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# A new criterion for the existence of KdV solitons in ferromagnets.

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## Abstract

The long-time evolution of the KdV-type solitons propagating in ferromagnetic materials is considered through a multi-time formalism, it is governed by all equations of the KdV Hierarchy. The scaling coefficients of the higher order time variables are explicitly computed in terms of the physical parameters, showing that the KdV asymptotic is valid only when the angle between the propagation direction and the external magnetic field is large enough. The one-soliton solution of the KdV hierarchy is written down in terms of the physical parameters. A maximum value of the soliton parameter is determined, above which the perturbative approach is not valid. Below this value, the KdV soliton conserves its properties during an infinite propagation time.

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Keywords : KdV solitons, ferromagnets, KdV Hierarchy, Higher order KdV.

# 1 Introduction

## 1.1 KdV-type solitons in ferromagnets

Electromagnetic wave propagation in ferromagnetic media is intrinsically nonlinear. It is therefore the matter of intensive research in the theoretical physics of the nonlinear waves. In the frame of the Maxwell-Landau model, analytical expressions describing solitary wave propagation out from any slowly envelope or long-wave approximation have been found [1]. These waves have also been studied numerically [2]. Envelope solitons have been studied from several theoretical approaches [4, 3]. There are many experiments regarding magnetostatic waves in thin films [5, 6, 7]. Long-wave type approximations allow to describe some features related to relativistic domain wall propagation [8, 9], but have also brought forward the existence of another type of wave, described by the Korteweg-de Vries (KdV) equation [10]. It has been shown that such a wave can be emitted by a transverse instability of the relativistic domain wall [2, 9]. The interaction between the two types of wave has also been studied [10].

The KdV model is obviously a rough approximation. In [10] where it has been first derived in this frame, anisotropy, damping and inhomogeneous exchange were neglected. Second it assumes that the wave depends on a single spatial coordinate (plane wave), and that the amplitude is weak enough, the wave length and the propagation distance large enough, so that the first order of the KdV approximation can be retained. A study taking into account three space dimensions, damping, and inhomogeneous exchange is published independently [11]. But the weakly nonlinear approximation itself may necessitate higher order corrections. The latter are independent from the former ones. Indeed, the wave is intrinsically nonlinear, and the weakly nonlinear approximation is forced by the introduction of a static field, to which the wave field can be compared. Even in the roughest approximation, the ratio between the two fields can become rather close to one. A derivation of the equations describing the evolution of the higher order terms has been derived using a multi-time formalism [12]. It allows to prove that a formal asymptotic expansion exists up to any order, with all its terms bounded [13], which is a first step in the mathematical justification of the convergence of the expansion. Following the idea by Kraenkel *et al.* [14, 15], the multi-times expansion for KdV uses the KdV Hierarchy. The evolution of the main term in the expansion relative to each higher order time variable is given by

the corresponding equation in the KdV Hierarchy. Regarding the main term only, all information about the particular physical situation considered is contained in scaling coefficients of the time variables. These coefficients can be computed. The aim of this paper is to give the value of these quantities, and to draw physical consequences from them. It is organized as follows: in section 2 we describe the perturbative scheme in the multi-time formalism. In section 3 we compute explicitly the time scaling coefficients. Conclusions can be drawn on the validity of the perturbative scheme considered as an asymptotic expansion, *i.e.* for a fixed number of terms, when the perturbative parameter becomes small enough. In section 4, we give the expression of the one-soliton solution of the complete KdV hierarchy. This gives information about the validity of the perturbative scheme considered as a series expansion, *i.e.* for a fixed value of the perturbative parameter and an infinite number of terms.

## 2 The multi-time formalism

### 2.1 The KdV mode

The evolution of the magnetization density  $\vec{M}$  in a magnetic field  $\vec{H}$  is described by the Landau equation

$$\partial_t \vec{M} = -\gamma \mu_0 \vec{M} \wedge \vec{H}_{eff}, \quad (1)$$

where  $\gamma$  is the gyromagnetic ratio ( $\gamma > 0$ ), and  $\mu_0$  the magnetic permeability in vacuum. The effective field  $\vec{H}_{eff}$  contains several terms giving account for the inhomogeneous exchange interaction, the effects of finite size and the anisotropy. Here we use the basic approximation:  $\vec{H}_{eff} = \vec{H}$ . Damping is also neglected.

The evolution of the magnetic field  $\vec{H}$  is described by the Maxwell equations. We assume that, regarding its dielectric properties, the material is perfectly linear and isotropic, and we denote by  $c$  the light velocity based on its dielectric constant  $\hat{\varepsilon}$ , *i.e.*  $c = 1/\sqrt{\hat{\varepsilon}\mu_0}$ . The Maxwell equations reduce then to

$$-\vec{\nabla} \left( \vec{\nabla} \cdot \vec{H} \right) + \Delta \vec{H} = \frac{1}{c^2} \partial_t^2 \left( \vec{H} + \vec{M} \right). \quad (2)$$

We replace below  $\vec{H}$ ,  $\vec{M}$  and  $t$  by the normalized quantities  $\gamma \mu_0 \vec{H}/c$ ,  $\gamma \mu_0 \vec{M}/c$  and  $ct$ . The constants  $\gamma \mu_0$  and  $c$  take then the value 1.

