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THE "FORGOTTEN" PROCESS : the emission stimulated by matter waves.

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Abstract: In a famous paper where he introduces the A and B coefficients, Einstein assumed that atomic decays of excited atoms can be stimulated by light waves. Here we also include atomic decays stimulated by atomic waves. It is necessary to change the Maxwell-Boltzmann statistics of thermal equilibrium into Bose-Einstein statistics and to introduce a coefficient C which complements the list of the coefficients introduced by Einstein.

Stimulated emission of light can be considered as the first step towards the laser. Similarly, stimulated production of matter waves can be considered as the basic phenomenon for an atom-laser.

Most of the results that we obtain here are not new. However, our method is very close to elementary classical physics and emphasizes the symmetry between electromagnetic and matter waves from various points of view. The level of the discussion is indirect for advanced undergraduate and possible direct for graduate.

1 Introduction

In 1916 and 1917, Einstein introduced the well known A and B coefficients [1]. For the first time the stimulated emission of photons by electromagnetic radiation was suggested. In the present paper we generalize the argument of Einstein. We assume that an atom has an internal structure and that a matter wave describes its space-time behavior. The decay of an atom into its ground state can happen spontaneously. It can also be stimulated by the electromagnetic waves or the matter waves which bathe the atom. The latter (decay stimulated by matter waves) has been "forgotten" until the 1990s when it appeared in the context of Bose Einstein Condensation (BEC) and the atom laser [2, 3, 4, 5].

In his papers, Einstein put forward "a derivation of Planck's formula [...] closely related to Wien's original argument". Our goal is different: we want to emphasize the symmetry between electromagnetic and matter waves. However, we will follow Einstein and derive a simple description of the "forgotten" process

via the introduction of a new coefficient C in addition to the Einstein coefficients A and B .

We first summarize the usual approach describing atom-photon interactions using Maxwell-Boltzmann statistics and the Einstein A and B coefficients. This is followed by a short summary of matter waves and their properties that are relevant to our purpose. In section 4 we restore the symmetry between electromagnetic and matter waves, which we then apply to the thermal equilibrium (section 5) leading to the derivation of transition probabilities for all processes. We obtain the usual Einstein coefficients and a "forgotten" coefficient from these probabilities in section 6 and conclude in section 7.

2 The usual approach

We consider two level atoms where E_1 and E_2 are the energy of the atom in each level (with $E_2 - E_1 = \hbar\omega_0 > 0$). The atoms are in an ideal electromagnetic cavity with perfectly reflecting walls. The thermal equilibrium, at temperature T , is achieved. The case of atoms in free space is obtained when the volume V of the cavity becomes infinite.

The electromagnetic energy is distributed among the different modes of the cavity. It is described by Planck's energy spectral density.

$$\rho_\varphi(\omega, T) = d_\varphi \times \frac{4\pi\nu^2}{c^3} \times \hbar\omega \times \frac{1}{e^{\hbar\omega/k_B T} - 1} \quad (1)$$

where k_B is Boltzmann's constant. Each term in $\rho_\varphi(\omega, T)$ has a precise meaning.

- The electromagnetic wave is characterized by a wave vector \vec{k} . We define $2\pi\nu = c \|\vec{k}\|$. The electromagnetic waves in an ideal cavity must satisfy boundary conditions. As a consequence, ν can take only resonant values: the number of possible values between ν and $\nu + d\nu$ is $d\mathcal{N}_\nu = \frac{4\pi\nu^2}{c^3} d\nu \times V$ where c is the speed of light in vacuum and V the volume of the cavity.

- The energy of a photon is $\hbar\omega$, where ω is the angular frequency. For an electromagnetic field the relation $\omega = 2\pi\nu$ holds true.

- Once \vec{k} is given, ω is known. There remain however $d_\varphi = 2$ polarizations. The first factor, d_φ in relation (1), is precisely the number of polarizations (the degeneracy of photonic states) of the electromagnetic waves. Therefore, in an ideal cavity, the number of modes in the interval $[\nu, \nu + d\nu]$ is

$$d\mathcal{N}_m = d_\varphi \times \frac{4\pi\nu^2}{c^3} d\nu \times V \quad (2)$$

- Finally, using Bose-Einstein statistics, the mean number of photons in a mode at thermal equilibrium is $\frac{1}{e^{\hbar\omega/k_B T} - 1}$.

The spectral energy density (1) represents the energy density per interval $d\nu$. It is sometimes useful to define a quantity $\tilde{\rho}_\varphi(\omega, T)$ that represents the energy density per corresponding interval $d\omega$, i.e.

$$\tilde{\rho}_\varphi(\omega, T) = \rho_\varphi(\omega, T) \frac{d\nu}{d\omega}. \quad (3)$$

For electromagnetic waves we have $\tilde{\rho}_\varphi(\omega, T) = \rho_\varphi(\omega, T)/2\pi$.

Thermal equilibrium in the cavity is secured by a connection to a thermal reservoir at temperature T . We assume the following properties:

A1 First we assume that the reservoir can exchange some energy with the atoms in order that the Maxwell-Boltzmann equilibrium law is fulfilled:

$$\frac{N_2}{N_1} = \frac{d_2}{d_1} e^{-\hbar\omega_0/k_B T} \quad (4)$$

where N_1 (or N_2) is the number of atoms with energy E_1 (or E_2) and d_1 (or d_2) is the degeneracy of level E_1 (or E_2).

A2 Second we assume that photons can be either absorbed or emitted by the reservoir to achieve the equilibrium spectral density, $\rho_\varphi(\omega, T)$, above.

A3 Finally we assume that once the thermal equilibrium is achieved, it remains without the help of the reservoir, although it is a statistical equilibrium where absorption and emission of photons, and excitation and decay of atoms, continue to happen.

