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LOCAL INDEX OF POTENTIAL OPERATORS
OF MONOTONE TYPE

SATYANAD KICHENASSAMY

Abstract. We prove the following result: if \( f = \varphi' \) is the Gâteaux derivative of a functional on a reflexive Banach space, and is demicontinuous of class \((S)_+\), then it has local index +1 at every isolated zero which is also a local minimum point for \( \varphi \). An application to the existence of multiple solutions of nonlinear partial differential equations is given.

1. INTRODUCTION AND STATEMENT OF RESULTS.

1.1 A topological result. We consider in this paper a reflexive Banach space \( X \) and an open bounded set \( G \subset X \), containing the origin. We may, by Trojanski’s theorem, renorm \( X \) so that it be locally uniformly convex as well as \( X^* \) (with the dual norm). Let \( J \) be the associated duality map from \( X \) to \( X^* \). F. E. Browder has constructed in [3] a degree function for maps of class \((S)_+\), demicontinuous from \( cl(G) \) to \( X^* \), which is normalized by \( J \) and reduces to the Leray-Schauder degree when \( X \) is a Hilbert space and the map considered is a compact perturbation of the identity (we recall this construction in §2.1).

Our main result (proved in §2.2) is the following:

**Theorem 1.** Assume that \( f : cl(G) \to X^* \) satisfies

(i) \( f \in (S)_+ \) and is demicontinuous
(ii) \( f \) is in \( G \) the Gâteaux derivative of \( \varphi : G \to \mathbb{R} \)
(iii) If \( x \in cl(G) \) and \( x \neq 0 \), then \( f(x) \neq 0 \)
(iv) If \( x \in G \), then \( \varphi(x) \geq \varphi(0) \).

Then,

\[
\text{deg}(f, G, 0) = +1.
\]

In case \( X \) is a Hilbert space (identified with \( X^* \)) and \( f = Id + \) compact, we recover results of P. H. Rabinowitz [11], K. Thews [15] and H. Amann [1] ([11] assumes that \( \varphi \in C^2 \) and [15], [1] that \( \varphi \in C^1(G; \mathbb{R}) \)). For the finite dimensional case (Poincaré-Hopf) see also e.g., J. Milnor [10] and E. H. Rothe [12].

139
1.2 An illustration. We now illustrate Theorem 1 by following multiplicity result: Let us first take \( g \in C^1(\mathbb{R}; \mathbb{R}) \) be such that

(i) \( g \) has \( m \) zeros \( 0 < a_1 < \cdots < a_m \),

(ii) ("Area condition") For \( k \geq 2 \) and \( F(s) := \int_0^s g(t)\,dt \)

\[
\sup \{ F(s) : 0 \leq s \leq a_{k-1} \} < F(a_k),
\]

(iii) \( g(0) > 0 \).

Let also \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \) \( (N \geq 1) \), \( 1 < p < \infty \). We then have:

**Theorem 2.** If \( g \) satisfies (i), (ii) and (iii) then the problem (P):

\[
Au := -\text{div} \left( (1 + |\nabla u|^2)^{(p-2)/2} \nabla u \right) = \lambda g(u) \quad \text{on } \Omega
\]

\[
u = 0 \quad \text{on } \partial\Omega
\]

has at least \( 2m - 1 \) weak solutions for all \( \lambda \) large enough.

This result contains theorems of K. J. Brown and H. Budin [5], P. Hess [6], J. Smoller and A. Wasserman [14] for semi-linear equations \( (p = 2) \). The case of radial solutions (in a more general setting including the case \( p \neq 2 \)) has been solved by J. Smoller and the author [7] by O.D.E. methods.

**Remarks:**

(1) The very classical "area condition" (ii) is necessary for radial solutions with maximum "close" to each \( a_k \) to exist (see e.g., [7]).

(2) Note that no growth condition on \( g \) is assumed; we may even take \( g = 0 \) far out. The problem is "sublinear."

This theorem will be proved in §3.

