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Generic singularities of minimax solutions to Hamilton–Jacobi equations*

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Abstract

Minimax solutions are weak solutions to Cauchy problems involving Hamilton–Jacobi equations, constructed from generating families quadratic at infinity of their geometric solutions. We give a complete description of minimax solutions and we classify their generic singularities of codimension not greater than 2.

Keywords : Minimax solution, Hamilton–Jacobi equation.
2000 MSC : 53D05, 53D10, 53D12, 58E05, 70H20.

1 Introduction

Hamilton–Jacobi equations play an important role in many fields of mathematics and physics, as for instance calculus of variations, optimal control theory, differential games, continuum mechanics and optics.

Let us consider a Cauchy Problem involving a Hamilton–Jacobi equation. For small enough time t the solution u is classically determined using the characteristic method. Although u is initially smooth, there exists in general a critical time beyond which characteristics cross. After this time, the solution u is multivalued and singularities appear. Therefore, the problem how to extract "true solutions" from multivalued solutions naturally arises.

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In 1991 Marc Chaperon proposed in [4] a geometric method to construct weak solutions to Hamilton–Jacobi equations, called minimax solutions (see also [13]). Their definition is based on generating families quadratic at infinity of Lagrangian submanifolds. The minimax solutions have the same analytic properties as the viscosity solutions, namely existence and uniqueness theorems hold. However, minimax solutions are in general different from viscosity solutions.

This paper is organized as follows. The first part is a survey on minimax solutions. In the second part we study the small codimension generic singularities of minimax solutions. Namely, we prove that the singularities of codimension not greater than 2 of minimax solutions and viscosity solutions are the same. This result may be useful to find examples in which minimax solutions have physical meaning.

The survey part of the paper is mainly based on a talk I have given for the "Trimester on Dynamical and Control Systems" at SISSA-ICTP in Trieste. I thank Andrei Agrachev for its kind invitation.

2 Minimax of functions quadratic at infinity

Let E be the space $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ a smooth function having a finite number if critical points. We suppose f *quadratic at infinity*, that is f is a non-degenerate quadratic form outside a compact set. For $\lambda \in \mathbb{R}$, we set

$$E^\lambda := \{\xi \in E : f(\xi) \leq \lambda\} \ .$$

Let us fix $a \in \mathbb{R}$, big enough for that $(-a, a)$ contains all the critical values of f. Then we set $E^{\pm\infty} := E^{\pm a}$.

Let us denote by ϱ_λ the restriction mapping $E \to E^\lambda$, inducing the homomorphism $\varrho_\lambda^* : \tilde{H}_*(E, E^{-\infty}) \to \tilde{H}_*(E^\lambda, E^{-\infty})$ between the relative reduced $\mathbb{Z}$-homology groups. The quotient space $E/E^{-\infty}$ is homeomorphic to $\mathbb{S}^q$, where q denotes the index of the quadratic part of f at infinity. Hence, the only non-trivial reduced homology group is $\tilde{H}_q(E, E^{-\infty}) \simeq \mathbb{Z}$. Fix a generator γ of this group.

Definition. The *minimax* of f is the real number

$$\min\max(f) := \inf\{\lambda \in \mathbb{R} : \varrho_\lambda^* \gamma \neq 0\} \ .$$

The infimum defining the minimax exists and it is finite, indeed $\varrho_\lambda^*\gamma = \gamma \neq 0$ and $\varrho_{-\lambda}^*\gamma = 0$ whenever $\lambda > a$; it does not depend on the choice of the generator γ. Moreover, the minimax of a function is a critical value, since by definition the topology of the sublevel sets E^λ changes when λ crosses the minimax value.

Since the minimax level is stable under small deformation of the function (see [2]), we may assume without loss of generality that f is generic. Genericity conditions mean that all the critical points of f are of Morse type and all its critical values are different. Therefore, every critical point has an *index*, which is the number of negative signs in the Morse normal form of the function at the critical point.

Morse Theory (see [9]) describes how the topology of the sublevels sets E^λ changes when λ changes. Namely, *E^λ is diffeomorphic to E^μ, provided that $[\lambda, \mu]$ does not contain critical values.* On the other hand, *if $[\lambda, \mu]$ contains only one critical value, then E^μ retracts on the space obtained from E^λ attaching to its boundary a cell of dimension equal to the index of the critical point realizing the critical value.* Hence, every generic function on E defines a cell decomposition of the space and there is a $1:1$ correspondence between its cells and the critical points of the function.