Assuming conservation of energy, two kinds of effects are possible:

1. Spontaneous mechanisms happen even when no radiation is present in the cavity. This is the case of a spontaneous decay of an atom from energy E_2 to energy E_1 with the emission of a photon whose angular frequency is ω_0 . The number of spontaneous decays per unit time is $A \times N_2$. In a large enough cavity two neighboring modes have very similar properties and their frequencies can be considered to belong to a continuum. In such a case, A is a characteristic of the atom alone, independent of the cavity dimensions. We do not consider spontaneous excitation where energy would be created from nothing.

2. Stimulated mechanisms happen only because some radiation is already present. Two mechanisms are actually possible: i) the absorption of a photon with energy $\hbar\omega_0$ by an atom which is therefore excited from energy E_1 to energy E_2 and ii) the emission of a photon with energy $\hbar\omega_0$ by an atom which decays from energy E_2 to energy E_1 . The number of events per unit time of each mechanism is $B_{(abs)} \times \rho_\varphi(\omega_0, T) N_1$ and $B_{(em)} \times \rho_\varphi(\omega_0, T) N_2$, where $\omega_0 = 2\pi\nu_0$.

The equilibrium condition is

$$AN_2 + B_{(em)}\rho_\varphi(\omega_0, T) N_2 = B_{(abs)}\rho_\varphi(\omega_0, T) N_1. \quad (5)$$

Within the preceding theoretical framework, one finds that the thermal equilibrium (Eq. (5) with Eqs. (1) and (4)) is possible at any temperature T if and only if

$$B_{(abs)} = \frac{d_2}{d_1} B_{(em)} \quad \text{and} \quad A = \frac{8\pi\nu_0^2}{c^3} \hbar\omega_0 B_{(em)} \quad (6)$$

Therefore the existence of spontaneous emission (*i.e.* $A \neq 0$), implies that absorption and stimulated emission also exist (*i.e.* $A \neq 0 \Rightarrow B_{(em)} \neq 0$ and $B_{(abs)} \neq 0$).

These results are well known and can be found in many textbooks [6, 7, 8].

3 Matter waves

Matter waves were introduced in de Broglie's thesis in 1924. At this time the photoelectric effect had already been interpreted (Einstein 1905) and the Compton effect observed (Compton 1923). Thus, it was already known that light could behave as a flow of particles called photons [9]. The merit of Louis de Broglie idea was to reverse the proposition and claim that particles such as electrons could behave like waves.

Since these early days, the similarity between light waves and matter waves never ceased to be emphasized, from the first observation of electronic interferences by Davisson and Germer in 1927 [10] to atom interferometry, and recently Bose-Einstein condensation of atoms, first achieved in 1995 [11].

Several examples can be given of matter waves and their applications: the electronic microscope and the Collela-Overhauser-Werner experiment with neutrons [12] are well known. More recently laser cooling of atoms has lead to the development of atom interferometry used, for example, for ultra precise inertial sensors such as gravimeters [13], gradiometers [14] and gyroscopes [15]. Ultimately the use of the forgotten process to produce Bose-Einstein condensates, *i.e.* coherent beams of matter waves with all atoms in the same mode, should result in atom lasers based on amplification of the matter fields via the forgotten process. These are expected to lead to significant improvements in matter wave interferometry and to new applications. Presently, what is often called an atom laser is a Bose-Einstein condensate which has been produced by other means.

Despite its complex structure, an atom can be considered as a particle with a mass M depending on its internal energy : $M = M_0 + E/c^2$ where E is the internal energy and M_0 the mass of the atom in its fundamental level. The total energy of the atom is then

$$\hbar\omega = \sqrt{(Mc^2)^2 + (c\hbar\vec{k})^2} \approx Mc^2 + \frac{\hbar^2\vec{k}^2}{2M} \quad (7)$$

where ω is its angular frequency, $\hbar\vec{k}$ its momentum and the non-relativistic approximation holds for $\hbar\|\vec{k}\|/M \ll c$.

An atom is generally characterized by its angular momentum, \vec{J} , which plays the role of an intrinsic spin. The value of \vec{J}^2 is $\hbar^2j(j+1)$. General results in quantum mechanics lead to the conclusion that $2j$ is an integer which depends on the internal state of the atom, and that the atom is a boson when j itself is an integer. This is the only case that we consider [16].

The energy can be degenerate. Therefore, for a given mass, the internal state is a multi-component vector which belongs to a d_a dimensional space.

Keeping the definition $2\pi\nu = c \left\| \vec{k} \right\|$, the fundamental difference with respect to photons is that now the relation between ω and ν is the dispersion relation obtained from Eq. (7)

$$\hbar\omega = \sqrt{(Mc^2)^2 + (2\pi\hbar\nu)^2}. \quad (8)$$

Let us imagine that matter waves can be trapped in an ideal cavity where they appear as a superposition of the resonant modes of the cavity. The number of resonance values of ν in the interval $[\nu, \nu + d\nu]$ is

$$d\mathcal{N}_\nu = \frac{4\pi\nu^2}{c^3} d\nu \times V \quad (9)$$

and the number of modes $d\mathcal{N}_m = d_a \times d\mathcal{N}_\nu$.

As we consider bosons only, the mean number of particles per mode at thermal equilibrium is given by Bose-Einstein statistics

$$\bar{n} = \frac{1}{e^{(\hbar\omega - \mu)/(k_B T)} - 1}$$

where μ is the chemical potential (see ref. [17] for instance).