2. Proof of Theorem 1.

2.1 Degree of a demicontinuous map of class \((S)_+\). Let \( G \) be an open bounded subset of a reflexive Banach space \( X \) and let \( f : cl(G) \to X^* \) be a mapping. We say that \( f \) is demicontinuous if it is continuous from the strong topology of \( X \) to the weak topology of \( X^* \). We say that \( f \) is of class \((S)_+\) and write \( f \in (S)_+ \) whenever for any sequence \( (x_j) \) in \( cl(G) \) converging weakly to \( x \) in \( X \), and such that

\[
\limsup \langle f(x_j), x_j - x \rangle \leq 0,
\]
one has $x_j \to x$ (strongly). Note that $J$ (duality mapping) is $(S)_+$ and demicontinuous.

We also define for $X_0 \subset X$ finite dimensional, the corresponding Galerkin approximation

\[(2) \quad f_0 = i_0^* f i_0\]

where $i_0 : X_0 \to X$ is the inclusion, $i_0^*$ its transpose. $J_0$ is defined in a similar way for the duality map $J$.

We define the degree of a mapping $X_0 \to X_0^*$ as the Brouwer degree normalized by $J_0$.

Assume now that $0 \notin f(\partial G)$, $f \in (S)_+$ and is demicontinuous. One can then show [3] that there exists a finite dimensional $X_f$ such that for any finite dimensional $X_0 \supset X_f$

(a) $\deg(f_0, G \cap X_0, 0)$ is well defined
(b) It does not depend on $X_0$.

We call the integer thus obtained $\deg(f, G, 0)$. For details and examples see [3], [4].

**Remarks:** (1) The class of mappings we consider is stable under compact perturbations. Thus if $X =$ Hilbert space, $J = Id$, we obtain indeed an extension of Leray-Schauder degree theory.

(2) In case $f$ is the Gâteaux derivative of $\varphi$ and $X_0$ is large enough, we might try to define $\deg(f_0, G \cap X_0, 0)$ in another way, namely by choosing an identification of $X_0$ with $\mathbb{R}^n$, and by using Brouwer degree in $\mathbb{R}^n$. That amounts to a normalization by the duality mapping $I_0$ for the Euclidean norm on $\mathbb{R}^n$ (while $J_0$ is the duality map for $X_0$ equipped with $\| \cdot \|_X$ restricted to $X_0$). Both viewpoints are equivalent because

(a) for every $x \neq 0$, $(J_0 x, x)$ and $(I_0 x, x)$ are $> 0$ and by homotopy, for every $R$,

$$\deg_{\mathbb{R}^n} (J_0, B_R, 0) = +1;$$

(b) $I_0$ and $J_0$ have degree 0 on every set which does not contain 0 in its closure.

This remark justifies our use of customary degree in $\mathbb{R}^n$ for Galerkin approximations.
2.2 Proof of Theorem 1. We proceed in 3 lemmas. We write $f = \varphi'$, $\varphi(0) = 0$, and fix $\epsilon > 0$ such that $B_{2\epsilon} \subset G$.

**Lemma 1.** There are $\alpha, \alpha', c, c' > 0$ such that

$$B_{\alpha} \subset (\varphi < c)_{\epsilon} \subset B_{\alpha'} \subset (\varphi < c')_{\epsilon} \subset B_{\epsilon/2},$$

where $(\varphi < c)_{\epsilon}$ stands for $B_{\epsilon} \cap (\varphi < c)$ and similarly for $c'$.

**Lemma 2.** There is a finite dimensional subspace $X_0$ of $X$ such that

$$\deg(f, G, 0) = \deg(f_0, X_0 \cap B_{\epsilon}, 0)$$

and

$$f_0 \neq 0 \text{ on } X_0 \cap [cl(B_{\epsilon}) \backslash B_{\alpha/2}].$$

**Lemma 3.** $\deg(f_0, X_0 \cap (\varphi < c')_{\epsilon}, 0) = +1$.

This will end the proof. Indeed, by (5), $f_0$ has no zero on $cl(X_0 \cap (B_{\epsilon} \backslash (\varphi < c'))_{\epsilon})$ and therefore

$$\deg(f, G, 0) = \deg(f_0, X_0 \cap B_{\epsilon}, 0) = \deg(f_0, X_0 \cap (\varphi < c')_{\epsilon}, 0) = +1.$$  

It remains to prove the lemmas.

