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Breather Solutions of the Nonlinear Wave Equation

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1 Introduction.

1.1 Generalities.

This paper is devoted to the problem:

$$\begin{cases} u_{tt} - u_{xx} + g(u) = 0 \\ u(x, t + T) = u(x, t); u(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty; u_t \not\equiv 0. \end{cases} \quad (1)$$

A solution of this problem is called a *breather*.

Throughout this paper, g is assumed to be entire with $g(0) = 0$ and $g'(0) = 1$.

We write

$$g(u) = \sum_{m=1}^{\infty} g_m u^m.$$

When $g(u) = \sin u$, one can write down an explicit breather for every period T greater than 2π ; it reads:

$$u_{SG} = 4 \arctan\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \frac{\cos(t\sqrt{1-\varepsilon^2})}{\cosh(\varepsilon x)}\right), \quad (2)$$

where ε is related to T by

$$\varepsilon^2 + (2\pi/T)^2 = 1.$$

This is connected to the Inverse Scattering Transform. Some solitons are therefore breathers.

Observe that (2) is given as a series in ε , $e^{-\varepsilon x}$ and $\cos(t\sqrt{1-\varepsilon^2})$, convergent on any domain of the form

$$\{|\varepsilon| \leq \frac{1}{\sqrt{2}} + \delta; \varepsilon x \geq A(\delta)\}$$

with a suitable function $A(\delta)$.

We sketch in an appendix some previous attempts at proving or disproving the existence of breathers, and give selected references.

We now list the results of the present work. Various improvements and other approaches will be considered in subsequent papers.

1.2 Results.

Our first two results (Theorems 1 and 2) show that one can construct and characterize two different types of *formal solutions* to our problem.

One is of the type

$$\sum_{k=1}^{\infty} \left(\varepsilon^k \sum_{\substack{0 \leq q \leq l \leq k; l \geq 1 \\ k \equiv q \equiv l \pmod{2}}} a_{kql} (1/\cosh(\varepsilon x))^l \cos(qt\sqrt{1-\varepsilon^2}) \right) \quad (3)$$

in powers of the parameter ε defined above.

The other has the form

$$\sum_{l=1}^{\infty} u_l(t, \varepsilon) e^{-l\varepsilon x}, \quad (4)$$

in powers of $e^{-\varepsilon x}$. These series are constructed in Sections 2 and 3.

We shall prove that the coefficients of the second series are polynomials in $\cos(t\sqrt{1-\varepsilon^2})$, but are *rational* in ε .

Theorem 3, stated and proved in Section 4, shows that (3) *does not* define an analytic function of ε , $e^{-\varepsilon x}$ and $\cos(t\sqrt{1-\varepsilon^2})$ on any domain of the form

$$\{|\varepsilon| \leq \frac{1}{\sqrt{2}} + \delta; x \geq A\}$$

unless $g(u)$ has one of the three forms:

$$C \sin(\alpha u),$$

$$C \sinh(\alpha u),$$

$$Cu.$$

Only the first gives rise to breathers, which are merely scaled versions of (2). The sine-Gordon equation is thereby singled out among non-linear wave equations.

The proof of Theorem 3 involves a detailed study of the series (4). We have in that connection written a program in "Mathematica" which generates this series. The program was run on a Sun 3 of the Academic Computing Facility at New York University. The results of this calculation reveal that the terms of

this series, as functions of the coefficients of g , are very strongly related to one another. We give in Section 4 some first results in that direction.

Theorem 4, stated and proved in Section 5, shows that the series (4) *does converge* for every fixed *real* value of ε , and large x , *provided* that $g(u) = \sum_{m \text{ odd}} g_m u^m$ with

$$|g_m| \leq \alpha^{(m-1)}/m!$$

for every odd m and some positive α . More precisely, convergence takes place when εx lies in the domain of convergence of the corresponding series for $g(u) = \sinh(\alpha u)/\alpha$, which reads

$$u_{SHG} := \frac{4i}{\alpha} \arctan \left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \frac{\cos(t\sqrt{1-\varepsilon^2})}{i \sinh(\varepsilon x)} \right).$$

This does not contradict the former result since convergence only takes place for *large* x .

When g is not odd, we cannot expect a similar result, since the functions $u_l(t, \varepsilon)$ have in general *real* poles for $\varepsilon^2 = 1/m^2$, m an integer — as opposed to pure imaginary poles for odd g .

We can draw the following consequences from these results:

1. The sine function does have a special status in this problem, and for reasons not directly related to the Inverse Scattering Transform or Bäcklund transformations. The introduction of ε as a new variable creates a degeneracy which prevents the existence of a solution holomorphic in the three variables ε , t , and $e^{-\varepsilon x}$.
2. Our convergence result means that when ε is kept fixed, there is in some cases of interest a *convergent* series solution for x large enough, while it would have seemed difficult, in view of the divergence result, to expect more than a series of asymptotic type.

Theorem 5, stated and proved in Section 6, shows that for any odd g , without growth conditions on the coefficients, there always exists a solution of our equation which tends to zero as x tends to $+\infty$ only, and which belongs to every H^s class in x and t . This follows from a somewhat detailed study of a stable manifold-type argument. Our treatment differs from others mentioned in the Appendix in that we establish period-independent estimates and actually single out one solution in a multi-dimensional stable manifold.

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2 First Formal Solution.

2.1 Statement of the result.

We shall construct in the present section a formal solution to the equation:

$$u_{tt} - u_{xx} + u + \sum_{m=2}^{\infty} g_m u^m = 0. \quad (5)$$

We seek the solution u in the form

$$u(x, t) = v(\varepsilon x, \sqrt{1 - \varepsilon^2} t, \varepsilon) \quad (6)$$

where ε is a new parameter, and v is a formal series:

$$v = \sum_{k=1}^{\infty} \varepsilon^k v_k. \quad (7)$$

Letting $\xi := \varepsilon x$ and $\tau := \sqrt{1 - \varepsilon^2} t$, we find that v satisfies the following equation:

$$v_{\tau\tau} + v - \varepsilon^2(v_{\tau\tau} + v_{\xi\xi}) + \sum_{m=2}^{\infty} g_m v^m = 0. \quad (8)$$

Our result is that under a simple condition on g , there are precisely two nonzero such series, if we require the functions v_k to be even in their arguments ξ and τ , decaying in ξ , and of period $2\pi/\sqrt{1 - \varepsilon^2}$ in t :

Theorem 1 *Assume that*

$$\lambda := \frac{5}{6}g_2^2 - \frac{3}{4}g_3 > 0. \quad (9)$$

Then (8) has a formal solution of the form

$$\sum_{k=1}^{\infty} \varepsilon^k v_k(\tau, \xi).$$

There are only two nonzero such series, determined by the following conditions:

1. *all v_k are 2π periodic in τ and tend to zero as $\xi \rightarrow \pm\infty$;*
2. *all v_k are even in τ and ξ .*

One then finds that

1. *$v_1 = \pm S \cos \tau$ with $S = \frac{\sqrt{2/\lambda}}{\cosh \xi}$.*

2. The functions v_k have the form

$$v_k = \sum_{0 \leq q \leq k} a_{kq}(S) \cos q\tau$$

where a_{kq} is a polynomial of degree $\leq k$, vanishes if $k + q$ is odd, has the same parity as k , and is always divisible by S .

