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# Evolution of heterogeneity in an agent-based market model.

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## Abstract

We propose an agent-based model of a single-asset financial market, described in terms of a small number of parameters. The joint effect of feedback and heterogeneity leads to price returns with statistical properties similar to the stylized facts observed in financial time series.

We investigate theoretically the evolution of heterogeneity in the model.

*Keywords:* agent-based model, financial markets, stylized facts, heterogeneity, feedback, asynchronous updating.

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There is a growing body of research on agent-based models of financial markets that explores the relation between market participants and the statistical properties of aggregate market variables such as prices and trading volume. Indeed, financial time series exhibit non-trivial and intriguing statistical features [3] which are not easy to model and even less to explain : volatility clustering and heavy tailed increments for example.

In a previous paper [7], we considered some methodological issues related to this approach. We argued that two main ingredients in agent-based models (and in real speculative markets) which lead to realistic behavior of prices and trading volume are heterogeneity in behavioral rules, resources or beliefs among market participants and feedback effects, the agents behavior determining the state of market variables which in turn influences these behaviors.

Let us recall the crucial ingredients of this model: threshold behavior of agents that leads to investor inertia; absence of exogeneous “fundamental price” process; absence of information asymetry; absence of “social interaction”.

A final element is endogeneous heterogeneity: heterogeneity of agents behavioral rules appears endogeneously due to the asynchronous updating scheme.

After describing the model, we will focus our study on this final ingredient and investigate the thermodynamic limit.

## 1 Description of the model.

Our model describes a market where a single asset, whose price is denoted by  $p_t$ , is traded by  $N$  agents. Trading takes place at discrete dates  $t = 0, 1, 2, \dots$ <sup>2</sup>. At each period, every agent receives public news about the asset’s performance and, using a subjective criterion, judge whether this news is significant. If the news is found to be significant, the agent places a buy or sell order, depending on whether the news received is pessimistic or optimistic. Prices then moved up or down according to excess demand. We now describe these ingredients in more precise terms.

### 1.1 Trading rules.

At each period, agents have the possibility to send an order to the market for buying or selling a unit of asset: denoting by  $\phi_i(t)$  the demand of the agent, we have  $\phi_i(t) = 1$  for a buy order and  $\phi_i(t) = -1$  for a sell order. We allow the value  $\phi_i(t)$  to be zero; the agent is then inactive at period  $t$ . The inflow of new information is modeled by a sequence of IID Gaussian random variables  $(\epsilon_t, t = 0, 1, 2, \dots)$  with  $\epsilon_t \sim N(0, D^2)$ .  $\epsilon_t$  represents the value of a common signal received by all agents at date  $t$ . The signal  $\epsilon_t$  is a forecast of the future return

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<sup>2</sup>Provided the parameters of the model are chosen in a certain range, we will be able to interpret these periods as “trading days”.

$r_t$  and each agent has to decide whether the information conveyed by  $\epsilon_t$  is significant, in which case she will place a buy or sell order according to the sign of  $\epsilon_t$ .

The trading rule of each agent  $i = 1, \dots, N$  is represented by a (time-varying) decision threshold  $\theta_i(t)$ . The threshold  $\theta_i(t)$  can be viewed as the agents (subjective) view on volatility. By comparing the signal to her threshold, the agent decides whether the news is significant enough to generate a trade ( $|\epsilon_t| > \theta_i(t)$ ):

$$\begin{aligned} & \text{if } \epsilon_t > \theta_i^+, \phi_i = 1 \\ & \text{if } \epsilon_t < \theta_i^-, \phi_i = -1 \\ & \text{otherwise } \phi_i = 0. \end{aligned} \tag{1}$$

This trading rule may be seen as a stylized example of threshold behavior: without sufficient external stimulus, an agent remains inactive and if the external signal is above a certain threshold, the agent will act. The corresponding demand generated by the agent is therefore given by:

$$\phi_i(t) = 1_{\epsilon_t > \theta_i} - 1_{\epsilon_t < -\theta_i}. \tag{2}$$

## 1.2 Price response to aggregate demand.

The aggregate excess demand is then given by:

$$Z_t = \sum_i \phi_i(t). \tag{3}$$

A non zero value of  $Z_t$  produces a change in the price, and the resulting log return is given by :

$$r_t = \ln \frac{p_t}{p_{t-1}} = g\left(\frac{Z_t}{N}\right). \tag{4}$$

where the price impact function  $g : \mathfrak{R} \rightarrow \mathfrak{R}$  is an increasing function with  $g(0)=0$ . We define the (normalized) market depth  $\lambda$  by:

$$g'(0) = 1/\lambda. \tag{5}$$

While most of the analysis below holds for a general price impact function  $g$ , in some cases it will be useful to consider a linear price impact:  $g(z) = z/\lambda$ .

