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# STEINNESS OF BUNDLES WITH FIBER A REINHARDT BOUNDED DOMAIN

KARL OELJEKLAUS & DAN ZAFFRAN

## 1. INTRODUCTION AND NOTATIONS

Stein manifolds can be characterized by the fact that they holomorphically embed in  $\mathbb{C}^N$  for some  $N$ . From that point of view it is obvious that if  $F$  and  $B$  are Stein manifolds, then the product  $E = F \times B$  is Stein. More generally, take a fiber bundle  $E$  with fiber  $F$  and with basis  $B$ , which we shall denote by  $E \xrightarrow{F} B$ . Is such an  $E$  necessarily Stein? That question was asked fifty years ago by J.-P. Serre (cf. [Ser]), and is often referred to as “the Serre problem” in the literature. A counterexample with  $F = \mathbb{C}^2$  was produced by H. Skoda in 1977 (cf. [Skoda]). On the other hand, many interesting “positive results” have been obtained (see below).

Following [Pfl-Zwo], we shall say that a Stein manifold  $F$  is of class  $\mathcal{S}$ , or  $F \in \mathcal{S}$  for short, when<sup>1</sup>: For any bundle  $E \xrightarrow{F} B$  with  $B$  Stein,  $E$  is Stein. K. Stein proved in [Stein] that if  $\dim F = 0$ , then  $F \in \mathcal{S}$ . Building on previous work by A. Hirschowitz, Y.T. Siu and N. Sibony (cf. [Hirsch], [Sib] and [Siu 1]), N. Mok proved in [Mok] that if  $\dim F = 1$ , then  $F \in \mathcal{S}$ . Skoda’s above result can be stated as:  $\mathbb{C}^2 \notin \mathcal{S}$ .

In this paper we focus on the case where  $F$  is a bounded domain  $D \subset \mathbb{C}^n$  (“domain” means connected open subset). Several results showing that large classes of bounded domains belong to  $\mathcal{S}$  have been proved (cf. [Siu 2], [Steh] and [Die-For]). Nevertheless, in [Cœ-Lœ], G. Cœuré and J.-J. Lœb produced, for each given  $A \in SL_2(\mathbb{Z})$  with  $|\text{trace } A| > 2$ , a non Stein bundle  $E_{CL} \xrightarrow{D_{CL}} \mathbb{C}^*$  whose fiber  $D_{CL}$  is a bounded Stein domain subset of  $(\mathbb{C}^*)^2$ . Thus  $D_{CL} \notin \mathcal{S}$ . Their  $D_{CL}$  has the following properties:

- $D_{CL}$  has the Reinhardt symmetry, i.e., it is invariant by the map  $(z_1, z_2) \mapsto (\alpha_1 z_1, \alpha_2 z_2)$  for any complex numbers  $\alpha_1$  and  $\alpha_2$  of modulus 1.
- $D_{CL}$  has an automorphism  $g$  of the form  $g(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d)$  with  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$ .

The second named author generalized their construction and gave a better understanding of those bundles. In [Zaf], a key point is the existence of a

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<sup>1</sup>conveniently enough, that letter honors simultaneously Serre, Sibony, Siu, Skoda, Stehlé and Stein...

“natural”  $g$ -equivariant open dense embedding  $D_{CL} \hookrightarrow \hat{D}_{CL}$ , where  $\hat{D}_{CL} \setminus D_{CL}$  is an infinite chain of rational curves. Roughly speaking, the non-Steinness of  $E_{CL}$  is “explained” by what happens near that infinite chain.

Our goal here is to answer the following “converse” question:  
Let  $D \subset \mathbb{C}^n$ , with  $n = 2$  or  $3$ , be *any* bounded Stein Reinhardt domain. Does  $D$  belong to  $\mathcal{S}$ ? In other words, does there exist a bundle  $E \xrightarrow{D} B$  with  $B$  Stein and  $E$  non-Stein? The answer is contained in theorem 1 below. It reveals that in dimension two, Cœuré-Lœb’s examples  $D_{CL}$  (for all  $A \in SL_2(\mathbb{Z})$ ) essentially are the only Reinhardt bounded domains not in  $\mathcal{S}$ : all other examples are provided by  $g$ -invariant subdomains of some  $D_{CL}$ . Moreover, it is easily checked that the interior of the closure in  $\hat{D}_{CL}$  of such a subdomain still contains the infinite chain of curves, so from the point of view of [Zaf], it is natural that those domains do not belong to  $\mathcal{S}$  (indeed, proofs in [Cœ-Lœ] and [Zaf] apply almost verbatim to show that they do not belong to  $\mathcal{S}$ ). We shall see that both methods and results become a bit more complicated in dimension three.

We also address the following question: given  $D$  bounded and Reinhardt not in  $\mathcal{S}$  and  $B$  Stein, can we give a characterization of the Steinness of a bundle  $E \xrightarrow{D} B$ ? For a two-dimensional  $D$ , we answer that question in theorem 2. For a higher dimensional  $D$ , we give a partial answer in theorem 3.

*We work throughout the article in the complex category. In other words, all manifolds and maps we deal with are holomorphic. By the word “bundle” we mean a locally trivial holomorphic fiber bundle. We shall also use the notations  $\mathcal{O}(E)$  for the set of holomorphic functions on  $E$ , and  $S^1$  and  $\Delta$  will respectively denote the unit circle and the unit disk in  $\mathbb{C}$ .*

*Notice that most proofs are postponed until the end of the paper, in section 4.*

We shall make use of several known results about a given bundle  $E \xrightarrow{D} B$  with  $B$  Stein and  $D \subset \mathbb{C}^n$  bounded and Stein. Namely:

- $E$  is a flat bundle (see [Kaup] or [Siu 2]). That means  $E$  can be defined by locally constant transition functions. All flat bundles can be constructed as follows:

Let  $\rho$  be a morphism  $\Pi_1(B) \rightarrow \text{Aut}(D)$ . Such a  $\rho$  induces a  $\Pi_1(B)$ -action on  $D$ . Denote by  $\tilde{B}$  the universal cover of  $B$ , and consider the diagonal action of  $\Pi_1(B)$  on  $\tilde{B} \times D$ . Define

$$E = \frac{\tilde{B} \times D}{\Pi_1(B)}.$$

Then the projection map  $\tilde{B} \times D \rightarrow \tilde{B}$  induces a map  $E \rightarrow B$  that turns  $E$  into a bundle with fiber  $D$ . The *structural group*  $G_{\text{struct}}(E)$  of  $E$  is by definition the image of  $\rho$ . That definition is quite improper because  $G_{\text{struct}}(E)$  does not only depend on the isomorphism class

of  $E$  as an  $F$ -bundle over  $B$ , but also on the  $\rho$  chosen. That subtlety won't matter for our purposes...

- $E$  is holomorphically separable (see [Siu 2]).
- $E$  is Stein if  $G_{struct}(E)$  is compact or contained in a compact group (see [Sib] and [Siu 2]).
- $E$  is Stein if  $G_{struct}(E)$  has finitely many connected components (see [Siu 2]).

Given  $g \in Aut(D)$ , there is exactly one bundle  $E \xrightarrow{D} \mathbb{C}^*$  with monodromy  $g$ . Namely,  $E = \frac{\mathbb{C} \times D}{\mathbb{Z}}$ , where the  $\mathbb{Z}$ -action is the “diagonal action” generated by

$$\begin{aligned} \tilde{g} : \mathbb{C} \times D &\rightarrow \mathbb{C} \times D \\ (z ; d) &\mapsto (z + 1 ; g(d)). \end{aligned}$$

We shall call that bundle the *complex suspension of  $g$* . It has infinite cyclic  $\mathcal{A}(E)$ , generated by  $g$ .

*For simplicity, we introduce the following notations and results assuming  $n = 2$ , but they all extend in the “obvious” way to any  $n \geq 2$ .*

By a well-known criterion for the Steinness of a Reinhardt domain (see [Nara]), the map

$$\begin{aligned} \text{“log”} : (\mathbb{C}^*)^2 &\rightarrow \mathbb{R}^2 \\ (z_1, z_2) &\mapsto (\log |z_1|, \log |z_2|) \end{aligned}$$

induces a one-to-one correspondence between Stein Reinhardt domains of  $(\mathbb{C}^*)^2$  and open convex subsets of  $\mathbb{R}^2$ .

Now take  $D \subset (\mathbb{C}^*)^2$  a *bounded* Stein Reinhardt domain. We shall denote by  $\log D$  the image of  $D$  by the above map. To make more explicit the one-to-one correspondence

$$\log D \leftrightarrow D,$$

remark that  $D$  can be recovered from  $\log D$  as the image of the “tube”  $\log D + i\mathbb{R}^2 \subset \mathbb{C}^2$  by the map “exp” :  $(w_1, w_2) \mapsto (z_1, z_2) = (e^{w_1}, e^{w_2})$ . Moreover  $\log D$  contains no affine line: otherwise  $\log D + i\mathbb{R}^2$  would contain a copy of  $\mathbb{C}$  on which “exp” would restrict to a non-constant map from  $\mathbb{C}$  to  $D$ , contradicting the boundedness of  $D$ . By [Zwo], the converse statement holds: for a Stein Reinhardt  $D$ , if  $\log D$  contains no affine line, then  $D$  is isomorphic to a bounded domain (we won't use that fact, though).

We denote by  $Aut(D)$  the group of automorphisms of  $D$ . By [Shim],

$$Aut(D) = Aut_{alg}^{\mathbb{R}}(D) \ltimes Aut_0(D),$$

where:

- $Aut_0(D)$  is the connected component of the identity,
- for each  $g \in Aut_{alg}(D)$ , there exist a unique  $A_g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$  and unique  $\alpha_1, \alpha_2 \in \mathbb{C}^*$  such that

$$g(z_1, z_2) = (\alpha_1 z_1^a z_2^b, \alpha_2 z_1^c z_2^d),$$

–  $Aut_{alg}^{\mathbb{R}}(D)$  denotes the subgroup of  $Aut_{alg}(D)$  of all  $g$ 's with real and positive  $\alpha_i$ 's. Thus  $Aut_{alg}(D) = Aut_{alg}^{\mathbb{R}}(D) \ltimes (S^1)^2$ , and by lemma 1.4,  $Aut_{alg}^{\mathbb{R}}(D)$  is a discrete group.

