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# EXTREMALITY FOR THE VAFA–WITTEN BOUND ON THE SPHERE

MARC HERZLICH

ABSTRACT. We prove that the round metric on the sphere has the largest first eigenvalue of the Dirac operator among all metrics that are larger than it. As a corollary, this gives an alternative proof of an extremality result for scalar curvature due to M. Llarull.

## 1. INTRODUCTION

C. Vafa and E. Witten have shown [17] that there exists a common upper bound for the smallest eigenvalue of all *twisted* Dirac operators on a given Riemannian manifold. The common upper bound they found depends strongly on the choice of the metric on the base. Using an analogous method, H. Baum [3] exhibited later an explicit upper bound for the first eigenvalue of the (untwisted) Dirac operator on an even-dimensional Riemannian manifold that can be sent on a sphere by a map of (high) non-zero degree. This bound depends on the Lipschitz norm of the map from the manifold to the round sphere.

The goal of this short note is to elaborate further on H. Baum's results [3]. Indeed, we prove below that some *optimal* upper bound can be obtained for the bottom of the spectrum of the (untwisted) Dirac operator of a large class of metrics on the sphere. This extends [3] in two different ways, first by providing an optimal bound, and secondly by extending it to odd dimensions.

**Theorem 1.** *Let  $g$  be any Riemannian metric and  $b$  be the round metric on the sphere  $\mathbb{S}^n$ . If  $g \geq b$ , then there is a eigenvalue of the Dirac operator of  $g$  in  $[-\frac{n}{2}, \frac{n}{2}]$ . Moreover, if no eigenvalue lies in the open interval  $]-\frac{n}{2}, \frac{n}{2}[$ , then  $g$  is isometric to the round metric  $b$ .*

As an interesting corollary, one gets an alternative proof of M. Llarull's extremality result for scalar curvature on the sphere [13]. Indeed, the classical first eigenvalue estimate for the Dirac operator due to Th. Friedrich [7] states that the smallest eigenvalue  $\lambda_1(g)$  (in absolute value) of the Dirac operator satisfies:

$$|\lambda_1(g)|^2 \geq \frac{n}{4(n-1)} \inf \text{Scal}_g .$$

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Hence one concludes:

**Theorem 2** (Llarull [13]). *Let  $g$  be any metric and  $b$  the round metric on the sphere  $\mathbb{S}^n$  such that  $g \geq b$ . Then*

$$\inf \text{Scal}_g \leq n(n-1),$$

*with strict inequalities if  $g$  is distinct from  $b$ .*

The proof of Theorem 1 is close to the original approach of Vafa and Witten, see also M. Atiyah [1] and H. Baum [3]. Since we look for optimal bounds, we have however to pay a special attention to all the estimates. Especially, this implies to have two different proofs in the even- and odd-dimensional cases (see [6] for another occurrence of this problem in a related context). The first one is obtained from an index argument, whereas the second needs a slightly more delicate proof using spectral flow considerations.

*Acknowledgements.* It was Maung Min-Oo who pointed out during a lecture in July 2001 [15] that Vafa–Witten was optimal on a round sphere and that a strong link might exist with Llarull extremality. This idea appears (in a somewhat more hidden way) on pages 84–85 in M. Gromov’s long and rich paper [10]. The author is then happy to acknowledge the origin of the ideas behind this note. He also thanks H el ene Davaux, Maung Min-Oo again, Andrei Moroianu and Uwe Semmelmann for useful comments on a first draft of this paper.

