangle \simeq \frac{2t\Delta t}{\pi} \int_0^\infty dx \frac{x R_0}{S_0(r\Delta t)^2 + S_2 x^2} \sin x. \quad (79)$$

For $r\Delta t$ small enough, the integral is a constant $\pi R_0/2S_2$, and m is then proportional to $\sqrt{\Delta t}$, which gives the mean field exponent $\beta = 1/2$ for the boundary magnetization. With the presence of a small boundary magnetic field, at T_c , we use equation (76) to compute directly the magnetization:

$$\langle \sigma_{1k} \rangle \simeq u \langle H_k L_k \rangle = \frac{ut}{\pi} \int_0^\pi d\theta \frac{(1 + \cos \theta) \sqrt{R(\theta)}}{S(\theta) + 2tu^2(1 + \cos \theta) \sqrt{R(\theta)}}. \quad (80)$$

The quantities R and S are easily evaluated at t_c :

$$\begin{aligned} R(\theta) &= 16C_0(1 - \cos \theta)(3 - \cos \theta), \\ S(\theta) &= 8C_0(1 - \cos \theta) \left[3 - \cos \theta + \sqrt{2(1 - \cos \theta)(3 - \cos \theta)} \right], \end{aligned}$$

with $C_0 = 17 - 12\sqrt{2}$. When u is small, the main contribution in (80) is when the denominator is small, or when θ is close to zero. In this case we have $\sqrt{R(\theta)} \simeq 4\sqrt{C_0}\theta$ and $S(\theta) \simeq 8C_0\theta^2$. Replacing these values in (80) we obtain in the limit where $u \ll 1$:

$$\langle \sigma_{1k} \rangle \simeq \frac{ut}{\pi} \int_0^\pi d\theta \frac{1}{\sqrt{C_0}\theta + 2tu^2} \simeq -\frac{2t}{\pi\sqrt{C_0}} u \log u. \quad (81)$$

These results agree with McCoy and Wu's paper (14) and we can extend the method to the case of an inhomogeneous boundary magnetic field, as we will see in the next section.

8. An example of inhomogeneous boundary magnetic field

The existence of a 1D action allows us to solve more general configurations of the boundary magnetic field. In this section, we illustrate that considering the simplest case of a non uniform field at the boundary: Between sites $(1, 1)$ and $(1, l)$ we impose a field h , and between sites $(1, l+1)$ and $(1, L)$ a field $-h$ (configuration noted C_b). The previous solution in the presence of a uniform boundary magnetic field (configuration C_a) has been presented in different publications (see for example (17), (18) and (19)) but to our knowledge the present configuration has not been yet exactly studied. Using our method, it appears that the evaluation of the PF $\mathcal{Q}(h; l)$ is a simple extension of the previous calculation.

Indeed, the product (73) depending on the field can be simply expressed as

$$\exp \left(\sum_{m < n} u_m u_n H_m H_n \right) = \exp \left(u^2 \sum_{m < n} H_m H_n - 2u^2 \sum_{m=1}^l H_m \sum_{n=l+1}^L H_n \right).$$

Then we have

$$\begin{aligned} \mathcal{Q}(h; l) &= \mathcal{Q}_0 \text{Tr} \left[e^{\mathcal{S}_{1D}} \left(1 - 2u^2 \sum_{m=1}^l \sum_{n=l+1}^L H_m H_n \right) \right], \\ &= \mathcal{Q}(h) \left(1 - 2u^2 \sum_{m=1}^l \sum_{n=l+1}^L \langle H_m H_n \rangle_{\mathcal{S}_{1D}} \right), \end{aligned}$$

since the other terms from the exponential expansion all include the square of linear Grassmann sums and are therefore zero.