In section 2, we assumed that the atoms are not able to be created or annihilated, only their internal energy could change by emission or absorption of photons. However, considering what we know about the origin of the Universe, we have to admit that atoms can be created too. Of course, the corresponding mechanism can be complicated and slow but annihilation and creation of atoms via exchange of energy with the thermal reservoir are possible as well as annihilation and creation of photons; this is a matter of principle not of order of magnitude. Thus the number of atoms in the cavity must be determined from the condition of thermal equilibrium. The same happens to the photons with the same consequence, *i.e.* $\mu = 0$ (see the black body radiation in [17]). Therefore :

$$\bar{n} = \frac{1}{e^{\hbar\omega/(k_B T)} - 1}. \quad (10)$$

Therefore, the spectral energy density is

$$\rho_a(\omega, T) = d_a \times \frac{4\pi\nu^2}{c^3} \times \hbar\omega \times \frac{1}{e^{\hbar\omega/(k_B T)} - 1}. \quad (11)$$

Analogously to the electromagnetic case one can introduce a corresponding energy density per interval of angular frequency $\tilde{\rho}_a(\omega, T)$ defined by Eq. (3). But now $d\nu/d\omega$ is obtained from the dispersion relation (8) and therefore

$$\tilde{\rho}_a(\omega, T) = \frac{\rho_a(\omega, T)}{2\pi} \frac{\hbar\omega}{\sqrt{(\hbar\omega)^2 - (Mc^2)^2}} = \frac{\rho_a(\omega, T)}{2\pi} \frac{\omega}{2\pi\nu}. \quad (12)$$

One can check that the relations (8) to (12) are the same for matter waves and for light waves when setting $M = 0$ and $d_a = d_\varphi = 2$ for the photon.

4 Restoring the symmetry between electromagnetic and matter waves

In section 2, the atoms and the photons have been considered from two very different points of view. For instance, the decay of an atom could be stimulated by the presence of the photons but the presence of the atoms did not produce any similar effect.

First, in order to restore the similarity between the atoms and the photons, let us modify the notation. Now, a is an atom in its fundamental state of energy E_a . This atom can absorb a photon φ , the corresponding excited state is $a\varphi$ whose internal energy is $E_{a\varphi}$. We assume that thermal equilibrium is driven by the chemical-like equation

$$a\varphi \rightleftharpoons a + \varphi \quad (13)$$

where φ , a and $a\varphi$ are bosons described by waves (i.e. electromagnetic waves and matter waves) which are trapped in an ideal cavity at temperature T .

An atom is considered as the quantum associated to a matter wave, therefore $N_{a\varphi}$ and N_a are now occupation numbers of matter-wave-modes, similar to the occupation number of the electromagnetic modes (i.e. the number of photons).

The mass of the atom depends on its internal energy. Thus we can interpret the change of the internal energy as the annihilation of an atom with the initial value of the mass and the creation of an atom with the final value of the mass.

The atoms and the photons are assumed to be trapped in an ideal cavity without losses. The cavity is coupled to a thermal reservoir at temperature T . We assume the following properties:

B1 The photons and the atoms occupy respectively electromagnetic-modes and matter-wave-modes. The equilibrium is achieved when the mean number of quanta per mode is given by the Bose-Einstein statistics $\bar{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1}$ i.e. we substitute Eq. (10) into Eq. (4).

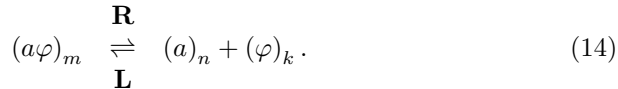
B2 Second we assume that photons and atoms can be either absorbed or emitted by the reservoir, in order to achieve the equilibrium spectral densities (1) and (11).

B3 Finally we assume that once thermal equilibrium is achieved, it remains without the help of the reservoir although it is in statistical equilibrium where annihilation and creation of photons and atoms, continue to happen.

From the old point of view the dissymmetry between matter and light lies in the difference between the assumptions **A1** and **A2** of section 2. Here, this dissymmetry has disappeared but the Maxwell-Boltzmann statistics used in assumption **A1** has been changed into Bose-Einstein relativistic statistics.

The various modes of the excited atoms $a\varphi$ are labelled by a set of indexes called m ; the modes of the atoms a are labelled by n and the modes of the photons φ by a set of indexes k . We use the notation $(a\varphi)_m$ for an excited atom $a\varphi$ in the mode m , and the similar notations $(a)_n$ and $(\varphi)_k$. With this notation

Eq. (13) becomes



Following assumption **B3** above, we assume that the equilibrium is achieved when, during a given arbitrary time, the number of reactions to the right (reaction **R**) is the same as the number of reactions to the left (reaction **L**). Moreover we accept the usual assumption that the number of reactions per unit time is proportional to the number of quanta in the modes involved.

We introduce the concentrations $[a\varphi]_m$, $[a]_n$ and $[\varphi]_k$ i.e. the number of atoms or photons per unit volume respectively in mode m , n and k . The volume of the cavity is V . We define α , $\beta_{(abs)}$, $\beta_{(em)}$ and γ :

- (i) The number of spontaneous reactions **R** per unit time is $\alpha \times [a\varphi]_m V$.
- (ii) The number of reactions **L** per unit time is $\beta_{(abs)} \times [\varphi]_k V \times [a]_n V$.
- (iii) Given an excited atom, $a\varphi$, in the mode m , we consider its decay (reaction **L**) stimulated by the presence of the photons in the mode k . The number of such reactions per unit time is $\beta_{(em)} \times [\varphi]_k V \times [a\varphi]_m V$.

With Maxwell-Boltzmann statistics where $M_0 c^2 \gg \hbar\omega_0$ and $Mc^2 \gg k_B T$, and within the "broad band" approximation [18], these mechanisms result in Eq. (5).