## 2. BACKGROUND MATERIAL

For any metric  $g$  on  $\mathbb{S}^n$ , we let  $\Sigma_g$  be its spin bundle. Using the idea introduced by J. P. Bourguignon and P. Gauduchon [4], one may identify the spin bundle  $\Sigma_g$  with the spin bundle  $\Sigma_b$  through a lift of the principal oriented orthonormal frame bundles isomorphism

$$H : P_{SO}\mathbb{S}^n(g) \longrightarrow P_{SO}\mathbb{S}^n(b)$$

induced by the unique symmetric positive definite map  $H$  such that

$$g(H\cdot, H\cdot) = b(\cdot, \cdot).$$

This allows to transfer the Clifford action relative to  $g$  on the bundle  $\Sigma_b$ . Denoting by  $\ell_b$  (resp.  $\ell_g$ ) :  $T\mathbb{S}^n \rightarrow \text{End}(\Sigma_b)$  the Clifford actions for  $b$  and  $g$ , they are related as:

$$\ell_b(\cdot) = \ell_g(H\cdot)$$

The Levi-Civita connection for any metric  $g$  can be transferred in the same way on  $\Sigma_b$  as a metric connection (but with torsion). In what follows, we will always assume that all spinor bundles on the sphere have been identified to those relative to  $b$ , as above.

The following facts are extracted from classical texts on spinor geometry, see also [2, 3] which are useful references.

In even dimensions  $n = 2m$ , the spin bundle  $\Sigma_g$  splits into two *half-spin* bundles  $\Sigma_g^+$  and  $\Sigma_g^-$ ; on the sphere  $\mathbb{S}^{2m}$ , the whole spin bundle, or equivalently the sum of both half-spin bundles, is a trivial bundle obtained by restricting the spinor bundle of  $\mathbb{R}^{2m+1}$  to the sphere; namely:

$$\mathbb{T} := \Sigma_b = \Sigma_b^+ \oplus \Sigma_b^- = \mathbb{S}^{2m} \times \mathbb{C}^{2^m}.$$

Moreover, projections on each factor  $\Sigma_b^\pm$  can be explicitly described at a point  $x$  on the sphere as:

$$\Pi_\pm(x) = \frac{1}{2}(1 \pm i\ell_0(x))$$

where  $\ell_0$  is the algebraic Clifford action of  $\mathcal{Cl}_{2m+1}$  on  $\mathbb{C}^{2^m}$  (as a result, none of the half-spin bundles is trivial). Using this expression, it is easy to relate the trivial connection  $\nabla^0$  (of flat space) acting on  $\mathbb{T}$  with the Levi-Civita connection of  $b$  acting on  $\Sigma_b^\pm$ . If  $\psi$  is any spinor field in  $\Sigma_b^\pm$ , then:

$$\nabla_X^\pm \psi = \Pi_\pm \nabla_X^0 \Pi_\pm \psi = \nabla_X^0 \psi - \frac{1}{2} \ell_0(x) \ell_0(X) \psi$$

and the full Levi-Civita connection for  $b$  on  $\Sigma_b = \Sigma_b^+ \oplus \Sigma_b^-$  is  $\nabla^b = \nabla^+ \oplus \nabla^-$  (note that actions  $\ell_0$  on vectors tangent to  $\mathbb{S}^n$  and  $\ell_b$  may be identified).

Moreover, the bundle  $\mathbb{T} \otimes \mathbb{T} = \Sigma_b \otimes \Sigma_b$  is the (complex) differential form bundle  $\Lambda^\bullet \mathbb{S}^n \otimes \mathbb{C}$  of the sphere. On this bundle, two different Clifford actions exist. The first is the usual one for twisted spinor bundles, with the Clifford algebra acting on the left factor of the tensor product; we will continue to denote this one below by  $\ell$ , as it is defined on a decomposed element  $\sigma \otimes \tau$  of  $\Sigma_b \otimes \Sigma_b$  by:

$$\ell_b(X)(\sigma \otimes \tau) = (\ell_b(X)\sigma) \otimes \tau.$$

But there is another, right-handed, one that we shall denote by  $r$  and which is defined by:

$$r_b(X)(\sigma \otimes \tau) = \sigma \otimes (\ell_b(X)\tau).$$

Both Clifford actions can be explicitly described when one identifies  $\Sigma_b \otimes \Sigma_b$  with  $\Lambda^\bullet \mathbb{S}^{2m}$  as follows: for any  $p$ -form  $\omega$ , and any 1-form  $\alpha$

$$\ell_b(\alpha^\sharp)\omega = \alpha \wedge \omega - i_{\alpha^\sharp}\omega, \quad r_b(\alpha^\sharp)\omega = (-1)^p (\alpha \wedge \omega + i_{\alpha^\sharp}\omega)$$