Let us recall that the *incidence coefficient* of two cells in this decomposition is the degree of the restriction of the attaching map from the higher dimension cell to the lower dimension cell (see [6]). The *incidence coefficient* $[\xi : \eta]$ of two critical points ξ and η of f is the incidence coefficient of the corresponding cells. Note that $[\xi : \eta] \neq 0$ implies that $\mathrm{ind}(\xi) = \mathrm{ind}(\eta) + 1$ and $f(\xi) > f(\eta)$. For example, the two critical points of $f(\xi) = \xi^3 - \varepsilon\xi$ have non-zero incidence coefficient for $\varepsilon > 0$.

The incidence coefficient of pairs of critical points leads to a natural partition of the critical set Σ of f in the following way. Denote by C_1 a pair of critical points realizing the minimum of the set

$$\{f(\xi) - f(\eta) : \xi, \eta \in \Sigma, [\xi : \eta] \neq 0\} \ .$$

Then define by induction C_{i+1} from $C_1, \ldots, C_i$ as a pair of critical points realizing the minimum of the set

$$\{f(\xi) - f(\eta) : \xi, \eta \in \Sigma \smallsetminus (C_1 \cup \cdots \cup C_i), [\xi : \eta] \neq 0\} \ .$$

In this way, we decompose the critical set into the disjoint union of pairs C_i and a set F which does not contains incident critical points, i. e. $[\xi : \eta] = 0$ for every $\xi, \eta \in F$.

Definition. Two critical points are *coupled* if they belong to the same pair C_i in the preceding decomposition; a critical point is *free* if it belongs to F.

In [2] we proved the following result.

Theorem 1. *Every generic function quadratic at infinity has exactly one free critical point, and its value is the minimax.*

3 Geometric and multivalued solutions

In this section we introduce the symplectic framework for the geometric and multivalued solutions of Hamilton–Jacobi equations. Let X be a closed manifold of dimension n, $\pi : T^*X \to X$ its cotangent bundle, endowed with the standard symplectic form $d\lambda$, where λ is the Liouville's 1-form. In local coordinates $T^*X = \{x, y\}$, we have $\pi(x, y) = x$ and $\lambda = y\ dx$. A dimension n submanifold of T^*X is said to be *Lagrangian* whenever $d\lambda$ vanishes on it. Two Lagrangian submanifolds are *Hamiltonian isotopic* if there exists a flow generated by a Hamiltonian field, transforming one into the other. Hamiltonian isotopies transform Lagrangian submanifolds into Lagrangian submanifolds. A *generating family* of a Lagrangian submanifold $L \subset T^*X$ is a smooth function $S : X \times \mathbb{R}^k \to \mathbb{R}$ such that

$$L = \left\{ (x, \partial_x S(x; \xi)) : \partial_\xi S(x; \xi) = 0 \right\} ,$$

where S verifies also the rank condition $\mathrm{rk}\left(\partial^2_{\xi\xi} S, \partial^2_{\xi x} S\right)|_{\partial_\xi S=0} = \max$.

Given a generating family S, the following operations give rise to new generating families T of the same Lagrangian submanifold:

1. *Addition of a constant:* $T(x; \xi) = S(x; \xi) + C$ for $C \in \mathbb{R}$;
2. *Stabilization:* $T(x; \xi, \eta) = S(x; \xi) + Q(\eta)$, where Q is a non-degenerate quadratic form;
3. *Diffeomorphism:* $T(x; \eta) = S(x; \xi(x, \eta))$, where $(x; \eta) \mapsto (x, \xi(x, \eta))$ is a fibered diffeomorphism with respect to the coordinate x.

Two generating families are *equivalent* if one can obtain one from the other by a finite sequence of the preceding operations. A generating family S is *quadratic at infinity* (gfqi) if there exists a non-degenerate quadratic form Q such that $S(x; \xi) = Q(\xi)$, whenever $|\xi|$ is big enough (uniformly in x).

Theorem 2 (Sikorav–Viterbo, [10]—[12]). *Every Lagrangian submanifold of T^*X, Hamiltonian isotopic to the zero section $\{(x;0) : x \in X\}$, admits a unique gfqi modulo the preceding equivalence relation.*

Remark. The Theorem still holds in the case of non-compact manifolds, provided that the projection of the Lagrangian submanifold into the base is $1:1$ outside a compact set.