Remarks:

1. We have thus obtained and characterized the series (3) announced in the Introduction.
2. $\lambda > 0$ for $g = u - u^3$ and for $g = -(1 + u) + (1 + u)^3/2$ (“ φ^4 -model;” the factor $1/2$ makes $g'(0)$ equal to one.).

2.2 Proof of Theorem 1.

The strategy is as follows:

We first compute the first three terms of our series, which will contain two arbitrary functions (§2.2.1). The computation of v_3 will impose the value of v_1 . We next show (§2.2.2) that the linearization of the equation determining v_1 has an inverse, mapping polynomials of the form $S^3P(S^2)$ to polynomials of the form $SQ(S^2)$. This will enable us to construct the coefficients v_k inductively (§2.2.3).

2.2.1 First three terms of the series.

Substitution of the series in (8) results in the following:

$$v_{1\tau\tau} + v_1 = 0;$$

$$v_{2\tau\tau} + v_2 + g_2v_1^2 = 0;$$

$$v_{3\tau\tau} + v_3 + 2g_2v_1v_2 + g_3v_1^3 - \Delta v_1 = 0,$$

where $\Delta := \partial_{\tau\tau} + \partial_{\xi\xi}$.

Since the functions v_k are even in both τ and ξ , it follows that there are functions $S(\xi), \sigma_2(\xi)$, even and decaying as $|\xi| \rightarrow \infty$, such that:

$$v_1 = S(\xi) \cos \tau;$$

$$v_2 = -(1/2)g_2S^2(1 - (1/3)\cos 2\tau) + \sigma_2 \cos \tau;$$

$$v_{3\tau\tau} + v_3 = (S''' - S + \lambda S^3) \cos \tau - (g_2^2/6 + g_3/4)S^3 \cos 3\tau - 2g_2\sigma_2S \cos^2 \tau.$$

Here, λ is the quantity defined in Theorem 1 and the prime stands for differentiation with respect to ξ .

Periodicity of v_3 requires that $v_{3\tau} + v_3$ be orthogonal to $\cos \tau$, which imposes

$$S'' - S + \lambda S^3 = 0.$$

This equation has, up to translation, exactly two nontrivial solutions tending to zero as $|\xi| \rightarrow \infty$, namely:

$$\pm \frac{\sqrt{2/\lambda}}{\cosh \xi}.$$

They are clearly even. Each of these will generate a formal solution to our problem.

We fix S to be one of these two functions from now on.

v_3 is then given by:

$$v_3 = \frac{S^3}{8}(g_2^2/6 + g_3/4) \cos 3\tau - g_2\sigma_2 S(1 - \frac{1}{3} \cos 2\tau) + \sigma_3 \cos \tau,$$

where σ_3 , like σ_2 is still unknown. Here again, as v_3 is even in τ , no term in $\sin \tau$ appears.

2.2.2 An auxiliary Schrödinger operator.

We here give some properties of the linearization of

$$S'' - S + \lambda S^3 = 0$$

at $S_0 = \pm \frac{\sqrt{2/\lambda}}{\cosh \xi}$. This linearization is the Schrödinger operator

$$L := \partial_\xi^2 - 1 + \frac{6}{\cosh^2 x},$$

which is a translate of the Schrödinger equation with “2-soliton” potential (one of the “Bargmann potentials”). As an operator on L^2 , it possesses two eigenvalues, one of them being zero. The other is 3, with eigenfunction S^2 . $dS/d\xi$ is clearly an eigenfunction for the eigenvalue zero, and it is simple, since decaying potentials are in the limit-point case. The essential spectrum is the half-line $(-\infty, -1]$. It easily follows that L , on the space of even, square-summable functions is *invertible*.

For the proof of Theorem 1, we shall need a more precise result:

Lemma: *Let P be a polynomial. The only even, square-summable solution of $L\sigma = S^3 P(S^2)$ has the form $\sigma = SQ(S^2)$, where Q is a polynomial. It is also the only even solution tending to zero as ξ tends to infinity.*

Proof: It suffices to find one solution of the required form, since we know that L is invertible on even functions. Now, for any integer $p \geq 0$, one has

$$L(S^p) = S^p(p^2 - 1 + (3 - p(p + 1)/2)\lambda S^2)$$

which implies the result.

Remarks:

1. There is a very simple way to use the preceding calculation to prove the invertibility of L on even functions. As the essential spectrum is bounded away from zero, it suffices to show that 0 is not an eigenvalue for L on even, square-summable functions. But if there were a nonzero σ such that $L\sigma = 0$, $\sigma'(0) = 0$, then

- (a) σ has a finite number of zeros (because σ'' and σ have the same sign far out), and none of them is 0;
- (b) if $\alpha_1, \dots, \alpha_r$ are the positive zeros of σ , in increasing order, and if, say, $\sigma(0) > 0$, then

$$\Sigma := S^3(S^2 - S^2(\alpha_1)) \dots (S^2 - S^2(\alpha_r))$$

has the same sign as σ , since S is decreasing; it lies in the range of L , by the Lemma.

But σ and Σ should then be L^2 -orthogonal, which is impossible. This shows that such a σ cannot exist.

2. An analogous argument shows that

$$\sigma'' - \sigma + \beta S^2 \sigma = 0$$

has no even, square-integrable solution if $\beta \neq p(p+1)\lambda/2$ for every odd integer p . (Compare [9, p. 60].)

2.2.3 Induction and end of proof.

The argument is in six Steps.

Step 1: Induction hypothesis.

Assume, by induction, that we have, for some $m \geq 4$, found $v_1 \dots v_{m-3}$, and that they have the form:

$$v_k = \sum_{\substack{0 \leq q \leq k \\ q \equiv k \pmod{2}}} a_{kq}(S) \cos q\tau, \quad (10)$$

where the a_{kq} are polynomials of degree k at most, of the same parity as k , and divisible by S : assume also that the relations obtained by setting to zero the coefficients of $\varepsilon, \dots, \varepsilon^{m-1}$ in (8) determine v_{m-2} and v_{m-1} as well, in the form

$$\sum_{q \neq 1} a_{kq}(S) \cos q\tau + \sigma_k \cos \tau$$

where $k = m - 1$ or $m - 2$, the a_{kq} are polynomials with the properties listed above, and $\sigma_{m-2}, \sigma_{m-1}$ are arbitrary functions of ξ .

The remainder of the present Section is devoted to showing that the above properties hold for $m = 4$, and that if they hold for some m , they also hold for $m + 1$. This will prove Theorem 1.

Step 2: Hypothesis holds for $m = 4$.

This follows from the expressions for v_1, v_2 and v_3 found above.

Step 3: Equation for v_m .

Setting to zero the coefficient of v_m in (8), we find that v_m solves

$$v_{m\tau\tau} + v_m = \Delta v_{m-2} - 2g_2(v_1 v_{m-1} + v_2 v_{m-2}) - 3g_3(v_1^2 v_{m-2}) + \phi_m(v_1, \dots, v_{m-3}). \quad (11)$$

In this equation, ϕ_m is a polynomial in its arguments. From the induction hypothesis, the right-hand side of (11) is a trigonometric polynomial, and we must ascertain

i) that the coefficient of $\cos j\tau$ is, for every $j \neq 1$, a polynomial in S of the right form,

ii) that the coefficient of $\cos \tau$ is zero, so that no secular term arises.