Using the Fourier transformation (74), we obtain :

$$\frac{\mathcal{Q}(h; l)}{\mathcal{Q}(h)} = 1 + \frac{4u^2}{L} \sum_{q=0}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D}}{Z_{q+\frac{1}{2}}(u)} \frac{\sin \theta_{q+\frac{1}{2}} l}{1 - \cos \theta_{q+\frac{1}{2}}}. \quad (82)$$

The free energy difference between the 2 field configurations C_b and C_a is positive and is equal to

$$-\beta f_{\text{int}} = \ln \left(1 + 4u^2 \frac{1}{L} \sum_{q=0}^{L/2-1} \frac{i\gamma_{q+\frac{1}{2}}^{2D} \sin \theta_{q+\frac{1}{2}} l}{Z_{q+\frac{1}{2}}(u) 1 - \cos \theta_{q+\frac{1}{2}}} \right). \quad (83)$$

This term is added to the total free energy (64) and for $l = L/2$, $\sin(\theta_{q+\frac{1}{2}} l)$ is simply equal to $(-1)^q$. If the transverse direction has an infinite size, we can use the expressions (67) to obtain

$$-\beta f_{\text{int}} = \ln \left(1 - 8u^2 \frac{1}{L} \sum_{q=0}^{L/2-1} \frac{(-1)^q \cot(\theta_{q+\frac{1}{2}}/2)}{\sqrt{R(\theta_{q+\frac{1}{2}}) Z_{q+\frac{1}{2}}(u)}} \right), \quad (84)$$

or more explicitly

$$-\beta f_{\text{int}} = \ln \left(1 - \frac{16tu^2}{L} \sum_{q=0}^{L/2-1} \frac{(-1)^q \cot(\theta_{q+\frac{1}{2}}/2)}{(1+t^2)(1-2t \cos \theta_{q+\frac{1}{2}} - t^2) + 4tu^2(1 + \cos \theta_{q+\frac{1}{2}}) + \sqrt{R(\theta_{q+\frac{1}{2}})}} \right).$$

In the 1D Ising model, for a system of L spins with periodic boundary conditions and with a field configuration C_b identical to the 2D boundary line with $l = L/2$, a similar result can be obtained, either by the matrix transfer method or with the grassmannian solution of the section 5. In the first case, we obtain

$$-\beta f_{\text{int}}^{1D} = \ln \left((1-t)^2 + \frac{8tu^2[4t(1-u^2)]^{L/2}}{(1+t - \sqrt{(1-t)^2 + 4tu^2})^L + (1+t - \sqrt{(1-t)^2 + 4tu^2})^L} \right) - \ln[(1-t)^2 + 4tu^2] \quad (85)$$

and in the thermodynamic limit this leads to

$$-\beta f_{\text{int}}^{1D} = \ln \left[\frac{(1-t^2)^2}{(1-t)^2 + 4tu^2} \right]. \quad (86)$$

With the Grassmannian fields, we obtain a different expression but rigorously identical to equation (85):

$$-\beta f_{\text{int}}^{1D} = \ln \left(1 - \frac{8tu^2}{L} \sum_{q=0}^{L/2-1} \frac{(-1)^q \cot(\theta_{q+\frac{1}{2}}/2)}{1+t^2 - 2t \cos(\theta_{q+\frac{1}{2}}) + 4tu^2 \cos(\theta_{q+\frac{1}{2}}/2)^2} \right). \quad (87)$$

In the thermodynamic limit and at zero temperature, f_{int}^{1D} is equal to $4J$ for any non zero value h , which is the energy difference between the 2 ground states of C_b and C_a , all spins following the magnetic field direction in both cases. In the 2D case however, if we suppose that the size is infinite in the transverse direction, for small values of the magnetic field the boundary spins point all in the same direction (imposed by the bulk spins, up for example) below a critical value of the field $h_c = J(1 + 4/L)$ and for the C_b configuration. Indeed the interaction between boundary spins and the ones in the bulk are strong enough that a small field $-h$ is not sufficient to reverse these spins. In this case $f_{\text{int}} = hL$ and is therefore extensive contrary to the 1D case. Above h_c , the field $-h$

is strong enough to reverse half of the spins that were up and therefore $f_{\text{int}} = h_c L$ due to the frustrated couplings with the neighboring spins in the bulk. Figure 4 represents the free energy f_{int}/L as a function of the boundary magnetic field for different values of the temperature. $L = 20$ and $h_c = 1.2J$. There is a change of the curve slope at $h = h_c$ and low temperature curves suggest the previous reversal picture.

Figure 5 represents the contribution f_{int}/L as a function of temperature for various values of the magnetic fields. For $h > h_c$ the curves saturate at zero temperature to the value $f_{\text{int}}/L = h_c$ as expected.