- (iv) Finally, we assume that the reaction **R** can also be stimulated by the presence of the atoms a in the mode n , which restores the symmetry between electromagnetic waves and matter waves. We call this mechanism the "forgotten" process [19]. The number of such reactions per unit time is $\gamma \times [a]_n V \times [a\varphi]_m V$.

5 Thermal equilibrium and its consequences

Compared to the year 1917 the conception of matter has changed dramatically. We will now explain why this conceptual change leaves the description of thermal equilibrium practically unchanged. However, we will emphasize the importance of the "forgotten" process.

Now we assume that equilibrium is achieved when the mean number of quanta per mode is given by Bose-Einstein statistics and when the number of reactions **R** per unit time is equal to the number of reactions **L**. Therefore

$$\begin{aligned} [a]_n V &= \frac{1}{e^{(\hbar\omega_a)/(k_B T)} - 1}, \\ [\varphi]_k V &= \frac{1}{e^{(\hbar\omega_\varphi)/(k_B T)} - 1}, \\ [a\varphi]_m V &= \frac{1}{e^{(\hbar\omega_{a\varphi})/(k_B T)} - 1}, \end{aligned} \quad (15)$$

and

$$\begin{aligned} \beta_{(abs)} \times [\varphi]_k V \times [a]_n V &= \alpha \times [a\varphi]_m V + \beta_{(em)} \times [\varphi]_k V \times [a\varphi]_m V \\ &+ \gamma \times [a]_n V \times [a\varphi]_m V. \end{aligned} \quad (16)$$

The energy of the atom a in the mode n is $\hbar\omega_a$, similarly $\hbar\omega_\varphi$ is the energy of the photon in the mode k and $\hbar\omega_{a\varphi}$ is the energy of the excited atom $a\varphi$ in the mode m . We assume that the energy ($\hbar\omega_a$, $\hbar\omega_\varphi$ or $\hbar\omega_{a\varphi}$) defines the mode except for the polarization of the light and the degeneracy of the internal energy of the atoms [20].

One can check that the equality (16) holds true at any temperature if and only if

$$\omega_{a\varphi} = \omega_a + \omega_\varphi \quad \text{and} \quad \beta_{(abs)} = \alpha = \beta_{(em)} = \gamma \quad (17)$$

The first relation in (17) expresses the conservation of energy while the other relations imply that the "forgotten" process does exist (*i.e.* $\gamma \neq 0$) because spontaneous decay is observed (because $\alpha \neq 0$).

Let us now verify that at thermal equilibrium the results above are practically the usual ones when the "forgotten" process is neglected. We consider Eq. (16) with $\beta_{(abs)} = \alpha = \beta_{(em)} = \gamma$. Then the sum of the last two terms is $\gamma \times [\varphi]_k V \times [a\varphi]_m V \left(1 + \frac{[a]_n}{[\varphi]_k}\right)$. Using Eq. (15) it is easy to check that $\frac{[a]_n}{[\varphi]_k} \ll 1$ even with extreme (impossible!) values of $\hbar\omega_\varphi$ and $k_B T$. Therefore the contribution $\gamma \times [a]_n V \times [a\varphi]_m V$ in Eq. (16) is completely negligible and the "forgotten" process does not play any significant role at thermal equilibrium.

Moreover one can easily calculate $N_{a\varphi}/N_a$ where $N_{a\varphi}$ (respectively N_a) is the mean number of atoms with energy $E_{a\varphi} = \hbar\omega_{a\varphi}$ (respectively $E_a = \hbar\omega_a$). It is the number of atoms in one mode with energy $\hbar\omega_{a\varphi}$ (respectively $\hbar\omega_a$) times the number of modes with such an energy. From the preceding assumptions we obtain

$$N_{a\varphi} = \frac{d_{a\varphi}}{e^{(\hbar\omega_{a\varphi})/(k_B T)} - 1} \quad \text{and similarly} \quad N_a = \frac{d_a}{e^{(\hbar\omega_a)/(k_B T)} - 1}. \quad (18)$$

Finally, using the relation $\hbar\omega_{a\varphi} \sim \hbar\omega_a \sim M_0 c^2 \gg k_B T$ we obtain

$$\frac{N_{a\varphi}}{N_a} \simeq \frac{d_{a\varphi}}{d_a} \frac{e^{(\hbar\omega_a)/(k_B T)}}{e^{(\hbar\omega_{a\varphi})/(k_B T)}} = \frac{d_{a\varphi}}{d_a} e^{-(\hbar\omega_{a\varphi} - \hbar\omega_a)/(k_B T)} \quad (19)$$

which is identical to Eq. (4) under the simple change of notation ($a\varphi \rightarrow 2$, $a \rightarrow 1$ and $\hbar\omega_{a\varphi} - \hbar\omega_a = \hbar\omega_0$).

We notice that at thermal equilibrium nothing is significantly modified if we use Maxwell-Boltzmann statistics instead of Bose-Einstein statistics and if we neglect the "forgotten" process. However, far from equilibrium this is not necessarily the case.

Now we can use standard methods to give an estimation of γ in the simplest case of a homogeneous line width.

Let α be the probability per unit time that an excited atom $a\varphi$, in a given mode m with given energy $\hbar\omega_{a\varphi}$, decays spontaneously into $a + \varphi$ where the photon φ is in the mode k with energy $\hbar\omega_\varphi$ and the atom a in the mode n with energy $\hbar\omega_a$. Let us define the probability per unit time, dp , that an excited atom decays spontaneously into a photon with angular frequency $\omega_\varphi \in [\omega_{\varphi k}, \omega_{\varphi k} + d\omega]$

and a ground state atom with angular frequency $\omega_a \in [\omega_{an} - d\omega, \omega_{an}]$ where $\omega_{a\varphi} = \omega_a + \omega_\varphi$. We then have

$$dp = \alpha \times d_\varphi d_a \times d\mathcal{N}_{a+\varphi} \quad (20)$$

where $d\mathcal{N}_{a+\varphi}$ is the number of pairs of resonance values (ν_a, ν_φ) that satisfy $\omega_\varphi \in [\omega_{\varphi k}, \omega_{\varphi k} + d\omega]$, $\omega_a \in [\omega_{an} - d\omega, \omega_{an}]$ and $\omega_{a\varphi} = \omega_a + \omega_\varphi$.