A Lagrangian submanifold $L \subset T^*X$ is *exact* if the Liouville's 1-form is exact on it. In this case, we can lift L to a Legendrian submanifold in the contact space J^1X of 1-jets over X. This lifting projects into a wave front in the space $J^0X \simeq X \times \mathbb{R} = \{x, z\}$ of 0-jets over X, defined up to shifts in the z-direction. If S is a gfqi of L, then the corresponding wave front is the graph of S

$$\{(x, S(x;\xi)) : \partial_\xi S(x;\xi) = 0\} \ .$$

Graphs of equivalent gfqi are equal up to a translation along the z-axis. The points for which the natural projection "forgetting z" of the graph into X is not a fibration form a stratified hypersurface in X, provided that L is generic.

We may now define generalized solutions to Hamilton–Jacobi equations. Let Q be a closed manifold. We consider the following Cauchy Problem on Q involving a Hamilton–Jacobi equation:

$$(CP) \begin{cases} \partial_t u(t,q) + H(t,q,D_q u(t,q)) = 0, & \forall\, t \in \mathbb{R}^+, q \in Q \\ u(0,q) = u_0(q), & \forall\, q \in Q \ . \end{cases}$$

The Hamiltonian $H : \bar{\mathbb{R}}^+ \times T^*Q \to \mathbb{R}$ is supposed of class $\mathscr{C}^2$ on $\mathbb{R}^+ \times T^*Q$ and continuous at the boundary; the initial condition $u_0 : Q \to \mathbb{R}$ is assumed to be of class $\mathscr{C}^1$.

Let $M := Q \times \mathbb{R}$ be the space-time manifold and $T^*M = \{t, q; \tau, p\}$ its cotangent bundle (endowed with the symplectic form $dp \wedge dq + d\tau \wedge dt$). We consider the flow $\Phi : \bar{\mathbb{R}}^+ \times T^*M \to T^*M$, generated by the Hamiltonian $\mathscr{H} : T^*M \to \mathbb{R}$ defined by $\mathscr{H} := \tau + H$, and the submanifold

$$\sigma := \big\{\big((0,q; -H(0,q,du_0(q)), du_0(q)\big) : q \in Q\big\} \subset \mathscr{H}^{-1}(0) \subset T^*M \ .$$

Definition. The *geometric solution* of (CP) is the submanifold

$$L := \bigcup_{t>0} \Phi^t(\sigma) \subset T^*M \ .$$

It turns out that the geometric solution to (CP) is an exact Lagrangian submanifold, contained into the hypersurface $\mathscr{H}^{-1}(0)$ and Hamiltonian isotopic (for finite times) to the zero section of T^*M (see [2]). Hence, the Sikorav–Viterbo Theorem guarantees that it has a unique gfqi modulo the equivalence relation. Up to a suitable constant, we may assume that the graph of the gfqi restrained at $t = 0$ is equal to σ. We denote by S such a gfqi.

Let us consider now the projection $\mathrm{pr} : T^*M \to T^*Q$, defined locally by $\mathrm{pr}(t, q; \tau, p) := (q, p)$. We call *geometric solution at time* t of (CP) the exact Lagrangian submanifold $L_t := \mathrm{pr} \circ \Phi^t(\sigma) \subset T^*Q$, which is an isochrone section of the global geometric solution. It is easy to check that L_t is Hamiltonian isotopic to the zero section, so, by the Uniqueness Theorem, its gfqi is $S_t(q; \xi) := S(t, q; \xi)$.

Definition. The graph of S (resp., S_t) will be called the *multivalued solution* (resp., *at time* t) to (CP).

Remark. It is possible to construct global generating families of geometric solutions as follows. The *action functional* $\int pdq - Hdt$ is a global generating family, whose parameters belong to an infinite dimension space. By a fix point method, proposed by Amann, Conley and Zehnder, one can obtain a true generating family (with finite dimensional parameters), see [3].

4 Minimax and viscosity solutions

In this section we introduce minimax solutions to Hamilton–Jacobi equations, and we discuss their properties and relations with viscosity solutions. Let S be the gfqi of the geometric solution to (CP). For every fixed point (t, q) in the space time, the function $S_{t,q}$ is quadratic at infinity, so we may consider its minimax critical values, defined in section 2.