The ‘long wave’ limit of a wave with negative helicity is considered. We introduce a small parameter  $\varepsilon$ , such that  $1/\varepsilon$  measures the length of the solitary wave and  $\varepsilon^2$  its amplitude. The magnetic field is expanded as

$$\vec{H} = \vec{H}_0 + \varepsilon^2 \vec{H}_2 + \dots, \quad (3)$$

and  $\vec{M}$  in an analogous way. Using the slow variables

$$\begin{cases} \xi = \varepsilon(x - Vt), \\ \tau_1 = \varepsilon^3 t, \end{cases} \quad (4)$$

it is shown first that this wave propagates at the velocity

$$V = \sqrt{(\alpha + \sin^2 \theta) / (\alpha + 1)}, \quad (5)$$

where  $\theta$  is the angle between the propagation direction and the applied field, and  $\alpha = H_0/M_0$  the ratio from the latter to the saturation magnetization. Second it is shown that the propagation of this type of ‘long waves’ is governed by the KdV equation [10]

$$\partial_{\tau_1} \varphi_2 + q \varphi_2 \partial_{\xi} \varphi_2 + r \partial_{\xi}^3 \varphi_2 = 0, \quad (6)$$

where  $q$  and  $r$  are real constants given by

$$q = \frac{3 \cos^2 \theta \sin^2 \theta \sqrt{1 + \alpha}}{2 (\alpha + \sin^2 \theta)^{3/2}}, \quad (7)$$

and

$$r = \frac{-1 \cos^4 \theta \sqrt{\alpha + \sin^2 \theta}}{2m^2 \sin^2 \theta (1 + \alpha)^{7/2}}. \quad (8)$$

$\varphi_2$  is the wave amplitude, related to the main component  $\vec{H}_2$  and  $\vec{M}_2$  of the wave magnetic field and the magnetization density through

$$\vec{H}_2 = \varphi_2 \vec{h}_1 \quad \text{and} \quad \vec{M}_2 = \varphi_2 \vec{m}_1, \quad (9)$$

where  $\vec{h}_1$  and  $\vec{m}_1$  are polarization vectors defined by

$$\vec{h}_1 = m (1 + \alpha) \sin \theta \begin{pmatrix} \frac{\sin \theta \cos \theta}{\alpha + \sin^2 \theta} \\ 1 \\ 0 \end{pmatrix}, \quad (10)$$

and

$$\vec{m}_1 = \frac{m(1+\alpha)\sin\theta\cos\theta}{\alpha+\sin^2\theta} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}. \quad (11)$$

(We use the normalization of [10, 12], introduced for computational convenience).

## 2.2 Higher order terms

Going further in the resolution of the perturbative scheme, it is seen that the field component of order  $j$  ( $j > 2$ ) writes

$$\vec{H}_j = \varphi_j \vec{h}_1 + \vec{h}_j^0(\varphi_2, \varphi_3, \dots, \varphi_{j-1}), \quad (12)$$

where  $\vec{h}_j^0$  is an explicit functional of the lower order amplitudes  $\varphi_2, \varphi_3$ , up to  $\varphi_{j-1}$ ,  $\vec{h}_1$  the polarization vector given above by (10), and  $\varphi_j$  is an higher order amplitude.  $\varphi_j$  satisfies a linearized KdV equation of the form

$$\begin{aligned} \partial_{\tau_1} \varphi_j + q \partial_\xi (\varphi_2 \varphi_j) + r \partial_\xi^3 \varphi_j = \\ \Xi_j(\varphi_2, \varphi_3, \dots, \varphi_{j-1}), \end{aligned} \quad (13)$$

where the right-hand-side (rhs) member  $\Xi_j$  is an explicit functional of the lower order amplitudes  $\varphi_2, \varphi_3$ , up to  $\varphi_{j-1}$ . The parity and homogeneity properties of the expansion allow to prove that half of these equations admit the zero solution, so that  $\varphi_j$  is non zero for even  $j$  only. Note that the inhomogeneous part  $\vec{h}_j^0$  of the  $j^{\text{th}}$  order magnetic field amplitude  $\vec{H}_j$  does not vanish for odd  $j$ .

We study the long-time propagation by considering the unbounded or secular solutions, and a multi-time expansion. Therefore we introduce a sequence of slower and slower temporal variables  $\tau_1 = \tau, \tau_2, \tau_3, \dots$  defined by  $\tau_j = \varepsilon^{2j+1}t$ . The propagation is governed by all equations of the KdV Hierarchy. In particular, the equation giving the evolution of the leading term  $\varphi_2$  with regard to the first higher order time variable  $\tau_2$  is derived as follows (more detail is given in [12]).  $\varphi_4$  is the amplitude of the first correction to the main term whose amplitude is  $\varphi_2$ . The equation that determines its evolution can be written in the form

$$\partial_{\tau_1} \varphi_4 + q \partial_\xi (\varphi_2 \varphi_4) + r \partial_\xi^3 \varphi_4 = -\partial_{\tau_2} \varphi_2 - r_2 \partial_\xi^5 \varphi_2 + \mathcal{O}_2, \quad (14)$$

where  $\mathcal{O}_2$  refers to an expression depending on  $\varphi_2$  without linear term, and  $r_2$  is a real coefficient. Some functions  $\varphi_4$  solutions of (14) are secular, *i.e.* grow linearly with the time  $\tau_1$ . Consider values of the time variable  $t$  about  $1/\varepsilon^5$ . Then the the time variable  $\tau_1 = \varepsilon^3 t$  take values about  $1/\varepsilon^2$ , and the secular term in  $\varphi_4$  becomes of order  $\varepsilon^2$  instead of  $\varepsilon^4$ , due to the factor  $\tau_1 \propto 1/\varepsilon^2$ . For times with this order of magnitude, this correction term must be taken into account in the expression of the main amplitude  $\varphi_2$ . In order to incorporate the correction into the evolution of the main amplitude  $\varphi_2$  with regard to the second order time variable  $\tau_2$ , we impose some condition on the rhs member of equation (14), so that  $\varphi_4$  remains bounded (or more exactly sublinear). The condition to be satisfied is thus that the equation (14) does not admit any secular solution. Through an explicit computation in the case where  $\varphi_2$  is the one-soliton solution of KdV, Kodama and Taniuti [16] have noticed that the secular-producing terms are the terms linear with regard to the solution of lowest order  $\varphi_2$ . The secular solutions  $\varphi_4$  will vanish thus if the linear terms vanish from the rhs member of the equation (14). To achieve this, we impose that  $\varphi_2$  satisfies some partial differential equation such that

$$\partial_{\tau_2} \varphi_2 = -r_2 \partial_{\xi}^5 \varphi_2 + \mathcal{O}_2. \quad (15)$$

We still need to determine the nonlinear terms of equation (15), represented by  $\mathcal{O}_2$ . They are not free but imposed by the compatibility condition between the KdV equation (6) and the equation (15), which is the Schwartz condition:  $\partial_{\tau_1} \partial_{\tau_2} \varphi_2 = \partial_{\tau_2} \partial_{\tau_1} \varphi_2$ . Kraenkel, Manna, and Pereira [15] have conjectured and checked on many examples that the only equation that possesses the same homogeneity properties as the rhs member of (14), and that satisfies this condition, is the second equation of what is called the KdV Hierarchy.