## 1.3 Updating of strategies.

As we noted above, the threshold  $\theta_i(t)$  represents the view of agent  $i$  on recent market volatility: these thresholds are updated by agents from time to time to reflect the amplitude of recent returns. Initially, we start from a population distribution  $F_0$  of thresholds:  $\theta_i(0), i = 1..N$  are positive IID variables drawn from  $F_0$ .

Updating of strategies is *asynchronous*: at each time step, any agent  $i$  has a probability  $s$  ( $s \in [0, 1]$ ) of updating her threshold  $\theta_i(t)$ . Thus, in a large population,  $s$  represents the fraction of agents updating their views at any period;  $1/s$  represents the typical time period during which an agent will hold a given view  $\theta_i(t)$ . If periods are to be interpreted as days,  $s$  is typically a small number  $s \simeq 10^{-1} - 10^{-3}$ .

When an agent updates her threshold, she sets it to be equal to the recently observed absolute return, which is an indicator of recent volatility  $|r_t| = |\ln \frac{p_t}{p_{t-1}}|$ . Introducing IID random variables  $u_i(t)$ ,  $i = 1..N, t \geq 0$  uniformly distributed on  $[0, 1]$ , which indicate whether agent  $i$  updates her threshold or not, we can write the updating rule as:

$$\theta_i(t) = 1_{u_i(t) < s} |r_t| + 1_{u_i(t) \geq s} \theta_i(t-1). \quad (6)$$

Here  $\epsilon_t$  represents randomness due to public news arrivals whereas the random variables  $u_i(t)$  represent idiosyncratic sources of randomness. This way of updating can be seen as a stylized version of various estimators of volatility based on moving averages or squared returns. It is also corroborated by a recent empirical study by Zovko & Farmer [10], who show that traders use recent volatility as a signal when placing orders.

Note that, given this random updating scheme, even if we start from an initially homogeneous population  $\theta_i(0) = \theta_0$ , heterogeneity creeps into the population through the updating process. In this sense, the heterogeneity of agents strategies is endogenous in this model and, as we will see below, evolves in a random manner.

## 2 Dynamics of heterogeneity.

Numerical simulations of the model [7] lead to series of returns with realistic ranges and realistic values of annualized volatility and with some regularities which match some empirical properties: excess volatility [9, 4]; mean-reverting stochastic volatility [6]; a leptokurtic distribution of returns with (semi-)heavy tails [5]; absence of autocorrelation of the returns; volatility clustering [8].

The origin of these properties can be understood with the following mechanism: suppose we are in a period of “low-volatility”; the amplitude  $|r_t|$  of returns is small. Agents who update their thresholds will therefore update them to small values, become more sensitive to news arrivals, thus generating higher excess demand and thus increasing the amplitude of returns.

The persistence of a low or high volatility period depends on how frequently agents update their thresholds, thus reacting to market activity. If  $s$  is the proportion of agents updating their thresholds at each period, the duration of such periods is of order  $1/s$ .

We will study first some limiting cases underlining the role of heterogeneity and feedback and then the general case with the thermodynamic limit.

## 2.1 Feedback without heterogeneity: $s = 1$ .

In the case where  $s = 1$ , all agents synchronously update their threshold at each period. Consequently, the agents have the same thresholds, given by the absolute return of the last period:

$$\theta_i(t) = |r_{t-1}|. \quad (7)$$

and will therefore generate the same order:  $Z_t = N\phi_1(t) \in \{0, -N, N\}$ .

So, the return  $r_t$  depends on the past only through the absolute return  $|r_{t-1}|$ :  $r_t = f(|r_{t-1}|, \epsilon_t) = g(1)1_{\epsilon_t > |r_{t-1}|} + g(-1)1_{\epsilon_t < -|r_{t-1}|}$ , a dependence structure typical of ARCH models, leading to uncorrelated returns and volatility clustering. In this case, the distribution of  $r_t$  conditional on  $|r_{t-1}|$  is actually a trinomial distribution:

$$r_t \in \{0, g(1), g(-1)\}. \quad (8)$$

Of course, this behavior is extreme since it implies that the agents have identical trading strategies and all agents are trading at each period, saturating market activity: it does not allow for market inertia. It also generates a trinomial distribution of returns which is not realistic. Simulation studies show that a similar behavior persists for  $1 - s \ll 1$ , leading to tri-modal distributions. This confirms our intuition that the updating probability  $s$ , which reflects the proportion of agents updating their choices at a given period, should be chosen small in order to guarantee the heterogeneity of the population.

