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Blow-up Surfaces
for
Nonlinear Wave Equations, I¹

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Abstract

We introduce a systematic procedure for reducing nonlinear wave equations to characteristic problems of Fuchsian type. This reduction is combined with an existence theorem to produce solutions blowing up on a prescribed hypersurface. This first part develops the procedure on the example $\square u = \exp(u)$; we find necessary and sufficient conditions for the existence of a solution of the form $\ln(2/\phi^2) + v$, where $\{\phi = 0\}$ is the blow-up surface, and v is analytic. This gives a natural way of continuing solutions after blow-up.

1. INTRODUCTION.

1.1. Results. We prove in this paper that if Σ is given in a neighborhood of $(x_0, t_0) \in \mathbf{R}^n \times \mathbf{R}$ by the equation $\phi(x, t) = 0$ where ϕ is analytic and $\nabla\phi \neq 0$

¹Comm. PDE, **18**, (3&4) 431–452 (1993).

then

$$u_{tt} - \Delta u = e^u \tag{1}$$

has a solution of the form

$$u = \ln(2/\phi^2) + v(x, t), \tag{2}$$

where v is holomorphic near (x_0, t_0) , if and only if the blow-up surface Σ satisfies:

$$\Sigma \text{ is space-like} \tag{3}$$

and

$$\begin{aligned} \Sigma \text{ has zero scalar curvature for the metric} \\ \text{induced by the Minkowski metric.} \end{aligned} \tag{4}$$

Such solutions are uniquely determined by the choice of one analytic function on Σ . (Th. 2, section 3).

This result is a consequence of the reduction of (1) to a Fuchsian equation:

$$\begin{cases} Tw_T + Aw = Tf(T, x, w, D_x w) \\ w(0) = w_0 \end{cases} \tag{5}$$

where w is a vector-valued function related to v , A is a constant matrix, and w_0 satisfies appropriate conditions. (see section 3). An existence theorem for such problems is given in Th. 1 (section 2).

REMARK 1. Motivation and previous results on this question are discussed in §1.2.

REMARK 2. The solution in the form (2) is defined on both sides of Σ . Further, if ϕ is given, $u - \ln(2/\phi^2)$ can be defined everywhere from its values on one side of Σ only, by analytic continuation. We therefore have a cogent way of continuing singular solutions beyond their blow-up set. The only arbitrariness lies in the choice of continuation of the singular term; we have chosen the most obvious one, and will discuss it elsewhere.

REMARK 3. It is easily checked on examples that the singular set for nonlinear wave equations can be entirely non-characteristic. However, in this particular case, the blow-up surface *is* characteristic for the Fuchsian equation associated with the equation and the given surface.

REMARK 4. The case when the curvature condition (4) is not satisfied, as well as more general nonlinearities, will be dealt with in Part II.

1.2. Literature. We list here a number of special cases and weaker versions of the above results that can be found in the literature.

a) *Exact Solutions.* Liouville [11] proved that all analytic solutions of

$$u_{xt} = e^u \tag{6}$$

are locally of the form

$$u = \ln \frac{2f'(x)g'(t)}{(f(x) + g(t))^2}, \tag{7}$$

where f and g are analytic functions; conversely, (7) gives a solution of (6) for all choices of f and g , which is valid in the maximal domain in which it makes sense. This formula, on which the results of §1.1 are easily checked, was the initial motivation for the present work. Equation (7) can also be recovered from the fact that (6) is connected to $u_{xt} = 0$ by a Bäcklund transformation.

Equation (6) has received some recent attention in view of possible applications to quantum field theory.

We note the following points, which may not be new:

1. (6) has complete solutions.² (It is known that complete solutions that are bounded below, with $u_x + u_t \geq \beta > 0$ do not exist (J. Keller [9])).

²That is, solutions which are defined and free of singularities in the whole x - t plane.

2. There is a complex-valued solution with an isolated branch point, namely

$$\ln \frac{2i}{(x + it)^2}.$$

The singularity of this solution does not “propagate.” One can think of it as single-valued solution defined on $\mathbf{R}^2 \setminus \{(0,0)\}$, with values in $\mathbf{C}/2\pi\mathbf{Z}$.

Motivated by various applications to superfluidity and superconductivity, among others, Grunland and Tuszynski [4] have obtained a number of exact solutions for equations $\square u = F(u)$, where F is either a polynomial, or $\sinh u$ or $\cosh u$, in low dimensions, using symmetry techniques. The singular behavior exhibited by these solutions corresponds to those found by the present methods.

b) *Expansion Techniques.* In their generalization of the Painlevé test, for the integrability of a given system by the Inverse Scattering Transform, Weiss, Tabor and Carnevale (WTC) [14] have considered solutions of soliton problems in the form

$$u = \sum_{k \geq 0} u_k(x, t) \phi(x, t)^{k-\nu} \quad (8)$$

where ν is required to be an integer. The given equation passes the test if such a formal series solves it, without constraint on ϕ ; convergence questions are not addressed. “When the Ansatz [(8)] is correct, the PDE is said to possess the Painlevé property and is conjectured to be integrable” [13]. The problem is said to be “conditionally integrable” if only special ϕ are allowed. Most of the equations known to be integrable by Inverse Scattering are also known to pass the test. In that case, with a little hindsight, one can often relate ϕ to the eigenvalue problem associated with the equation. A general explanation of the many connections unraveled by this method, and a proof of the validity of the test in all cases, are not known.