(musical isomorphism  $\sharp$  referring, as always, to  $b$ ). For more information on these points, the reader is referred to the book [12] (see also the paper [16] where a concise account of these facts is given). The reader should also be careful about tensor product connections, as they might not coincide: for instance,  $\nabla^b \otimes 1 + 1 \otimes \nabla^b$  is the Levi-Civita connection on differential forms on the sphere whereas  $\nabla^0 \otimes 1 + 1 \otimes \nabla^0$  is the trivial connection induced from  $\mathbb{R}^{2m+1}$ .

In odd dimensions  $n = 2m - 1$ , the Clifford algebra is isomorphic to the sum of two copies of the matrix algebra  $\text{End}(\mathbb{C}^{2^{m-1}})$ . Hence it has two inequivalent representations which lead to two different Clifford bundles  $\Sigma_b$  and  $\Sigma'_b$  on the sphere

$\mathbb{S}^{2m-1}$ . They are equivalent as bundles with structure group  $\text{Spin}(2m-1)$  but the Clifford actions differ. Their sum is again a trivial bundle:

$$\mathbb{T} := \Sigma_b \oplus \Sigma'_b = \mathbb{S}^{2m-1} \times \mathbb{C}^{2^m} ;$$

as above, it is the restriction to the sphere of the full spinor bundle of flat space  $\mathbb{R}^{2m}$ . This bundle itself splits in two trivial subbundles of rank  $2^{m-1}$

$$\mathbb{T} = \mathbb{T}^+ \oplus \mathbb{T}^-.$$

Here we have to take care that these bundles are different from the bundles  $\Sigma_b$  and  $\Sigma'_b$  on the sphere already alluded to a few lines above: for instance,  $\mathbb{T}^+$  and  $\mathbb{T}^-$  are exchanged when Clifford multiplied by vectors, whereas  $\Sigma_b$  and  $\Sigma'_b$  are preserved by the Clifford action. Notice that, contrarily to what happens if  $n = 2m$ , the bundles  $\mathbb{T}^\pm$  are preserved by the flat connection  $\nabla^0$ . As final remark, we recall that  $\mathbb{T} \otimes \mathbb{T}$  is again isomorphic to the full differential form bundle of  $\mathbb{R}^{2m}$ .

From now on and to avoid confusion, we will use notations  $\mathbb{T}$ ,  $\Sigma_b$  and  $\Sigma_g$  in the following sense : they will always denote the corresponding bundle relative to the metric  $b$ , but endowed either with the trivial connection  $\nabla^0$ , or the Levi-Civita connection  $\nabla^b$ , or  $\nabla^\pm$ , of  $b$  on spinors, or the Levi-Civita connection  $\nabla^g$  of  $g$  on spinors.

### 3. THE PROOF: EVEN-DIMENSIONAL CASE

We consider the bundle  $S = \Sigma_g \otimes \Sigma_b^+$ ; as spin bundles for different metrics always are identified, this means that we take the tensor product of the whole spin bundle  $\Sigma_b$  with the half-spin  $\Sigma_b^+$ , but endowed with the *non-trivial* tensor product connection  $\nabla^g \otimes 1 + 1 \otimes \nabla^+$ .

We now apply Atitah-Singer index theorem to the twisted Dirac operator

$$(3.1) \quad \mathcal{D}^{g+} : \Sigma_g^+ \otimes \Sigma_b^+ \longrightarrow \Sigma_g^- \otimes \Sigma_b^+.$$

One of the most important consequences of index theory is the topological invariance of the index. An easy computation yields the index of the following model ('round') Dirac operator:

$$\text{ind} (\mathcal{D}^{b+} : \Sigma_b^+ \otimes \Sigma_b^+ \longmapsto \Sigma_b^- \otimes \Sigma_b^+) = 2$$

(up to orientation reversing), so that the index of the Dirac operator  $\mathcal{D}^{g+}$  built with the connection  $\nabla^g \otimes 1 + 1 \otimes \nabla^+$  on the twisted spin bundle  $S = \Sigma_g \otimes \Sigma_b^+$  is also 2. Its kernel is then non-zero.