Definition. The *minimax solution* to (CP) is the function defined by

$$u(t, q) := \min \max(S_{t,q}) \ .$$

Remark. For a Cauchy Problem (CP) on a non-compact manifold, as for instance for $Q = \mathbb{R}^n$, we define the minimax solution as follows. Let $(H_n)_n$ be a sequence of smooth Hamiltonians with compact support in the variables q. Then, at any Cauchy Problem defined by H_n and the given initial condition,

we may associate its minimax solution U_n, defined as above. By definition, the minimax solution of (CP) is the limit of the solutions U_n for $n \to \infty$.

Theorem 3 (Chaperon, [4]). *The minimax solutions are weak solutions*[1] *to (CP), Lipschitz on finite times, which not depend on the choice of the gfqi.*

We end this section discussing the relations between minimax and viscosity solutions. Viscosity solutions to Hamilton–Jacobi equations have been introduced by Crandall and Lions in [5], following the generalization proposed by Kruzhkov of the Hopf's formulas.

Theorem 4 (Joukovskaia, [8]). *The minimax and the viscosity solutions of (CP) are equal, provided that the Hamiltonian defining the equation is convex or concave in the variables p.*

Izumiya and Kossioris constructed in [7] the viscosity solution beyond its first critical time. Actually, they proved that the Lagrangian graph of the viscosity solution is not always contained into the geometric solution of the equation. Hence, minimax and viscosity solutions are in general different.

5 Characterization of minimax solutions

In order to describe a useful characterization of minimax solutions (first presented in [2]), we start introducing some standard notations concerning unidimensional wave fronts.

Let $J^0\mathbb{R} \simeq \mathbb{R}^2 = \{(q, z)\}$ be the space of 0-jets over $\mathbb{R}$ and $\pi_0 : J^0\mathbb{R} \to \mathbb{R}$ the natural fibration $(q, z) \mapsto q$. A *wave front* in $J^0\mathbb{R}$ is the projection of an embedded Legendrian curve in the contact space $J^1\mathbb{R} \simeq \mathbb{R}^3 = \{(q, z, p)\}$ under the projection $\pi_1 : (q, z, p) \mapsto (q, z)$. The only singularities of generic wave fronts are semicubic cusps and transversal self-intersections. Two embedded Legendrian curves in $J^1\mathbb{R}$ are *Legendrian isotopic* if there exists a smooth path joining them in the space of the embedded Legendrian curves. In this case their projections are also called isotopic. A front is *long* if it is the graph of a smooth function outside a compact set of $\mathbb{R}$; it is *flat* if its

[1] A weak solution to (CP) is a continuous and almost everywhere derivable function, which solves in these points the Hamilton–Jacobi equation; moreover, its restriction at $t = 0$ is equal to the initial condition.

tangent lines are nowhere vertical. A *section* of a flat front is a connected maximal subset which is the graph of a piecewise $\mathscr{C}^1$ function; a *branch* is a smooth section. The generic perestroikas occurring to flat wave fronts under isotopies among flat fronts are the following, illustrated in figure 1: *cusp birth/death*, *triple intersection*, *cusp crossing*.

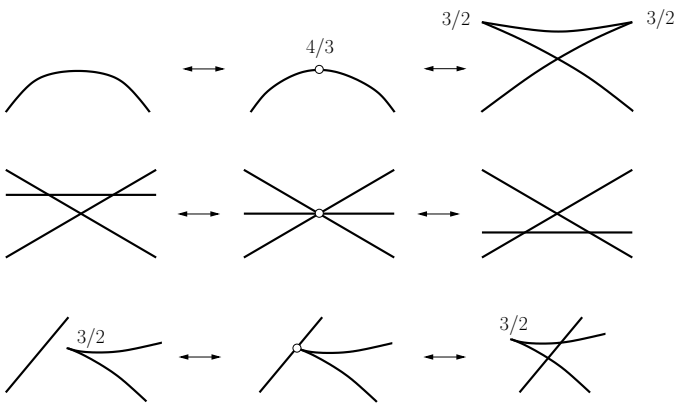

Figure 1: Generic singularities of flat fronts under Legendrian isotopies.