Weiss [13] obtained by this procedure the curvature condition for complex-valued solutions of the sine-Gordon equation in n dimensions; $\exp(iu)$ then possesses an expansion of the form (8). The question of the convergence of this expansion is not addressed.

We will come back, in the subsequent parts of this paper, on the application of the present technique to the WTC method.

Kobayashi and Nakamura [7] show that, in a suitable Sobolev space, a solution to hyperbolic equations of the form $Lu = R(u, Du)$, with R rational, L linear, can be found near a prescribed blow-up surface provided that a formal solution $(\phi + i0)^{-k/p} \sum_{l \geq 0} u_l (\phi + i0)^{l/p}$ can be *uniquely constructed to all orders*. Such a series, however, does not exist in general, and the appropriate statement will be considered in subsequent parts of this paper.

c) *Fuchsian Equations.* The existence theorem for nonlinear Fuchsian equations given in this paper follows essentially the argument of Baouendi and Goulaouic [1a,b], and is in the spirit of Ovsjannikov's version of the Cauchy-Kowalewska theorem. The nonlinear result also follows from a little-known paper of Rosenbloom [12], who uses a majorant method.

d) *Blow-up time and blow-up boundary.* Although a number of sufficient conditions for the occurrence of blow-up are known, very few results on the mechanism of blow-up are available. Caffarelli and Friedman [2,3] show that, for equations modeled on $\square u = u^p$, smooth initial data with appropriate bounds lead to a blow-up set of class C^1 in one space dimension; the result extends, with additional hypotheses, to 2 and 3 space dimensions. Estimates on the time where a singularity first occurs ("blow-up time") are found in Hörmander [5], John [6] and Lindblad [10] (see these papers for more references); they consider the Cauchy problem with initial data $O(\varepsilon)$ and find the order of the blow-up time as $\varepsilon \rightarrow 0$.

2. FUCHSIAN EQUATIONS.

2.1. The problem. We consider the construction of solutions to

$$(t\partial_t + A(x, t))u = tf(t, x, u, D_x u), \quad (9)$$

where x belongs to an open set in \mathbf{R}^n and $t \in \mathbf{R}$. u is throughout this section a vector-valued function, and A is a square matrix of corresponding size. We often write D for D_x .

We show that this problem has one and only one solution holomorphic near $x = 0, t = 0$, if A and f are holomorphic near $(0, 0)$ and $(0, 0, 0, 0)$ respectively and

$$j + A(x, t) \text{ is invertible for every } j = 0, 1, 2, \dots$$

If the last condition is satisfied only for large enough values of j , we show that a solution exists if and only if f and A satisfy finitely many constraints, in which case the solution depends on finitely many arbitrary functions.

Our result is summarized as follows.

Theorem 1. *The Fuchsian problem (9) has a holomorphic solution near $(0, 0)$ if and only if it has a formal solution.*

The rest of §2 is organized as follows. After a brief study of formal solutions of (9) of the form $\sum_{j \geq 0} u_j t^j$ in §2.2, we reduce the situation to the case when all the eigenvalues of A have positive real parts (§2.3). In §2.4, we give an inverse for $t\partial_t + A$ and prove simple estimates. We finally define, for the case when all the eigenvalues of A have positive real parts, an iteration procedure to find the unique holomorphic solution of (9), and prove its convergence. This will complete the proof of Theorem 1.

2.2. Structure of formal solutions. We consider formal series $u = \sum_{j \geq 0} u_j(x) t^j$ which solve (9). The functions u_j are holomorphic in x . For any

such expression r , we denote by $\{r\}_j$ the coefficient of t^j in the expansion of r in increasing powers of t . Insertion of the series into (9) readily gives $Au_0 = 0$ and, for $j \geq 1$, taking $A = A(x)$ as we may (see §2.3), we find

$$(j + A)u_j = \{f(x, t, u, D_x u)\}_{j-1}.$$

The r.h.s. is computed using the expansion of f ; it depends only on u_0, \dots, u_{j-1} . In particular,

$$\{f(x, t, v, D_x v)\}_{j-1} = \{f(x, t, u, D_x u)\}_{j-1}$$

if $(u - v) = O(t^j)$. The above equation for u_j can be solved uniquely if and only if $(j + A)$ is invertible. If it is not, u_0, \dots, u_{j-1} must satisfy conditions, and u_j involves arbitrary functions of x . These functions may, in turn, be constrained by other equations arising from the calculation of some u_k , $k > j$. We may summarize these facts as follows.