**Proof of Lemma 1:** For $0 < a < \epsilon$ let

$$m_a = \inf \{\varphi(x) : a \leq \|x\| \leq \epsilon\}.$$

We claim that $m_a > 0$. Indeed $m_a \geq 0$ and if $m_a = 0$, there is a sequence $(x_n)$ in $cl(B_{\epsilon}) \backslash B_a$ such that

$$\varphi(x_n) \leq n^{-2}.$$  

By a result due to Ekeland, as $cl(B_{2\epsilon})$ is a complete metric space, one can find another sequence $(y_n)$ such that for each $n$

$$\|x_n - y_n\| \leq \frac{1}{n}; \quad \|y_n\| \leq 2\epsilon$$

$$\varphi(y_n) \leq \varphi(x_n)$$

(9) For $y \neq y_n$ and $\|y\| \leq 2\epsilon$, $\varphi(y) > \varphi(y_n) - \frac{\|y - y_n\|}{n}$.  

These properties ensure that for $n^{-1} < a/2$,

\begin{align}
(10) \quad \|y_n\| & \geq \frac{a}{2} \\
(11) \quad \|f(y_n)\| & \leq \frac{1}{n}.
\end{align}

(Indeed (9) implies that for every $z \in X$, for $t$ small, as $t \to 0$ one has

$$
\varphi(y_n + tz) - \varphi(y_n) + \frac{\|tz\|}{n} = t \left[(f(y_n), z) + \frac{\|z\|^2}{n}\right] + o(t) \geq 0.
$$

Now $\|y_n\| \leq 2\varepsilon$. Therefore one may (after extraction of a subsequence) assume that

$$
\exists \ y \in X; \ y_n \rightharpoonup y \text{ and } f(y_n) \to 0.
$$

$f$ being $(S)_+$, this implies that $y_n \to y$, and $f(y) = 0$. Now $y \neq 0$ because of (10). This is a contradiction.

We have proved that $m_a > 0$. Now take $c' < m_{c'/2}$ and (as $(\varphi < c')$ is a neighborhood of 0) $\alpha'$ such that $B_{\alpha'} \subset (\varphi < c') \cap B_c \subset B_{c/2}$. Repeat the operation: take $c < m_{\alpha'/2}$ and $B_\alpha \subset (\varphi < c)_{\alpha'/2}$. (1) follows.

PROOF OF LEMMA 2: As $f$ has no zeros on $G \setminus B_\varepsilon$ we may restrict our attention to $\text{deg}(f, B_\varepsilon, 0)$. We know that (4) holds for $X_0 \supset X_f$. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the set of all finite dimensional subspaces of $X$ containing $X_f$. If (5) were false the following sets would be nonempty for every $\lambda$:

$$
V_\lambda = \{ x \in \text{cl}(B_\varepsilon) \setminus B_{\alpha/2} : (f(x), x) = 0 \text{ and } v \in X_\lambda \}
$$

(Indeed zeros of the Galerkin approximation of $f$ in $X_\lambda$ belong to $V_\lambda$).

$V_\lambda$ being bounded, its weak closure is weakly compact. The sets $\text{wk} - \text{cl}(V_\lambda)$ enjoy the finite intersection property. Therefore there is an $x$ in $V_\lambda$ such that

$$
\bigcap_{\lambda \in \Lambda} \text{wk} - \text{cl}(V_\lambda).
$$

Now let $w \in X$. We wish to prove that $(f(x), w) = 0$. For that purpose take $X_\lambda$ containing both $x$ and $w$. Now (Eberlein-Šmulian) we may find a sequence $x_j \to x$ with $x_j$ belonging to $X_\lambda$. We have

\begin{align}
(13) \quad (f(x_j), x_j - x) & = 0 \quad (f(x_j), w) = 0
\end{align}
for all \( j \). As \( f \in (S)_+ \), \( x_j \to x \) and as \( f \) is demicontinuous, (11) gives

(14) \((f(x), w) = 0\).