For  $g \in Aut_{alg}(D)$ , we shall denote by  $f_g$  the map

$$\begin{aligned} f_g : \log D &\rightarrow \log D \\ p &\mapsto A_g p + b_g, \end{aligned}$$

where  $b_g = (\log |\alpha_1|, \log |\alpha_2|)$ . The correspondence  $g \leftrightarrow f_g$  is one-to-one between  $Aut_{alg}^{\mathbb{R}}(D)$  and the group of affine automorphisms of  $\log D$ . Remark that  $f_{g^{-1}}(p) = f_g^{-1}(p) = A_g^{-1}p - A_g^{-1}b_g$ .

Define

$$\mathcal{A}(D) = \{A_g : g \in Aut_{alg}^{\mathbb{R}}(D)\} \subset GL_2(\mathbb{Z}).$$

It is useful to think of  $\mathcal{A}(D)$  as “the set of matrices that act on  $D$ ”.

For a given bundle  $E \xrightarrow{D} B$ ,

$$\mathcal{A}(E) = \left\{ A_g : g \in Aut_{alg}^{\mathbb{R}}(D) \cap (Aut_0(D).G_{struct}(E)) \right\} \subset \mathcal{A}(D).$$

It is useful to think of  $\mathcal{A}(E)$  as “the set of matrices that are used to build  $E$ ”.

For any group of matrices  $\mathcal{A}$ , we denote

$$Spec_{\mathbb{C}} \mathcal{A} = \bigcup_{A \in \mathcal{A}} Spec_{\mathbb{C}} A.$$

We can now state the main results of this paper. We point out the importance of the set  $Spec_{\mathbb{C}} \mathcal{A}(D)$  to study whether a domain  $D$  belongs to  $\mathcal{S}$  or not. Theorem 1 below in the case  $n = 2$  is the main result of [Pf-Zwo]. Our proof for that case is simpler.

**Theorem 1.** *A bounded Stein Reinhardt domain  $D \subset \mathbb{C}^n$  with  $n = 2$  or 3 belongs to  $\mathcal{S}$  if and only if  $Spec_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ . In other words:*

- For  $n = 2$ ,  $D \notin \mathcal{S}$  if and only if there exists  $A \in \mathcal{A}(D)$  with  $Spec_{\mathbb{C}} A = \{\lambda, \frac{1}{\lambda}\}$ ,  $\lambda \in \mathbb{R} \setminus \{+1, -1\}$ .
- For  $n = 3$ ,  $D \notin \mathcal{S}$  if and only if (cf. lemma 1.2) there exists  $A \in \mathcal{A}(D)$  such that either
  - (a)  $Spec_{\mathbb{C}} A = \{\lambda_1, \lambda_2, \lambda_3\}$  with  $\lambda_i$ 's pairwise distinct and real, or
  - (b)  $Spec_{\mathbb{C}} A = \{1, \lambda, \frac{1}{\lambda}\}$  with  $\lambda \in \mathbb{R} \setminus \{+1, -1\}$ .

**Theorem 2.** *Let  $D \subset \mathbb{C}^n$ , with  $n = 2$  or 3, be a bounded Stein Reinhardt domain. Let  $E$  be a bundle over a Stein basis, and with fiber  $D$ . Then  $E$  is Stein if and only if  $Spec_{\mathbb{C}} \mathcal{A}(E) \subset S^1$ .*

**Theorem 3.** *Let  $D \subset (\mathbb{C}^*)^n$  be a bounded Stein Reinhardt domain, with  $n \geq 2$ . Let  $E$  be a bundle over a Stein basis, and with fiber  $D$ . If  $Spec_{\mathbb{C}} \mathcal{A}(E) \subset S^1$ , then  $E$  is Stein.*

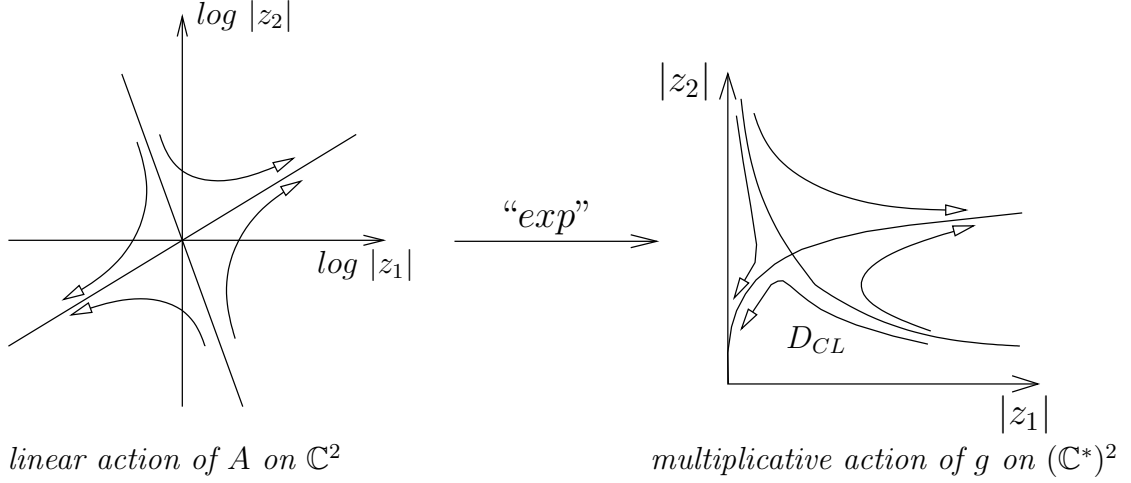
Theorem 1 naturally leads to the following

**Question.** For an arbitrary  $n$ , does a bounded Stein Reinhardt domain  $D \subset \mathbb{C}^n$  belong to  $\mathcal{S}$  if and only if  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ ?

We believe the answer to be “yes”.

### Geometric description of domains not in $\mathcal{S}$

For  $n = 2$ . It follows from theorem 1 that if  $D \notin \mathcal{S}$ , then  $D$  has an automorphism  $g$  of the form  $g(z_1, z_2) \mapsto (z_1^a z_2^b, z_1^c z_2^d)$  with  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$  and  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \not\subset S^1$ . Up to replacing  $A$  by  $A^2$ , we can assume that  $A \in SL_2(\mathbb{Z})$  and  $\text{Spec}_{\mathbb{C}} A = \{\lambda, \frac{1}{\lambda}\}$ ,  $\lambda \in (1, +\infty)$ . Then no entry of  $A$  is zero, therefore  $g$  maps both axes to  $(0, 0)$ , so  $D$  must be a subset of  $(\mathbb{C}^*)^2$ . Moreover the action of  $g$  on  $(\mathbb{C}^*)^2$  is well-understood:



The eigenspaces of  $A$  split  $\mathbb{R}^2$  into four quadrants, which correspond to four  $g$ -invariant “quadrant domains” in  $(\mathbb{C}^*)^2$  that are depicted above. Any quadrant domain is sent isomorphically to a bounded domain by a well-chosen automorphism of  $(\mathbb{C}^*)^2$ . The domain  $D_{CL}$  appearing (as the fiber of a non Stein bundle) in [Cœ-Lœ] is a quadrant domain of the matrix  $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ . Now, boundedness of  $D$  implies that  $D$  must be contained in a quadrant domain (otherwise  $\log D$ , being  $A$ -invariant, would contain an affine line).

Finally we conclude that for  $n = 2$ ,  $D \notin \mathcal{S}$  if and only if  $D$  is a  $g$ -invariant Stein subdomain of a quadrant domain of some matrix  $A \in SL_2(\mathbb{Z})$ .

Such  $D$ ’s are in one-to-one correspondence with convex  $A$ -invariant open subsets of a quadrant of  $A$ . Examples are provided by the quadrant domains themselves, and also by taking  $D$  such that  $\log D$  is the interior of the convex hull of the  $A$ -orbit of a point in  $\mathbb{R}^2$  not belonging to an eigenspace of  $A$ .

For  $n = 3$ . As opposed to the two-dimensional case, there are examples of domains not in  $\mathcal{S}$ , but not included in  $(\mathbb{C}^*)^3$ : the product of the unit disk and any two-dimensional domain not in  $\mathcal{S}$  is easily checked not to be

in  $\mathcal{S}$ . Besides those trivial “product examples”, domains invariant by a matrix that falls in case (b) of theorem 1 are some sort of a product with a two-dimensional domain  $D'$  not in  $\mathcal{S}$  (see section 3.2).

Let  $A$  be a matrix in  $SL_3(\mathbb{Z})$  satisfying condition (a) of theorem 1, with eigenvectors  $X_1, X_2$  and  $X_3$ . Examples of domains not in  $\mathcal{S}$  are:

- (1) Any  $D$  such that  $\log D$  is an octant delimited by the  $X_i$ 's. This example is the three-dimensional generalization of [Cœ-Lœ] and [Zaf].
- (2) Another basic example is given as follows: pick a point  $p = x_1X_1 + x_2X_2 + x_3X_3$  with  $x_1x_2x_3 \neq 0$ . Then take  $D$  such that  $\log D$  is the interior of the convex hull of  $\{A^k p : k \in \mathbb{Z}\}$ .
- (3) In some cases, starting with a  $\log D$  from the above example, then reflecting it through a coordinate plane, then taking the convex hull gives another convex  $A$ -invariant open subset of  $\mathbb{R}^3$  that still corresponds to a *bounded* domain. This produces examples not contained in an octant delimited by the  $X_i$ 's, that have no counterpart in the case of a two-dimensional  $D$ .

**Remark.** We've given a fairly precise description of all domains not in  $\mathcal{S}$  of dimension two, but a complete answer should also tell when two domains in the list are actually isomorphic. Actually, it follows from the literature about Inoue-Hirzebruch surfaces that considering “even Dloussky matrices” (defined in [Zaf]) already yields all examples of  $D$ 's not in  $\mathcal{S}$ . That (easy) classification question will be answered, as a byproduct, in a forthcoming paper.