Lemma. *The existence of a formal solution of (9) in increasing powers of t can be ascertained in a finite number of operations, and the formal solution involves a finite number of arbitrary functions if it exists. The coefficient u_j is uniquely determined from u_0, \dots, u_{j-1} if $(j + A)$ is invertible.*

We assume in the rest of §2 that a formal solution U has been constructed, and proceed to prove its convergence. It may happen that *no* holomorphic solution exists, as in the case of equation

$$(t\partial_t - 1)u = t.$$

2.3. Preliminary reduction. We prove here that one may assume

1. $A = A(x)$,
2. All the eigenvalues of A have real parts greater than 0.

The first property is clear: $(A - A(x, 0))$ can be divided by t , and can therefore be lumped into the right-hand side of (9).

As for the second one, let us define v by

$$u = U_0(x) + U_1(x)t + \cdots + U_m(x)t^m + v(x, t)t^m,$$

where U_0, \dots, U_m are coefficients of the given formal solution, v is assumed to be holomorphic, and u is the putative holomorphic solution.

Let us write the equation satisfied by v . We find, using the fact that A is now independent of t ,

$$mt^m v + t^m(t\partial_t + A(x))v + \sum_{j=0}^m (j + A)U_j t^j - tf = 0.$$

But from §2.2, we know that $(j + A)U_j$ is the coefficient of t^{j-1} in the expansion of $f(x, t, U, DU)$, which coincides with

$$\{f(x, t, u, Du)\}_{j-1},$$

for $j \leq m$, since u formally agrees with the formal solution upto order $m - 1$ inclusive. Therefore,

$$\sum_{j=0}^m (j + A)U_j t^j - tf(x, t, u, Du) = t^{m+1}F(x, t, v, Dv),$$

where F is analytic in x, t, v, Dv . The function v therefore satisfies an equation of the same type as u , with A replaced by $A + m$. Taking m large enough proves the claim.

We assume that A is independent of t and that all the eigenvalues of A have real parts greater than 0 in what follows. Also, by introducing the components of Du as additional dependent variables, one may in a standard fashion assume f is linear in Du .

2.4. Inverse of $t\partial_t + A$. We consider the equation

$$(t\partial_t + A)u = tg(t)$$

where A is independent of t . It has one solution holomorphic in t if the eigenvalues of A have real parts greater than 0, namely

$$u = Hg := \int_0^1 t\sigma^A g(\sigma t) d\sigma, \quad (10)$$

with $\sigma^A = \exp(A \ln \sigma)$. u is holomorphic in t if g is. If g depends analytically on x as well, so does u . For $t \leq 1$ (and every fixed x), we have, for some $C_1 > 0$,

$$|u(t)| + |Au| + |t\partial_t u| + |\partial_t(tu)| \leq C_1 \sup_{[0,t]} |g(s)|. \quad (11)$$

and

$$|(t\partial_t + 1)Hg| \leq C_1 \sup_{[0,t]} |g(s)|. \quad (12)$$

2.5. Iteration. We assume $|t| \leq 1$ throughout. We define a sequence of analytic functions $\{v_j(x, t)\}_{j \geq 0}$ by $v_0 = 0$ and

$$v_{j+1} = f(t, x, Hv_j, DHv_j) := F[t, Hv_j] \quad (13)$$

for $j \geq 0$. D is short for D_x . Recall that f is now linear in Du .

We use the following two norms. The first is defined on functions of x :

$$\|u\|_s := \sup\{|u(x)| : x \in \mathbf{C}^n, d(x, \Omega) < s\} \quad (14)$$

where Ω is a small ball about 0 in \mathbf{R}^n so that f is holomorphic for $d(x, \Omega) < 2s_0$ and $|u| < 2R$. The second is on functions of x and t :

$$|u|_a := \sup_{0 < s < s_0, |t| < a(s_0 - s)} \left\{ \|u(\cdot, t)\|_s (s_0 - s) \left[1 - \frac{|t|}{a(s_0 - s)}\right]^{1/2} \right\}. \quad (15)$$

We prove that G defined by

$$G[u] := F[t, Hu] \quad (16)$$

is a contraction on a set of the form $\{|u|_{2a} < \rho/a\}$ for a sufficiently small. It will follow that (i) $\|Hv_j\|_s < R$ for all j and for $|t| < a(s_0 - s)$, making the

iteration possible, (ii) the iteration converges and provides the desired solution. The procedure being very similar to that of [1b], we merely outline the steps.

The basic estimate is the following.

Lemma. *There is an $a_0 > 0$ such that if $a < a_0$ and $j \geq 1$, then all the v_j are well-defined for $|t| < a(s_0 - s)$ and*

$$|v_{j+1} - v_j|_a \leq \frac{1}{2}|v_j - v_{j-1}|_a. \quad (17)$$

PROOF. It suffices to consider $t \geq 0$. The letter C denotes various constants independent of s, j and a . Writing $F[t, u] = a(t, v)Dv + b(t, v)$, we find, using Cauchy's inequality, that if $\|u\|_s(t) < R$ and $\|v\|_s(t) < R$, then for $0 < s' < s$,

$$\begin{aligned} \|F[t, u] - F[t, v]\|_{s'} &\leq \|a(t, u) - a(t, v)\|_{s'} \|Du\|_{s'} + \|b(t, u) - b(t, v)\|_{s'} \\ &\quad + \|a(t, v)\|_{s'} \|Du - Dv\|_{s'} \end{aligned} \quad (18)$$

$$\leq C\|u - v\|_s / (s - s'). \quad (19)$$

Next, one finds, by inspection, that if $t < a(s_0 - s)$, then

$$\|Hu(t)\|_s \leq \int_0^1 \|u(\tau)\|_s d\tau \leq Ka|u|_a \quad (20)$$

for some $K > 0$.