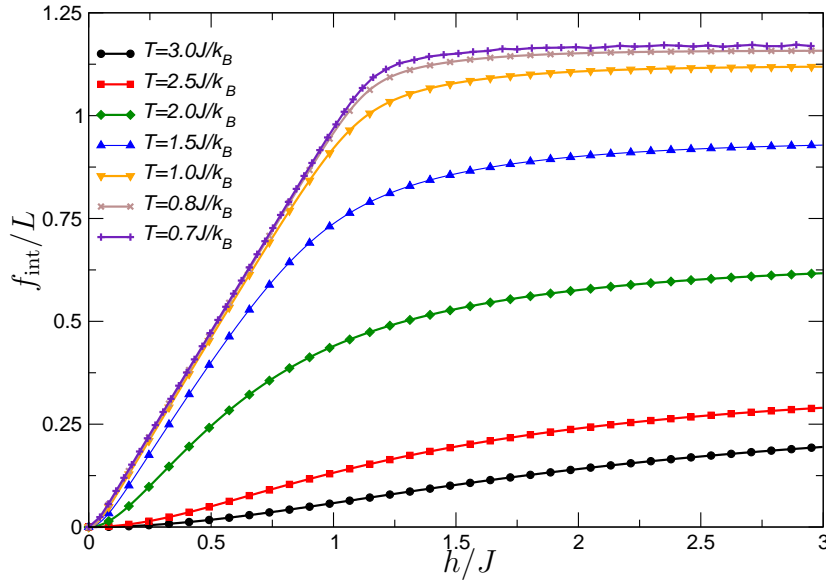


Figure 4. Boundary free energy f_{int}/L as function of h for $L = 20$, $h_c = 1.2$, for various values of the temperature. Notice the transition at $h = h_c$ corresponding to the reversing of half the boundary spins.

To check the low temperature behavior of f_{int} , we can perform an expansion of equation (84) for $h \ll 1$ and $T \ll 1$. In this case, we assume that $t \simeq 1$ and $u \simeq 1 - 2 \exp(-2h/k_B T)$. We obtain

$$-\beta f_{\text{int}} \simeq \ln \left(1 - 4u^2 \frac{1}{L} \sum_{q=0}^{L/2-1} \frac{(-1)^q \cot(\theta_{q+\frac{1}{2}}/2)}{(1+u^2) - (1-u^2) \cos \theta_{q+\frac{1}{2}}} \right). \quad (88)$$

The sum inside the logarithm can be computed using equations (85) and (87), with $t = 1$. We obtain

$$-\beta f_{\text{int}} = \ln \left(\frac{2(1-u^2)^{L/2}}{(1-u)^L + (1+u)^L} \right) \simeq -\beta h L \quad (89)$$

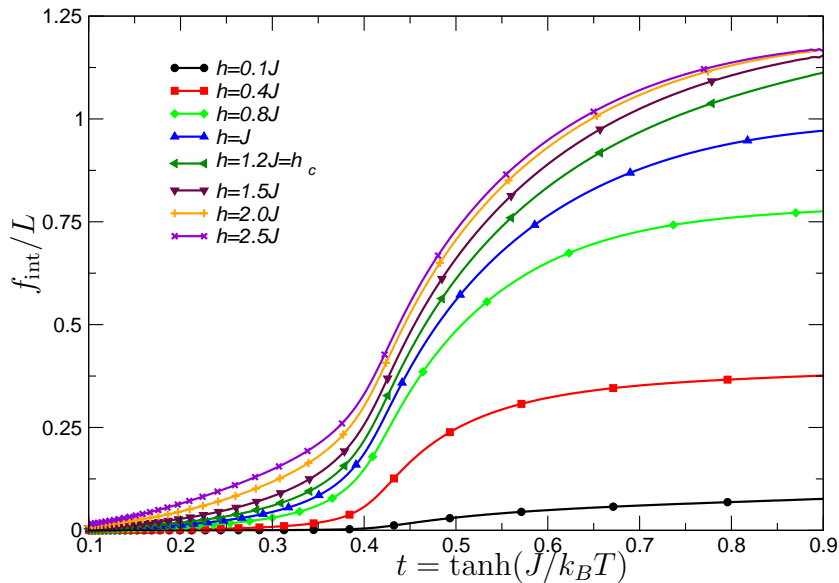


Figure 5. Boundary free energy f_{int}/L for $L = 20$, $h_c = 1.2$, as a function of $\tanh(J/k_B T)$ for various values of the magnetic field h/J .