We come back now to the Cauchy Problem (CP) in the case $Q = \mathbb{R}$. We assume that its geometric solution is a generic Lagrangian curve. A multivalued solution Σ of (CP) at a generic time is a flat wave front, isotopic to $\{(q, 0) : q \in \mathbb{R}\}$. Up to replace our Cauchy Problem with a sequence of problems approximating it, we may assume also that Σ is a long wave front, and that it is the graph of a gfqi, denoted by S, of the geometric solution. Hence, every branch of the multivalued solution is realized by a critical point of $S_{t,q}$ (depending on q as a parameter).

Definition. The *index* of a branch is by definition the index of its corresponding critical point minus the index of the quadratic part of S.

The index of a branch does not depend on the choice of the gfqi. The non-compact branch of Σ has index 0, as well as the minimax section. Moreover, the index changes by $+1$ (resp., -1) passing through a positive cusp[2] (resp., negative cusp). A double point of Σ is called *homogeneous* whenever it is the intersection of two branches having the same index.

[2] A cusp is *positive* (resp., *negative*) if, following the orientation of the front, we pass from a branch to the other according to the coorientation defined by the positive z direction.

The partition of the critical set of a generic function quadratic at infinity we give in section 2 leads, by Theorem 1, to a decomposition of the multivalued solution into the graph μ of the minimax solution and the pairs X_i of coupled sections, formed by the sections corresponding to coupled critical points of the gfqi:

$$\Sigma = \mu \cup \bigcup_i X_i \ .$$

Indeed, when q runs on $\mathbb{R}$, every pair of coupled critical points of $S_{t,q}$ moves on the front along two sections, forming a closed curve.

The curves X_i are homeomorphic to a disc boundary, they have exactly two cusps and no self-intersections. They are almost everywhere smooth and only continuous at homogeneous double points of the multivalued solution (see [2]).

Any homogeneous double point of an oriented multivalued solution defines in it a closed connected curve. Such a curve is called a *triangle* whenever it has exactly two cusps. The double point is the *vertex* of the triangle. Let T be a triangle of Σ. Fix an arbitrary small ball centered at the vertex of T, intersecting only the two branches containing the double point. We denote by $\Sigma - T$ a long flat front equal to $\Sigma \setminus T$ outside the ball and smooth inside it (see figure 2).

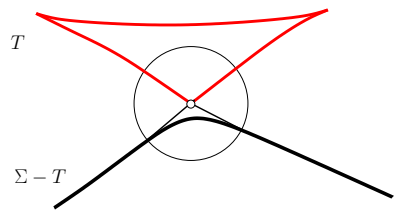

Figure 2: The front $\Sigma - T$.

Definition. A triangle T of Σ is *vanishing* if there exists a Legendrian isotopy joining Σ and $\Sigma - T$ among long flat fronts without homogeneous triple intersections.

The following Theorem (see [2]) provides an effective method to simplify recursively a given multivalued solution. After a finite number of steps, we get a front, which is actually the graph of the minimax solution to the initial problem.

Theorem 5. *Let Σ be a multivalued solution and $\mu \cup X_1 \cup \cdots \cup X_N$ its decomposition into minimax section and coupled sections. Then Σ is smooth or has a vanishing triangle T among the curves X_i. In this case, the decomposition of $\Sigma - T$ is induced by that of Σ. In particular, the minimax of Σ and $\Sigma - T$ are equal outside the ball centered at the vertex of the vanishing triangle.*

Example. Consider the multivalued solution Σ depicted in figure 3. The homogeneous double points B, C, D, F, G and I are not verteces of triangles. The triangles T_E and T_L of verteces E and L are vanishing, while those of verteces A, H and M are not. Indeed, to cut off T_A and T_M we must pass through a self-tangency, while to cut off T_H we must pass through a homogeneous triple intersection at F. The minimax of Σ is equal, outside two arbitrary small balls centered at E and L, to the minimax of $\Sigma' := \Sigma - T_E - T_L$. Now, B and I define two vanishing triangles T_B and T_I; $\Sigma' - T_B - T_I$ is smooth, so it is the minimax section of Σ outside two arbitrary small balls centered at B and I.