Furthermore, for

$$|u|_a < R/(2Ka) \text{ and } |v|_{2a} < R/(4Ka), \quad (21)$$

we have, for $t < a(s_0 - s)$,

$$\|G[u] - G[v]\|_s \leq C \int_0^t \frac{\|u(\tau) - v(\tau)\|_{s(\tau)}}{s(\tau) - s} d\tau \quad (22)$$

where

$$s(\tau) = (s + s_0 - \tau/a)/2$$

(see [1b, Lemma 3], where we may take $\varepsilon = 1$). Substituting this value of $s(\tau)$ and estimating $\|u(\tau) - v(\tau)\|_{s(\tau)}$ in terms of $|u - v|_a$, we find

$$|G(u) - G(v)|_a \leq Ca|u - v|_a. \quad (23)$$

Since $G(0)$ is a known analytic function, we also have for any b ,

$$|G(u)|_b \leq Cb|u|_b + C. \quad (24)$$

Therefore, the set $\{|u|_{2a} < R/(4Ka)\}$ is mapped into itself by G , and G defines on this set a contraction in the norm $|\cdot|_a$, provided a is taken sufficiently small. The sequence $\{v_j\}$ therefore converges for this norm to a function analytic for $t < a(s_0 - s)$. Applying H to this limit gives the solution to the Fuchsian equation under consideration.

This ends the proof of Theorem 1.

REMARK 5. Theorem 1 contains the Cauchy-Kowalewska Theorem as a special case: for instance, if $u = \sum_{j \geq 0} u_j t^j$ solves a first-order system of Cauchy-Kowalewska type, $t^{-1}(u - u_0)$ solves a Fuchsian equation and we recover the fact that the solution contains one arbitrary function.

3. REDUCTION TO A FUCHSIAN PROBLEM.

We prove in this section that the search for solutions of (1) of the form (2) is equivalent to a Fuchsian problem of the type studied in the previous section. The existence of a formal solution is equivalent to the conditions (3) and (4), and if they are satisfied, a solution exists for each choice of a holomorphic function on Σ .³

To see this, we will first take ϕ as new time variable, and show one can assume that $\phi = t - \psi(x)$. Apart from convenience in the calculation, this assumption has the effect of associating to every singular solution a *unique* expansion of the form (2).

Let us show that such a choice is licit. Let us substitute $\ln(2/\phi^2) + v(x, t)$ for u into (1), and multiply the result by ϕ^2 . A short calculation reveals that if we let ϕ tend to zero in the result, we are left with the equation

$$\sum_{i=1}^n \phi_{x^i}^2 - \phi_t^2 = -\exp(v) \tag{25}$$

on Σ . This proves

1. Σ is space-like,
2. $\phi_t \neq 0$ on Σ .

Σ therefore has locally an equation $t = \psi(x)$, and it is easily verified that $\phi/(t - \psi(x))$ is analytic and non-zero. We may thus replace ϕ by $t - \psi(x)$ at the expense of changing v .

We assume from now on that $\phi(x, t) = t - \psi(x)$.

3.1. Change of variables. Let $x^0 = t$ and introduce new variables X^0, \dots, X^n by

$$X^0 = T := \phi(x, t), \quad X^i = x^i \quad (i = 1, \dots, n). \tag{26}$$

³Some of the formulae in this section are somewhat lengthy to derive, and the necessary details have been grouped in the Appendix, for the convenience of the reader.

The derivatives transform according to

$$\partial_a = L_a^{a'} \partial_{a'}, \quad (27)$$

where the summation convention is used,

$$\partial_a = \partial/\partial x^a, \quad \partial_{a'} = \partial/\partial X^{a'},$$

and a, a' run from 0 to n , while i, i' and *all* other Latin or Greek indices run from 1 to n . Clearly, $L_0^{0'} = \phi_t$, $L_i^{0'} = \phi_{x^i} = \phi_i$, $L_0^{i'} = 0$ and $L_i^{i'} = \delta_i^{i'}$, where δ is the Kronecker symbol. Therefore, (1) becomes

$$e^u = [(\phi_t \partial_T)^2 - \sum_i (\delta_i^{i'} \partial_{i'} + \phi_i \partial_T)^2] u. \quad (28)$$

REMARK 6. We use throughout the summation convention on repeated indices *one of which is in the upper position and the other in the lower position*. Thus, there is summation in $a^i b_i$ but not in $a_i b_i$. More generally, all indices are raised and lowered using the Kronecker delta. We let $\psi_i = \partial_i \psi$, $\psi_{ij} = \partial_{ij} \psi \dots$, and $\psi^i = \delta^{ik} \psi_k$, $\psi^{ij} = \delta^{ik} \delta^{jl} \psi_{kl} \dots$. In particular, $\sum_{1 \leq i \leq n} (\psi_i)^2 = \psi^i \psi_i$. This convention enables us to keep track of terms in lengthy calculations such as that of §A.3.