For the photons, the number of resonant values of ν_φ , $d\mathcal{N}_\varphi$, over the bandwidth $d\omega$ are given (c.f. Eq. (1)) by the number of possible values of ν , i.e. $\frac{4\pi\nu^2}{c^3}d\nu V$, and $2\pi\nu = \omega$ which holds true for photons, therefore

$$d\mathcal{N}_\varphi = \frac{\omega_\varphi \nu_\varphi d\omega}{\pi c^3} V. \quad (21)$$

Similarly for the ground state atom, the number of resonant values of ν_a over the bandwidth $d\omega$ are given by $\frac{4\pi\nu^2}{c^3}d\nu V$, but with the relation between ν and ω given in Eq. (8). This leads to

$$\begin{aligned} d\mathcal{N}_a &= \frac{\omega_a \nu_a d\omega}{\pi c^3} V = \frac{\omega_a \sqrt{(\hbar\omega_a)^2 - (M_0 c^2)^2} d\omega}{2\pi^2 c^3 \hbar} V \\ &\simeq \frac{(M_0 c^2) \sqrt{2M_0 c^2 E_k} d\omega}{2\pi^2 c^3 \hbar^2} V \end{aligned} \quad (22)$$

where $E_k = \hbar\omega_a - M_0 c^2$ is the kinetic energy of a , and where we have used $\hbar\omega_a + M_0 c^2 \simeq 2M_0 c^2$.

Comparing Eq. (21) to Eq. (22) we note that even in extreme conditions $d\mathcal{N}_a$ is much bigger than $d\mathcal{N}_\varphi$. For example, with $E_k \approx k_B T \approx 10^{-13}$ eV, $M_0 c^2 \approx 1$ GeV and $\hbar\omega_\varphi \approx 20$ eV we have $d\mathcal{N}_a \approx 10^4 d\mathcal{N}_\varphi$. As a result the number of possible energy pairs $d\mathcal{N}_{a+\varphi}$ is entirely determined by $d\mathcal{N}_\varphi$ because for each ω_φ there exists a ω_a such that $\omega_{a\varphi} = \omega_a + \omega_\varphi$ (but not vice-versa), so we have $d\mathcal{N}_{a+\varphi} \simeq d\mathcal{N}_\varphi$. Under these conditions Eq. (20) becomes

$$dp = \alpha \times d_\varphi d_a \times \frac{\omega_\varphi \nu_\varphi d\omega}{\pi c^3} V. \quad (23)$$

On the other hand, under the assumption that the atom is at rest and the assumption of isotropy, the spontaneous emission of a photon is characterized by a function $f_\varphi(\omega_\varphi - \omega_{\varphi 0})$ which defines the line shape of the emitted light. The quantity $f_\varphi(\omega_\varphi - \omega_{\varphi 0}) d\omega$ is interpreted as the probability that the emitted photon displays an angular frequency $\omega_\varphi \in [\omega_{\varphi k}, \omega_{\varphi k} + d\omega]$. Therefore

$$dp = \frac{1}{t_{sp}} f_\varphi(\omega_\varphi - \omega_{\varphi 0}) \quad (24)$$

where t_{sp} is the time constant which characterizes the spontaneous emission (i.e. $1/t_{sp} = A$, the Einstein coefficient).

It is then straightforward to obtain α by eliminating dp between Eqs. (23) and (24)

$$\alpha = \beta_{abs} = \beta_{em} = \gamma = \frac{\pi c^3}{d_\varphi d_a \omega_\varphi \nu_\varphi t_{sp}} \frac{1}{t_{sp}} f_\varphi(\omega_\varphi - \omega_{\varphi 0}) \times \frac{1}{V}. \quad (25)$$

One can now calculate the probability per unit time, $W_{(em)}$, of stimulated decay of an excited atom $a\varphi$ in mode m into a photon φ in mode k and a ground state atom a in any mode. First we notice that the decay due to the photons in mode k , towards a special given mode of a , has a probability per unit time $\beta_{(em)} \times [\varphi]_k V$ (see the property (iii) of section 4 above). Therefore, the probability of a decay towards the various modes n of a which display the same energy is

$$W_{(em)} = d_a \times \beta_{(em)} \times [\varphi]_k V = \frac{\pi c^3}{d_\varphi \omega_\varphi \nu_\varphi t_{sp}} \frac{1}{t_{sp}} f_\varphi(\omega_\varphi - \omega_{\varphi 0}) \times \frac{u_k(\omega_\varphi)}{\hbar \omega_\varphi} \quad (26)$$

where we have introduced the electromagnetic energy density of the mode k defined as $u_k(\omega_\varphi) = [\varphi]_k \times \hbar \omega_\varphi$.

Similarly the probability per unit time, $W_{(abs)}$, of absorption of a photon φ in mode k by a ground state atom a in mode n to form an excited atom $a\varphi$ in any mode is

$$W_{(abs)} = d_{a\varphi} \times \beta_{(em)} \times [\varphi]_k V = \frac{d_{a\varphi}}{d_a} \frac{\pi c^3}{d_\varphi \omega_\varphi \nu_\varphi t_{sp}} \frac{1}{t_{sp}} f_\varphi(\omega_\varphi - \omega_{\varphi 0}) \times \frac{u_k(\omega_\varphi)}{\hbar \omega_\varphi}. \quad (27)$$

Expressions (26) and (27) are well known in the usual laser theory and can be found in many textbooks (e.g. [6]).