The KdV Hierarchy is the following family of equations [17]:

$$\partial_{T_n} v = \partial_X \mathcal{L}^n v \quad (n \text{ integer}), \quad (16)$$

where  $\mathcal{L}$  is a recurrence operator, defined by

$$\mathcal{L} = -\frac{1}{4} \partial_X^2 - v + \frac{1}{2} \int^X dX (\partial_X v). \quad (17)$$

For  $n = 1$ , it is the KdV equation, with a normalization that differs from that of (6) ( $q = \frac{3}{2}$ ,  $r = \frac{1}{4}$ ). We identify both using the relations

$$v = \frac{q}{6r} \varphi_2 \quad , \quad X = \xi \quad \text{and} \quad T_1 = 4r\tau_1. \quad (18)$$

For  $n = 2$ , the equation of the Hierarchy (16) writes as

$$\partial_{T_2} v = \frac{1}{16} \partial_X^5 v + \frac{5}{4} (\partial_X v) \partial_X^2 v + \frac{5}{8} v \partial_X^3 v + \frac{15}{8} v^2 \partial_X v. \quad (19)$$

An important property is the existence of the  $\tau$  Hirota function [17], that is a function of all variables  $(X, T_1, T_2, \dots)$ , related to  $v$  by

$$v(X, T_1, T_2, \dots) = 2 \partial_X^2 \ln \tau(X, T_1, T_2, \dots)$$

(avoid any confusion between the  $\tau$  Hirota function and the time variables  $\tau_j$ ). The existence of  $\tau$  ensures that a solution  $v$  of the system yielded by all equations of the Hierarchy exists, and thus that the Schwartz condition is satisfied at any order. After an adequate choice of the proportionality constant that connects the time variables of order 2, the variable  $\tau_2$  of our expansion and the variable  $T_2$  of the Hierarchy, that we write as

$$T_2 = -16r_2\tau_2, \quad (20)$$

the evolution equation to be satisfied by  $\varphi_2$  is

$$\frac{-1}{16r_2} \partial_{\tau_2} \varphi_2 = \partial_\xi \mathcal{L}^2 \varphi_2. \quad (21)$$

This way, the linear terms have been removed from the equation (15). It remains to justify that this procedure, that removes all linear terms from the rhs member of the linearized KdV equation, assuming it polynomial with regard to the solution of KdV, ensures that the solution of the linearized equation is bounded [18]. The KdV equation admits an infinite sequence of conserved densities we denote by  $\mathcal{A}_j$ , an expression of which can be found in [16]. It has been proven in [18] that the secular-producing terms are the terms proportional to  $\partial_\xi \mathcal{A}_j$ . Further the relations existing between the conserved densities  $\mathcal{A}_j$ , and the recurrence operator  $\mathcal{L}$ , defined by (17), that allows to write the Hierarchy, allow to show that the procedure by Kraenkel *et al.*, initially intended to remove the linear terms, exactly removes all these secular-producing terms.

Otherwise, the rhs member of the linearized KdV equation that governs the evolution of  $\varphi_6$  involves  $\varphi_4$ , solution of (14). It is thus necessary to see whether, when a solution of the linearized KdV equation itself is used in the rhs member, which part of it is secular-producing, and which part is not. This is

not too difficult. Indeed, this solution is given by its expansion on the basis of the squares  $\Phi_k$  of the Jost functions related to the KdV equation [16, 18], and we have characterized the fact that a source term is secular-producing or not by some criterion, that involves the coefficients of this expansion and their  $t$ -dependency. It remains a last point to be studied: the dependency of the higher order terms with regard to the higher order times. We shown that it is governed by a linearized KdV Hierarchy [13]. Finally, we have been able to justify that the higher order terms are not secular-producing, and to prove that the formal expansion contains bounded terms only.

### 3 Time scales

The generalization of the above procedure to an arbitrary order  $n \geq 2$  yields the equation

$$\frac{-1}{(-4)^n r_n} \partial_{\tau_n} \varphi_2 = \partial_\xi \mathcal{L}^n \varphi_2, \quad (22)$$

which governs the evolution of the main amplitude  $\varphi_2$  with regard to the higher order time variable  $\tau_n$ .  $\mathcal{L}$  is defined by the above formula (17). The scaling coefficient  $r_n$  is defined by  $r_1 = r$  and the recurrence formula

$$r_{n+1} = \sum_{\substack{(\alpha_j)_{1 \leq j \leq n-1}, k \geq 0 \\ (\sum_{j=1}^{n-1} 2j\alpha_j) + k = 2n+3}} \Xi((\alpha_j)_{1 \leq j \leq n-1}, k) \prod_{j=1}^{n-1} (-r_j)^{\alpha_j}. \quad (23)$$