## 2.2 Heterogeneity without feedback: $s=0$ .

In the case where  $s = 0$ , no updating takes places: the trading strategies, given by the thresholds  $\theta_i$ , are unaffected by the price behavior and the *feedback* effect is not present anymore. Heterogeneity is still present: the distribution of the thresholds remains identical to what it was at  $t=0$ . The return  $r_t$  depends only on  $\epsilon_t$ :

$$r_t = g\left(\frac{1}{N} \sum_{i=1}^N 1_{\epsilon_t > \theta_i} - 1_{\epsilon_t < -\theta_i}\right) = F(\epsilon_t). \quad (9)$$

We conclude therefore that the returns are IID random variables, obtained by transforming the Gaussian IID sequence  $(\epsilon_t)$  by the nonlinear function  $F$  given in (12), whose properties depend on the (initial) distribution of thresholds  $(\theta_i, i = 1..N)$ . The log-price then follows a (non-Gaussian) random walk and the model does not exhibit volatility clustering.

The two limiting cases above show that, in order to obtain the interesting statistical properties, it is necessary to have  $0 < s \ll 1$ : both feedback and heterogeneity are essential ingredients. Let us now turn to the general case.

### 2.3 General remarks.

Define  $a_k = g(k/N)$  for  $k = -N..0..N$  and consider the finite sets

$$E = \{a_k, k = -N..0..N\}, E_+ = \{g(k/N), k = 0..N\} = \{0, a_1, ..a_N\} \quad (10)$$

The returns take their value in  $E$  while the thresholds  $\theta_i(t)$  take their values in  $E_+$ . Let us start by noting that the law of the thresholds  $(\theta_i(t), i = 1..N)$  only depends on their values at  $t-1$ . We have with a probability 1-s:

$$\theta_i(t+1) = \theta_i(t) \quad (11)$$

and with probability s

$$\theta_i(t+1) = |r_t| = |g(\frac{1}{N} \sum_i [1_{\epsilon_t > \theta_i} - 1_{\epsilon_t < -\theta_i}])|. \quad (12)$$

In the extreme cases where  $s=0$  or  $s=1$ , we observed that the return  $r_t$  was in fact a Markov chain. This is not true in general: as seen in the above relations, the return  $r_t$  not only depends on  $r_{t-1}$  but also on the states of the agents  $\theta_i(t)$ . However it is readily observed from the above that:

**Property 1**  $[\theta_i(t), i = 1..N]$  is a Markov chain in  $E_+^N$ .

More interestingly, given that agents are indistinguishable and only the empirical distribution of threshold values affects the returns, defining  $N_k(t)$  as the (random) number of agents with a threshold smaller than the  $k$ -th value  $N_k(t) = \sum_{i=1}^N 1_{[0, a_k]}(\theta_i(t))$  one can actually show that the Markovian dynamics can be entirely described by  $(N_k(t), k = 0..N)$ :

**Property 2 (Evolution of heterogeneity)**  $(N_k(t), k = 0..N)_{t=0,1..}$  evolves as a Markov chain in  $\{0, \dots, N\}^N$ .

Note that  $N(t) = (N_k(t), k = 0..N)$  is none other than the (cumulative) population distribution of the thresholds. The fact that  $N(t)$  itself follows a Markov chain means that the population distribution of thresholds is a *random measure* on  $\{0, \dots, N\}$ , which is characteristic of disordered systems. In this case, even if we start from a deterministic set of values for the initial thresholds (even identical ones), the population distribution will evolve. By contrast with some models of disorder which have been used as analogies for systems of economic agents [2], here the disorder is endogeneous and is generated by the random updating mechanism.