On the other hand, the "forgotten" process leads to new results. We calculate the probability per unit time, W_f , that an excited atom $a\varphi$ in mode m undertakes the forgotten process i.e. that it decays into a ground state atom a in mode n and a photon φ in any mode. The decay due to the atoms a in mode n , towards a special given mode of φ , has a probability per unit time $\gamma \times [a]_n V$ (see the property (iv) of section 4 above). Therefore, the probability of a decay towards the various modes k of φ which have the same energy is

$$W_f = d_\varphi \times \gamma \times [a]_n V = \frac{\pi c^3}{d_a \omega_\varphi \nu_\varphi t_{sp}} \frac{1}{t_{sp}} f_\varphi(\omega_\varphi - \omega_{\varphi 0}) \times \frac{u_n(\omega_a)}{\hbar \omega_a} \quad (28)$$

where $u_n(\omega_a) = [a]_n \times \hbar \omega_a$ is the energy density of the ground state atoms in mode n .

At thermal equilibrium we have $[a]_n \ll [\varphi]_k$ (from Eq. (15)) which, using Eqs. (26) and (28), implies $W_f \ll W_{(em)}$, i.e. the probability for an excited atom to decay via stimulation by a matter wave is much smaller than via stimulation by an electromagnetic wave, and the "forgotten" process is practically negligible. However, this may not be the case far from equilibrium. For example, one can imagine a situation where one particular mode $[a]_n$ is strongly populated. This would lead to decay preferentially towards that mode via the "forgotten" process, which in turn increases the population of that mode, and so on. That kind of mechanism is the basis of the experimental proposal for an atom laser in [4].

6 The Einstein and the "forgotten" coefficients

Now, on one hand we consider that the frequency width of the electromagnetic spectrum of the radiation in the cavity is much larger than the width of the natural decay spectrum of the atom (broad band approximation). On the other hand we model the spectrum of the radiation in the cavity by monochromatic radiation with angular frequencies $\omega_{\varphi\ell}$ where ℓ is an integer and $\omega_{\varphi\ell+1} - \omega_{\varphi\ell} = \delta\omega_{\varphi}$. The function $f_{\varphi}(\omega_{\varphi\ell} - \omega_{\varphi 0})$ has a maximum for $\omega_{\varphi\ell} \simeq \omega_0$; it is negligible for $|\omega_{\varphi\ell} - \omega_{\varphi 0}| > \Delta\omega$ with $\Delta\omega \ll \omega_{\varphi 0}$, and it fulfills the condition $\int_0^{\infty} f_{\varphi}(\omega_{\varphi\ell} - \omega_{\varphi 0})d\omega = 1$. Assuming $\delta\omega_{\varphi} \ll \Delta\omega$ we can write :

$$\sum_{\ell} f_{\varphi}(\omega_{\varphi\ell} - \omega_{\varphi 0})\delta\omega_{\varphi} \simeq 1 \quad . \quad (29)$$

We consider a given initial state characterized by the number $N_{a\varphi}$ of atoms $a\varphi$ with angular frequency $\omega_{a\varphi}$, and the number N_a of atoms a with angular frequency ω_a . We use the preceding results to calculate the number dN of photons which are emitted or absorbed during a time interval dt , through the various processes.

Let us give the final results before we outline the derivation. One finds

(i) for spontaneous emission : $dN_{sp} = A N_{a\varphi} dt = \frac{1}{t_{sp}} N_{a\varphi} dt$,

(ii) for absorption : $dN_{abs} = B_{(abs)} \rho_{\varphi}(\omega_{\varphi 0}) N_a dt$ with

$$B_{(abs)} = \frac{d_{a\varphi}}{d_a d_{\varphi}} \times \frac{c^3}{4\pi\hbar\omega_{\varphi 0}\nu_{\varphi 0}^2} \frac{1}{t_{sp}} \quad (30)$$

(iii) for emission stimulated by the photons :

$dN_{em} = B_{(em)} \rho_{\varphi}(\omega_{\varphi 0}) N_{a\varphi} dt$ with

$$B_{(em)} = \frac{1}{d_{\varphi}} \times \frac{c^3}{4\pi\hbar\omega_{\varphi 0}\nu_{\varphi 0}^2} \frac{1}{t_{sp}} \quad (31)$$

(iv) for the forgotten process (emission stimulated by the atoms) :

$dN_f = C\rho_a(\omega_{a0}) N_{a\varphi} dt$ with

$$C = \frac{1}{d_a} \times \frac{c^3}{4\pi\hbar\omega_{a0}\nu_{a0}^2} \frac{1}{t_{sp}} \quad (32)$$

where $\omega_{a0} = \omega_{a\varphi} - \omega_{\varphi 0}$.

$B_{(abs)}$ and $B_{(em)}$ are the well known Einstein coefficients but C is a new one.