3.2. Expansion. We seek a solution in the form

$$\ln \frac{2}{\phi^2} + v(X, T),$$

where $v = \sum_{k \geq 0} v_k(X) T^k$.

Let us write

$$u = \ln(2/\phi^2) + v_0(X) + v_1(X)T + w(X, T)T^2.$$

Substitution into (28) leads to the following equation

$$\{T^2((1 - |D\psi|^2)\partial_T^2 + 2\psi^i \delta_i^{i'} \partial_{i'T} - \Delta' + (\Delta\psi)\partial_T)w$$

$$\begin{aligned}
& + T(4\psi^i \delta_i^{i'} \partial_{i'} w + 2(\Delta\psi)w) + (1 - |D\psi|^2)(2w + 4Tw_T) \} \\
& + \{2\psi^i \delta_i^{i'} \partial_{i'} v_1 - T\Delta' v_1 + v_1 \Delta\psi\} - \Delta' v_0 - \frac{2}{T} \Delta\psi \\
= & \frac{2}{T^2} \{e^{v_0}(1 + Tv_1 + T^2(w + v_1^2/2) + T^3 g(X, T, w)) - (1 - |D\psi|^2)\}. \quad (29)
\end{aligned}$$

In this equation, Δ (resp. Δ') stands for the Laplacian in the x^i (resp. $X^{i'}$) variables, and subscripts stand for derivatives. $|D\psi|^2$ stands for $\sum_i \psi_i^2$. We have used the fact that for every i ,

$$(\delta_i^{i'} \partial_{i'} + \phi_i \partial_T) \psi_i = \partial_{ii} \psi.$$

Note that Δ and Δ' have the same effect on functions of X alone. The function g is holomorphic in its arguments, and involves v_0 and v_1 as well.

3.3. The curvature condition. Identifying like powers of T will enable us in this section to

1. Compute v_0 and v_1 .
2. Derive a condition on ϕ , and show that $w(X, 0)$ can be chosen arbitrarily.

Once this has been done, one can recursively compute all the coefficients in the expansion of w in powers of T . We will not go through this here, since we will show directly the existence of w using the result of §2.

The coefficient of T^{-2} in (29) gives:

$$v_0 = \ln(1 - |D\psi|^2). \quad (30)$$

The coefficient of T^{-1} gives:

$$v_1 = -\Delta\psi / (1 - |D\psi|^2). \quad (31)$$

The coefficient of T^0 does not contain w and its vanishing implies a constraint on the blow-up surface. After some calculation, the condition reduces to

$$(1 - |D\psi|^2)[(\Delta\psi)^2 - \psi^{ij} \psi_{ij}] - 2\psi^{ij} \psi_j [\psi_{ik} \psi^k - \psi_i \Delta\psi] = 0. \quad (32)$$

For $n = 1$, this is an identity.

For $n = 2$, it reduces to the equation $\psi_{11}\psi_{22} - \psi_{12}^2 = 0$,

There is a simple geometric meaning to relation (32): Σ is endowed with a Riemannian metric from the ambient Minkowski space (with metric $|dx|^2 - dt^2$); using the x 's as coordinates on Σ , we can find (see §A.3) the induced metric tensor,

$$g_{ij} = \delta_{ij} - \psi_i\psi_j,$$

the corresponding Christoffel symbols,

$$\Gamma_{ij}^k = -\frac{\psi^k\psi_{ij}}{1 - |D\psi|^2},$$

and the curvature tensor,

$$\begin{aligned} R_{ijk}{}^l &= (\psi_{jk}\psi_i{}^l - \psi_{ik}\psi_j{}^l)/(1 - |D\psi|^2) \\ &\quad + (\psi_{jk}\psi_{mi} - \psi_{ik}\psi_{mj})\psi^m\psi^l/(1 - |D\psi|^2)^2. \end{aligned} \quad (33)$$

It follows that the l.h.s. of (32) equals $-(1 - |D\psi|^2)^2 R$, where $R = g^{ik}R_{ijk}{}^j$ is the scalar curvature.

This proves the curvature condition on the blow-up surface.

3.4. Fuchsian equation for w . If the curvature condition holds, there is no formal obstruction to the existence of a formal series for w .

After some reduction, w turns out to solve an equation of the form

$$\begin{aligned} \{(1 - |D\psi|^2)T\partial_T(T\partial_T + 3) - T^2\Delta'\}w = \\ Tg_1(X, T, w) + Th^{i'}(X, T, w)\partial_{i'}w + T^2h^0(X, T, w)w_T. \end{aligned} \quad (34)$$

Recall that $\partial_{i'} = \partial/\partial X^{i'}$.