It is the second condition that determines σ_{m-2} . As all we know from (11) is the value of $v_{m\tau\tau} + v_m$, v_m will be determined up to the addition of a term $\sigma_m(\xi) \cos \tau$, which will be computed only in the calculation of v_{m+2} .

We examine the above two points in order, thereby finishing the proof of the Theorem.

Step 4: Examination of the coefficient of $\cos j\tau$ for $j \neq 1$.

We show here that the right-hand side of (11) contains only powers of S of the same parity as m and that it contains S^2 (resp. S) as a factor for m even (resp. odd):

All terms in the right-hand side of (11) come either from Δv_{m-2} or from the expansion of some product of the form

$$v_{r_1}^{\alpha_1} \dots v_{r_s}^{\alpha_s} \quad (12)$$

where

$$s \geq 1, \alpha_i \geq 1, \sum_i \alpha_i r_i = m, 2 \leq \sum \alpha_i \leq m.$$

Under the induction hypothesis, such a product can only be a combination of terms of the form $S^q \cos l\tau$, where q and l have the same parity as $\sum_i \alpha_i r_i$, that is, as m itself.

On the other hand, the Δv_{m-2} term has the required form, since

$$(S^m)'' = S^m(m^2 - m(m+1)\lambda S^2/2).$$

Finally, we must check the powers of S which occur:

Δv_{m-2} can contain $S^q \cos l\tau$ only if $q \geq 1$ (in case l is odd) or $q \geq 2$ (in case l is even) because of the induction hypothesis.

For the remaining terms, we shall distinguish three cases according to the number s of factors in (12):

1. $s = 1$: This corresponds to the term $-g_m v_1^m$, which has a factor S^m . As $m \geq 2$, this term has the required form.
2. $s = 2$: This corresponds to terms $v_k v_{m-k}$. If m is even, they have S^2 as a factor, by the induction hypothesis. If m is odd, either k or $m - k$ is odd, and the other even; these terms therefore contain at least $S \cdot S^2$ as a factor.
3. $s \geq 3$: A term generated by such a product contains at least S to the power $\alpha_1 + \dots + \alpha_s$ which is greater than or equal to s , and therefore greater than or equal to 3: This term contains S^3 as a factor.

Step 5: Examination of the coefficient of $\cos \tau$.

The equation expressing the vanishing of the coefficient of $\cos \tau$ in the right-hand side of (11) reads

$$L\sigma_{m-2} = S^3 \varphi_m(S^2) \quad (13)$$

where L is the operator studied in Section 2.2, and φ_m is a polynomial of degree less than or equal to $(m - 3)/2$ obtained by collecting terms in $\cos \tau$ in the expansion of ϕ_m . $L\sigma_{m-2}$ comes from the substitution of v_1 and v_2 in (11). Note that $v_{m\tau\tau} + v_m$, as given by (11), contains only even cosines for even m ; we must therefore have $\varphi_m = 0$ for even m .

It now follows from the results of §2.2 that (13) has a unique even and decaying solution, which has the required form.

Step 6: End of proof.

We have seen that the induction defined above generates the coefficients v_k recursively, and that they are uniquely determined by the assumptions of decay and parity. The proof is now complete.

2.3 Other means of generating the formal series.

We briefly mention here two other means of constructing a formal solution in increasing powers of ε :

(a) Solve the Cauchy problem for (8) with data $(S(\xi), 0)$, S even, decaying. Let

$$T(S, \varepsilon) = v_\tau(\pi)/\varepsilon^2.$$

Then a direct calculation leads to

$$T(S, 0) = \text{const.}(S'' - S + \lambda S^3),$$

so that $T_S(S, 0)$ is invertible, by the results of §2.2. The existence of a formal solution to $T(S, \varepsilon) = 0$ in powers of ε follows easily.

(b) Let $F(v, \varepsilon)$ be the right-hand side of (8). Let Q denote the projection on functions orthogonal to $\cos \tau$, P being the complementary projection. Write

$$v(\xi, \tau) = S(\xi) \cos \tau + \varepsilon w(\xi, \tau),$$

where

$$\int_0^{2\pi} w \cos \tau \, d\tau = 0.$$

One can then see that the operator

$$\tilde{F}(v, \varepsilon) = QF/\varepsilon^2 + PF/\varepsilon$$

is well-defined. We consider it as a function of the variables S , w , and ε . When ε equals 0, $\tilde{F} = 0$ reduces to

$$\begin{cases} S'' - S + \lambda S^3 = 0 \\ w_{\tau\tau} + w = -g_2 S^2 \cos^2 \tau, \end{cases}$$

as it should.

Once again the linearization of $\tilde{F} = 0$ when $\varepsilon = 0$ is invertible at the nonzero solutions, and the existence of a formal solution in powers of ε follows. This second approach was suggested by techniques used for the water wave problem (see [2]).

Both approaches lead to a formal series by the same process, which may be worth mentioning. When an equation takes the form $T(u, \varepsilon) = 0$, where $T(u_0, 0) = 0$ and $T_u(u_0, 0)$ is invertible in some sense, then, under fairly general conditions, one can show that $T(u, \varepsilon) = 0$ has a formal solution in increasing powers of ε . One can put this observation in a more precise form, which however will not be needed in the present paper.

3 Second Formal Solution.

3.1 Statement of the result.

The preceding section has shown that one can achieve a formal solution of our problem in increasing powers of ε , and that the series thus obtained must have the form (3), which, we recall, is:

$$\sum_{k \geq 1} \varepsilon^k \left[\sum_{\substack{0 \leq q \leq k \\ q \equiv k \pmod{2}}} \cos q\tau \left(\sum_{\substack{q \leq l \leq k \\ l \equiv k \pmod{2}}} a_{kql} (\lambda/2)^l S^l(\xi) \right) \right] \quad (14)$$

where

$$\xi = \varepsilon x \text{ and } \tau = t\sqrt{1 - \varepsilon^2}.$$

This form suggests that there exists a formal solution in increasing powers of $e^{-\xi}$. The next Theorem shows that such is indeed the case.

Theorem 2 *For every g , there exists a formal solution of the corresponding nonlinear wave equation of the form*

$$\sum_{l=1}^{\infty} u_l(\tau, \varepsilon) e^{-l\xi}, \quad (15)$$

with coefficients u_l analytic in ε . The coefficients u_l can then be taken rational in ε , and polynomial in $\cos \tau$.

The first coefficient u_1 must have the form

$$A(\varepsilon) \cos(\tau + \theta)$$

and all u_l can be computed recursively once A and θ are given.

Remarks:

1. This series has the form (4) announced in the Introduction. Observe that no conditions on g or parity assumptions are needed here.
2. This series yields, as we shall see, more information than (3). We have generated its first terms using a Symbolic Manipulation language (Mathematica). Some interesting points revealed by this study are developed at the end of the next Section.
3. A similar series in increasing powers of S can of course be constructed, but it is more cumbersome to compute explicitly. It is again determined by its first term.
4. Multiplying the function $A(\varepsilon)$ by an analytic function which is positive for small real ε amounts to a translation in ξ .