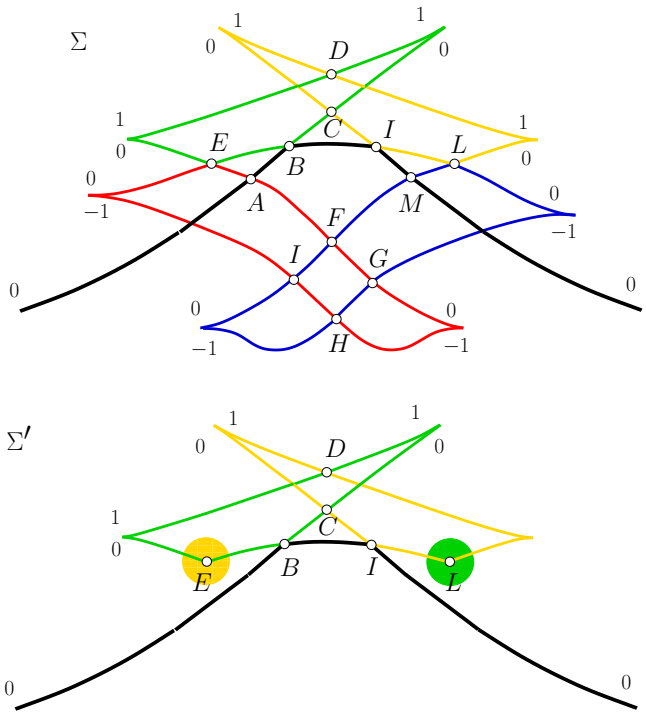

Figure 3: The multivalued solutions Σ and Σ'.

6 Singularities of minimax solutions

In this section we classify the generic singularities of small codimension arising in minimax solutions to Hamilton–Jacobi equations. This classification is made with respect to the Left–Right equivalence, fibered on the time direction, defined below. Let us consider two functions $f, g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. We denote by t the coordinate in $\mathbb{R}$ and by $q = (q_1, \ldots, q_n)$ the coordinate in $\mathbb{R}^n$.

Definition. Let a, b be two points of the space-time $\mathbb{R} \times \mathbb{R}^n$. The germ of f at a and the germ of g at b are said to be *equivalent* if there exists two diffeomorphism germs φ and ψ such that $\psi \circ f \circ \varphi^{-1} = g$ and φ is fibered with respect to the time axis: $\varphi(t, q) = \big(T(t), Q(t, q)\big)$.

A *singularity* of a function germ for this equivalence relation is its equivalence class. A Cauchy Problem on $Q = \mathbb{R}^n$ is said to be *generic* if its geometric solution is generic as Lagrangian submanifold; its minimax solution is also called *generic*.

In [8], Joukovskaia proved that the singular set of any generic minimax solution (formed by the points where the solution is not $\mathscr{C}^1$) is a closed stratified hypersurface, diffeomorphic at any point to a semi-algebraic hypersurface.

Theorem 6. *Let u be a generic minimax solution and (t, q) a point belonging to a stratum of codimension $c = 1$ or 2 of its singular set. Then the germ of u at (t, q) is equivalent to one of the map germs in the table below.*

c	*Normal form*
1	$\lvert q_1 \rvert$
2	$\min\{\lvert q_1 \rvert, t\}$
2	$\min\{Y^4 - tY^2 + q_1 Y : Y \in \mathbb{R}\}$

The images of these normal forms (in the case $n = 2$) are depicted in figure 4. Note that these singularities are stable. These map germs are also the normal forms of the generic singularities (of codimension not greater than 2) of viscosity solutions (whose classification has been done by Bogaevski in [1]).

Joukovskaia studied in [8] the generic singularities of minimax functions, defined as minimax levels of Lagrangian submanifold's gfqi which are not necessarily geometric solutions of Hamilton–Jacobi equations. To prove Theorem 6, we will show that some singularities in Joukovskaia's list do not arise in minimax solutions. In order to do this, let us recall her Theorem.

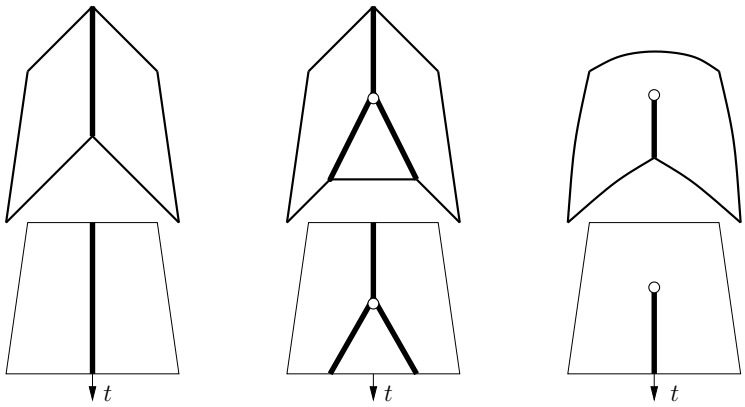

Figure 4: Small codimension generic singularities of minimax solutions.