We shall make use of the following auxiliary results, proved in section 4:

**Lemma 1.1.** *The only possible rational eigenvalues for a matrix  $A \in GL_n(\mathbb{Z})$  are  $+1$  or  $-1$ .*

**Lemma 1.2.** *Let  $A \in GL_3(\mathbb{Z})$ . Then exactly one of the following holds:*

- [1]  $\text{Spec}_{\mathbb{C}} A \subset \{+1, -1\}$ ,
- [2] $_{\mathbb{C}}$   $\text{Spec}_{\mathbb{C}} A = \{\pm 1, \lambda, \frac{1}{\lambda}\}$  with  $\lambda \in \mathbb{C} \setminus \{+1, -1\}$  and  $|\lambda| = 1$ ,
- [2] $_{\mathbb{R}}$   $\text{Spec}_{\mathbb{C}} A = \{\pm 1, \lambda, \frac{\pm 1}{\lambda}\}$  with  $\lambda \in \mathbb{R} \setminus \{+1, -1\}$ ,
- [3]  $\text{Spec}_{\mathbb{C}} A = \{\lambda_1, \lambda_2, \lambda_3\}$  with  $\deg_{\mathbb{Q}} \lambda_i = 3$  for  $i = 1 \dots 3$ .

We shall write for short  $GL_3(\mathbb{Z}) = [1] \cup [2]_{\mathbb{C}} \cup [2]_{\mathbb{R}} \cup [3]$  (and that is a disjoint union).

**Corollary 1.3.** *If  $A \in GL_3(\mathbb{Z})$  doesn't have pairwise distinct eigenvalues, then  $\text{Spec}_{\mathbb{C}} A \subset \{+1, -1\}$ .*

**Lemma 1.4.** *The set of matrices  $\mathcal{A}(D)$  is a subgroup of  $GL_n(\mathbb{Z})$ , and the map*

$$\begin{array}{ccc} \psi : \text{Aut}_{\text{alg}}^{\mathbb{R}}(D) & \rightarrow & \mathcal{A}(D) \\ g & \mapsto & A_g \end{array}$$

*is a group isomorphism.*

The following known result is in some sense a generalization of Bézout's theorem (which corresponds to the case  $n = 2$ ).

**Lemma 1.5.** *The vector  $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{Z}^n$  is unimodular (i.e.,  $\gcd(v_1, \dots, v_n) = 1$ )*

*if and only if there exists a matrix in  $SL_n(\mathbb{Z})$  of the form:*

$$\begin{bmatrix} v_1 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ v_n & * & \cdots & * \end{bmatrix}.$$

**Lemma 1.6.** *Let  $D \subset \mathbb{C}^n$  be a Stein Reinhardt domain, with  $n \geq 2$ . Denote  $\pi(z_1, \dots, z_n) = (z_2, \dots, z_n)$ . Then  $\pi(D)$  is a Stein (Reinhardt) domain of  $\mathbb{C}^{n-1}$ .*

**Lemma 1.7.** *Let  $E \xrightarrow{D} B$  be a bundle with fiber a Reinhardt domain  $D \subset (\mathbb{C}^*)^n$ . Assume there exists  $v \in \mathbb{Z}^n$  such that for all  $A \in \mathcal{A}(E)$ ,  $Av = v$ . Then there exists a “quotient bundle”  $E' \xrightarrow{D'} B$  such that:  $D' \subset (\mathbb{C}^*)^{n-1}$ ;  $E'$  is Stein if and only if  $E$  is Stein; and for any  $A' \in \mathcal{A}(E') \subset GL_{n-1}(\mathbb{Z})$ ,  $A'$  is the automorphism of  $\frac{\mathbb{Z}^n}{\langle v \rangle_{\mathbb{Z}}} \approx \mathbb{Z}^{n-1}$  induced by some  $A \in \mathcal{A}(E)$ . In particular,  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E') \subset \text{Spec}_{\mathbb{C}} \mathcal{A}(E)$ .*

The following result is due to Burnside (cf. [Steinb] p.33).

**Lemma 1.8.** *Let  $V$  be a finite-dimensional vector space over a field  $F$ . Let  $S$  be a semi-group of endomorphisms of  $V$  acting irreducibly on  $V$ . If the elements of  $S$  have only finitely many different traces, then  $S$  itself is finite.*

**Lemma 1.9.** *Let  $E \xrightarrow{D} B$  be a flat bundle and  $H \subset G_{\text{struct}}(E)$  be a subgroup of finite index. Then there exists a flat bundle  $E' \xrightarrow{D} B$  with  $G_{\text{struct}}(E') = H$  and such that  $E'$  is Stein if and only if  $E$  is Stein.*

**Lemma 1.10.** *Let  $D \subset (\mathbb{C}^*)^n$  be a bounded Stein Reinhardt domain, and  $E \xrightarrow{D} B$  be a bundle with  $B$  Stein. If  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E)$  is finite, then  $E$  is Stein.*

**Lemma 1.11.** *Let  $\mathcal{A}$  be a subgroup of  $GL_n(\mathbb{Z})$  with  $n \geq 2$ . If  $\text{Spec}_{\mathbb{C}} \mathcal{A} \subset S^1$ , then  $\text{Spec}_{\mathbb{C}} \mathcal{A}$  is finite.*

## 2. PROOF OF THEOREM 1 FOR $D \subset \mathbb{C}^2$ .

The following two results prove theorem 1 for  $n = 2$ .

**Proposition 2.1.** *Let  $D \subset (\mathbb{C}^*)^2$  be a bounded Stein Reinhardt domain. Then  $D \in \mathcal{S}$  if and only if  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ .*



**Proof.** “Only if” part: Suppose there exists  $A \in \mathcal{A}(D)$  and  $\text{Spec}_{\mathbb{C}} A \not\subset S^1$ . Up to replacing  $A$  by  $A^2$ , we can assume that  $\det A = 1$ . As  $1 \notin \text{Spec}_{\mathbb{C}} A$ , up to conjugation by an automorphism of  $(\mathbb{C}^*)^2$ , we can assume (see p.9 for more details) that  $D$  has an automorphism of the form  $g(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d)$  with  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$  and  $\text{Spec}_{\mathbb{C}} A = \{\lambda, \frac{1}{\lambda}\}$ ,  $\lambda \neq 1$ . Let  $E \xrightarrow{D} \mathbb{C}^*$  be the complex suspension of  $g$ . Then the method of [Cœ-Lœ] applies almost directly to show that  $E$  is not Stein. The only (easy) extra work is to check that  $\log D$  contains an  $\mathbb{R}$ -orbit of the form  $\{A^t \begin{bmatrix} x \\ y \end{bmatrix} : t \in \mathbb{R}\}$  for some  $\begin{bmatrix} x \\ y \end{bmatrix} \in \log D$  (see section 3 for more details). The proof from [Zaf] can be adapted as well. We conclude that  $D \notin \mathcal{S}$ .

(Remark for later use: Denote  $E' \subset E$  the restriction of  $E$  above a centered annulus in  $\mathbb{C}$ . As was already noticed in [Cœ-Lœ], their method of construction of analytic disks in  $E$  can easily be adapted to  $E'$ .)

*Conversely:* Suppose that  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ . Then by lemma 1.11,  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D)$  is finite. Take a bundle  $E \xrightarrow{D} B$  with  $B$  Stein. As  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E) \subset \text{Spec}_{\mathbb{C}} \mathcal{A}(D)$ ,  $E$  is Stein by lemma 1.10. This proves that  $D \in \mathcal{S}$ .  $\square$

**Proposition 2.2.** *If  $D \subset \mathbb{C}^2$  with  $D \not\subset (\mathbb{C}^*)^2$ , then  $D \in \mathcal{S}$  and  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ .*

**Proof.** If  $D \cap \{z_i = 0\}$  is nonempty for  $i = 1, 2$ , then by [Shim] p. 412, Lemma 1,  $\text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  is finite. Therefore  $\text{Aut}(D)$  has only finitely many connected components, so  $D \in \mathcal{S}$  by [Siu 2]. As any  $g \in \text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  has finite order, we also know that  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ .

Else we can assume without loss of generality that  $D \cap \{z_1 = 0\} \neq \emptyset$  and  $D \cap \{z_1 = 0\} = \emptyset$ . Then any  $g \in \text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  must have the form  $g(z_1, z_2) = (\alpha_1 z_1 z_2^k, \alpha_2 z_2^{\pm 1})$ . As  $D$  is bounded and stable by  $g^n$  for any  $n \in \mathbb{Z}$ , we get  $\alpha_2 = 1$ , so the second component of  $g^n(z_1, z_2)$  has modulus independent from  $n$ . Now, for the first component to remain bounded for any  $(z_1, z_2) \in D$ , we must have  $\alpha_1 = 1$  and  $k = 0$ . So  $\text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  is finite and, as above,  $D \in \mathcal{S}$  and  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ .  $\square$

### 3. PROOF OF THEOREM 1 FOR $D \subset \mathbb{C}^3$ .

**3.1. For  $D \subset (\mathbb{C}^*)^3$ , if  $\mathcal{A}(D) \subset [1] \cup [2]_{\mathbb{C}}$ , then  $D \in \mathcal{S}$ .**

As  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ , it follows from lemmas 1.11 and 1.10 that  $D \in \mathcal{S}$ .

**3.2. For  $D \subset (\mathbb{C}^*)^3$ , if  $\mathcal{A}(D) \cap [2]_{\mathbb{R}} \neq \emptyset$ , then  $D \notin \mathcal{S}$ .**

Let  $g \in \text{Aut}_{\text{alg}}(D)$  be such that  $A = A_g \in [2]_{\mathbb{R}}$ . Up to taking  $A^2$  instead of  $A$ , we can assume that  $\text{Spec}_{\mathbb{C}} A = \{1, \lambda, \frac{1}{\lambda}\}$ , with  $\lambda \in \mathbb{R} \setminus \{+1, -1\}$ . As  $1 \in \text{Spec}_{\mathbb{Q}} A$ , there exists  $v \in \mathbb{Q}^3$  such that  $Av = v$ . Up to multiplying by some integer, we can assume that  $v \in \mathbb{Z}^3$ .

Now define  $E \xrightarrow{D} \mathbb{C}^*$  as the complex suspension of  $g$ . Lemma 1.7 applies here and yields a bundle  $E' \xrightarrow{D'} \mathbb{C}^*$  with fiber  $D' \subset (\mathbb{C}^*)^2$ , and  $\mathcal{A}(E')$  being

generated by  $A' \in SL_2(\mathbb{Z})$  with  $\text{Spec}_{\mathbb{C}} A' = \{\lambda, \frac{1}{\lambda}\}$  and  $\lambda \neq \pm 1$ . We saw that  $E'$  is not Stein in the proof of proposition 2.1. Therefore  $D \notin \mathcal{S}$ .