3.2 Proof of Theorem 2.

As in the proof of Theorem 1, one sees immediately that each u_l satisfies a differential equation involving the terms previously computed in the right-hand side. The problem is to show that these equations are uniquely solvable under our present assumptions.

Let us write the equation in scaled form:

$$(1 - \varepsilon^2)u_{\tau\tau} - \varepsilon^2 u_{\xi\xi} + u + g_2 u^2 + g_3 u^3 + \dots = 0.$$

Let

$$M_l := (1 - \varepsilon^2)\partial_\tau^2 + (1 - l^2\varepsilon^2).$$

We then find that u_1 is determined by the following equation:

$$(1 - \varepsilon^2)(u_{\tau\tau} + u) = 0. \quad (16)$$

Without loss of generality, we may take

$$u_1 = A(\varepsilon) \cos \tau.$$

One can see by induction that A will only appear in u_l through a factor A^l . For the sake of simplicity, we therefore take in what follows $A(\varepsilon) = 1$.

The other coefficients u_l satisfy :

$$M_l u_l = \sum_{\substack{0 \leq q \leq l \\ q \equiv l \pmod{2}}} \alpha_{lq}(\varepsilon) \cos q\tau, \quad (17)$$

obtained by setting to zero the coefficient of $e^{-l\xi}$ in (8).

Here, the right-hand side of the last equation comes from the expansion of

$$-g_2 u^2 - \dots - g_l u^l.$$

It follows that

$$u_l = \sum_{l,q} \alpha_{lq} \cos q\tau / (1 - q^2 - \varepsilon^2(l^2 - q^2)). \quad (18)$$

By induction, one shows that u_l is always rational in ε .

Remark: An equation of the form $M_l u_l = f$ has at most one solution which is analytic (or even continuous) in ε . It is this circumstance which prevents the occurrence of undetermined terms as in Section 2. Indeed, such an indeterminacy occurs only for isolated values of ε , for a given l . Note that if g is odd, there is at most one 2π -periodic solution at all, even without this assumption.

3.3 Relation between the two formal series.

As the convergence of the series (15) will be dealt with in Section 5, we will now simply point out some connections between the two formal solutions that we now have at our disposal.

First of all, as we already remarked, Theorem 1 sets restrictions on g ; Theorem 2 does not.

The existence of the second series clearly amounts to saying that one can exchange the order of the first two summations in (14). But Theorem 2 does not follow from Theorem 1 since it is not clear that performing the summation in k , which leads to an infinite series, will leave us with a finite, let alone simply computable, result.

Theorem 2 does not imply Theorem 1 either, for two different reasons:

1. We may always write (15) as a series in powers of S , but as before, it is not clear that expanding the u_i and collecting the powers of ε will leave us with a finite result.
2. Even if this were possible, it depends on the choice of the function $A(\varepsilon)$. To recover (3), we must choose the expansion of A in such a way that the power of ε in any term of the series is not less than the power of S in the same term. One can always achieve this *formally*, but if A thus restricted cannot be given by a convergent series, we recover (3), but A is not analytic any more, and (14) doesn't make sense.

4 Divergence of the First Formal Solution.

4.1 Statement of the result.

Theorem 3 *The series (3) cannot be absolutely convergent on any region of the form*

$$|\varepsilon| < \frac{1}{\sqrt{2}} + \delta; |e^{-\xi}| < \varrho; |\cos \tau| < 1 + \varrho$$

for any choice of the positive numbers δ and ϱ , and define a breather for these values of ε , unless g is one of the following:

$$\sin(\alpha u)/\alpha, \sinh(\alpha u)/\alpha, u.$$

Remarks:

1. The series in question does converge on the domain given in Theorem 3 for the three exceptional functions we have listed, as the explicit form of the corresponding solutions show, provided that ϱ is chosen small enough with respect to δ . When $g = u$, the series (4) actually reduces to its first term.
2. Among the three exceptional functions g listed in the Theorem, only the first gives rise to breathers. For the other two, the series (3) converges only for $|\xi|$ large enough. Such a behaviour is actually typical, as is shown by Theorem 4 below.

4.2 Proof of Theorem 3.

We will first prove that the assumptions of Theorem 3 imply that the series (15) converges for large ξ and for *complex* values of ε of modulus less than or equal to $\sqrt{2}^{-1} + \delta$. This, in turn, will lead to the determination of the coefficients of g .

4.2.1 Reduction to the second series.

The hypothesis of the Theorem implies that u is a holomorphic function of the three variables ε , $e^{-\xi}$, $\cos \tau$ in the domain indicated. This means that u can also be expanded in increasing powers of $e^{-\xi}$. We know from Theorem 2 that such a series is entirely determined by its first term, which, in the notation of (14), reads

$$\sum_k a_{k11} \varepsilon^k \cos \tau.$$

Let us call this $a(\varepsilon) \cos \tau$. By assumption, a is analytic, and is real for real ε . We may take it positive for positive ε . It is not identically zero since otherwise u itself would be.

We claim that a cannot vanish for $\varepsilon \neq 0$.

Indeed, if $a(\varepsilon_0) = 0$, (where ε_0 may be complex), as the terms of the series in powers of $e^{-\xi}$ can be found recursively by the procedure of Section 3, we reach the conclusion that $u(x, t, \varepsilon_0) \equiv 0$. Therefore, $w := \partial u / \partial \varepsilon(x, t, \varepsilon_0)$ is a solution of:

$$w_{\tau\tau} + w - \varepsilon_0^2 \Delta w = 0.$$

Our assumptions on u show that w has a Fourier expansion $\sum_j w_j \cos j\tau$, and that for every j ,

$$(1 - j^2)w_j - \varepsilon_0^2(\partial_\xi^2 - j^2)w_j = 0.$$

This equation has only one solution tending to 0 as $|\xi|$ tends to infinity, namely zero. The argument extends by induction to all derivatives of u with respect to ε at $\varepsilon = \varepsilon_0$. As u is assumed to be analytic, we must have $u \equiv 0$. This contradiction shows that a never vanishes.

Now $v(\xi - \log a(\varepsilon), \tau, \varepsilon)$ is again analytic in $e^{-\xi}$ and $\varepsilon \neq 0$, and represents a breather. As v is a function of $e^{-\xi}$, this solution is single-valued in $\{0 < |\varepsilon| < (1/\sqrt{2}) + \delta\}$. It has a convergent series of the form (15), but now with $u_1 = \cos \tau$.

4.2.2 End of Proof.

We now know that (15) is convergent, with $u_1 = \cos \tau$. The coefficients u_2, u_3, \dots can then be computed recursively. These coefficients should also be analytic in ε . But from Section 3, we also know that they are merely rational in ε . u has therefore potential poles for all the values of ε such that there exist q and l , nonnegative integers of the same parity, satisfying

$$l \geq q \text{ and } h(l, q) := 1 - q^2 - \varepsilon^2(l^2 - q^2) = 0. \quad (19)$$

We shall express that for these poles not to occur, some relations must be satisfied by the coefficients of g .

To avoid a pole for $\varepsilon = 1/2$ in u_2 , we must have, as is found by computing u_2 ,

$$g_2 = 0.$$

There is no condition on g_3 . If it is zero, positive, negative, we shall see that g must equal u , $\sinh(\alpha u)/\alpha$, $\sin(\alpha u)/\alpha$, with $|g_3| = \alpha^2/6$.