These results allow us also to study quite precisely the effect of an inhomogeneous magnetic field on the spins inside the bulk for a finite transverse size system or a fixed ratio between the sizes of the two directions (square or rectangular system). In this case, the previous domain wall that appears for $h > h_c$ due to the reversal of half of the boundary spins can propagate or jump inside the bulk for sufficiently high temperature in order to lower the free energy. This may cause the total magnetization to cancel if this wall separates two regions of equal number of opposite spins. An exhaustive study will be published in a forthcoming paper.

9. Conclusion

In this article we have presented an application of the Plechko's method to the 2D Ising model in the case of a inhomogeneous boundary magnetic field. We show that this model can be mapped onto a 1D model with a Gaussian Grassmannian action similar to the case of the 1D Ising in a magnetic field. Results have been obtained for the homogeneous case, in order to validate our results with the ones obtained by McCoy and Wu(13), and an extension is made to a special inhomogeneous case where a magnetic interface is imposed. This case is a generalization of the homogeneous case, and an expression of the boundary partition function in term of a 1D action is given. Contrary to the 1D case, the interface free energy appears to be extensive, proportional to the system

size L . A simple argument gives a zero temperature critical field h_c to be the field above which a domain wall appears on the boundary and may eventually propagate inside the bulk at higher temperature. This generalization of Plechko's method for the Ising model with non uniform boundary magnetic field appears to be simple and straightforward. Further developments are possible. In particular, the expression for the bulk-boundary correlation functions can be computed. It appears however that this method is not efficient in the case of the 2D Ising model with homogeneous magnetic field in the bulk, or for the 3D Ising model since operations using mirrors symmetries seem not to be sufficient to write a Grassmannian quadratic action. The mapping of the boundary region onto a 1D action could be used to obtain some results in other cases, for example it can be a starting point for studying the properties of random magnetic fields on the boundary. For the 2D Ising case, random fields on the boundary are found to be marginal (20) and computations may be made based on the 1D action (61) using possibly random matrix theory.

Acknowledgments

We acknowledge Francois Delduc, Peter C.W. Holdsworth and Jean Richert for useful discussions.

References

- [1]E. Ising 1925 *Z. Physik* **31** 253
- [2]L. Onsager 1944 *Phys. Rev.* **65** 117
- [3]S. Samuel 1980 *J. Math. Phys.* **21** 2806,2815,2820
- [4]K. Nojima 1998 *Int. J. Mod. Phys. B* **12** 1995
- [5]V. N. Plechko 1985 *Theo. Math. Phys.* **64** 748
- [6]V. N. Plechko 1988 *Physica A* **152** 51
- [7]A.B. Zamolodchikov 1989 *Adv. Stud. Pure Math.* **19** 641
- [8]A.B. Zamolodchikov 1989 *Int. J. Mod. Phys. A* **4** 4235
- [9]B.M. McCoy, T.T. Wu 1973 *The Two-Dimensional Ising Model*, Harvard University Press
- [10]T. T. Wu 1966 *Phys. Rev.* **149** 380
- [11]B. M. McCoy and T. T. Wu 1967 *Phys. Rev.* **155** 438
- [12]H. Cheng and T. T. Wu 1967 *Phys. Rev.* **164** 719
- [13]B. M. McCoy and T. T. Wu 1967 *Phys. Rev.* **162** 436
- [14]B. M. McCoy and T. T. Wu 1968 *Phys. Rev.* **174** 546
- [15]R. Chatterjee 199 *Nucl. Phys. B* **468** 439
- [16]M. Nakahara 1999 *Geometry, topology and physics*, 2nd edition, Institute of Physics Publishing
- [17]H. Au-Yang and M. Fisher 1980 *Phys. Rev. B* **21** 3956
- [18]M.L. Rubio Puzzo and E.V. Albano 2002 *Phys. Rev. B* **66** 104409
- [19]A. I. Bugrij and V. N. Shadura 1990 *Phys. Lett. A* **150** 171
- [20]J. L. Cardy 1991 *J. Phys. A: Math. Gen.* **24** L1315