The sequence of time variables  $\tau_1, \tau_2, \tau_3, \dots$  involved by the multiple time formalism are thus affected by the sequence of scaling coefficients  $r_1, r_2, r_3, \dots$ . The equations of the KdV Hierarchy are 'universal', not specific to the physical situation considered. The time scaling coefficients contain thus most physical data about the time evolution of the wave. Further, they are of interest regarding the convergence of the asymptotic series. They are computed using recurrence formula (23), together with the results of [12] listed in the appendix. The first coefficients read as follows:

$$r_2 = \frac{\gamma^3 V^9}{8(1+\alpha)^2 m_t^4 \mu} \left[ \begin{array}{l} 8 + (4\alpha - 13)\gamma + (3\alpha + 6)\gamma^2 \\ + \alpha(4\alpha - 10)\gamma^3 + 4\alpha(1 - \alpha)\gamma^4 \end{array} \right], \quad (24)$$

$$r_3 = \frac{-\gamma^4 V^{13}}{16(1+\alpha)^3 m_t^6 \mu^2} \left[ \begin{aligned} &40 + (32\alpha - 88)\gamma + (8\alpha^2 + 12\alpha + 67)\gamma^2 \\ &+ (68\alpha^2 - 132\alpha - 20)\gamma^3 + (40\alpha^3 - 143\alpha^2 + 122\alpha + 2)\gamma^4 \\ &- \alpha(32\alpha^2 - 70\alpha + 32)\gamma^5 + \alpha^2(16\alpha^2 - 56\alpha + 6)\gamma^6 \\ &- 8\alpha^2(4\alpha^2 - 8\alpha + 1)\gamma^7 + 16\alpha^3(\alpha - 1)\gamma^8 \end{aligned} \right], \quad (25)$$

in which

$$\gamma = 1 - \frac{1}{V^2}, \quad \mu = 1 + \alpha\gamma, \quad m_t = m \sin \theta. \quad (26)$$

The expressions of the higher order coefficients can be obtained in the same way, but are too complicated to be written down here; numerical computation is more convenient.

The coefficients  $r_1 = r, r_2, \dots$  up to  $r_5$  are plotted on figure 1, against

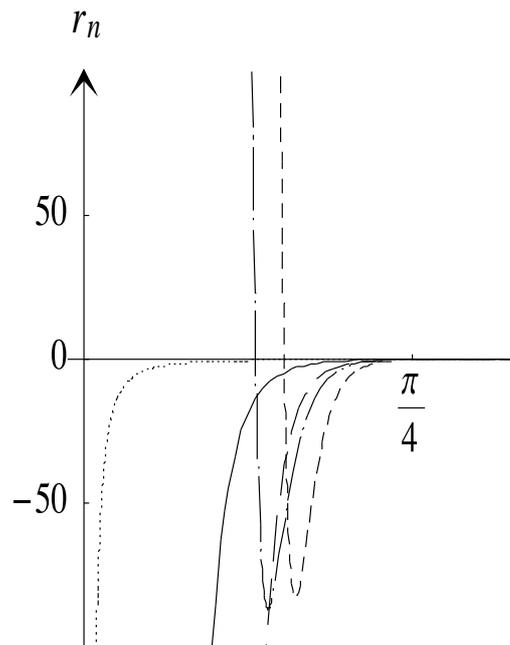


Figure 1: Plot of the five first time scaling coefficients  $r_1, \dots, r_5$  against the angle  $\theta$  between the propagation direction and the exterior field. The rescaled magnetic induction is  $m = 1$ , and the parameter determining the strength of the exterior field is  $\alpha = 0.5$ . Dotted line:  $r_1$ , solid line:  $r_2$ , large dasheding:  $r_3$ , dashed-dotted line:  $r_4$ , short dasheding:  $r_5$ .

the value of the angle  $\theta$  between the propagation direction and the external field, for a given value of the parameter  $\alpha$  that determines the magnitude of this field. Notice the annulation and sign change of the coefficients  $r_4$  and  $r_5$ , about 0.41 and 0.48 radians respectively. This marks a change in the behaviour of the corresponding corrections. When  $r_4$  is zero,  $\varphi_2$  is constant with regard to  $\tau_4$ , thus the 3<sup>rd</sup> order correction is in fact valid at order 4, regarding its time dependency.

It is seen that the  $r_n$  take very small values when  $\theta$  is close to  $\pi/2$ , and very large values when  $\theta$  is small. In the limiting case where the propagation direction is orthogonal to the external field ( $\theta = \pi/2$ ), the velocity  $V$  is 1, thus  $\gamma = 0$ , and the coefficients  $q$  and  $r$  vanish, so that the KdV equation (6) is replaced by

$$\partial_\tau \varphi = 0. \quad (27)$$

Thus  $\varphi$  is constant with time at first order, which means *a priori* that the wave evolves much slower than in the general case, at least for an order of magnitude. Recall that this order of magnitude is determined by the perturbative parameter  $\varepsilon$ , related to the wave amplitude and typical length. The wave propagates without deformation, to within a quantity of higher order in  $\varepsilon$ , up to times about  $T/\varepsilon^3$ , instead of times about  $T/\varepsilon$ , as usual in the long wave approximation. The higher order equations simplified by this trivial time evolution of the main term, yield also an approximation valid up to  $T/\varepsilon^3$ , for some finite  $T$ . Let us precise the influence of the scaling coefficients on the time validity range of the higher order approximations. We denote by  $L_0$  some typical length of the wave. The dimensionless space variable is  $\xi/L_0 = \varepsilon(x - Vt)/L_0$ . The reference length for  $x$  is chosen with the order of magnitude of  $\varepsilon L_0$ , in such a way that, as  $x$  takes values as large as  $1/\varepsilon$  with respect to this reference length,  $\xi$  is about  $L_0$ .  $V$  is close to 1, thus taking the same value as a reference time (recall that  $t$  has already been rescaled into  $ct$ ) is coherent with the asymptotic expansion. The higher order time variables adapted to the expansion are, rather than the  $\tau_n$ , the variables  $T_n$  of the KdV Hierarchy (22), written under its normalized form

$$\partial_{T_n} v = \partial_\xi \mathcal{L}^n v \quad (n \text{ integer}). \quad (28)$$