Let

$$z = \begin{pmatrix} w \\ T\partial_T w \\ T\partial_X w \end{pmatrix}.$$

Eq. (34) then reads

$$\begin{aligned}
T\partial_T z_1 &= z_2; \\
T\partial_T z_2 + 3z_2 &= Tg_1(X, T, z_1) + Th^{i'}(X, T, z_1)\partial_{i'}z_1 + Th^0(X, T, z_1)z_2 \\
&\quad + T\delta^{ii'}\partial_{i'}z_{2+i}/(1 - |D\psi|^2); \\
T\partial_T z_{2+i} &= T(\partial_{i'}z_1 + \partial_{i'}z_2),
\end{aligned}$$

which has the form studied in §2 where A is the $(n+2) \times (n+2)$ matrix

$$\begin{pmatrix}
0 & -1 & 0 & \cdots & 0 \\
0 & 3 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}.$$

Its eigenvalues are 0 and 3. The kernel of A is $n+1$ -dimensional. However, the only relevant case is the one in which the last $n+1$ components of z vanish for $T=0$. Therefore, only one arbitrary function which, in our case, is $w(X, 0)$, is allowed in the formal solution. Conversely, any solution of this system satisfies

$$z_2 = T\partial_T z_1,$$

and

$$T\partial_T(z_{2+i} - T\partial_{i'}z_1) = 0$$

and therefore, any solution with $z_k = 0$ initially for $k \geq 1$ gives rise to a solution of (34). We have proved:

Theorem 2. *Equation (1) has a solution of the form (2) if and only if Σ is space-like with zero curvature for the induced metric. The solution is uniquely determined by one holomorphic function on Σ .*

We have proved the result described in §1.1.

APPENDIX.

We give in this section some of the calculations that were omitted in the text.

A.1. Change of variables. If $X^0 = T = \phi(x, t)$,

$$\partial_t = \phi_t \partial_T, \quad \partial_i = \delta_i^{i'} \partial_{i'} + \phi_i \partial_T,$$

and

$$\square u = \left((\phi_t \partial_T)^2 - \sum_i (\delta_i^{i'} \partial_{i'} + \phi_i \partial_T)^2 \right) u,$$

so that, if $u = \ln(2/\phi^2) + v$, $e^u = (2/T^2)e^v$ and

$$\begin{aligned} \square(\ln(2/T^2) + v) &= \square v + \phi_t \partial_T(-2\phi_t/T) - \sum_i \partial_i \left(\frac{-2\phi_i}{T} \right) \\ &= \square v - \frac{2}{T} \left\{ \phi_t \partial_T \phi_t - \sum_i (\delta_i^{i'} \partial_{i'} \phi_i + \phi_i \partial_T \phi_i) \right\} + \frac{2}{T^2} \{ \phi_t^2 - \sum_i \phi_i^2 \}. \end{aligned}$$

Since v and ϕ are holomorphic, we find

$$e^v = \phi_t^2 - \sum_i \phi_i^2 + O(\phi),$$

which gives Eq. (25). This allows us to let $\phi = t - \psi(x)$, as explained in §3.1.

A.2. Expansion. Let us define w off Σ by

$$u = \ln\left(\frac{2}{\phi^2}\right) + v_0(X) + v_1(X)T + w(X, T)T^2.$$

We find

$$\begin{aligned} e^u &= \frac{2e^{v_0}}{T^2} \left\{ 1 + (v_1 T + w T^2) + \frac{1}{2}(v_1 T + w T^2)^2 + \right. \\ &\quad \left. + \frac{T^3}{2}(v_1 + T w)^3 \int_0^1 (1 - \sigma)^2 \exp[\sigma T(v_1 + T w)] d\sigma \right\}; \end{aligned}$$

$$\begin{aligned} \square &= \partial_T^2 - \sum_i (\delta_i^{i'} \partial_{i'} - \psi_i \partial_T)^2; \\ \square(\ln \frac{2}{T^2}) &= \frac{2}{T^2} - \sum_i \partial_i \left(\frac{2\psi_i}{T} \right) \\ &= \frac{2}{T^2} (1 - |D\psi|^2) - \frac{2}{T} \Delta\psi. \end{aligned}$$

Note that this calculation used the relation $\partial_i T = -\psi_i$.