Assume u_1, \dots, u_{l-1} have been found, for some $l \geq 4$, and that they have no pole of modulus $\leq 1/\sqrt{2}$ if and only if

$$g_2 = \dots = g_{[(l-1)/2]} = 0, \text{ and } g_{2m+1} = \frac{(\text{sgn}(g_3))^{m-1} \alpha^{2m}}{(2m+1)!} \quad (20)$$

for $2m+1 \leq l-1$.

We claim that this assertion is true with l replaced by $l+1$.

Let us write the equation giving u_l in the form

$$M_l u_l = (M_l u_l)_0 + \varepsilon^{l-1} \cos^l \tau ((g_l)_0 - g_l). \quad (21)$$

Here, the subscript 0 refers to the corresponding quantities where g and u have been replaced by the exceptional values given by (20).

Observe now that the last term in (21) contributes, if $g_l \neq (g_l)_0$, a non-zero term in $\cos 3\tau$ if l is odd, and a constant term if l is even. Each of them leads to a pole in u_l , since $(u_l)_0 = M_l^{-1} (M_l u_l)_0$ has *no* pole. The pole in question occurs for $\varepsilon^2 = -8/(l^2 - 9)$ for odd l , $\varepsilon^2 = 1/l^2$ for even l . For $l \geq 4$,

$$1/l^2 < 1/2,$$

and for odd $l \geq 4$,

$$8/(l^2 - 9) \leq 1/2;$$

both types of poles lie in the range $|\varepsilon| < \sqrt{2}^{-1} + \delta$.

It follows that u_l must have a pole in the forbidden range, unless $g_l = (g_l)_0$, in which case $u_l = (u_l)_0$

This ends the proof of Theorem 3.

4.3 Further remarks.

We have seen that the vanishing of the pole corresponding to $q = 0$ or 3 for every l suffices to determine the coefficients g_k one by one. But there are in principle other poles, coming from the higher harmonics. That all these conditions are automatically satisfied for the sine-Gordon equation is a remarkable fact. It suggests that the conditions corresponding to the vanishing of these poles are strongly interrelated. We list in Table 1 some of these conditions, the more complicated of which we obtained with the help of a computer. In this Table, we assume that g is *odd*, with $g_3 = -1/6$, and we make use of the following abbreviations:

$$(A) : 1 - 120g_5 = 0;$$

l	q	Pole	Condition
7	1	$i/\sqrt{2}$	(A)
	3	$i/\sqrt{2}$	(A)
		$i/\sqrt{5}$	(B)
	5	$i/\sqrt{2}$	(A)
		i	(C)
9	3	i	(C)
		$i/\sqrt{2}$	$(31 - 2880g_5)(1 - 120g_5) = 0$
		$i/\sqrt{5}$	(B)
		$i/\sqrt{9}$	$2500 - 338305g_5 + 1109760g_5^2 - 1239210g_7 - 1354752g_9 = 0$
	5	i	(C)
		$i/\sqrt{2}$	$(1 - 192g_5)(1 - 120g_5) = 0$
		$i\sqrt{3}/7$	$125 - 26985g_5 + 1739520g_5^2 + 115830g_7 + 746496g_9 = 0$
		$i/\sqrt{5}$	(B)
		7	$i\sqrt{3}/2$
	i		(C)
	$i/\sqrt{2}$		$1 - (120g_5)^2 = 0$

Table 1: Conditions on g

$$(B) : 1 - 129g_5 - 378g_7 = 0;$$

$$(C) : g_5 + 42g_7 = 0.$$

Observe from the Table that, for instance, the conditions arising from poles i and $i/\sqrt{5}$ imply those arising from $i/\sqrt{2}$. Similarly, those from poles i , $i/3$ and $i/\sqrt{3}/2$ imply that $g_5 = 1/120$ or $3/10$; those from i , $i/3$ and $i/\sqrt{5}$ imply that $g_5 = 1/120$ or $589/7680$; those from i , $i/\sqrt{5}$ and $i/\sqrt{3}/2$ imply that $g_5 = 1/120$ or $1/7680$.

The recurrence of some of the conditions is, however, no surprise: if a pole has not been cancelled in, say, the computation of u_7 , it will reappear in u_9 , and the conditions for its disappearance must contain those omitted in the preceding step of the calculation.

5 Convergence of the Second Formal Solution.

5.1 Statement of the result.

The preceding Section has shown that an expansion of a breather in increasing powers of ε cannot converge in the domain in which the corresponding series for the sine-Gordon equation does. We found indeed that the solution would

then also admit an expansion in powers of $e^{-\varepsilon x}$, which exhibits poles for finite values of ε . This fact could not have been found by the study of asymptotics as ε tends to zero.

For the φ^4 model, where $g(u) = -(1+u) + (1+u)^3/2$, it follows from (19) that poles on the real axis always occur.

But poles are all on the imaginary axis if g is *odd*. This suggests that when g is odd, the series (15) might be convergent, at least for large ξ , provided that ε is real and *kept fixed*. This is precisely the content of the next Theorem, stated and proved below.

Theorem 4 *Assume that g is odd and that its coefficients satisfy:*

$$|g_m| \leq \frac{\alpha^{m-1}}{m!} \quad (22)$$

for all m , with some $\alpha > 0$. Then (8) has a solution given by a series

$$\sum_{l=1}^{\infty} u_l(\tau, \varepsilon) e^{-l\xi}$$

convergent in the same domain as the corresponding series where all g_m are replaced by $\alpha^{m-1}/m!$.

Remarks:

1. The majorant series mentioned in the Theorem is αu_{SHG} , where u_{SHG} was defined in §1.2, rearranged after expanding $1/\cosh \xi$ in increasing powers of $e^{-\xi}$.
2. It follows from the next Section that there is, up to translation, only one solution which decays exponentially as $|\xi| \rightarrow \infty$ and involves odd cosines only. This means that the above function is *the only candidate for an exponentially decaying breather*.

5.2 Proof of Theorem 4.

We shall prove the Theorem by a majorant method.

Step 1: Preliminaries.

We recall that Theorem 3 constructed a formal solution

$$\sum_{l \geq 1} u_l e^{-l\xi}$$

and that this is equivalent to having a formal solution in powers of S , or $1/\cosh \xi$. We fix now $u_1 = \varepsilon$ and prove the convergence of the resulting series.

If g leads to a breather $u(x, t)$ for some value of the period, then $\alpha g(u/\alpha)$ admits the breather $\alpha u(x, t)$, and it satisfies (22) with $\alpha = 1$. We therefore assume from now on that $\alpha = 1$.

Step 2: The case $g = \sinh$.