The differential recurrence operator  $\mathcal{L}$  is as in (17), with  $v = \frac{q}{6r} \varphi_2$ . The variable  $T_n$  reads then:

$$T_n = -(-4)^n r_n \tau_n = -(-4)^n r_n \varepsilon^{2n+1} t \quad (29)$$

$T_n$  must be about 1 for large values of  $t$ . This necessitates a smaller value of  $\varepsilon$  when the coefficient  $r_n$ , or rather  $(-4)^n r_n$ , is large. For each  $n$ ,  $\varepsilon$  must be compared to the reference value  $\varepsilon_n$  defined as follows:  $\varepsilon_n$  is the value of  $\varepsilon$  in (29) such that, when  $T_0 = \varepsilon t$  is equal to  $L_0$ ,  $|T_n|$  has the same value. It yields

$$\varepsilon_n = \frac{1}{2r_n^{1/2n}}. \quad (30)$$

The approximation involving the first  $n$  time variables (for all written terms) is valid for  $t \leq T/\varepsilon^{2n+1}$ , for some finite  $T$  with the order of magnitude of unity in the initial unit ( $\varepsilon L_0 v$ ). Taking the scaling coefficients into account, the approximation will rather be valid for  $|T_n| \leq T$ , that is:

$$|t| \leq \frac{T}{4^n r_n \varepsilon^{2n+1}} = \frac{T}{\varepsilon} \left( \frac{\varepsilon_n}{\varepsilon} \right)^{2n} \quad (31)$$

Notice that it is in fact necessary that  $|T_p| \leq T$  for all  $p \leq n$ , what implies some conditions on the variations of  $\varepsilon_p$  with relation to  $p$ . According to (31), when  $\varepsilon_n$  takes large values, the propagation can be described over a long distance even if the order  $n$  is relatively low and the value of the perturbative parameter  $\varepsilon$  close to 1. The higher order time variables make sense only if  $\varepsilon$  is smaller than the  $\varepsilon_n$ , and long time propagation can be described only if the ratio  $\varepsilon/\varepsilon_n$  is very small. These conditions will be hardly fulfilled when  $\varepsilon_n$  becomes small.

The five first  $\varepsilon_n$  are plotted on figure 2, against the angle  $\theta$ , and on figure 3, against the ratio  $\alpha$  that determines the magnitude of the external field. If the extrapolation of the few computed terms is valid, the sequence  $\varepsilon_n$  seems to be bounded with regard to  $n$ , although its terms grow up as  $\theta$  tends to  $\pi/2$ . Further, this bound is not excessively small when  $\theta$  is not smaller than  $10^\circ$  or  $15^\circ$ . For smaller values of  $\theta$ , the  $\varepsilon_n$  are so small that the KdV approximation can be valid only for excessively low intensities, and the higher orders will never appear.

When  $\theta$  approaches  $\pi/2$ , the  $\varepsilon_n$  become large. Then the approximation yielded by the KdV equations is valid during a very long time. The pulse behaviour will be correctly described by them even if the small perturbative parameter  $\varepsilon$  takes values rather close to 1. At the limit  $\theta = \pi/2$ , the modulation described by the KdV equation itself arises only at a very slow rate. A typical dependency of the  $\varepsilon_n$  with regard to the strength of the external field is shown on figure 3. The  $\varepsilon_n$  grow slowly with  $\alpha$ . Thus a strong external field enhances the validity of the KdV approximation, and increases

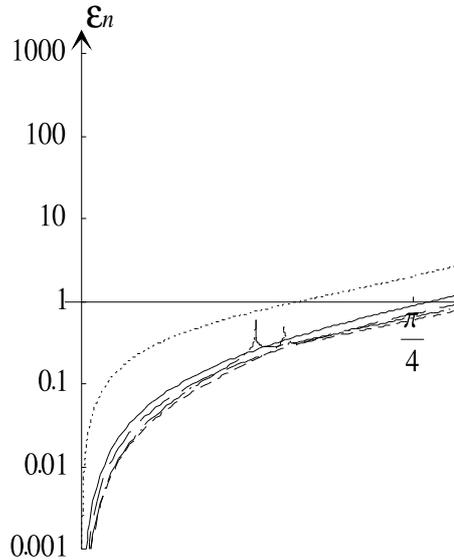


Figure 2: Logarithmic plot of the five first  $\varepsilon_n$ , reference values for the perturbative parameter  $\varepsilon$ , built from the time scaling coefficients  $r_n$ , against the angle  $\theta$  between the propagation direction and the exterior field. The value of the constants and the legend are the same as in figure 1.

duration along which it can be expected to describe the physics. However this effect is much weaker than the dependency with regard to the direction of the external field and the angle  $\theta$ .

## 4 The soliton of the hierarchy

The time scaling coefficients studied in the previous paragraph have given an insight into the convergence of the perturbative expansion as an asymptotic behaviour for small values of the perturbative parameter  $\varepsilon$ , for a fixed number  $n$  of corrective terms. We are also able to get some insight into the convergence of the series when  $n$  tends to infinity and  $\varepsilon$  is fixed, through the computation of the one soliton solution of the complete KdV hierarchy. As mentioned above, all equation of the KdV hierarchy are compatible together, in the sense that for a given initial data, a function  $v(X, T_1, T_2, T_3, \dots)$  satisfying equation (16) for any value of  $n$  can be found. This solution can be found using the inverse scattering transform (IST) method, at least in

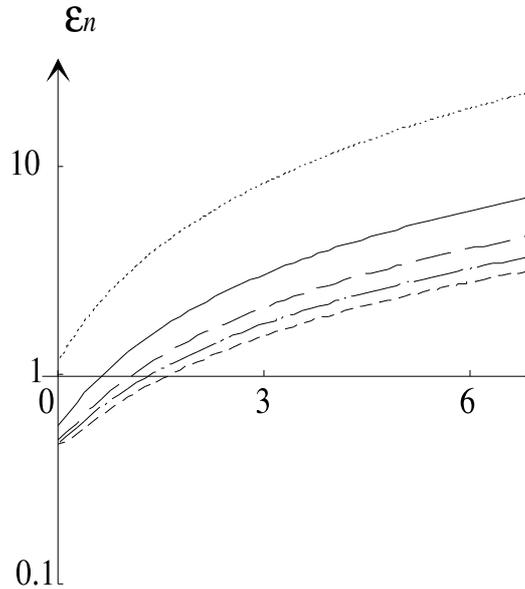


Figure 3: Logarithmic plot of  $\varepsilon_1, \dots, \varepsilon_5$ , against the parameter  $\alpha$  that determines the strength of the exterior field. The rescaled magnetic induction is  $m = 1$ , and the angle between the propagation direction and the exterior field is  $\theta = \pi/4$ . The legend is the same as in figure 1.