We next find, since $\partial_i g = \partial_{i'} g$ if $g = g(X)$, so that $\Delta g = \Delta' g$,

$$\begin{aligned}\square v_0(X) &= -\Delta v_0, \\ \square(v_1(X)T) &= -\sum_i (\delta_i^{i'} \partial_{i'} - \psi_i \partial_T) [(\partial_i v_1)T - \psi_i v_1] \\ &= -(\Delta v_1)T + v_1 \Delta \psi + 2\psi_i \partial_i v_1. \\ \square(wT^2) &= T^2 w_{TT} + 4Tw_T + 2w - \sum_i \partial_{ii} w.\end{aligned}$$

Now, for fixed i , ∂_{ii} can be expressed as follows (we write $\partial_{i'}$ for $\delta_i^{i'} \partial_{i'}$ for convenience):

$$\begin{aligned}(\partial_{i'} - \psi_i \partial_T)^2 &= (\partial_{i'} - \psi_i \partial_T) \partial_{i'} - \partial_i (\psi_i \partial_T) \\ &= (\partial_{i'} - \psi_i \partial_T) \partial_{i'} - (\Delta \psi) \partial_T - \psi_i \partial_{iT} \\ &= \partial_{i'}^2 - \psi_i \partial_{T i'} - (\Delta \psi) \partial_T - \psi_i (\partial_{i'} - \psi_i \partial_T) \partial_T \\ &= \partial_{i'}^2 - 2\psi_i \partial_{T i'} + |D\psi|^2 \partial_T^2 - (\Delta \psi) \partial_T.\end{aligned}$$

Therefore,

$$\square = (1 - |D\psi|^2) \partial_T^2 - \partial_{i'}^2 + 2\psi_i \partial_{T i'} + (\Delta \psi) \partial_T, \quad (35)$$

and

$$\square(wT^2) = (1 - |D\psi|^2)(T^2 w_{TT} + 4Tw_T + 2w) \quad (36)$$

$$+ T^2(-\Delta' + 2\psi_i \partial_{T i'} + (\Delta \psi) \partial_T)w + T[2(\Delta \psi)w + 4\psi^i \partial_{i'} w]. \quad (37)$$

Putting these together, we can write down the equation satisfied by w :

$$\begin{aligned}& T^2 \{ (1 - |D\psi|^2) w_{TT} - \Delta' w + 2\psi_i w_{T i'} + (\Delta \psi) w_T \} \\ & + T \{ 4(1 - |D\psi|^2) w_T + 4\psi^i \delta_i^{i'} \partial_{i'} w + 2w \Delta \psi - \Delta v_1 \} \\ & + \{ 2(1 - |D\psi|^2) w + v_1 \Delta \psi + 2\psi^i \partial_i v_1 - \Delta v_0 \} \\ & \qquad \qquad \qquad - \frac{2}{T} \Delta \psi + \frac{2}{T^2} (1 - |D\psi|^2) \\ & = \frac{2}{T^2} e^{v_0} \left\{ 1 + T(v_1 + Tw) + \frac{1}{2} T^2 (v_1 + Tw)^2 \right. \\ & \left. + \frac{1}{2} T^3 (v_1 + Tw)^3 \int_0^1 (1 - \sigma)^2 \exp[\sigma T(v_1 + Tw)] d\sigma \right\}.\end{aligned} \quad (38)$$

This is Eq. (29) with

$$g(X, T, w) = \frac{1}{2}(2v_1w + Tw^2) + \frac{1}{2}T^3(v_1 + Tw)^3 \int_0^1 (1 - \sigma)^2 \exp[\sigma T(v_1 + Tw)] d\sigma.$$

A.3. Curvature condition. Equations (30) and (31) now follow immediately. We compute here the coefficient $\alpha(X)$ of T^0 in the expansion of the difference between the l.h.s. and the r.h.s. of (38); we will prove that

$$\alpha(X) = 2R, \tag{39}$$

where R is the scalar curvature of Σ .

Inspection of (38) gives

$$\alpha(X) = v_1 \Delta \psi - \Delta v_0 + 2\psi^i \partial_i v_1 - (1 - |D\psi|^2)v_1^2.$$

This expression is proportional to the l.h.s. of (32). Indeed, using (30) and (31) for v_0 and v_1 in (31), we find, writing $\psi_i \psi^i$ for $|D\psi|^2$, that⁴

$$\begin{aligned} \alpha &= -2(\Delta\psi)^2/(1 - \psi_i \psi^i) + \partial_j \left[\frac{2\psi^{jk} \psi_k}{(1 - \psi_i \psi^i)} \right] - 2\psi^j \partial_j [\Delta\psi/(1 - \psi_i \psi^i)] \\ &= (1 - \psi_i \psi^i)^{-2} \left\{ 4\psi^{jk} \psi_k \psi_{ji} \psi^i - 4\psi^j \psi_{ij} \psi^i (\Delta\psi) \right\} \\ &\quad (1 - \psi_i \psi^i)^{-1} \left\{ -2\psi^j \psi^k_{jk} + 2\psi^{jk} \psi_k + 2\psi^{jk} \psi_{jk} - 2\Delta\psi \right\} \\ &= 4(1 - \psi^i \psi_i)^{-2} [\psi^{jk} \psi_k (\psi_{jl} \psi^l - \psi_j \Delta\psi)] \\ &\quad - 2(1 - \psi^i \psi_i)^{-1} [(\Delta\psi)^2 - \psi^{jk} \psi_{jk}]. \end{aligned}$$

This justifies Eq. (32).

Let us now relate this quantity to the scalar curvature.