**3.3. For  $D \subset (\mathbb{C}^*)^3$ , if  $\mathcal{A}(D) \cap [3] \neq \emptyset$ , then  $D \notin \mathcal{S}$ .**

Let  $g \in \text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  be such that  $A = A_g \in [3]$ . We know that  $g$  on  $D$  corresponds on  $\log D$  to

$$f_g : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

By lemma 1.2,  $1 \notin \text{Spec}_{\mathbb{C}} A$ . Therefore  $A - I$  is an invertible  $3 \times 3$  matrix, and it is easy to check that  $p = -(A - I)^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$  is a fixed point of  $f_g$ , and (therefore) the translation  $T_p : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of vector  $p$  conjugates  $f_g$  to  $A$  (i.e.,  $T_p^{-1} f_g T_p = A$ ). It follows that up to conjugation by the  $(\mathbb{C}^*)^3$  automorphism  $(z_1, z_2, z_3) \mapsto (e^{p_1} z_1, e^{p_2} z_2, e^{p_3} z_3)$ , we can assume that  $g$  is such that  $f_g \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  for any  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ .

**Lemma 3.1.** *The eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $A$  are real.*

**Proof.** As  $\mathbb{R}^3$  is an odd-dimensional real vector space, we shall assume without loss of generality that  $\lambda_1 \in \mathbb{R}$ . We know from lemma 1.2 that  $\lambda_1 \neq 1$ . Up to replacing  $A$  by  $A^{-1}$ , we can assume that  $\lambda_1 > 1$ .

Now we prove by contradiction that  $\lambda_2$  is real. Suppose that  $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$ . As  $\det A = \pm 1$ , we must have  $|\lambda_2| < 1$ . Denote by  $X_1$  a  $\lambda_1$ -eigenvector, and by  $X_2$  and  $X_3$  the imaginary and real parts of a (necessarily complex)  $\lambda_2$ -eigenvector. The matrix of  $f_g$  with respect to the basis  $\{X_1, X_2, X_3\}$  is:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix},$$

where  $a + ib = \lambda_2$  (as  $\lambda_2 \notin \mathbb{R}$ ,  $b$  must be nonzero). As  $\log D$  is open, we can pick a point  $q = x_1 X_1 + x_2 X_2 + x_3 X_3$  in  $\log D$  with  $x_1 \neq 0$ . Then it is not hard to check that the convex hull of  $\{A^k q : k \in \mathbb{Z}\}$  (which is in  $\log D$ ) must contain the half-space  $\{a_1 X_1 + a_2 X_2 + a_3 X_3 : a_1 x_1 > 0\}$ . This is impossible because  $\log D$  contains no affine line. So  $\lambda_2$  and (therefore)  $\lambda_3$  are real.  $\square$

Up to taking  $A^{\pm 2}$  instead of  $A$ , and renaming the eigenvalues, we can assume that  $0 < \lambda_3 < \lambda_2 < 1 < \lambda_1$ .

Let  $O^+$  denote the octant  $\{\sum_j a_j X_j / \text{for all } j, a_j > 0\}$ . Again, we shall use coordinates with respect to the basis  $\{X_1, X_2, X_3\}$ , namely we write “ $(p_1, p_2, p_3)$ ” instead of “ $\sum_j p_j X_j$ ”.

The next two lemmas help us understand the shape of  $\log D$ :

**Lemma 3.2.** *If  $p = (p_1, p_2, p_3) \in \log D \cap O^+$  then, for any  $t \geq 0$ ,  $(t + p_1, p_2, p_3) \in \log D \cap O^+$  and  $(p_1, p_2, t + p_3) \in \log D \cap O^+$ .*

**Proof.** Recall that  $0 < \lambda_3 < \lambda_2 < 1 < \lambda_1$ . Denote  $(x_n, y_n, z_n) := A^n p = (\lambda_1^n p_1, \lambda_2^n p_2, \lambda_3^n p_3)$ .

When  $n \rightarrow +\infty$ ,  $y_n$  and  $z_n$  tend to zero, whereas  $x_n$  tends to  $+\infty$ . As a consequence, the angle between the line  $(p, A^n p)$  and the line  $(p, p + (1, 0, 0))$  tends to zero. Now take a ball  $B$  in  $\log D$ , centered at  $p$ . Fix  $t \geq 0$ . Because of the above facts, the convex hull of  $\{A^n p\} \cup B$  will contain  $(t + p_1, p_2, p_3)$  as soon as  $n$  is big enough. That proves the first part of the statement.

When  $n \rightarrow -\infty$ ,  $x_n$  tends to zero, and  $y_n$  and  $z_n$  tend to  $+\infty$ , with  $y_n/z_n$  tending to zero. Therefore the angle between the line  $(p, A^n p)$  and the line  $(p, p + (0, 0, 1))$  tends to zero. The second part of the statement follows as in the above case.  $\square$

**Lemma 3.3.** *There exists  $r \in \log D$  such that  $\{A^t r, t \in \mathbb{R}\}$  is contained in  $\log D$ .*

**Proof.** We shall make repeated use of lemma 3.2 and of the inequalities  $\lambda_1 > 1$ ,  $\lambda_2 < 1$ ,  $\lambda_3 < 1$ . Take any  $q = (q_1, q_2, q_3) \in \log D$ .

*First case:* Assume that  $q \in O^+$ . Then  $V_1 := (\lambda_1 q_1, q_2, q_3)$  is in  $\log D$ , and therefore  $V_2 := (\lambda_1 q_1, q_2, \lambda_3^{-1} q_3)$  is in  $\log D$  as well. On the other hand,  $Aq = (\lambda_1 q_1, \lambda_2 q_2, \lambda_3 q_3)$  is in  $\log D$ , so  $V_3 := (\lambda_1 q_1, \lambda_2 q_2, q_3)$  is in  $\log D$ , and so is  $V_4 := (\lambda_1 q_1, \lambda_2 q_2, \lambda_3^{-1} q_3)$ .

Multiplying the first components by  $\lambda_1$  we get four other points in  $\log D$ :  $V_5 := (\lambda_1^2 q_1, q_2, q_3)$ ,  $V_6 := (\lambda_1^2 q_1, q_2, \lambda_3^{-1} q_3)$ ,  $V_7 := (\lambda_1^2 q_1, \lambda_2 q_2, q_3)$ , and  $V_8 := (\lambda_1^2 q_1, \lambda_2 q_2, \lambda_3^{-1} q_3)$ .

Let  $r := V_2$ . Remark that  $V_7 = Ar$ . As  $\log D$  is convex, it contains the parallelepiped  $\mathcal{P}$  with vertices  $V_1, V_2, \dots, V_8$ . Thus, if  $\lambda_1 q_1 \leq x_1 \leq \lambda_1^2 q_1$ ,  $q_2 \leq x_2 \leq \lambda_2 q_2$ , and  $\lambda_3^{-1} q_3 \leq x_3 \leq q_3$ , then  $(x_1, x_2, x_3)$  belongs to  $\mathcal{P}$ , and therefore to  $\log D$ . It follows that for any  $t$  between 0 and 1,  $A^t r = (\lambda_1^t \lambda_1 q_1, \lambda_2^t q_2, \lambda_3^t \lambda_3^{-1} q_3)$  belongs to  $\log D$ . As  $\log D$  is invariant by  $A$ , we get that  $A^t r$  belongs to  $\log D$  for any real  $t$ .

*Second case:* If  $q \notin O^+$ , take a linear automorphism  $B$  of  $\mathbb{R}^3$  such that for all  $j$ ,  $B(X_j) = \pm X_j$  (thus  $A^t B = B A^t$  for any  $t \in \mathbb{R}$ ), and such that  $Bq \in O^+$ . Now apply the first case to get in  $B(\log D)$  an  $\mathbb{R}$ -orbit, which is sent to an  $\mathbb{R}$ -orbit in  $\log D$  by  $B^{-1}$ .  $\square$

We now construct a non-Stein bundle  $E \xrightarrow{D} \mathbb{C}^*$  by adapting the construction of G. Cœuré and J.-J. Lœb. Lemma 3.3 is an important ingredient for the construction of analytic disks as in [Cœ-Lœ].

We consider the “tube above  $\log D$ ”:

$$V := \left\{ \sum_j u_j X_j \mid (Im u_1, Im u_2, Im u_3) \in \log D \right\} \subset \mathbb{C}^3.$$

We consider the action of  $G_{\mathbb{R}} = \mathbb{R} \ltimes \mathbb{R}^3$  on  $\mathbb{C} \times \mathbb{C}^3$  given by

$$\begin{aligned} \mathbb{C} \times \mathbb{C}^3 \times G_{\mathbb{R}} &\rightarrow \mathbb{C} \times \mathbb{C}^3 \\ ((z, \sum_j u_j X_j), (t, X)) &\mapsto (t + z, X + \sum_j \lambda_j^t u_j X_j). \end{aligned}$$

Define  $\Omega := \mathbb{C} \times V$ . By construction,  $\Omega$  is invariant by the action of  $G_{\mathbb{Z}} := \mathbb{Z} \ltimes \mathbb{Z}^3 \subset G_{\mathbb{R}}$ .

Now take  $E := \Omega/G_{\mathbb{Z}}$  (cf. [Cœ-Lœ]). Remark that  $E$  is just the complex suspension of  $g$ . We shall prove that  $E$  is not Stein by constructing a family of analytic disks  $\{d_R : \bar{\Delta} \rightarrow E\}_{R>1}$  such that  $d_R(0)$  tends to infinity in  $E$  (topologically) when  $R \xrightarrow{>} 1$ , whereas  $d_R(\partial\bar{\Delta})$  remains in a compact subset of  $E$  independent from  $R$ . Here  $\Delta$  (resp.  $\bar{\Delta}$ ) denotes the open (resp. closed) unit disk in  $\mathbb{C}$ , and  $\partial\bar{\Delta}$  denotes their boundary.

Fix a real number  $R > 1$ . In what follows, it is useful to keep in mind that the maps  $f$ ,  $g_1, g_2, g_3$  and  $\tilde{d}$  depend on  $R$ .

Define  $f : \Delta \rightarrow \mathbb{C}$  by

$$f(z) = \log i \frac{R+z}{R-z}.$$

Then  $0 < \text{Im } f < \pi$ , and  $f$  is continuous on  $\bar{\Delta}$ .

Thus we can define  $g_1, g_2$  and  $g_3$  as holomorphic functions  $\Delta \rightarrow \mathbb{C}$  such that on  $\partial\bar{\Delta}$ ,  $\text{Im } g_j = \lambda_j^{\mu_1^{-1} \text{Re } f}$  for  $j = 1, 2, 3$ , where  $\mu_1$  denotes  $\log \lambda_1$ .