principle. Indeed, all equations of the hierarchy are completely integrable by means of the IST method. Furthermore, they can all be described in the IST formalism using the same spectral problem ([19], p. 96), which ensures their compatibility. The scattering data  $(R_+(k), D_{+,j}, k_j)$  (see [19], p. 141 *sq.*, for the precise definition of these quantities) are defined in the same way for all equations, only their time evolution differ for each time variable  $T_n$ . These time evolution is given by ([19], p. 149)

$$R_+(k, T_n) = R_+(k, 0)e^{\Omega_n(k)T_n}, \quad (32)$$

$$D_{+,j}(T_n) = D_{+,j}(0)e^{\Omega_{n,j}T_n}, \quad (33)$$

$$k_j(T_n) = k_j(0). \quad (34)$$

The index  $n$  refers to the  $n^{\text{th}}$  equation of the hierarchy. The evolution factors are  $\Omega_{n,j} = \Omega_n(k_j)$ , and  $\Omega_n(k) = -i\omega_j(2k)$ , where  $\omega_j(k)$  is the dispersion relation of the  $n^{\text{th}}$  equation of the hierarchy linearized. Its seen from relation (34) that the discrete spectrum  $(k_j)$  is constant with regard to any of the time variables  $T_n$ . Therefore the number of solitons and their characteristics

are not modified by the higher order time evolution. The evolution of the spectral data with regard to all the higher order time variables can then be written as a single exponential factor for each spectral component,

$$R_+(k, T_1, T_2, \dots) = R_+(k, 0, 0, \dots) \exp \left( \sum_{n=1}^{\infty} \Omega_n(k) T_n \right). \quad (35)$$

From the expression (16-17) of the equations of the hierarchy, we find that

$$\omega_n(k) = \frac{-k^{2n+1}}{4^n}. \quad (36)$$

For a value of the spectral parameter  $k$  belonging to the discrete spectrum,  $k = k_j = i\kappa_j$  with  $\kappa_j$  real, we get

$$\Omega_{n,j} = 2(-1)^{n+1} \kappa_j^{2n+1}. \quad (37)$$

Using the definition (29) of the time variable  $T_j$ , we get the following expression of the complete time evolution factor:

$$\sum_{n=1}^{\infty} \Omega_{n,j} T_n = \Omega_j t, \quad \text{with} \quad \Omega_j = \sum_{n=1}^{\infty} (2\varepsilon \kappa_j)^{2n+1} r_n. \quad (38)$$

Obviously formula (38) is valid only if the power series converges. Notice that the coefficients of the latter are the time scaling coefficients  $r_n$ . For a one-soliton solution, the above formulas show that the introduction of a sequence of higher order time variables and of all equations of the KdV hierarchy yield nothing but a renormalization of the soliton speed. This result can be also found by direct computation as follows. By definition, the one-soliton solution propagates without deformation, at least with regard to the first time variable  $T_1$ . It can thus be written under the form  $v = v(X + \lambda T_1)$ . Then using the KdV equation, *i.e.* equation (16) with  $n = 1$ , we see that  $v$  is an eigenvector of the recurrence operator  $\mathcal{L}$  defined by (17), with the eigenvalue  $\lambda$ . We deduce easily the  $T_n$ -dependency of  $v$ , it is given by  $v = v(X + \lambda^n T_n)$ . We find this way the expression of the one-soliton solution of the complete hierarchy:

$$v = 2b^2 \operatorname{sech}^2 b \left( X + \sum_{n=1}^{\infty} (-b^2)^n T_n \right), \quad (39)$$

using the normalized variables. In the case of magnetic solitons, it can be written using the physical variables as

$$\vec{H}_w = \frac{12r}{q} \vec{h}_1 \beta^2 \operatorname{sech}^2 \beta(x - \mathcal{V}t), \quad (40)$$

where

$$\mathcal{V} = V + \sum_{n=1}^{\infty} 4^n \beta^{2n} r_n. \quad (41)$$

$V$  is the velocity given by (5),  $\vec{h}_1$  the polarization vector given by (10). The wave magnetic field  $\vec{H}_w$  is related to the previously defined field components through

$$\vec{H} = \vec{H}_0 + \varepsilon^2 \vec{H}_2 + \dots \simeq \vec{H}_0 + \vec{H}_w. \quad (42)$$

The dimensional soliton parameter  $\beta$  is related to the normalized soliton parameter  $b$  through  $\beta = \varepsilon b$ . Computation of the one-soliton from the IST formalism allows to identify the soliton parameter  $b$  to the single discrete eigenvalue  $\kappa_1$ . This way we check that the relative soliton velocity  $(\mathcal{V} - V)$  given by (41) is equal to  $\Omega_1/(2\beta)$ , using the expression (38) of the evolution factor  $\Omega_1$ .

The soliton speed is thus given by a power series of the soliton parameter  $\beta$ , whose coefficients are essentially the time scaling coefficients  $(r_n)_{n \geq 1}$ . Obviously if this series diverges so does the whole perturbative scheme. Reciprocally, the convergence of the series defining the velocity should favour that of the perturbative scheme, although the latter is by no means proven. Writing the power series which defines  $\mathcal{V}$  as

$$\mathcal{V} = V + \sum_{n=1}^{\infty} \left( \frac{\beta}{\varepsilon_n} \right)^{2n}, \quad (43)$$

we see that it converges when

$$\beta < \beta_M = \liminf_{n \rightarrow \infty} \varepsilon_n \quad (44)$$

and diverges for larger values of the soliton parameter  $\beta$ . Therefore the limit of the sequence  $\varepsilon_n$  for large  $n$  gives us a maximal value  $\beta_M$  of the soliton parameter  $\beta$ , above which we know that the perturbative scheme does not converge when increasing the number of terms. Physically, this lack

of convergence means that the KdV soliton will be destroyed by some effects which cannot be taken into account using the perturbative approach.