Let us now consider the tensor product bundle  $\Sigma_g \otimes \mathbb{T}$ , endowed this time with the connection  $\nabla^g \otimes 1 + 1 \otimes \nabla^0$ . As the pair  $(\mathbb{T}, \nabla^0)$  is a *trivial* flat bundle on the sphere, the spectrum of the twisted Dirac operator attached to this connection is the same as the spectrum of the Dirac operator on  $\Sigma_g$ , but with each eigenvalue repeated  $2^m$  times its multiplicity. Applying standard perturbation theory, one gets the first part of the statement of Theorem 1 if one is able to bound the difference

$$\bar{L} := \mathcal{D}^{g0} - \mathcal{D}^{g+}.$$

This is easily computed on a decomposed section  $\sigma \otimes \tau$ . If  $\{e_i\}$  is any  $g$ -orthonormal basis on the sphere,

$$\begin{aligned} \bar{L}(\sigma \otimes \tau) &= \sum_{i=1}^n \ell_g(e_i) (\nabla_{e_i}^g \sigma \otimes \tau + \sigma \otimes \nabla_{e_i}^0 \tau - \nabla_{e_i}^g \sigma \otimes \tau - \sigma \otimes \nabla_{e_i}^+ \tau) \\ &= \frac{1}{2} \sum_i \ell_g(e_i) \sigma \otimes \ell_b(x) \ell_b(e_i) \tau \\ &= \frac{1}{2} r_b(x) L(\sigma \otimes \tau), \end{aligned}$$

where  $L$  is the (pointwise) linear map of vector bundles given on a spinor field  $\psi$  in  $\Sigma_g \otimes \Sigma_b^+$  by

$$L\psi = \sum_i \ell_g(e_i) r_b(e_i) \psi.$$

We now choose a  $b$ -orthonormal basis  $\{\varepsilon_i\}$  that diagonalizes  $H$ . Eigenvalues  $\mu_i$  live in  $]0, 1]$  and  $\{e_i = \mu_i \varepsilon_i\}$  is a  $g$ -orthonormal basis, and moreover,

$$L\psi = \sum_i \mu_i^{-1} \ell_b(e_i) r_b(e_i) \psi = \sum_i \mu_i \ell_b(\varepsilon_i) r_b(\varepsilon_i) \psi.$$

Using the explicit expressions of  $\ell_b$  and  $r_b$  on differential forms, one gets

$$\ell_b(\varepsilon_k) r_b(\varepsilon_k) (\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}) = \begin{cases} (-1)^p \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} & \text{if } k \in \{i_1, \dots, i_p\}, \\ (-1)^{p+1} \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} & \text{if not,} \end{cases}$$

and this eventually leads to

$$L(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}) = (-1)^p \left( \sum_{j \in \{i_1, \dots, i_p\}} \mu_j - \sum_{j \notin \{i_1, \dots, i_p\}} \mu_j \right) \varepsilon_1 \wedge \cdots \wedge \varepsilon_p$$

for  $p$  between 0 and  $n$ . Ordering the eigenvalues in decreasing order  $\mu_1 \geq \cdots \geq \mu_n$ , the maximal absolute value of the eigenvalues of  $L$  at fixed  $p$  is obviously achieved by

$$\left| \sum_{1 \leq j \leq p} \mu_j - \sum_{j \geq p+1} \mu_j \right|$$

for each  $p$ . Since  $0 < \mu_j \leq 1$  for each  $j$ , it is obvious that the suprema of all such expressions for any  $p$  is bounded by  $n$  and that equality is achieved if and only if  $\mu_j = 1$  for each  $j$ . This concludes the proof of Theorem 1.  $\square$

## 4. THE PROOF: ODD-DIMENSIONAL CASE

Our treatment of the odd-dimensional case is much inspired by [1, 6]. Starting from the Dirac operator on the spin bundle  $\Sigma_g$ , our goal is to find a flat twisting bundle endowed with a non-flat connection whose associated twisted Dirac operator has a non-zero kernel. As index considerations are useless in odd dimensions, we have to rely on a spectral flow argument.