The induced metric on Σ is

$$g_{ij} = \delta_{ij} - \psi_i \psi_j.$$

The Christoffel symbols are

⁴Recall that the summation convention is used throughout, and indices are raised and lowered using the Kronecker δ .

$$\begin{aligned}
\Gamma_{ij}^k &= (1/2)g^{km}\{\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\} \\
&= (1/2)\left(\delta^{km} + \psi^k \psi^m / (1 - |D\psi|^2)\right) \\
&\quad \{-\partial_i(\psi_j \psi_m) - \partial_j(\psi_i \psi_m) + \partial_m(\psi_i \psi_j)\} \\
&= (1/2)\left(\delta^{km} + \psi^k \psi^m / (1 - |D\psi|^2)\right)\{-2\psi_m \psi_{ij}\} \\
&= -\left\{\psi^k \psi_{ij} + \psi^k \psi_{ij} \psi^m \psi_m / (1 - |D\psi|^2)\right\} \\
&= -\psi^k \psi_{ij} / (1 - |D\psi|^2).
\end{aligned}$$

It follows that for any covariant field ω_j ,

$$\nabla_i \omega_j = \partial_i \omega_j + \psi^k \omega_k \psi_{ij} / (1 - |D\psi|^2).$$

We compute the curvature tensor from the relation

$$(\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k) = R_{ijk}{}^l \omega_l.$$

First,

$$\begin{aligned}
\nabla_i \nabla_j \omega_k &= \\
&= \partial_i(\nabla_j \omega_k) - \Gamma_{ij}^l \nabla_l \omega_k - \Gamma_{ik}^l \nabla_j \omega_l \\
&= \partial_i \left\{ \partial_j \omega_k + \psi_{jk} \psi^h \omega_h / (1 - \psi^m \psi_m) \right\} \\
&\quad + [\partial_l \omega_k + \psi_{lk} \psi^h \omega_h / (1 - \psi^m \psi_m)] \psi^l \psi_{ij} / (1 - \psi^m \psi_m) \\
&\quad + [\partial_j \omega_l + \psi_{jl} \psi^h \omega_h / (1 - \psi^m \psi_m)] \psi^l \psi_{ik} / (1 - \psi^m \psi_m).
\end{aligned}$$

Since we know that all the derivatives of ω must cancel in the final result, we have, letting $M = 1 - \psi^m \psi_m$,

$$\begin{aligned}
R_{ijk}{}^l \omega_l &= \partial_i(\psi_{jk} \psi^l / M) \omega_l - \partial_j(\psi_{ik} \psi^l / M) \omega_l \\
&\quad + \psi^l \psi_{ik} \psi_{jl} \psi^\rho \omega_\rho / M^2 - \psi^l \psi_{jk} \psi_{il} \psi^\rho \omega_\rho / M^2,
\end{aligned}$$

and therefore,

$$R_{ijk}{}^l = \partial_i[\psi_{jk} \psi^l / M] - \partial_j[\psi_{ik} \psi^l / M] + (\psi_{ik} \psi_{j\rho} \psi^\rho \psi^l - \psi_{jk} \psi_{i\rho} \psi^\rho \psi^l) / M^2.$$

The derivatives of M give rise to -2 times the last two terms. The net result is

$$R_{ijk}{}^l = [\psi_{jk}\psi_i{}^l - \psi_{ik}\psi_j{}^l]/M + [\psi_{jk}\psi^\rho\psi_{i\rho}\psi^l - \psi_{ik}\psi_{j\rho}\psi^\rho\psi^l]/M^2.$$

Therefore,

$$R_{ik} = [\psi_i{}^l\psi_{lk} - \psi_{ik}\Delta\psi]/M + [(\psi_{lk}\psi^l)(\psi_{\rho i}\psi^\rho) - \psi_{ik}\psi_{l\rho}\psi^\rho\psi^l]/M^2.$$

Finally,

$$\begin{aligned} R &= R_{ik}(\delta^{ik} + \psi^i\psi^k/M) \\ &= [\psi^{il}\psi_{il} - (\Delta\psi)^2]/M + [(\psi_{lk}\psi^l)(\psi_{kl}\psi^k) - \psi^i\psi^k\psi_{ik}\Delta\psi]/M^2 \\ &\quad + [(\psi_{li}\psi^l)(\psi^{\rho i}\psi_\rho) - (\Delta\psi)\psi_{l\rho}\psi^l\psi^\rho]/M^2 \\ &\quad + \{(\psi_{lk}\psi^l\psi^k)(\psi_{i\rho}\psi^i\psi^\rho) - (\psi_{ik}\psi^i\psi^k)(\psi_{l\rho}\psi^l\psi^\rho)\}/M^3 \\ &= [\psi^{il}\psi_{il} - (\Delta\psi)^2]/M + 2\{(\psi^\rho\psi_{i\rho})(\psi_\sigma\psi^{i\sigma}) - \psi^\rho\psi^\sigma\psi_{\rho\sigma}\Delta\psi\}/M^2. \end{aligned}$$

The relation $\alpha = 2R$ follows.

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