This holds for every \( w : f(x) = 0 \). On the other hand \( \|x_j\| \to \|x\| \), so that

(15) \( \frac{\alpha}{2} \leq \|x\| \leq \epsilon \).

But \( f \) has no zero on \( \text{cl}(B_{\epsilon}) \setminus B_{\alpha/2} \). (5) is therefore true.

**Proof of Lemma 3:** We recall that \( i_0 \) is the inclusion \( X_0 \to X \). Let \( \varphi_0 = \varphi i_0 \). Notice that \( \varphi_0 \) is of class \( C^1 \) on \( X_0 \).

We are now dealing with maps defined on \( X_0 \). Identify \( X_0 \) with \( \mathbb{R}^n \). One may clearly find, as in Lemma 1, \( \alpha, \alpha', c, c', \theta > 0 \) such that

(16) \( B_{\alpha} \subset (\varphi_0 < c - \theta) \); \( (\varphi_0 < c + \theta) \subset B_{\alpha'} \),

(17) \( B_{\alpha'} \subset (\varphi_0 < c' - \theta) \); \( (\varphi_0 < c' + \theta) \subset B_{\alpha/2} \).

Let us approximate \( \varphi_0 \) in \( C^1 \) norm by a Morse function \( \psi \) in \( B_{3\epsilon/2} \) (generic situation). If \( \|\varphi_0 - \psi\|_{C^1} < \theta \) then, by (16) — (17), \( \psi \) satisfies (3).

As \( \|f(x)\| \) is bounded below for \( \alpha/2 \leq \|x\| \leq \epsilon \) we may assume that all critical points of \( \psi \) are of norm \( \leq \alpha \). Now the gradient flow \( \eta \) defined on \((\psi < c')_\epsilon \) by

\[
\eta(0, x) = x,
\]

\[
\frac{d\eta(t, x)}{dt} = -\psi'(\eta(t, x))
\]
deforms \((\psi < c')_\epsilon \) into \((\psi < c) \) is some finite time \( T \). But this deformation stays within \( B_\epsilon \) because \( \psi \) decreases along trajectories and there is no continuous curve from \((\psi < c')_\epsilon \) to \( \{\|x\| \geq \epsilon \} \) (because of (3)). As \((\psi < c) \subset B_\alpha \), the map \( \zeta \) defined by

\[
\zeta(t, x) = \eta(t, x) \quad \text{for } 0 \leq t \leq T
\]

\[
\zeta(t, x) = \frac{(2T - t)\eta(T, x)}{T} \quad \text{for } T \leq t \leq 2T
\]
is a deformation of \((\psi < c')_\epsilon \) to a point. By the Poincaré-Hopf theorem, Lemma 3 follows.
3. Proof of Theorem 2.

The problem is to find $2m - 1$ critical points of

$$(18) \quad \varphi(\lambda, u) = \int_{\Omega} \left[ \frac{(1 + |\nabla u|^2)^{p/2}}{p} - \lambda F(u) \right] dx$$

defined on $W_0^{1,p}(\Omega)$. This functional is Gâteaux differentiable and its gradient is demicontinuous of class $(S)_+$ by classical arguments of the theory of monotone operators.

We assume that all solutions are isolated — otherwise there is nothing to prove. We first obtain $m$ variational solutions $u_1, \ldots, u_m$ which minimize truncated functions $\varphi_1, \ldots, \varphi_m$. The point is then to find $m - 1$ more solutions, which is achieved thanks to Theorem 1.