Our choice of bundle is

$$\Sigma_g \otimes \mathbb{T}^+ = \Sigma \otimes \mathbb{C}^{2^{m-1}}$$

(notations as in section 2), so that, as above, the Dirac operator  $\mathcal{D}^{g0}$  associated to  $\nabla^g \otimes 1 + 1 \otimes \nabla^0$  on  $\Sigma_g \otimes \mathbb{T}^+$  has the same spectrum as  $\mathcal{D}^g$  on  $\Sigma_g$  (up to multiplicity). Fix  $e$  unit in  $\mathbb{R}^{2m}$  and let for each  $x$  in  $\mathbb{S}^{2m-1}$ :

$$u(x) = \text{the projection of } e \cdot x \cdot \in \mathbb{C}l_{2m} \text{ in } \text{End}(\mathbb{T}^+) = \text{End}(\mathbb{C}^{2^{m-1}}).$$

As each  $x$  is skew-hermitian and of square  $-1$ ,  $u(x)$  is unitary for each  $x$  and one can define the unitarily (gauge) equivalent connection

$$\nabla^u = u^{-1} \circ \nabla^0 \circ u$$

on  $\mathbb{T}^+$ . The path of connections

$$t \in [0, 1] \longmapsto \nabla^g \otimes 1 + 1 \otimes (t\nabla^0 + (1-t)\nabla^u)$$

gives rise to a path of Dirac operators  $\mathcal{D}^t$  on  $\Sigma_g \otimes \mathbb{T}^+$  with unitarily equivalent operators at  $\{t=0\}$  and  $\{t=1\}$ . The spectral flow of this family can be computed using the index theorem on  $\mathbb{S}^{2m-1} \times \mathbb{S}^1$ , applied to the Dirac operator acting on the spinor bundle  $\Sigma_b$  twisted by the bundle obtained by identifying  $\mathbb{T}^+$  at  $\{t=0\}$  and  $\{t=1\}$  through  $u$ , and is non-zero (see [5] for instance).

As a result, there exists a value of  $t$  in  $[0, 1]$  for which the kernel of  $\mathcal{D}^t$  is non-zero. Suppose  $t \leq 1/2$ , then, as in the previous section, what is needed to conclude is an estimate of the difference

$$\mathcal{D}^t - \mathcal{D}^{g0}$$

(in case  $t > 1/2$ , one should use  $u^{-1} \circ \mathcal{D}^{g0} \circ u$  rather than  $\mathcal{D}^{g0}$ ). An easy computation shows that this difference of operators acts as the following linear map on  $\Sigma_g \otimes \mathbb{T}$  seen as a sub-bundle of the whole differential form bundle  $\mathbb{T} \otimes \mathbb{T}$  of  $\mathbb{R}^{2m}$ :

$$L_t = r_b(tx) \circ \left( \sum_{i=1}^{2m-1} \mu_i \ell_b(\varepsilon_i) r_b(\varepsilon_i) \right)$$

(same notations as in the previous section). The end of the proof is then entirely analogous to that in even dimensions.  $\square$

## 5. FINAL COMMENTS

It is known that scalar curvature extremality extends to a large family of Riemannian manifolds; among these, one can find complex projective spaces, Kähler manifolds of positive Ricci curvature, non-negatively curved locally symmetric spaces (see the foundational [10, §6] and [8, 9, 11, 13, 14]). In view of Theorem 2, it seems then natural to ask the following obvious questions:

**Question 1.** Can one find other examples of eigenvalue-extremal metrics ?

**Question 2.** In particular, if  $g$  is a metric on  $\mathbb{C}\mathbb{P}^m$  (with  $m$  odd, so that it is spin) that is larger than the Fubini-Study metric  $f$ , does one have  $|\lambda_1(g)| \leq |\lambda_1(f)|$  with equality if and only if  $g$  is isometric to  $f$  ?

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