For values of the soliton parameter  $\beta$  below the limit  $\beta_M$ , we get a renormalized soliton speed, *a priori* valid for an infinite propagation time. The boundness of all terms in the perturbative scheme proves that for a given propagation time and a given number of terms, this soliton gives a good approximation of the real impulsion for small enough values of  $\varepsilon$ , *i.e.* of  $\beta$ . We can reasonably conjecture that small enough can be understood here as less than the limiting value  $\beta_M$  of  $\beta$ . Physically it means that magnetic KdV solitons with parameter smaller than  $\beta_M$  should conserve their properties during a long propagation time.

According to figures 2 and 3, the  $\varepsilon_n$ , thus also their limit  $\beta_M$ , depend on the physical parameters, and specially on the angle  $\theta$  between the propagation direction and the applied field. An example of computation showing the convergence of the velocity series is drawn on figure 4 as a function of this angle. It is seen that, for  $\theta$  close to  $\pi/2$ , the first approximation (KdV) gives almost the exact speed, while for small angles the series diverges. We denote by  $\theta_M$  the value of  $\theta$  for which  $\beta_M$  is equal to the fixed value of  $\beta$ . When  $\theta < \theta_M$ , the series does not converge, and the whole perturbative approach is not valid. To compute  $\theta_M$  for the figure 4, we have approximated  $\beta_M$  by  $\varepsilon_5$ . When  $\theta > \theta_M$ , if we consider only the soliton speed, the KdV approximation will correctly describe the wave evolution. More precisely, the KdV equation itself will give an acceptable description above some value  $\theta_t$  of the angle  $\theta$ , while this first order approximation needs to be corrected by higher order terms below  $\theta_t$  (notice that the threshold value  $\theta_M$  is precisely defined, while  $\theta_t$  is only an order of magnitude depending on the accuracy required). It is reasonable to think that the same kind of conclusion holds in a more general situation, involving several solitons and radiation.

## 5 Conclusion

The multiple time formalism has been applied to the study of the propagation of KdV solitons in ferromagnetic media. According to this formalism, the dependency of the higher order terms with respect to the first order time variable is given by linearized KdV equations, while the dependency of the main term with regard to the higher order time variables is governed by all equations of the KdV Hierarchy. The latter are determined by the

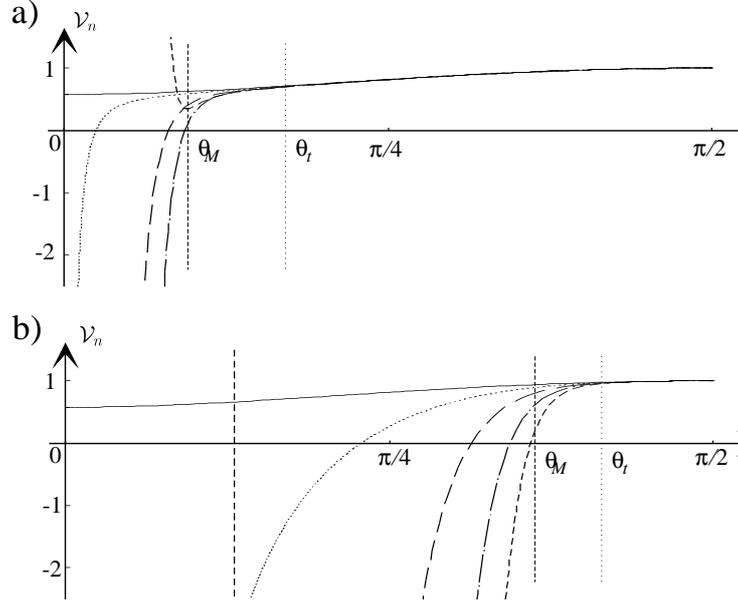


Figure 4: Plot of the five first approximate values  $\mathcal{V}_n = V + \sum_{p=1}^n (\beta/\varepsilon_n)^{2p}$  of the soliton velocity  $\mathcal{V}$ , against the angle  $\theta$  between the propagation direction and the exterior field. The rescaled magnetic induction is  $m = 1$ , and the parameter determining the strength of the exterior field is  $\alpha = 0.5$ . Solid line:  $n = 0$ , dotted line:  $n = 1$ , large dashed:  $n = 2$ , dashed-dotted line:  $n = 3$ , short dashed:  $n = 4$ . For a soliton parameter  $\beta = 0.1$  (a),  $\beta = 1.5$  (b).

requirement that the linear terms in the rhs of the linearized KdV equation vanish. This yields scaling coefficients for the higher order time variables of the KdV Hierarchy, which contain most physical information concerning the wave evolution. Explicit computation of these scaling coefficients shows in particular that the approximation yielded by the KdV model gives a good account for the physical behavior of the wave during long propagation times when the angle between the propagation direction and the external field is large enough. The time during which the pulse is correctly described by the KdV equation falls to zero when they are parallel. Mathematically, the perturbative parameter  $\varepsilon$  is infinitely small, while it takes a finite value in a physical situation. The approximation is valid only if this finite value is small enough. The corresponding range of the perturbative parameter  $\varepsilon$  is usually determined in a rather empirical way. The present study gives some theoretical insight into this question, through a physical interpretation of the

time scaling coefficients.