We show that by a suitable translation in ξ , we can obtain from u_{SHG} a convergent series solution to

$$u_{tt} - u_{xx} + \sinh u = 0$$

with first term $\varepsilon e^{-\xi} \cos \tau$. u_{SHG} is equal to

$$\begin{aligned} & 4i \arctan \left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \frac{\cos \tau}{i \sinh \xi} \right) \\ &= 4 \sum_{l \geq 0} e^{-(2l+1)\xi} \sum_{0 \leq s \leq l} \varepsilon^{2s+1} \left(\frac{2 \cos \tau}{\sqrt{1-\varepsilon^2}} \right)^{2s+1} \\ &= 8 \sum_{l \geq 0} e^{-(2l+1)\xi} \sum_{0 \leq r \leq l} \cos(2r+1)\tau \\ & \quad \left[\sum_{r \leq s \leq l} \left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \right)^{2s+1} \binom{2s+1}{s-r} \right] \\ &:= \sum_{0 \leq s \leq l} b_{ls}(\varepsilon) e^{-(2l+1)\xi} \cos(2s+1)\tau. \end{aligned} \tag{23}$$

The series

$$\sum_{l,s} b_{ls} e^{-(2l+1)\xi} \cos(2s+1)\tau$$

is absolutely convergent in the domain

$$\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \frac{1}{|\sinh \xi|} < 1.$$

As the coefficient of $e^{-\xi}$ in this series is

$$8\varepsilon \cos \tau / \sqrt{1-\varepsilon^2},$$

if we set

$$a(\varepsilon) := \log(\sqrt{1-\varepsilon^2}/8),$$

we find

$$u_{SHG}(\xi - a(\varepsilon), \tau, \varepsilon) = \sum b_{ls} e^{-(2l+1)\xi} (\sqrt{1-\varepsilon^2}/8)^{2l+1} \cos(2s+1)\tau$$

and this series now converges for

$$\left| \frac{\varepsilon/\sqrt{1-\varepsilon^2}}{\sinh(\xi - a(\varepsilon))} \right| < 1.$$

Note that such a region includes the set

$$\{\xi \geq c, 0 \leq \varepsilon \leq d\}$$

for $1/c$ and d small enough.

Let now

$$\bar{a}_{ls}(\varepsilon) := b_{ls}(\sqrt{1-\varepsilon^2}/8)^{2l+1}.$$

The series (15) for $g = \sinh$ is then equal to

$$\sum_{l,s} \bar{a}_{ls} e^{-(2l+1)\xi} \cos(2s+1)\tau.$$

The quantities \bar{a}_{ls} are all nonnegative for ε nonnegative and fixed.

Step 3: Majorant method and end of proof.

Take g as in the Theorem, and consider the series (15) with, we recall, $u_1 = \varepsilon$.

We shall write it as

$$\sum_{0 \leq s \leq l} a_{ls}(\varepsilon) e^{-(2l+1)\xi} \cos(2s+1)\tau.$$

We have $a_{00} = \bar{a}_{00} = \varepsilon$. Let us show by induction on l that $|a_{ls}| \leq \bar{a}_{ls}$ for $s = 0, 1, \dots, l$.

Assume that the result holds for all integers less than or equal to $l-1$; u_l is then determined by an equation of the form

$$L_{2l-1}u_{2l-1} = -g_3(\dots) - \dots - g_{2l-1}(\dots).$$

All terms on the right-hand side are obtained by multiplying $-g_q$, for some q , by a linear combination with *nonnegative* coefficients, of terms $a_{ps} \cos(2s+1)\tau$ already computed. To obtain u_l , one must divide each of them by $h(2l+1, 2s+1)$ as given by (19), since

$$L_{2l+1} \cos(2s+1)\tau = h(2l+1, 2s+1) \cos(2s+1)\tau.$$

As s is never greater than l , we always have $h < -\varepsilon^2$ if $l \geq 3$; in particular, h is negative. Therefore, $|a_{ls}|$ is no greater than the expression obtained by replacing every g_m by $1/m!$ for odd m , and every a_{ps} by \bar{a}_{ps} , for $p < l$. The result of this substitution is precisely \bar{a}_{ls} .

This is the desired result. The formal series solution is dominated term by term by the corresponding series for the ‘‘sinh-Gordon’’ equation, and therefore converges in the same domain.

6 Decaying solution in H^s .

6.1 Statement of the result.

We assume in this Section that g is an *odd* entire function. We shall prove the following result.

Theorem 5 *Let s be a real number greater than $3/2$. For every T greater than, but close enough to 2π , there exists, up to translation and sign, a unique T -periodic solution of*

$$u_{tt} - u_{xx} + g(u) = 0$$

which tends exponentially to zero as x tends to $+\infty$, is odd in $\cos(2\pi t/T)$ and is of class H^s in x and t .

The proof will be achieved by the application of the contraction mapping theorem. The method follows closely the corresponding argument for ordinary differential equations, with the difference that we shall follow the dependence of the estimates on T and s .

The next section introduces the notation and defines the map, the fixed points of which are the desired solutions. The third is devoted to the action of nonlinear functions on H^s spaces. We shall re-derive there a few classical results; the explicit constants in the inequalities given here may be new. The two final sections contain the iteration argument and the proof of our result.

6.2 Setting and notations.

As in the preceding paragraphs, we introduce the parameter

$$\varepsilon := \sqrt{1 - (2\pi/T)^2}$$

and the scaled variables ξ and τ . The value of T is fixed for the rest of this Section.

We consider the space $X \subset H^s \times H^{s-1}$ (for some $s > 3/2$), which is defined as the set of pairs

$$\begin{bmatrix} u \\ v \end{bmatrix}$$

of functions of a variable τ that are 2π -periodic and have only odd harmonics in their Fourier expansions. We let

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X^2 = |u|_s^2 + \varepsilon^2 |v|_{s-1}^2,$$
$$A = \begin{bmatrix} 0 & 1 \\ \varepsilon^{-2}((1 - \varepsilon^2)\partial_\tau^2 + 1) & 0 \end{bmatrix}.$$

where we have used the definition

$$|u|_s^2 := \sum_j |u_j|^2 (1+j^2)^s$$

if $u = \sum_j u_j \cos j\tau$.

Every element U of X admits the decomposition

$$U = U_- + U_+ + \sum_{j \text{ odd}, j \geq 3} U_j, \quad (24)$$

where

$$U_- = \begin{bmatrix} a \\ -a \end{bmatrix} \cos \tau, \quad U_+ = \begin{bmatrix} b \\ b \end{bmatrix} \cos \tau, \quad U_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix} \cos j\tau.$$

One derives easily:

$$\|U\|_X^2 = \|U_+ + U_-\|_X^2 + \sum_j \|U_j\|_X^2$$

and

$$\frac{2\varepsilon^2}{2+\varepsilon^2} (\|U_+\|_X^2 + \|U_-\|_X^2) \leq \|U_+ + U_-\|_X^2 \leq 2(\|U_+\|_X^2 + \|U_-\|_X^2). \quad (25)$$

Writing

$$\omega_j = \sqrt{(1-\varepsilon^2)j^2 - 1},$$

we obtain

$$e^{A\xi} U_\pm = e^{\pm\xi} U_\pm;$$

$$e^{A\xi} U_j = \begin{bmatrix} \cos(\omega_j \xi / \varepsilon) & (\varepsilon / \omega_j) \sin(\omega_j \xi / \varepsilon) \\ -(\omega_j / \varepsilon) \sin(\omega_j \xi / \varepsilon) & \cos(\omega_j \xi / \varepsilon) \end{bmatrix} U_j,$$

for all odd $j \geq 3$. It follows that

$$\|e^{A\xi} U_j\|_X^2 \leq C \{ (u_j)^2 [(1+j^2)^s + (1+j^2)^{s-1} \omega_j^2] + \varepsilon^2 (v_j)^2 [(1+j^2)^s / \omega_j^2 + (1+j^2)^{s-1}] \}. \quad (26)$$

Now

$$\left\| \sum_j U_j \right\|_X^2 = \sum_j (u_j)^2 (1+j^2)^s + \varepsilon^2 (v_j)^2 (1+j^2)^{s-1},$$

and if

$$\varepsilon^2 \leq \varepsilon_0^2 < 8/9,$$

there is a positive number $\kappa < 1$ such that

$$\kappa(1+j^2) \leq \omega_j^2 = (1-\varepsilon^2)j^2 - 1 \leq 1+j^2.$$

It follows that for every j ,

$$\|e^{A\xi}U_j\|_X^2 \leq (1 + 1/\kappa) \|U_j\|_X^2. \quad (27)$$

Observe that this estimate is *independent* of ε .