The fraction of agents with a  $\theta_i$  smaller than a given value  $\theta$  at time  $t$  is:

$$F_{N,t}(\theta) = \frac{1}{N} \Theta(\theta - \theta_i(t)). \quad (13)$$

which in the thermodynamic limit tends to the cumulative of the distribution of the thresholds at time  $t$  if the law of large numbers(LLN) is valid :

$$F_t(\theta) = Prob(\theta_i(t) < \theta). \quad (14)$$

At time  $t$  the returns are

$$r_t = g\left(\frac{1}{N} \sum_i [1_{\epsilon_t > \theta_i} - 1_{\epsilon_t < -\theta_i}]\right) = g([F_{N,t}(\epsilon_t) - F_{N,t}(-\epsilon_t)]) = g(\text{sgn}(\epsilon_t) F_{N,t}(|\epsilon_t|)). \quad (15)$$

In the thermodynamic limit if the LLN is valid, one can describe the evolution of the distribution of the thresholds with this equation :

$$f_{t+1}(\theta) = (1 - s) f_t(\theta) + s \delta_{|r_t|, \theta}. \quad (16)$$

with  $|r_t| = |g(\text{sgn}(\epsilon_t) F_t(|\epsilon_t|))|$  We then get the evolution equation for  $F$ :

$$F_{t+1}(\theta) = (1 - s) F_t(\theta) + s \Theta(\theta - |r_t|). \quad (17)$$

And we have :

$$f_t = (1 - s)^t f_0 + \sum_{j=1}^t s^j (1 - s) \delta_{|r_{t-j}|}. \quad (18)$$

One can say from this analysis that the LLN is valid if the measured distribution of the thresholds matches after a time greater than  $-\frac{1}{\ln(1-s)}$  the distribution  $F$  computed with the above equation.

A quick look at figure 1 reveals that the evolution of occupation numbers of a given threshold computed with the asymptotic equation is similar to the one with the multi-agent simulation. The asymptotic equation is then a good candidate for determining the stationary distribution of the markovian process. A finer look is needed and will be investigated in an other work.

Note finally that the occupation numbers decays exponentially in time and increases through upward “jumps”: this behavior is actually similar to that of a class of stochastic volatility models, introduced by Barndorff-Nielsen & Shepard [1] and successfully used to describe various econometric properties of returns.

### 3 Conclusion.

We have presented a parsimonious agent-based model which is capable of reproducing the main empirical stylized facts observed in returns of financial assets. We focused our study on the heterogeneity and its dynamics in the model. More precisely, we derived an asymptotic equation for the evolution of the ditribution of the thresholds. A full exploration of the parameter phase space and of the complex dynamics of heterogeneity will be investigated in other works.

To quote Brian Arthur, “economics is changing currently from an emphasis on equilibrium and homogeneity to an emphasis on the formation of pattern and



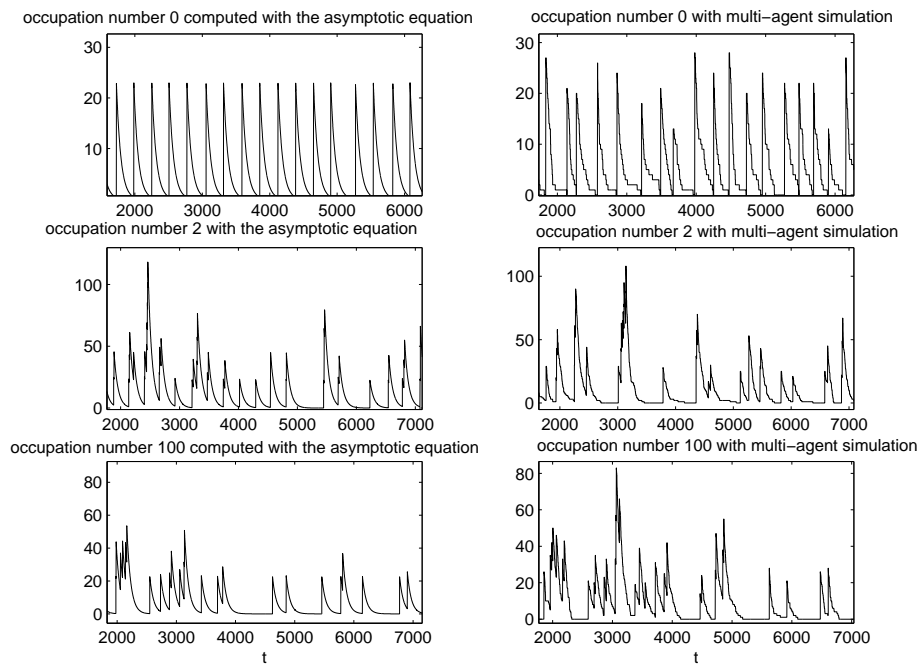


Figure 1: Evolution of occupation numbers of threshold  $\theta_k$ ,  $k=0,2,100$ , computed both with the asymptotic equation and with multi-agent simulation ( $N = 1500$ ).

heterogeneity". Thus, economics needs different tools and approaches. We hope that the present work will contribute to this objective.

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