Define a map  $\tilde{d} : \Delta \rightarrow \mathbb{C} \times \mathbb{C}^3$  by

$$\tilde{d}(t) := (\mu_1^{-1} f(t), \sum_j g_j(t) r_j X_j),$$

where  $r = (r_1, r_2, r_3)$  is given by lemma 3.3. Notice that  $\tilde{d}$  is continuous on  $\bar{\Delta}$ .

**Lemma 3.4.** *For any  $t \in \bar{\Delta}$ ,  $\tilde{d}(t) \in \Omega$ .*

**Proof.** Recall that  $\tilde{d}(t) \in \Omega$  if and only if  $(r_1 \text{Im } g_1(t), r_2 \text{Im } g_2(t), r_3 \text{Im } g_3(t)) \in \log D$ .

First assume that  $t \in \partial\bar{\Delta}$ .

Denote  $(\text{Im } g_1(t)r_1, \text{Im } g_2(t)r_2, \text{Im } g_3(t)r_3)$  by  $\tilde{d}'(t)$ . By definition of the  $g_j$ 's,  $\tilde{d}'(t) = (\lambda_1^\alpha r_1, \lambda_2^\alpha r_2, \lambda_3^\alpha r_3)$  for some  $\alpha \in \mathbb{R}$ , thus  $\tilde{d}'(t)$  is on the  $\mathbb{R}$ -orbit of  $r$ , which is in  $\log D$  by the lemma 3.3.

Now for  $s \in \Delta$ , the componentwise harmonicity of  $\tilde{d}'$  allows us to write in vector form

$$\tilde{d}'(s) = \int_{S^1} \tilde{d}'(t) P_s(t),$$

where  $P_s$  is the Poisson kernel at  $s$ . As  $\int_{S^1} P_s = 1$ , the above shows that  $\tilde{d}'(s)$  is in the convex hull of  $\tilde{d}'(S^1)$ , therefore is in  $\log D$ .  $\square$

We define an analytic disk  $d_R : \Delta \rightarrow E$  by

$$d_R = q_{\mathbb{Z}} \circ \tilde{d},$$

where  $q_{\mathbb{Z}}$  is the quotient map  $\Omega \rightarrow E$ . Remark that  $d$  is continuous on  $\bar{\Delta}$ .

**Lemma 3.5.** *There exists a compact set  $K$  in  $E$ , independent from  $R$ , such that  $d_R(\partial\bar{\Delta}) \subset K$ .*

**Proof.** It is not hard to check that  $\tilde{d}(t)$  is equal up to action by  $G_{\mathbb{R}}$  to

$$(i\mu_1^{-1} \text{Im } f(t), i \sum_j r_j \lambda_j^{-\mu_1^{-1} \text{Re } f(t)} \text{Im } g_j(t) X_j).$$

When  $t$  is in  $\partial\bar{\Delta}$ , the above becomes

$$(i\mu_1^{-1} \text{Im } f(t), i \sum_j r_j X_j).$$

As  $0 < \text{Im } f < \pi$ , there exists a compact set  $\tilde{K} \subset \Omega$  independent from  $R$ , such that  $\tilde{d}(\partial\bar{\Delta}) \subset \tilde{K}.G_{\mathbb{R}}$ . Then  $d_R(\partial\bar{\Delta})$  is included in  $K = q_{\mathbb{Z}}(\tilde{K}).(G_{\mathbb{R}}/G_{\mathbb{Z}})$ , which is compact because  $G_{\mathbb{R}}/G_{\mathbb{Z}}$  is compact.  $\square$

On the other hand,

$$\tilde{d}(0) = (\mu_1 \frac{\pi}{2}, \sum_j g_j(0) r_j X_j),$$

and  $\text{Im } g_1(0) = \int_{S^1} \text{Im } g_1 = \int_{S^1} \lambda_1^{\mu_1^{-1} \text{Re } f} = \int_{S^1} e^{\text{Re } f}$ , which tends to  $+\infty$  when  $R$  tends to 1 (cf. [Cœ-Lœ]).

By construction of  $E$ , a point  $q_{\mathbb{Z}}(z; (z_1, z_2, z_3))$  in  $E$  tends to infinity (topologically) if and only if either  $\text{Im } z$  or  $\text{Im } (z_1, z_2, z_3)$  tends to infinity. Therefore when  $R$  tends to 1,  $d_R(0) = q_{\mathbb{Z}}(\tilde{d}(0))$  tends to infinity in  $E$ . By the maximum principle (for functions  $\Delta \rightarrow \mathbb{C}$ ), any function on  $E$  will be bounded on the sequence  $\{d_{1+1/n}(0)\}_{n \in \mathbb{N}}$ , which proves that  $E$  is not holomorphically convex, therefore  $E$  is not Stein.

Conclusion:  $D \notin \mathcal{S}$ .

**3.4. The case of  $D \not\subset (\mathbb{C}^*)^3$ .** Up to action on  $\mathbb{C}^3$  by an algebraic automorphism, we can assume that  $D$  is in Shimizu's normal form (in his notations, that means  $D = \tilde{D}$ ; see [Shim] p. 410). Remark: That realization of  $D$  may not be bounded.

**Proposition 3.6.** *If  $D \subset \mathbb{C}^3$  intersects several coordinate hyperplanes, then  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$  and  $D \in \mathcal{S}$ .*

**Proof.** If  $D \cap z_i = 0$  is nonempty for  $i = 1, 2$  and 3, then by [Shim] p. 412, Lemma 1,  $\text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  is finite. Therefore  $\text{Aut}(D)$  has only finitely many connected components, and  $D \in \mathcal{S}$  by [Siu 2]. As any  $g \in \text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  has finite order, we also know that  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ .

Without loss of generality, we can now assume that  $D \cap \{z_1 = 0\}$  and  $D \cap \{z_2 = 0\}$  are both nonempty and  $D \cap \{z_3 = 0\}$  is empty. (In Shimizu's notations, we must have  $\sum_{i=1}^s n_i = 2$ ,  $t = s + 1$ ,  $n_t = 1$ .)

Take  $g \in \text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$ . Then  $g^2$  must be of the form:

$$g^2(z_1, z_2, z_3) = (\alpha_1 z_1 z_3^{\mu_1}, \alpha_2 z_2 z_3^{\mu_2}, \alpha_3 z_3).$$

In particular,  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ . As  $\text{proj}_3(D)$  is bounded (in Shimizu's notations, projection of  $D$  on  $\mathbb{C}^{n_{s+1}} \times \dots \times \mathbb{C}^{n_t}$  is always bounded) and  $g^2$ -invariant,  $\alpha_3 = 1$ . Now, for any  $(z_1, z_2, z_3) \in D$ ,  $\text{proj}_3^{-1}(z_3)$  is bounded and  $g^2$ -invariant, so  $\alpha_1 = \alpha_2 = 1$  and  $\mu_1 = \mu_2 = 0$ . It follows that  $\text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  is finite. Therefore, again,  $D \in \mathcal{S}$ .  $\square$

We can now assume (until the end of section 3.4) that  $D \cap \{z_1 = 0\}$  is nonempty, and  $D \cap \{z_2 = 0\}$  and  $D \cap \{z_3 = 0\}$  are both empty.

Then any  $g \in \text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  must be of the form:

$$(*)_{\text{alg}} \quad g(z_1, z_2, z_3) = (\alpha_1 z_1 z_2^{\mu_2} z_3^{\mu_3}, \alpha_2 z_2^a z_3^b, \alpha_3 z_2^c z_3^d),$$

with  $A'_g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$ . Denote by  $\mathcal{A}'$  the group  $\{A'_g \in GL_2(\mathbb{Z}) : g \in \text{Aut}_{\text{alg}}^{\mathbb{R}}(D)\}$ . Remark that  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) = \{1\} \cup \text{Spec}_{\mathbb{C}} \mathcal{A}'$ , so  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$  if and only if  $\text{Spec}_{\mathbb{C}} \mathcal{A}' \subset S^1$ .

**Proposition 3.7.** *If  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \not\subset S^1$ , then  $D \notin \mathcal{S}$ .*

**Proof.** Take  $g \in \text{Aut}_{\text{alg}}^{\mathbb{R}}(D)$  with  $\text{Spec}_{\mathbb{C}} A_g \not\subset S^1$ . By  $(*)_{\text{alg}}$ ,  $g$  induces an automorphism  $g'$  of  $D' = D \cap \{z_1 = 0\}$ , and  $\text{Spec}_{\mathbb{C}} A'_{g'} = \{\lambda, \frac{1}{\lambda}\}$  with  $\lambda \in \mathbb{R} \setminus \{+1, -1\}$ .

Define bundles  $E \xrightarrow{D} \mathbb{C}^*$  and  $E' \xrightarrow{D'} \mathbb{C}^*$  as complex suspensions of  $g$  and  $g'$  respectively. The map  $D' \hookrightarrow D$  induces a map  $E' \hookrightarrow E$  realizing  $E'$  as a closed submanifold. By the results of section 2, we know that  $E'$  is not Stein, so  $E$  can't be Stein. Conclusion:  $D \notin \mathcal{S}$ .  $\square$

Therefore we shall assume until the end of section 3.4 that

$$\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1.$$

Our assumptions ( $z_1$  vanishes somewhere on  $D$  but  $z_2$  and  $z_3$  don't) imply by [Shim] that there can be three cases: (Shimizu's  $n_{s+1}, \dots, n_t$  must all be 1, otherwise by [Shim] p. 411 we would have an  $U(2)$ -action on  $D$ , and  $D$  would intersect at least two coordinate hyperplanes.)

- (1)  $D$  is bounded, and  $\text{Aut}_0(D)$  is only  $(S^1)^3$ . (This is Shimizu's  $D = \tilde{D}_2$  case.)
- (2)  $D = \{(z_1, z_2, z_3) : z_1 \in \mathbb{C}, (z_2 e^{q_2|z_1|^2}, z_3 e^{q_3|z_1|^2}) \in D \cap \{z_1 = 0\}\}$ , with  $q_2, q_3$  non-negative real numbers and  $q_2 q_3 \neq 0$ .
- (3)  $D = \{(z_1, z_2, z_3) : |z_1| < 1, (z_2(1 - |z_1|^2)^{-q_2/2}, z_3(1 - |z_1|^2)^{-q_3/2}) \in D \cap \{z_1 = 0\}\}$ , with  $q_2, q_3$  non-negative real numbers (Shimizu's notations for  $q_2$  and  $q_3$  are  $p_1^2$  and  $p_1^3$ ).