We also have immediately

$$\|e^{A\xi}U_\pm\|_X \leq e^{\pm\xi} \|U_\pm\|_X. \quad (28)$$

We next define an operator M acting on the space

$$Y \subset C([0, +\infty), X)$$

consisting of those continuous X -valued functions making the quantity

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_Y := \sup_{\xi > 0} e^{\beta\xi} \left\| \begin{bmatrix} u(\xi) \\ v(\xi) \end{bmatrix} \right\|_X$$

finite; β is a given positive number.

Fix the number β between 0 and 1, as well as a real number a to be taken sufficiently small later. Define M as follows:

$$\begin{aligned} M(U)(\xi) &= e^{-\xi} \begin{bmatrix} a \\ -a \end{bmatrix} \cos \tau \\ &\quad + \int_0^\xi e^{A(\xi-s)} F_-(U)(s) ds \\ &\quad - \int_\xi^\infty e^{A(\xi-s)} [F_+(U) \\ &\quad + \sum_{j \geq 3} F_j(U)](s) ds, \end{aligned} \quad (29)$$

where

$$F(U) = \begin{bmatrix} 0 \\ g_3 u^3 + g_5 \varepsilon^2 u^5 + \dots \end{bmatrix} = \begin{bmatrix} 0 \\ h(\varepsilon u) u^3 \end{bmatrix}.$$

and $F = F_+ + F_- + \sum_j F_j$ is the decomposition (24) applied to $F(U)$.

Remarks:

1. The convergence of the improper integral will be shown in §6.4 below; it follows from the fact that M sends Y to itself.
2. If $M(U) = U$, then writing $U = \begin{bmatrix} u \\ v \end{bmatrix}$, one finds that

$$U_\xi = AU + F(U)$$

and therefore that εu solves (8) and that $v = u_\xi$. As u has, by construction, exponential decay, we have achieved the desired solution.

3. The uniqueness of the solution will follow from the fact that, as we prove below, the map M is a contraction near 0 in Y .

Before we proceed, we shall need a few estimates on products in H^s . They are proved in the following section.

6.3 Products in H^s .

Let H^s denote the ordinary Sobolev space of 2π periodic functions of a variable τ . It is well-known that this space is an algebra for $s > 1/2$. We establish here this fact with an explicit s -dependence of the constants.

If u has the Fourier expansion

$$u = \sum_{j=-\infty}^{+\infty} u_j e^{ij\tau},$$

we define its H^s norm by

$$|u|_s^2 = \sum_j (1 + j^2)^s |u_j|^2.$$

(The notation is here slightly different from that in the other paragraphs of this paper.)

We prove our estimates for C^∞ functions only, which as usual, implies the corresponding estimates for non-smooth functions, by a regularization argument.

If u, v are of class C^∞ , we have

$$(uv)_j = \sum_k u_k v_{j-k}.$$

As for every x, y , and positive α

$$(1 + y^2)^\alpha + (1 + (x - y)^2)^\alpha \geq (1 + x^2/4)^\alpha,$$

we have

$$(1 + j^2)^{s/2} \leq 2^s [(1 + k^2)^{s/2} + (1 + (j - k)^2)^{s/2}].$$

It follows that

$$\begin{aligned} |(uv)_j|^2 (1 + j^2)^s &\leq \left(\sum_k 2^s [(1 + k^2)^{s/2} u_k v_{j-k} + u_k (1 + (j - k)^2)^{s/2} v_{j-k}] \right)^2 \\ &\leq 2^{2s+1} \left[\left(\sum_k (1 + k^2)^s u_k^2 \right) \left(\sum_k v_k^2 \right) \right. \\ &\quad \left. + \left(\sum_k (1 + (j - k)^2)^s v_{j-k}^2 \right) \left(\sum_k u_k^2 \right) \right]. \end{aligned} \quad (30)$$

On the other hand,

$$\sum_k u_k^2 \leq |u|_s^2 \left(\sum_k (1+k^2)^{-s} \right) \leq |u|_s^2 (1+2\zeta(2s)), \quad (31)$$

with

$$\zeta(s) = \sum_{n \geq 1} n^{-s}.$$

If $s > 1/2$, we find

$$|uv|_s \leq 2^{s+1} \sqrt{1+2\zeta(2s)} |u|_s |v|_s. \quad (32)$$

Remark: One can slightly improve this estimate if u and v contain only odd harmonics, since the sum in (31) is taken over odd values of k only.

We now have, for every $m \geq 1$ and s bounded away from $1/2$,

$$|u^m|_s \leq (2^s C)^m |u|_s^m.$$

It follows that

$$|g(u)|_s \leq \text{const.} \sum_m |g_m| (|u|_s (2^s C))^m := \Phi(2^s |u|_s). \quad (33)$$

If $g(u)$ contains u^p as a factor, we obtain

$$|g(u)|_s \leq |u|_s^p \Psi_s(2^{s/2} |u|_s).$$

6.4 Construction of the decaying solution.

We show here that $M(U) = U$ can be solved by the Banach fixed point theorem in the space Y we have defined.

To prove this, we estimate, for U and V in Y ,

$$\begin{aligned} \|M(U) - M(V)\|_X(\xi) &\leq \int_0^\xi e^{-(\xi-s)} \|F_-(U) - F_-(V)\|_X(s) ds \\ &\quad + \int_\xi^\infty e^{(\xi-s)} \|(F_+ + \sum_j F_j)(U) \\ &\quad - (F_+ + \sum_j F_j)(V)\|_X(s) ds. \end{aligned} \quad (34)$$