Theorem 7 (Joukovskaia, [8]). *Any generic minimax function on* $\mathbb{R} \times \mathbb{R}^n$ *is equivalent, at any point in a codimension* 2 *stratum of its singular set, to one of the map germs listed in Theorem* 6 *(for* $c = 2$*) or to one of the following map germs:*

(a) $(t; q_1, \ldots, q_n) \mapsto \max\left\{t, -|q_1|\right\}$;

(b) $(t; q_1, \ldots, q_n) \mapsto \min\left\{|q_1|, \max\{-|q_1|, t\}\right\}$.

The images of the map germs (a) and (b) are shown in figure 5.

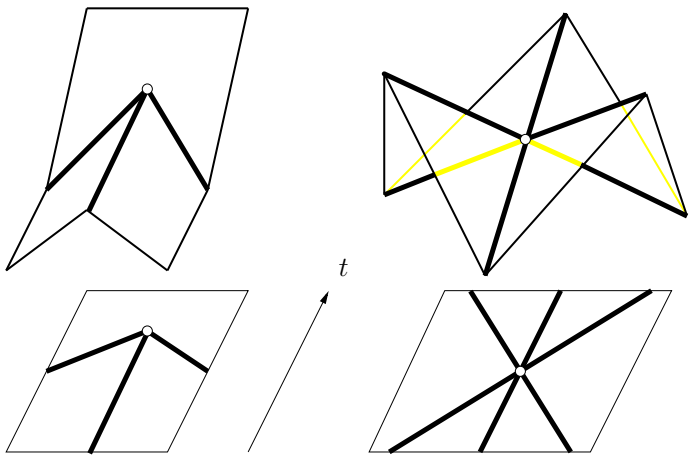

Figure 5: Further generic singularities of minimax functions.

Proof of Theorem 6. We may assume $n = 1$ without loss of generality. Let us consider a multivalued solution having a singularity (a) or (b). In both cases, there are three 0-index branches crossing at the singular point (settled at the origin).

Fix an arbitrary small neighborhood U of the origin in $\mathbb{R}$; then for every small enough time $t < 0$, the isochrone multivalued solution at this time takes, over the corresponding section of U, the configuration illustrated in the left part of figure 6; moreover, all the other sections of the front's decomposition are smooth over U. The minimax section μ has three jumps in A, B and C, shrinking to the homogeneous triple point at the origin as t goes to 0.

Actually, the jumps A and C belong to the same section, denoted by α in the figure. By Theorem 5 we may recursively eliminate the vanishing

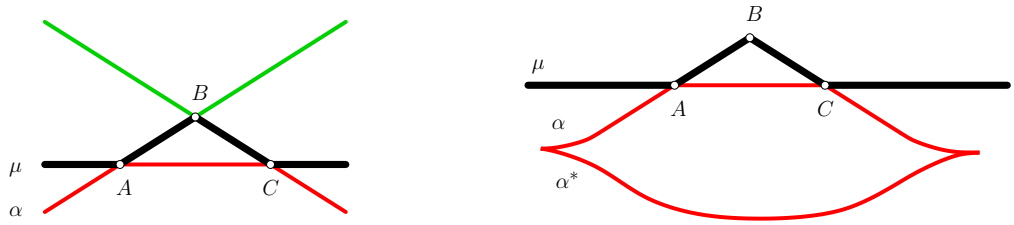

Figure 6: Multivalued solution configurations near the singular point.

triangles in the multivalued solution changing neither the minimax section nor the other pairs of coupled critical points. Hence, after a finite number of such operations, the front is composed by the minimax solutions, the section α and its coupled section α^*, see the right part of figure 6.

This front is not isotopic to the trivial front $\{(q, 0) : q \in \mathbb{R}\}$. Hence, multivalued solutions have neither singular points of type (a) nor (b). Theorem 6 is now proven. □

Remark. In other words, both singularities (a) and (b) imply the existence of a cycle in the graph associated to a decomposition of an isochrone multivalued solution of the Cauchy Problem. This is impossible, since all the graphs associated to admissible decompositions of isochrone multivalued solutions are trees, due to Theorem 5.

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