**If  $D$  is in case (1)**

Then any  $g$  in  $\text{Aut}(D)$  has the form  $(*)_{\text{alg}}$  with  $\alpha_1, \alpha_2, \alpha_3$  in  $\mathbb{C}^*$ .

Denote  $P_{23}(z_1, z_2, z_3) = (z_2, z_3)$ ,  $D_{23} = P_{23}(D)$ , and  $\hat{D} = \{(z_1, z_2, z_3) : z_1 \in \mathbb{C}, (z_2, z_3) \in D_{23}\}$ .

Take  $E \xrightarrow{D} B$  with Stein  $B$ . The  $\text{Aut}(D)$ -action on  $D$  induces actions on  $\hat{D}$  and  $D_{23}$  such that the sequence of maps  $D \hookrightarrow \hat{D} \rightarrow D_{23}$  is  $\text{Aut}(D)$ -equivariant. Then we get a sequence of maps of bundles above  $B$ :

$$\begin{array}{ccccc} E & \hookrightarrow & \hat{E} & \twoheadrightarrow & E_{23} \\ & \searrow D & \downarrow \hat{D} & \swarrow D_{23} & \\ & & B & & \end{array}$$

Moreover the map  $\hat{D} \rightarrow D_{23}$  turns  $\hat{D}$  into a (trivial) line bundle over  $D_{23}$ , and that bundle structure is preserved by the  $\text{Aut}(D)$ -action on  $\hat{D}$ . Therefore  $\hat{E}$  is a line bundle above  $E_{23}$ .

Now, as  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E_{23}) \subset \text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ , it follows from section 2 that  $E_{23}$  Stein. Therefore  $\hat{E}$  is Stein by [Mok]. By local triviality of our bundles,  $E$  is a locally Stein open subset of  $\hat{E}$ , so  $E$  is Stein by Docquier-Grauert theorem.

Conclusion:  $D \in \mathcal{S}$ .

**If  $D$  is in case (2) or (3)**

The huge formulas in [Shim] p. 411 tell us that in cases (2) and (3), any  $g_0 \in \text{Aut}_0(D)$  is of the form:

$$(*)_0 \quad g_0(z_1, z_2, z_3) = \left( h_1(z_1), z_2 h_2(z_1), z_3 h_3(z_1) \right),$$

where  $h_1$  can be any automorphism of  $\mathbb{C}$  in case (2) and any automorphism of the unit disk in case (3), and  $h_2$  and  $h_3$  are two holomorphic functions determined by  $h_1$ .

**Lemma 3.8.** *For any  $g \in \text{Aut}_{alg}^{\mathbb{R}}(D)$ , the integers  $\mu_1$  and  $\mu_2$  in  $(*)_{alg}$  are both zero.*

**Proof.** Take  $g \in \text{Aut}_{alg}^{\mathbb{R}}(D)$  (written as in  $(*)_{alg}$ ), and  $g_0 \in \text{Aut}_0(D)$  (written as in  $(*)_0$ ) with  $h_1$  not a homogeneous function.

Direct computation yields:

$$gg_0g^{-1}(z_1, z_2, z_3) = \left( \alpha_1 h_1(\alpha_1^{-1} z_1 X) X^{-1} h_2^{\mu_2}(\alpha_1^{-1} z_1 X) h_3^{\mu_3}(\alpha_1^{-1} z_1 X), \right. \\ \left. z_2 h_2^{a+c}(\alpha_1^{-1} z_1 X), z_3 h_3^{b+d}(\alpha_1^{-1} z_1 X) \right)$$

with  $X = \alpha_2^{\mu_2 d - \mu_3 c} \alpha_3^{-\mu_2 b + \mu_3 a} z_2^{-\mu_2 d + \mu_3 c} z_3^{\mu_2 b - \mu_3 a}$ . As  $\text{Aut}_0(D)$  is a normal subgroup of  $\text{Aut}(D)$ ,  $gg_0g^{-1}$  belongs to  $\text{Aut}_0(D)$ , so the above expression must be of the form  $(*)_0$ . In particular, the first component of  $gg_0g^{-1}(z_1, z_2, z_3)$  does not depend on  $z_2$  or  $z_3$ . As  $h_1$  is not homogeneous,  $X$  itself must be independent of  $z_2$  or  $z_3$ . Therefore  $\mu_2 = \mu_3 = 0$ .  $\square$

Now  $(*)_{alg}$  can be stated more precisely as:

$$(*)'_{alg} \quad g(z_1, z_2, z_3) = (\alpha_1 z_1, \alpha_2 z_2^a z_3^b, \alpha_3 z_2^c z_3^d).$$

As  $\text{Spec}_{\mathbb{C}} \mathcal{A}(D) \subset S^1$ ,  $\text{Spec}_{\mathbb{C}} \mathcal{A}' \subset S^1$ . Therefore, by lemmas 1.11 and 1.8, there are two possibilities:

1.  $\mathcal{A}'$  is finite.

Then by  $(*)'_{alg}$ ,  $\mathcal{A}(D)$  is finite, so  $D \in \mathcal{S}$  by [Siu 2].

2. There exists  $v = (p, q) \in \mathbb{Q}^2$  such that for any  $A' \in \mathcal{A}'$ ,  $A'v = v$ .

Up to division or multiplication by some integer, we can assume that  $v$  belongs to  $\mathbb{Z}^2$  and is unimodular. By lemma 1.5, there exists  $(p', q')$  such that  $\begin{bmatrix} p & p' \\ q & q' \end{bmatrix}$  belongs to  $SL_2(\mathbb{Z})$ . Consider the automorphism

$$h : \mathbb{C} \times (\mathbb{C}^*)^2 \rightarrow \mathbb{C} \times (\mathbb{C}^*)^2 \\ (z_1, z_2, z_3) \mapsto (z_1, z_2^p z_3^{p'}, z_2^q z_3^{q'}).$$

We replace  $D$  by  $h(D)$ . The automorphisms get conjugated by  $h$ , so any  $g_0$  and  $g$  in  $\text{Aut}_0(D)$  and  $\text{Aut}_{alg}^{\mathbb{R}}(D)$  now take the forms:

$$(*)'_0 \quad g_0(z_1, z_2, z_3) = (h_1(z_1), h_2^{p+q}(z_1), h_3^{p'+q'}(z_1)), \\ (*)''_{alg} \quad g(z_1, z_2, z_3) = (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3 z_2^*).$$

Take a bundle  $E \xrightarrow{D} B$  with  $B$  Stein.

Denote  $P_{12}(z_1, z_2, z_3) = (z_1, z_2)$ ,  $P_1(z_1, z_2) = (z_1)$ ,  $D_{12} = P_{12}(D)$ ,  $D_1 = P_1(D_{12})$ ,  $\hat{D} = \{(z_1, z_2, z_3) : (z_1, z_2) \in D_{12}, z_3 \in \mathbb{C}\}$  and  $\hat{D}_{12} = \{(z_1, z_2) : z_1 \in D_1, z_2 \in \mathbb{C}\}$ . These domains are Stein by lemma 1.6. The forms  $(*)'_0$  and  $(*)''_{alg}$  show that we get an induced  $\text{Aut}(D)$ -action on  $D_{12}$ ,  $D_1$ ,  $\hat{D}$  and  $\hat{D}_{12}$  such that the sequence of maps  $D \hookrightarrow \hat{D} \twoheadrightarrow D_{12} \hookrightarrow \hat{D}_{12} \twoheadrightarrow D_1$  is equivariant. So we get a corresponding sequence of maps of bundles above  $B$ :

$$\begin{array}{ccccccc} E & \hookrightarrow & \hat{E} & \twoheadrightarrow & E_{12} & \hookrightarrow & \hat{E}_{12} \twoheadrightarrow E_1 \\ & & \searrow & & \downarrow & & \swarrow \\ & & & & B & & \end{array} \quad \begin{array}{c} D \\ D_1 \end{array}$$

In analogy with case (1) above, we see that  $\hat{E}$  and  $\hat{E}_{12}$  are line bundles above  $E_{12}$  and  $E_1$  respectively. Now, as  $D_1$  is one dimensional,  $E_1$  is Stein by [Mok]. Same reasoning as in case (1) shows that  $E$  is Stein.

Conclusion:  $D \in \mathcal{S}$ .

#### 4. POSTPONED PROOFS

##### Theorem 2 p.4:

**Proof.** During the proof theorem 1, we already proved that if  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E) \subset S^1$ , then  $E$  is Stein.

Now assume that  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E) \not\subset S^1$ .

Then by definition of  $\mathcal{A}(E)$  there exists a continuous loop  $\gamma^0 : S^1 \rightarrow B$  such that  $\rho(\gamma^0) = g$  with  $\text{Spec}_{\mathbb{C}} A_g \not\subset S^1$ .



As  $B$  is Stein, we can embed it (holomorphically, therefore) real-analytically into some  $\mathbb{R}^N$ . So we consider  $\gamma^0$  as a map  $S^1 \rightarrow \mathbb{R}^N$ .

By a convolution argument, we get a  $C^\infty$  approximation  $\gamma^\infty$  of  $\gamma^0$ , arbitrarily close to  $\gamma^0$  in uniform distance. Then  $\gamma^\infty$ 's Fourier series truncated to high enough order gives a real-analytic approximation  $\gamma^\omega$  of  $\gamma^\infty$ , arbitrarily close to  $\gamma^\infty$  in uniform distance. Choosing  $\gamma^\omega$  close enough to  $\gamma^0$  ensures that the projection on  $B$  by closest distance is well-defined and real-analytic.

Thus we can assume the  $\gamma$  we started with to be real-analytic. Now, we can extend  $\gamma$  to a holomorphic map

$$\gamma^{\mathcal{O}} : U = \{z \in \mathbb{C} : 1 - \varepsilon < |z| < 1 + \varepsilon\} \rightarrow B$$

for some positive  $\varepsilon$ . Then by construction the bundle  $E' = (\gamma^{\mathcal{O}})^*(E)$  has an infinite cyclic  $\mathcal{A}(E')$  generated by  $g$ .