Now, for every U , $\|U\|_X$ can be estimated from the decomposition (24) and inequalities (25). We thus obtain:

$$\|M(U) - M(V)\|_Y \leq C_1 \varepsilon^{-1} \sup_{\xi > 0} \left(\int_0^\xi e^{-(\xi-s)} e^{-\beta s} \|F(U) - F(V)\|_Y ds \right)$$

$$\begin{aligned}
& + \int_{\xi}^{\infty} [1 + e^{(\xi-s)}] e^{-\beta s} \|F(U) - F(V)\|_Y ds \Big) e^{\beta \xi} \\
\leq & C_1 \varepsilon^{-1} \|F(U) - F(V)\|_Y \sup_{\xi \geq 0} e^{\beta \xi} \\
& \times \left(e^{-\xi} \frac{e^{(1-\beta)\xi}}{1-\beta} + e^{\xi} \frac{e^{-(1+\beta)\xi}}{1+\beta} \right), \tag{35}
\end{aligned}$$

so that

$$\|M(U) - M(V)\|_Y \leq \frac{2C_1/\varepsilon}{1-\beta^2} \|F(U) - F(V)\|_Y.$$

Now, by Section 6.2, if we write

$$\begin{aligned}
U &= \begin{bmatrix} u \\ u' \end{bmatrix}, \\
V &= \begin{bmatrix} v \\ v' \end{bmatrix},
\end{aligned}$$

we have

$$\begin{aligned}
\|F(U) - F(V)\|_X &= \varepsilon |u^3 h(\varepsilon u) - v^3 h(\varepsilon v)|_{s-1} \\
&= \varepsilon \left| \int_0^1 f(u + t(v-u))(v-u) dt \right|_{s-1}, \tag{36}
\end{aligned}$$

where

$$f(\xi) := d(\xi^3 h(\varepsilon \xi))/d\xi.$$

It follows that, using the results of §6.3,

$$\begin{aligned}
\|F(U) - F(V)\|_X &\leq \varepsilon C_2 (|u|_{s-1}, |v|_{s-1}) |v-u|_{s-1} 2^s \\
&\leq \varepsilon C_3(s, \|U\|_X, \|V\|_X) \|U - V\|_X, \tag{37}
\end{aligned}$$

where C_3 is an increasing function of its arguments. As $\| \cdot \|_X \leq \| \cdot \|_Y$,

$$\|M(U) - M(V)\|_Y \leq (2/(1-\beta^2)) C_1 C_3(s, \|U\|_Y, \|V\|_Y) \|U - V\|_Y. \tag{38}$$

Observe now that the map M is a contraction on a sufficiently small ball in Y , uniformly in ε , because $f(0) = 0$ ensures that C_3 tends to zero with $\|U\|_Y + \|V\|_Y$. If therefore

$$M(0) = \begin{bmatrix} a \\ -a \end{bmatrix} \cos \tau$$

is small enough, the iterates $(M^n(0))_{n \geq 1}$ converge as $n \rightarrow \infty$ to a fixed point of M .

This ends the construction of the decaying solution.

6.5 Uniqueness.

We prove here the uniqueness part of the Theorem. From the results of the previous section, we know that there is one solution in a suitably small neighborhood of the origin in Y . This solution clearly does not depend on the choice of the number β introduced in §6.4, but it might depend on a . We show here that different values of a correspond to the translates of the solution obtained in §6.4.

Let us write M_a for M , to stress its dependence on a .

If $M_a(U) = U$, where U lies in Y for some value of β , then the translates $U_\lambda = U(\cdot + \lambda)$ tend to zero in X (and Y), and therefore $V_\lambda = U_\lambda - M_0(U_\lambda)$ also tends to zero. But as U_λ and $M_0(U_\lambda)$ both solve

$$dU/d\xi = AU + F(U_\lambda),$$

we have

$$dV_\lambda/d\xi = AV_\lambda$$

using the definition of M_a . As V_λ decays exponentially, it follows that

$$V_\lambda = \begin{bmatrix} a(\lambda) \\ -a(\lambda) \end{bmatrix} \cos \tau.$$

Furthermore, $a(\lambda)$ is continuous and tends to zero as λ tends to infinity. It follows that for λ large enough, V_λ is the unique fixed point of $M_{a(\lambda)}$. As $U \neq 0$, $a(\lambda) \neq 0$ for large λ , and $a(\lambda) = a$ has a solution for a small enough and of the same sign as $a(\lambda)$.

This proves the uniqueness, up to sign and translation, of the small, odd, exponentially decaying solutions of the problem.

7 Appendix: Bibliographical Remarks.

We here briefly discuss the literature on the problem. We have limited ourselves to the works that have a direct relation to our results. The contributions are grouped according to the method they use.

1) The authors of [5, 11] and others first observed that the first terms of a formal expansion in powers of ε could be found for the φ^4 -model, in which one takes

$$g(u) = \frac{1}{2}(-(1+u) + (1+u)^3).$$

This suggested the existence of breathers for an equation other than sine-Gordon. This result is explained by our Theorem 1. Our second formal expansion, however, is not to be found in the literature. Endeavors have been made to combine this result with a Fourier expansion with a view to obtain a contradiction at the formal level: In [7, 8], by truncating the Fourier series

and using numerical integration of the remaining equations, one tries to extract information by “matching” a solution tending to zero at $+\infty$ with one tending to zero at $-\infty$. It is found there that the coefficients of g should be restricted, but that the sine does not satisfy these restrictions. In [10], an attempt is made to use the method of matched asymptotic expansions to produce a correction to the solution of order $e^{-C/|\varepsilon|}/\varepsilon$, such as would mean that the solution cannot be even in x , despite the formal solution being even to all orders. Numerical integration of approximate solutions is also used, together with several truncations where in particular 2π is substituted for the period T — a case in which (see below) breathers such as considered there do not exist.

Both of these arguments are motivated by phenomena that are found in other problems. The first is reminiscent of calculations of the splitting of separatrix loops in some ordinary differential equations, where in particular the successes of Melnikov’s method are well-known. The second is motivated by the relevance of “transcendentally small” corrections to the solutions of some recent problems.

2) A number of attempts at finding breathers numerically have been made (see e.g. [1, 6, 13, 12, 17]). One way to generate such solutions is to investigate a “head-on” collision of two “kink” solutions (solutions having different limits as x goes to plus or minus infinity). Such kinks are easy to write down for the φ^4 -model, since time-independent kinks exist. One finds that way long-lived “quasi-breather” modes. They appear to fade away after some time.

3) Energy estimates can be used to obtain bounds on possible breathers, as well as a limited number of nonexistence results. We mention here [4] which proves under mild decay assumptions that breathers, if they exist, must have period greater than or equal to 2π , [15] which shows nonexistence in a number of cases (all these cases have, in our notation, $\lambda \leq 0$, so that even formal solutions do not exist in these cases). McKean (unpublished) also obtained some uniqueness results by related methods; he also investigated the possibility of finding a “separated” solution $f(h_1(x)h_2(t))$, as in the sine-Gordon case; he finds that no other wave equation has breathers of that form.

4) It is natural to try to perturb the sine-Gordon breather itself [3]. If one sets

$$g(u) = \sin u + \alpha h(u),$$

and tries a perturbation expansion in powers of α ,

$$u = u_{SG} + \alpha u_1 + \dots,$$

one finds that u_1 cannot be found unless h satisfies some conditions. Concrete (nonexistence) results in that direction are in preparation ([3]).

5) Attempts have been made to view the problem as a dynamical system with x as “time variable”, whereby the problem becomes one of finding an orbit homoclinic to zero. We have mentioned formal arguments in that direction earlier. A. Weinstein observed that the stable manifold theorem gives, for every value of the period greater than, and close to, 2π , a three-dimensional manifold

of solutions tending to zero as x goes to $+\infty$ (but not necessarily $-\infty$). The dimension of this stable manifold increases with the period. It contains possible breathers, together with time-independent solutions. Several authors, see e. g. [14], using classical ideas from dynamical systems, show that solutions dominated by $c/\cosh x$, with c small enough must vanish identically. Such results apply to the sine-Gordon equation as well: the constant c depends on the period.

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