We now detail the derivation of the above expressions. Let us for instance calculate $B_{(em)}$. We assume that, at a given angular frequency ω_{φ} , the two electromagnetic polarizations display the same energy density $u_k(\omega_{\varphi})$. Therefore the number of photons produced during dt by the stimulated emission due to the electromagnetic radiation at angular frequency ω_{φ} is $d_{\varphi} W_{(em)} N_{a\varphi} dt$ where $W_{(em)}$ is given by Eq. (26). The stimulated emission due to the radiations at the various frequencies $\omega_{\varphi\ell}$ results in the number of emitted photons

$dN_{(em)} = \sum_{\ell} d_{\varphi} W_{(em)} N_{a\varphi} dt$. The function $f_{\varphi}(\omega_{\varphi} - \omega_{\varphi 0})$ is a quickly varying function of ω_{φ} while $u_k(\omega_{\varphi \ell})$ is slowly varying within the framework of the broad band approximation. More precisely, we assume $u_k(\omega_{\varphi \ell}) \simeq u_k(\omega_{\varphi 0})$, $\omega_{\varphi \ell} \simeq \omega_{\varphi 0}$ and $\nu_{\varphi \ell} \simeq \nu_{\varphi 0}$, for $\omega_{\varphi 0} - \Delta\omega < \omega_{\varphi \ell} < \omega_{\varphi 0} + \Delta\omega$. Therefore we can use these expressions for $u_k(\omega_{\varphi \ell})$, $\omega_{\varphi \ell}$ and $\nu_{\varphi \ell}$ to calculate $dN_{(em)}$. We obtain

$$\begin{aligned} dN_{(em)} &= \sum_{\ell} d_{\varphi} \times W_{(em)} N_{a\varphi} dt \\ &= \sum_{\ell} d_{\varphi} \times \frac{1}{d_{\varphi}} \frac{\pi c^3}{\hbar \omega_{\varphi \ell}^2 \nu_{\varphi \ell} t_{sp}} u_k(\omega_{\varphi \ell}) f_{\varphi}(\omega_{\varphi \ell} - \omega_{\varphi 0}) N_{a\varphi} dt \\ &\simeq \frac{1}{d_{\varphi}} \frac{\pi c^3}{\hbar \omega_{\varphi 0}^2 \nu_{\varphi 0} t_{sp}} d_{\varphi} u_k(\omega_{\varphi 0}) \sum_{\ell} f_{\varphi}(\omega_{\varphi \ell} - \omega_{\varphi 0}) N_{a\varphi} dt \end{aligned} \quad (33)$$

The spectral energy density is $\frac{d_{\varphi} u_k(\omega_{\varphi 0})}{\delta\omega_{\varphi}} = \tilde{\rho}_{\varphi}(\omega_{\varphi 0})$ where $\tilde{\rho}_{\varphi}$ is defined by relation (3) then

$$dN_{(em)} = \frac{1}{d_{\varphi}} \frac{\pi c^3}{\hbar \omega_{\varphi 0}^2 \nu_{\varphi 0} t_{sp}} \tilde{\rho}_{\varphi}(\omega_{\varphi 0}) \sum_{\ell} f_{\varphi}(\omega_{\varphi \ell} - \omega_{\varphi 0}) \delta\omega_{\varphi} N_{a\varphi} dt. \quad (34)$$

Finally, using Eqs. (29) and (3) we find

$$dN_{(em)} = B_{(em)} \rho_{\varphi}(\omega_{\varphi 0}) N_{a\varphi} dt \quad (35)$$

with

$$B_{(em)} = \frac{1}{d_{\varphi}} \times \frac{c^3}{4\pi \hbar \omega_{\varphi 0} \nu_{\varphi 0}^2 t_{sp}}. \quad (36)$$

This expression for $B_{(em)}$ can be found in many textbooks such as [6] where it is calculated in a way very similar to the one presented above. It is also derived in [8].

Expression (30) for $B_{(abs)}$ is obtained by the same method and we will not detail it here.

To calculate C we express dN_f using expression (28) for W_f . In this expression, $\omega_a = \omega_{a\varphi} - \omega_{\varphi \ell}$ is a function of $\omega_{\varphi \ell}$ because the mode m , of the initial state is given. We assume that the d_a atomic modes at the same frequency ω_a display the same energy density $u_n(\omega_a)$ so

$$\begin{aligned} dN_f &= \sum_{\ell} d_a \times W_f N_{a\varphi} dt \\ &= \sum_{\ell} \frac{1}{d_a} \frac{\pi c^3}{\omega_{\varphi \ell} \nu_{\varphi \ell} t_{sp}} f_{\varphi}(\omega_{\varphi \ell} - \omega_{\varphi 0}) \times \frac{d_a u_n(\omega_a)}{\hbar \omega_a} N_{a\varphi} dt. \end{aligned} \quad (37)$$

We assume that ω_a and $u_n(\omega_a)$ are two slowly varying functions of $\omega_{\varphi \ell}$, then, with the conditions that we have already assumed, we find

$$dN_f = \frac{1}{d_a} \frac{\pi c^3}{\omega_{\varphi 0} \nu_{\varphi 0} t_{sp}} \times \frac{d_a u_n(\omega_{a0})}{\hbar \omega_{a0}} \sum_{\ell} f_{\varphi}(\omega_{\varphi \ell} - \omega_{\varphi 0}) N_{a\varphi} dt$$

$$= \frac{1}{d_a} \frac{\pi c^3}{\omega_{\varphi 0} \nu_{\varphi 0}} \frac{1}{t_{sp}} \times \frac{1}{\hbar \omega_{a0}} \frac{d_a u_n(\omega_{a0})}{\delta \omega_{\varphi}} N_{a\varphi} dt. \quad (38)$$

It is not obvious whether to introduce the spectral density $\rho_a(\omega_{a0})$ of the atoms because $\delta \omega_{\varphi}$ which appears in the expression above is not the angular frequency separation between two neighboring atomic rays.