The one-soliton solution of the complete KdV hierarchy has been written down as a function of the physical parameters. The soliton velocity writes as a power series of the soliton parameter, involving the sequence of the time scaling coefficients. We get a maximum value of the soliton parameter, above which the perturbative series diverges. Then the KdV approximation, even with corrective terms, does not describe correctly the physics. If the soliton parameter is below the threshold, the long-distance effect of the higher order corrections is only a modification of the soliton speed, and the physical system behaves qualitatively as the KdV model. It has been observed that the validity domain of KdV-type asymptotics is often much larger than predicted by the mathematical analysis. The above conclusions can partially explain this observation: the KdV-type behaviour is qualitatively correct in the whole validity domain of the infinite KdV hierarchy expansion, which is expected to be much larger.

# Appendix

We list in this appendix the formulas needed for the computation of the time scaling coefficients  $r_n$ . These formulas are proven in [12].  $r_n$  is given by equation (23) with

$$\begin{aligned} \Xi((\alpha_j)_{j \geq 1}, k) &= \frac{-1}{\Lambda} \left[ V \vec{m} \cdot \vec{m}((\alpha_j)_{j \geq 1}, k-1) \right. \\ &\quad \left. - \sum_{i \geq 1} \vec{m} \cdot \vec{m}((\alpha_j - \delta_{i,j})_{j \geq 1}, k-1) \right], \end{aligned} \quad (45)$$

where

$$\vec{m} = \begin{pmatrix} m_x \\ m_t \\ 0 \end{pmatrix}. \quad (46)$$

$\vec{m}((\alpha_j), k)$  is deduced according to

$$\tilde{u}((\alpha_j)_{j \geq 1}, k, l) = \begin{pmatrix} \vec{e}((\alpha_j)_{j \geq 1}, k, l) \\ \vec{h}((\alpha_j)_{j \geq 1}, k, l) \\ \vec{m}((\alpha_j)_{j \geq 1}, k, l) \end{pmatrix}, \quad (47)$$

from the following recurrence formulas. For all  $l \geq 1$ ,

$$\tilde{u}((0), 0, l) = \tilde{u}_1. \quad (48)$$

For all  $k$  and  $l \geq 1$ ,

$$\tilde{u}((0), k, l) = S(V \vec{m}((0), k-1, l)). \quad (49)$$

For all  $(\alpha_j)_{j \geq 1} \neq (0)$  and  $l \geq 1$ ,

$$\begin{aligned} \tilde{u}((\alpha_j)_{j \geq 1}, 0, l) &= \\ &\sum_{i \geq 1} \Phi(\tilde{u}((\alpha_j - \delta_{i,j})_{j \geq 1}, 0, l)) \cdot \end{aligned} \quad (50)$$

For all  $(\alpha_j)_{j \geq 1} \neq (0)$ ,  $k, l \geq 0$ ,

$$\begin{aligned} \tilde{u}((\alpha_j)_{j \geq 1}, k, l) &= \\ &S(V \vec{m}((\alpha_j)_{j \geq 1}, k-1, l)) \\ &+ \sum_{i \geq 1} \left[ \Phi(\tilde{u}((\alpha_j - \delta_{i,j})_{j \geq 1}, 0, l)) \right. \\ &\quad \left. - S(\vec{m}((\alpha_j - \delta_{i,j})_{j \geq 1}, k-1, l)) \right]. \end{aligned} \quad (51)$$

In (48-51),  $\delta_{i,j}$  is the Kronecker symbol,  $V$  is given by (5).  $\Phi$  is defined by

$$\Phi(u) = S(\alpha \vec{m} \wedge \vec{m}_I(u)) + u_I(u), \quad (52)$$

where  $S$  is the  $9 \times 3$  matrix

$$S = TL^{-1} \quad \text{with} \quad T = \begin{pmatrix} -\frac{1}{V}R_x \\ I \\ -\Gamma \end{pmatrix}. \quad (53)$$

$I$  is the three-dimensional unity matrix, and

$$L^{-1} = \frac{1}{\mu m_x m_t} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m_t \\ m_x & 0 & 0 \end{pmatrix}. \quad (54)$$

$u_I$  is the linear operator in  $\mathbb{R}^9$  defined by

$$u_I \left( \begin{pmatrix} \vec{E} \\ \vec{H} \\ \vec{M} \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{V}\vec{E} \\ 0 \\ \vec{m}_I \left( \begin{pmatrix} \vec{E} \\ \vec{H} \\ \vec{M} \end{pmatrix} \right) \end{pmatrix}, \quad (55)$$

with

$$\vec{m}_I \left( \begin{pmatrix} \vec{E} \\ \vec{H} \\ \vec{M} \end{pmatrix} \right) = \frac{1}{V}(\vec{H} + \vec{M}) + \frac{1}{V^2}R_x \vec{E} \quad (56)$$

$R_x$  is the  $3 \times 3$  matrix

$$R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad (57)$$

The first term of the sequence  $\tilde{u}((\alpha_j)_{j \geq 1}, k, l)$  is given by  $\tilde{u}_1 = T\vec{h}_1$ , where  $\vec{h}_1$  is the polarization vector defined by (10), which also reads

$$\vec{h}_1 = \begin{pmatrix} \mu m_x \\ (1 + \alpha)m_t \\ 0 \end{pmatrix}. \quad (58)$$

The quantity  $\Lambda$  in (45) is given by

$$\Lambda = V \vec{m} \cdot \vec{\Phi}_m(\tilde{u}_1), \quad (59)$$

where  $\Phi_m$  is the  $m$ -component of  $\Phi$  defined by (52), according to

$$\Phi = \begin{pmatrix} \vec{\Phi}_e \\ \vec{\Phi}_h \\ \vec{\Phi}_m \end{pmatrix}. \quad (60)$$

We use the shortcuts

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad m_x = m \cos \theta, \quad (61)$$

and  $\gamma$ ,  $\mu$ ,  $m_t$  given by (26). The expression (5) of the velocity yields the relation

$$\mu m_x^2 + \gamma(1 + \alpha)m_t^2 = 0, \quad (62)$$

which is useful to simplify the expressions.

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