Therefore  $E'$  is just the restriction to an annulus of the complex suspension bundle of  $g$ . The methods of sections 2 and 3 are easily adapted to prove that  $E'$  is not Stein (cf. remark in the proof of proposition 2.1 p.7).

Now we prove by contradiction that  $E$  is not Stein:

As  $E'$  is not holomorphically convex, there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $E'$ , tending topologically to infinity, on which every  $f' \in \mathcal{O}(E')$  remains bounded. The projection of  $(x_n)$  on  $U$  must be contained in a compact subset of  $U$ : otherwise there would exist  $f' \in \mathcal{O}(E')$  (pulled back from  $U$ ) unbounded on  $(x_n)$ . Therefore, up to taking a subsequence, we can assume that the projection of  $(x_n)$  on  $U$  is converging in  $U$ . Thus, sequence  $(y_n)_{n \in \mathbb{N}}$  in  $E$  defined by  $y_n = \gamma_*^{\mathcal{O}}(x_n)$  tends to infinity in  $E$ . If  $E$  were Stein, there would exist  $f \in \mathcal{O}(E)$  unbounded on  $(y_n)$ , and  $f' = (\gamma^{\mathcal{O}})^*(f) \in \mathcal{O}(E')$  would be unbounded on  $(x_n)$ , which is impossible. Conclusion:  $E$  is not Stein.  $\square$

### Theorem 3 p.4:

**Proof.** The theorem follows from lemmas 1.11 and 1.10.  $\square$

### Lemma 1.1 p.6:

**Proof.** Let  $\lambda \in \mathbb{Q} \cap \text{Spec}_{\mathbb{C}} A$ . Denote by  $P_A$  the characteristic polynomial of  $A$ . Then  $P_A(\lambda) = 0$ , but as  $P_A$  is monic and has integer coefficients, any rational root of  $P_A$  must be in  $\mathbb{Z}$ . Thus  $\lambda \in \mathbb{Z}$ . As  $\frac{1}{\lambda} \in \text{Spec}_{\mathbb{C}} A^{-1}$ , we obtain similarly  $\frac{1}{\lambda} \in \mathbb{Z}$ . Therefore  $\lambda = +1$  or  $\lambda = -1$ .  $\square$

### Lemma 1.2 p.6:

**Proof.** Denote  $\text{Spec}_{\mathbb{C}} A = \{\lambda_1, \lambda_2, \lambda_3\}$ . Let  $P_A$  be the characteristic polynomial of  $A$ . Let  $P_{\lambda_1} \in \mathbb{Q}[X]$  be the minimal polynomial of  $\lambda_1$  over  $\mathbb{Q}$ .

- If  $\deg P_{\lambda_1} = 1$ . Then  $\lambda_1 \in \mathbb{Q}$ , so by lemma 1.1,  $\lambda_1 = \pm 1$ . Therefore  $P_A$  can be written  $P_A \stackrel{(*)}{=} (X - \lambda_1)Q$  with  $Q \in \mathbb{Q}[X]$  and  $\deg Q = 2$ .
  - If  $\lambda_2 = \pm 1$ ,  $\det A = 1$  implies  $\lambda_3 = \pm 1$  and we are in case [1].

- If  $\lambda_2 \neq \pm 1$ . Then  $\lambda_2 \notin \mathbb{Q}$ , and by (\*),  $Q(\lambda_2) = 0$  (remark also that necessarily  $\deg_{\mathbb{Q}} \lambda_2 = 2$  and  $\deg_{\mathbb{Q}} \lambda_3 = 2$ ).

Denoting  $\lambda = \lambda_2$ ,  $\det A = \pm 1$  gives  $\text{Spec}_{\mathbb{C}} A = \{\lambda_1 = \pm 1, \lambda_2 = \lambda, \lambda_3 = \frac{\pm 1}{\lambda}\}$ .

- \* If  $\lambda \in \mathbb{R}$ , we are in case  $[2]_{\mathbb{R}}$ .
- \* If  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , then  $\lambda_3$  must equal  $\bar{\lambda}_2$  (as complex roots of the real polynomial  $P_A$ ). As  $\lambda_3 = \frac{-1}{\lambda}$  is impossible (it would imply  $\lambda \bar{\lambda} = -1$ ), we get  $\lambda_3 = \frac{1}{\lambda}$ , which implies  $\lambda \bar{\lambda} = 1$ , so we are in case  $[2]_{\mathbb{C}}$ .
- If  $\deg P_{\lambda_1} = 2$ . Then  $\lambda_1 \notin \mathbb{Q}$ . As  $P_A(\lambda_1) = 0$ , one can write  $P_A = P_{\lambda_1} R$  with  $R \in \mathbb{Q}[X]$  and  $\deg R = 1$ . Therefore  $P_A$  has a root in  $\mathbb{Q}$ , which must be  $\pm 1$ . The same arguments as above show that we are in cases  $[2]_{\mathbb{C}}$  or  $[2]_{\mathbb{R}}$ , and that  $\deg_{\mathbb{Q}} \lambda_i \leq 2$  for  $i = 1 \dots 3$ .
- If  $\deg P_{\lambda_1} = 3$ . Then  $\deg_{\mathbb{Q}} \lambda_i = 3$  for  $i = 1 \dots 3$  because none of the above cases can occur (there we had  $\deg_{\mathbb{Q}} \lambda_i \leq 2$  for  $i = 1 \dots 3$ ). So we are in case  $[3]$ .

□

### Lemma 1.3 p.6:

**Proof.** Suppose that  $\lambda$  is a multiple root of  $P_A$ .

Assume  $A$  falls in case  $[3]$ . Then as  $P'_A(\lambda) = 0$  and  $\deg P'_A = 2$ , we get that  $\deg_{\mathbb{Q}} \lambda \leq 2$ , which is impossible.

When  $A$  falls in case  $[2]_{\mathbb{C}}$  or  $[2]_{\mathbb{R}}$ , it is easy to check that the condition  $\lambda \neq \pm 1$  prevents  $P_A$  from having a multiple root. Therefore  $A$  falls in case  $[1]$ . □

### Lemma 1.5 p.7:

**Proof.** Assume that  $A$  is in  $SL_n(\mathbb{Z})$ , and let  $v$  be its first column. Remark that for any  $N \in SL_n(\mathbb{Z})$  and any unimodular  $w$ ,  $Nw$  is still unimodular (because  $Nw = dw'$  implies  $w = dN^{-1}w'$ ). In particular

$$v = A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is unimodular.

Now let  $v_i$  be the non-zero entry of  $v$  with smallest absolute value. Take  $j \neq i$ , and replace  $v_j$  by  $v_j \bmod v_i$ . That amounts to multiplying  $v$  by a matrix  $M_1$  in  $SL_n(\mathbb{Z})$  (namely  $M_1$  has the form  $I_n + q[\delta_{ji}]_{1 \leq i, j \leq n}$ ). Doing that for all  $j \neq i$ , we get a new vector, still unimodular, with smaller entries (except for entries that were already zero). Repeating that procedure we

get

$$M_k \dots M_2 M_1 v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_p$$

( $e_p$  is the  $p^{\text{th}}$  standard vector). Thus  $M = (M_k \dots M_1)^{-1}$  is such that  $M e_p = v$ , so  $M$ 's  $p^{\text{th}}$  column is just  $v$ . If  $p \neq 1$ , swapping  $M$ 's first and  $p^{\text{th}}$  columns, and multiplying the second column by  $-1$  (to keep determinant  $+1$ ) yields the matrix we were looking for.  $\square$

**Lemma 1.6 p.7:**

**Proof.** This lemma is proved, as a particular case, in the next proof.  $\square$

**Lemma 1.7 p.7:**

**Proof.** Up to dividing by some integer, we can assume that  $v$  is unimodular. The matrix  $M$  given by lemma 1.5 yields an algebraic automorphism  $h$  of  $(\mathbb{C}^*)^n$  by the usual correspondence. Now, up to conjugation by  $h$ , we can assume that any  $g \in G_{\text{struct}}(E)$  is of the form:

$$g(z_1, \dots, z_n) = (\alpha_1 z_1 z_2^* \dots z_n^*, \alpha_2 z_2^* \dots z_n^*, \dots, \alpha_n z_n^* \dots z_n^*).$$

Therefore the  $S^1$ -action on  $D$ :

$$\begin{array}{ccc} S^1 \times D & \rightarrow & D \\ (\theta ; (z_1, \dots, z_n)) & \mapsto & (\theta z_1, \dots, z_n) \end{array}$$

commutes with the structural group  $G_{\text{struct}}(E)$ , so we get an induced  $S^1$ -action on  $E$ .

On the other hand, the projection map

$$\begin{array}{ccc} \pi : & D & \rightarrow (\mathbb{C}^*)^{n-1} \\ & (z_1, \dots, z_n) & \mapsto (z_2, \dots, z_n) \end{array}$$

is  $G_{\text{struct}}(E)$ -invariant, with the action on  $\pi(D)$  being *defined* by  $g(\pi(z_1, \dots, z_n)) = \pi(g(z_1, \dots, z_n))$ .

Therefore  $\pi : D \rightarrow D' = \pi(D)$  induces a bundle map  $q$ :

$$\frac{\tilde{B} \times D}{\Pi_1(B)} = \begin{array}{ccc} E & \xrightarrow{q} & E' \\ & \searrow & \swarrow \\ & B & \end{array} = \frac{\tilde{B} \times D'}{\Pi_1(B)}$$

Now consider the following equivalence relation on  $E$ :  $x \sim y$  when  $f(x) = f(y)$  for all  $S^1$ -invariant holomorphic functions on  $E$ . Then by [Heinz] we know that  $E/\sim$  is a complex space, which is Stein if  $E$  is Stein.

We now prove that  $q : E \rightarrow E'$  is a realization of that quotient, i.e.,  $x \sim y$  if and only if  $q(x) = q(y)$ .

- If  $q(x) \neq q(y)$  then as  $E'$  is holomorphically separable (cf. section 1), there exists  $f \in \mathcal{O}(E')$  such that  $f(q(x)) \neq f(q(y))$ . Then  $f \circ q \in \mathcal{O}(E)$  separates  $x$  and  $y$ . By construction, the  $S^1$ -orbits are contained in the fibers of  $q$ , so  $f \circ q$  is  $S^1$ -invariant. Thus  $x \not\sim y$ .
- If  $q(x) = q(y)$  then  $x$  and  $y$  both belong to the same fiber of  $q$ , which is a one-dimensional annulus (connectedness of that fiber follows from the convexity of  $\log(D \cap (\mathbb{C}^*)^n)$ ).