We proceed in 3 steps:

**STEP 1: $m$ variational solutions:** Let, for every $k \geq 1$, $g_k = g$ on $[0, a_k]$, extended so that it be $C^0, \geq 0$ on $]-\infty, 0]$, $\leq 0$ on $[a_k, +\infty[$; we may assume that $g_k$ and $g_{k+1}$ differ only in $[a_k, a_{k+1}]$. Let also

$$F_k(s) := \int_{0}^{s} g_k(t)dt.$$  

We minimize

$$(19) \quad \varphi_k(\lambda, u) = \int_{\Omega} \left[ \frac{(1 + |\nabla u|^2)^{p/2}}{p} - \lambda F_k(u) \right] dx.$$  

The minimum is attained at some $u_k$ ($\varphi_k$ is w.l.s.c. coercive). We show that this function is a solution of (P):

**Lemma 4.** All critical points $u$ of $\varphi_k$ satisfy $u < a_k$ and are positive solutions of (P).

**Proof:** (For $p = 2$ see [2] and [6].) It suffices to show $u < a_k$. We have $u \leq a_k$ by the maximum principle. Moreover by standard regularity results, $u$ is bounded in $C^1[9]$ and therefore we may assume that (P) reads:

$$A'u = -\text{div} (\rho(|\nabla u|^2)\nabla u) = g_k(u)$$
with $\rho$ bounded. Now as $g_k(a_k) = 0$ if $\mu$ is large enough, $u \to g_k(u) + \mu u$ is increasing and

$$(A' + \mu)(a_k - u) \geq 0,$$

and the strong maximum principle (which follows from Harnack’s inequality [9]) shows that $u < a_k$ everywhere.

We now show that the $u_k$'s are distinct.

**Lemma 5.** For $k < m$ and $\lambda$ large enough,

$$\varphi_{k+1}(\lambda, u_{k+1}) < \varphi_k(\lambda, u_k).$$

**Proof:** We exhibit $u$ such that $0 < u < a + \epsilon$ and

if $0 \leq v \leq a_k$ then $\varphi(\lambda, u) < \varphi(\lambda, v)$.

Let $\delta > 0$, $\Omega_\delta = \{x : d(x, \partial \Omega) < \delta\}$, so that $|\Omega_\delta| \to 0$, and $w_\delta \in C^\infty_0(\Omega)$ such that $0 \leq w_\delta \leq a_k$ and $w_\delta = a_k$ for $x \in \Omega \setminus \Omega_\delta$ (e.g., regularize $a_k \chi_{d(x, \partial \Omega) \geq \delta/2) \cap \Omega}$).

We have by (ii)

$$F(a_{k+1}) - \max_{0 \leq s \leq a_k} F(s) = \tau > 0$$

so that if $0 \leq v \leq a_k$,

$$\int_{\Omega} F(w_\delta)dx \geq \int_{\Omega \setminus \Omega_\delta} (F(v) + \tau) dx - \int_{\Omega_\delta} \max_{[0, a_{k+1}]} |F|dx$$

$$\geq \int_{\Omega} F(v)dx + \tau|\Omega| - 2 \max_{[0, a_{k+1}]} |F||\Omega_\delta|$$

$$\geq \int_{\Omega} F(v)dx + \tau|\Omega|/2$$

for $\delta$ small enough. Let $u = w_\delta$. Then

$$\varphi(\lambda, u) - \varphi(\lambda, v) \leq \int_{\Omega} \left[ \frac{(1 + |\nabla u|^2)^{p/2}}{p} - \frac{\lambda \tau}{2} \right] dx < 0$$

for $\lambda$ large enough. Lemma 5 is proved.

**Step 2:** Degree evaluation: Consider $f_k = \varphi_k$. Let $S_k = \{u \in W_0^{1,p}(\Omega) : f_k(u) = 0\}$. 


Lemma 6. Let $1 \leq k \leq m$.