In the bandwidth $\delta \omega_{\varphi}$, around ω_{a0} , the number of resonant atomic angular frequencies is given by expression (22): $\delta \mathcal{N}_a = \frac{\omega_{a0} \nu_{a0}}{\pi c^3} \delta \omega_{\varphi} \times V$. In the same bandwidth there is only one resonant frequency for the photons (the consequence of the definition of $\delta \omega_{\varphi}$) so $\delta \mathcal{N}_{\varphi} = \frac{\omega_{\varphi 0} \nu_{\varphi 0}}{\pi c^3} \delta \omega_{\varphi} \times V = 1$ (c.f. (21)). Therefore $\delta \mathcal{N}_a = \frac{\omega_{a0} \nu_{a0}}{\omega_{\varphi 0} \nu_{\varphi 0}}$. The spectral energy density is $\tilde{\rho}_a(\omega_{a0}, T) = \frac{d_a u_n(\omega_{a0}) \delta \mathcal{N}_a}{\delta \omega_{\varphi}} = \frac{d_a u_n(\omega_{a0})}{\delta \omega_{\varphi}} \frac{\omega_{a0} \nu_{a0}}{\omega_{\varphi 0} \nu_{\varphi 0}}$. Substituting this into Eq. (38) and using the definition (12) of $\tilde{\rho}_a$ we finally obtain

$$dN_f = C \rho_a(\omega_{a0}) N_{a\varphi} dt \quad (39)$$

with

$$C = \frac{1}{d_a} \times \frac{c^3}{4\pi \hbar \omega_{a0} \nu_{a0}^2} \frac{1}{t_{sp}}. \quad (40)$$

One can notice the similarities between the expressions for C and $B_{(em)}$.

A permanent state is achieved when $dN_{sp} + dN_{em} + dN_f = dN_{abs}$. Therefore, at thermal equilibrium, Eq. (5) becomes

$$A N_{a\varphi} + B_{(em)} \rho_{\varphi}(\omega_{\varphi 0}, T) N_{a\varphi} + C \rho_a(\omega_{a0}, T) N_{a\varphi} = B_{(abs)} \rho_{\varphi}(\omega_{\varphi 0}, T) N_a. \quad (41)$$

One can easily check that the expressions obtained above guarantee the validity of Eq. (41) at any temperature when $N_{a\varphi}$ and N_a satisfy Bose-Einstein statistics.

It is also possible to derive the conservation of energy and the coefficients $B_{(em)}$, $B_{(abs)}$ and C directly from Eq. (41) where $N_{a\varphi}$ and N_a fulfill Bose-Einstein rather than the Maxwell-Boltzmann statistics. However, in this paper, we chose to emphasize the elementary underlying processes providing, for example, the transition probabilities $W_{(em)}$, $W_{(abs)}$ and W_f (Eqs. (26), (27) and (28)) as functions of the observed electromagnetic line-shape $f_{\varphi}(\omega_{\varphi} - \omega_{\varphi 0})$.

7 Conclusion

We have derived a general description of atom-photon interactions of the form $a\varphi \rightleftharpoons a + \varphi$ that includes the "forgotten" process of decay stimulated by the matter waves a , additionally to the well known spontaneous decay and the decay stimulated by the electromagnetic waves φ . Our aim was to provide a description based on fundamental elementary principles, and to emphasize throughout this work the symmetry between atomic and matter waves. To do so, we have followed standard textbook descriptions of the involved known processes (in terms of transition probabilities, energy densities, observed line shapes, and Einstein coefficients) but applied them to also derive analogous expressions for the "forgotten" process that cannot yet be found in textbooks.

Throughout this work the various mechanisms have been considered for the case of thermal equilibrium. But, similarly to the well known photonic processes, our results (in particular the transition probability W_f and coefficient C of the "forgotten" process) are quite general and remain valid far away from equilibrium. Non-equilibrium conditions (population inversions etc.) are the fundamental ingredients of lasers, and one expects the same to hold true for atom lasers i.e. sources of coherent matter waves. Atom lasers based on matter wave amplification have not yet been built, but several proposals for a practical realization based on the "forgotten" process can be found in the literature [2, 4, 5]. Indeed, any practical realization will necessarily involve conditions far from thermal equilibrium, as can easily be seen from Eq. (15). For example, a single atom a inside a cavity corresponds to $[a]_n V = 1$ and therefore the thermal equilibrium condition (15) is only satisfied for temperatures $T \simeq 10^{13} K$ which are impossible to attain in practice.

It is easy to generalize the results that we have obtained to the general bosonic case $ab \rightleftharpoons a + b$ where ab , a and b are massive or massless bosons of any kind (photons, atoms, molecules, etc.). If we assume that the same concepts are valid for the decay of an atom and for the chemical reaction $ab \rightleftharpoons a + b$, i.e. the concept of line shape where $f_\varphi(\omega - \omega_{\varphi 0})$ becomes $f_b(\omega - \omega_{b0})$ and the concept of stimulated and spontaneous reactions, then all our considerations apply to the general bosonic case. Moreover if we assume that in the same elementary angular frequency interval $d\omega$, the number of resonance frequencies is much higher for b than for a , the developments of the preceding sections (sections 3 to 6) remain unchanged, except for the substitution $\varphi \rightarrow b$ and $\nu_\varphi = \omega_\varphi/2\pi \rightarrow \nu_b = \sqrt{(\hbar\omega_b)^2 - (M_b c^2)^2}/2\pi\hbar$ where M_b is the mass of b in its fundamental state. Under these conditions, the formulae (26), (27), (28), (30), (31), and (32) hold true for the general bosonic case, and the photon case is simply recovered when setting $M_b = 0$ and $d_b = 2$. This simple generalization further stresses the symmetry between electromagnetic and matter waves as now the atom-photon interaction is just a special case of a general bosonic interaction.

If a perfect symmetry exists between electromagnetic and matter waves, it may be relevant to raise the question of the physical nature of holography with atomic waves, a surely premature question for the time being.

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