Any  $S^1$ -invariant function on  $E$  is constant on each circle of that annulus, hence constant on the whole annulus (by the isolated zeros theorem). Thus  $x \sim y$ .

We still need to show that if  $E'$  is Stein, then so is  $E$ . Assume  $E'$  is Stein. Extending the  $G_{struct}(E)$ -action from  $D$  to  $\hat{D} = \{(z_1, \dots, z_n) : z_1 \in \mathbb{C}^*, (z_2, \dots, z_n) \in D'\}$ , we get an equivariant map  $D \hookrightarrow \hat{D}$ , and (therefore) a corresponding injective bundle map:

$$\begin{array}{ccc} E & \hookrightarrow & \hat{E} = \frac{\tilde{B} \times \hat{D}}{\Pi_1(B)} \\ & \searrow & \swarrow \\ & B & \end{array}$$

As  $\hat{E}$  is a  $\mathbb{C}^*$ -principal bundle over  $E'$ , it is Stein by the theorem of Matsushima and Morimoto. By local triviality of our bundles,  $E$  is locally Stein in  $\hat{E}$ . Therefore  $E$  is Stein by the Docquier-Grauert theorem.  $\square$

**Lemma 1.4 p.6:**

**Proof.** Take  $A_g$  and  $A_{g'}$  in  $\mathcal{A}(D)$ . It is easy to check that  $A_{gg'}$  is just  $A_g A_{g'}$ . The inverse matrix  $A_g^{-1}$  comes from  $g^{-1}$  by the above remark. Therefore  $\mathcal{A}(D)$  is a group and  $\psi$  is a morphism, which is onto by definition of  $\mathcal{A}(D)$ .

Moreover,  $\psi$  is one-to-one:

$$\begin{aligned} \text{Assume } A_g = A_{g'}. \text{ Then } f_g^{-1} f_{g'}(p) &= f_g^{-1}(A_{g'} p + b') \\ &= A_g^{-1} A_{g'} p + A_g^{-1} b' - A_g b \\ &= p + A_g^{-1}(b' - b). \end{aligned}$$

If  $b' - b \neq 0$ , then  $A_g^{-1}(b' - b) \neq 0$ , so  $f_g^{-1} f_{g'}$  is a non-trivial translation of  $\mathbb{R}^3$  which induces an automorphism of  $\log D$ . This is impossible because  $\log D$  contains no affine line.

Thus  $b' = b$ . That proves the injectivity of  $\psi$ .  $\square$

**Lemma 1.9 p.7:**

**Proof.** As  $E$  is flat, it is given by a morphism  $\rho : \Pi_1(B) \rightarrow \text{Aut}(D)$  (see p.2). Take  $G := \rho^{-1}(H)$ . Then the bundle

$$E' = (\tilde{B} \times D)/G \xrightarrow{D} B$$

has  $G_{\text{struct}}(E') = H$  and is Stein if and only if  $E$  is Stein, because it is a finite cover of  $E$ .  $\square$

**Lemma 1.10 p.7:**

**Proof.** We know by [Shim] that any  $g \in G_{\text{struct}}(E)$  has the form

$$(*) \quad g(z_1, \dots, z_n) = (\alpha_1 z_1^{a_{11}} \dots z_n^{a_{1n}}, \dots, \alpha_n z_1^{a_{n1}} \dots z_n^{a_{nn}})$$

with  $(a_{ij})_{1 \leq i, j \leq n} \in \mathcal{A}(E) \subset GL_n(\mathbb{Z})$  and  $(\alpha_1, \dots, \alpha_n) \in (\mathbb{C}^*)^n$ .

As  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E)$  is finite, it follows from lemma 1.8 that either  $\mathcal{A}(E)$  is finite, or there exists an  $\mathcal{A}(E)$ -stable  $\mathbb{Q}$ -subspace  $\{0\} \subsetneq V_1 \subsetneq \mathbb{Q}^n$ . In the latter case, call  $\mathcal{A}_1 \subset GL(V_1)$  the group of linear automorphisms induced by  $\mathcal{A}(E)$ . Being contained in  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E)$ ,  $\text{Spec}_{\mathbb{C}} \mathcal{A}_1$  is finite, so we can apply Burnside's result again to  $\mathcal{A}_1$  acting on  $V_1$ . We repeat that procedure until we get a positive-dimensional subspace  $V_k \subsetneq \mathbb{Q}^n$  such that  $\mathcal{A}_k$  is finite (notice that if  $\dim V_k = 1$ , then finiteness of  $\text{Spec}_{\mathbb{C}} \mathcal{A}_k$  implies finiteness of  $\mathcal{A}_k$ ). Then a finite index subgroup of  $\mathcal{A}(E)$  (namely, the kernel of  $\mathcal{A}(E) \rightarrow \mathcal{A}_k$ ) fixes  $V_k$  pointwise. By lemma 1.9, we can assume that  $\mathcal{A}(E)$  fixes  $V_k$  pointwise.

Pick  $v \in V_k \cap \mathbb{Z}^n$ . Lemma 1.7 applies and yields a bundle  $E' \xrightarrow{D'} B$  with:

- $E'$  is Stein if and only if  $E$  is,
- $\dim D' = n - 1$  and  $D'$  is Stein by lemma 1.6,
- any  $g \in G_{\text{struct}}(E')$  still has the form  $(*)$  (with  $n - 1$  variables),
- $\text{Spec}_{\mathbb{C}} \mathcal{A}(E')$  is a subset of  $\text{Spec}_{\mathbb{C}} \mathcal{A}(E)$ , and is therefore finite.

Even though  $D'$  may be unbounded, the above properties allow us to repeat the procedure from the beginning. This sets up an induction on  $n$ , so we only need to prove the theorem for  $n = 1$ . But in that case, as the fiber is of dimension one,  $E$  is Stein by [Mok].  $\square$

**Lemma 1.11 p.7:**

**Proof.** Assume  $\text{Spec}_{\mathbb{C}} \mathcal{A} \subset S^1$ . We prove finiteness of the set of all characteristic polynomials of all matrices in  $\mathcal{A}$ .

Take any  $A \in \mathcal{A}$ , and let  $P$  be its characteristic polynomial. By assumption, all roots of  $P$  have modulus 1. As  $\deg P = n$ ,  $P$  has  $n + 1$  coefficients. Each of them is an integer of modulus not greater than  $n$ , because it is given by an elementary symmetric expression in the roots of  $P$ . Therefore the set of polynomials has cardinal at most  $(n + 1)^{2n+1}$ .

Conclusion:  $\text{Spec}_{\mathbb{C}} \mathcal{A}$  is finite.  $\square$

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## REFERENCES

- [Cœ-Lœ] **G. Cœuré, J.-J. Lœb:** “A counterexample to the Serre problem with a bounded domain of  $\mathbb{C}^2$  as fiber”. *Ann. of Math.*, 122 (1985), 329–334.
- [Die-For] **K. Diederich, J. Fornaess:** “Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions”. *Inventiones Math.* 39, 129–141 (1977).
- [Heinz] **P. Heinzner:** “Kompakte Transformationsgruppen Steinscher Räume” [Compact transformation groups of Stein spaces]. *Math. Ann.* 285 (1989), no. 1, 13–28.
- [Hirsch] **A. Hirschowitz:** “Domaines de Stein et fonctions holomorphes bornées”. *Math. Ann.* 213 (1975), 185–193.
- [Kaup] **W. Kaup:** “Reelle Transformationsgruppen und invariante Metriken auf komplexen Räume”. *Invent. Math.* 3 (1967), 43–70.
- [Mok] **N. Mok:** “Le problème de Serre pour les surfaces de Riemann”. *C. R. Acad. Sci. Paris Sér. A-B* 290 (1980), no. 4, A179–A180.
- [Nara] **R. Narasimhan:** “Several complex variables”. *Chicago Lectures in Mathematics*. University of Chicago Press, Chicago, IL, 1995.
- [Pfl-Zwo] **P. Pflug, W. Zwonek:** “The Serre problem with Reinhardt fibers”. *Ann. Inst. Fourier*, 54, 1 (2004), 129–146.
- [Ser] **J.-P. Serre:** “Quelques problèmes globaux relatifs aux variétés de Stein”. *Colloque sur les fonctions de plusieurs variables*, Bruxelles, 1953, 57–68.
- [Shim] **S. Shimizu:** “Automorphisms of bounded Reinhardt domains”. *Japan. J. Math.* (N.S.) 15 (1989), no. 2, 385–414.
- [Sib] **N. Sibony:** “Fibrés holomorphes et métrique de Carathéodory”. *C. R. Acad. Sci. Paris Sér. A* 279 (1974), 261–264.
- [Siu 1] **Y.T. Siu:** “All plane domains are Banach-Stein”. *Manuscripta Math.* 14 (1974), 101–105.
- [Siu 2] **Y.T. Siu:** “Holomorphic fiber bundles whose fibers are bounded Stein domains with zero first Betti number”. *Math. Ann.* 219 (1976), no. 2, 171–192.
- [Siu 3] **Y.T. Siu:** “Pseudoconvexity and the problem of Levi”. *Bull. Am. Math. Soc.* 84 (1978), 481–512.
- [Skoda] **H. Skoda:** “Fibrés holomorphes à base et à fibre de Stein”. *Invent. Math.* 43 (1977), no. 2, 97–107.
- [Steh] **J.-L. Stehlé:** “Fonctions plurisousharmoniques et convexité holomorphe de certains fibrés analytiques”. *C. R. Acad. Sci. Paris Sér. A* 279 (1974), 235–238.
- [Stein] **K. Stein:** “Überlagerungen holomorph-vollständiger komplexer Räume”. *Arch. Math.* 7 (1956), 354–361.
- [Steinb] **R. Steinberg:** “Conjugacy classes in algebraic groups”. Notes by Vinay V. Deodhar. *Lecture Notes in Mathematics*, Vol. 366. Springer-Verlag, Berlin-New York, 1974.
- [Zaf] **D. Zaffran:** “Serre problem and Inoue-Hirzebruch surfaces”. *Math. Ann.* 319 (2001), no. 2, 395–420.

- [Zwo] **W. Zwonek:** “On hyperbolicity of pseudoconvex Reinhardt domains”. Arch. Math. (Basel) 72 (1999), no. 4, 304–314.