(i) For any $R$ large enough,
\[ \deg(f, B_R(0), 0) = +1 \]

(ii) For $k \geq 2$, there is an $\epsilon_k > 0$ such that if $0 < \epsilon \leq \epsilon_k$,
\[ \deg(f_k, S_{k-1} + B_\epsilon, 0) = +1 \]

(iii) For $k \geq 1$,
\[ \deg(f_k, S_k + B_\epsilon, 0) = +1. \]

Proof: The theory of §2 applies.

First note that $S_k + B_\epsilon$ is open and bounded because if $v \in S_k$, then $Av$ and $\|v\|$ are bounded. We omit in (i) the dependence of $\varphi$ and $f$ on $\lambda$.

(i) We have for $v \in X = W^{1,p}_0(\Omega)$
\[ (Av, v) \geq \|v\|^p. \]

Therefore if $J$ is the normalizing duality map and if we define $A_t$ by
\[ A_t v = tf(v) + (1 - t)J(v) \]
we have for $\|v\| = R$
\[ (A_t v, v) \geq tR^p - CR + (1 - t)R^2 > 0 \]
for $R$ large enough. This proves (by homotopy invariance) that
\[ \deg(f, B, 0) = \deg(A_t, B, 0) = \deg(J, B, 0) = +1. \]

(ii) (a) We consider the following homotopy: let for $v \in X$ and $0 \leq t \leq 1$
\[ A'_t v := tf_{k-1}(v) + (1 - t)f_k(v). \]

Let $v \in S_{k-1} + B_\epsilon$. We claim that if $\epsilon$ is small enough then $\deg(A'_t, S_{k-1} + B_\epsilon, 0)$ is defined for all $t$. Indeed if this were false
\[ \forall n, \exists v_n \in X, \exists t_n \in [0, 1] \]
such that

\[ 0 < d(v_n, S_{k-1}) < \frac{1}{n} \]

and

\[ t_n f_{k-1}(v_n) + (1 - t_n) f_k(v_n) = 0. \]

Now \( A'_1 v_n = \lambda h(v_n) \) with \( h \equiv 0 \) outside \([a_{k-1}, a_k]\). We have, by the maximum \( H \) principle, \( v_n \leq a_k \), and \( v_n \) must take values \( > a_{k-1} \) (because \( v_n \notin S_{k-1} \)). Thus there is an \( x_n \) such that

\[ a_{k-1} < v_n(x_n) \leq a_k \]

and (modulo a subsequence),

\[ v_n \to v \in S_{k-1} \]

\[ v_n \to v \text{ uniformly (by regularity [9])} \]

\[ x_n \to x \in cl(\Omega). \]

Now \( a_{k-1} \leq v(x) < a_k \) by Lemma 5. This is a contradiction. Therefore

\[ \deg(f_k, S_{k-1} + B_\epsilon, 0) = \deg(f_{k-1}, S_{k-1} + B_\epsilon, 0). \]

We are thus reduced to the proof of (iii).

(iii) By excision,

\[ \deg(f_k, S_k + B_\epsilon, 0) = \deg(f_k, B_R, 0) \]

for \( R \) large enough. By (i), this quantity equals +1.

The Lemma is proved.

STEP 3: Non variational solutions: Let \( k \geq 2 \). \( u_k \) is by assumption isolated. Assume that \( S_k = S_{k-1} \cup \{u_k\} \). For \( \epsilon \) small enough (so that \( B_\epsilon(u_k) \cap (S_{k-1} + B_\epsilon) = \emptyset \)) and \( R \) large enough, by the additivity property of the degree and Theorem 1 on \( B_\epsilon(u_k) \)

\[ \deg(f_k, B_R \setminus [(S_{k-1} + cl(B_\epsilon)) \cup \{u_k\} + cl(B_\epsilon)], 0) \]

\[ = \deg(f_k, B_R, 0) - \deg(f_k, S_{k-1} + B_\epsilon, 0) - \deg(f_k, \{u_k\} + B_\epsilon, 0) \]

\[ = -1. \]

It follows that \( S_k \neq S_{k-1} \cup \{u_k\} \).

The theorem is proved.

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