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► **To cite this version:**

Andrei Moroianu, Uwe Semmelmann. Twistor Forms on Riemannian Products. 2007. hal-00002174v2

HAL Id: hal-00002174

<https://hal.science/hal-00002174v2>

Preprint submitted on 16 May 2007 (v2), last revised 9 Sep 2008 (v3)

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TWISTOR FORMS ON RIEMANNIAN PRODUCTS

ANDREI MOROIANU AND UWE SEMMELMANN

ABSTRACT. We study twistor forms on products of compact Riemannian manifolds and show that they are defined by Killing forms on the factors. The main result of this note is a necessary step in the classification of compact Riemannian manifolds with non-generic holonomy carrying twistor forms.

Keywords: Twistor forms, Killing forms, Riemannian products, conformal vector fields.
2000 Mathematics Subject Classification. Primary 53C29, 58J50.

1. TWISTOR FORMS ON RIEMANNIAN MANIFOLDS

A *twistor p -form* on a Riemannian manifold (M^n, g) is a smooth section ψ of $\Lambda^p T^*M$ whose covariant derivative only depends on its differential $d\psi$ and codifferential $\delta\psi$. More precisely, ψ satisfies the equation

$$\nabla_X \psi = \frac{1}{p+1} X \lrcorner d\psi - \frac{1}{n-p+1} X \wedge \delta\psi, \quad (1)$$

for all vector fields X . Note that here, and in the remaining part of this note, we identify vectors and 1-forms using the metric.

If the p -form ψ is in addition coclosed (*i.e.* $\delta\psi = 0$), then it is called a *Killing p -form*. For a comprehensive introduction to twistor forms, see [4].

A few years ago, a program of classification of twistor forms on compact manifolds was started. By the de Rham decomposition theorem, every simply connected Riemannian manifold is a Riemannian product of *irreducible* manifolds. Moreover, the Berger holonomy theorem implies that any simply connected irreducible Riemannian manifold is either symmetric or has holonomy SO_n , U_m , SU_m , Sp_k , $\mathrm{Sp}_k \cdot \mathrm{Sp}_1$, G_2 or Spin_7 . Killing forms on symmetric spaces were studied in [1]. Twistor forms on Kähler manifolds (covering the holonomies U_m , SU_m , and Sp_k) were described in [2], and Killing forms on quaternion-Kähler manifolds (holonomy $\mathrm{Sp}_k \cdot \mathrm{Sp}_1$) or Joyce manifolds (holonomies G_2 or Spin_7) were studied in [3] and [5] respectively. In Theorem 2.1 below, we prove that the general case (twistor forms on a Riemannian product of compact manifolds) reduces to the study of Killing forms on the factors. By the discussion above, besides the case of generic holonomy (SO_n), all other cases are fully understood.

2. THE MAIN RESULT

Let $M = M_1 \times M_2$ be the Riemannian product of two compact Riemannian manifolds M_1 and M_2 of dimensions m and n respectively. The aim of this note is to prove the following result:

Theorem 2.1. *Every twistor form on M is a sum of forms of the following types: parallel forms, pull-backs of Killing forms on M_1 or M_2 , and Hodge duals of them.*

Proof. We define the differential operators

$$d_1 = \sum_{i=1}^m e_i \wedge \nabla_{e_i}, \quad d_2 = \sum_{j=1}^n f_j \wedge \nabla_{f_j},$$

where $\{e_i\}$ and $\{f_j\}$ denote local orthonormal basis of the tangent distributions to M_1 and M_2 . Using the Fubini theorem, we easily see that the adjoint operators to d_1 and d_2 are

$$\delta_1 = - \sum_{i=1}^m e_i \lrcorner \nabla_{e_i}, \quad \delta_2 = - \sum_{j=1}^n f_j \lrcorner \nabla_{f_j}.$$

The following relations are straightforward:

$$\begin{aligned} d &= d_1 + d_2, & \delta &= \delta_1 + \delta_2, & (d_1)^2 &= (d_2)^2 = (\delta_1)^2 = (\delta_2)^2 = 0, \\ 0 &= d_1 d_2 + d_2 d_1 = \delta_1 \delta_2 + \delta_2 \delta_1, & 0 &= d_1 \delta_2 + \delta_2 d_1 = \delta_1 d_2 + d_2 \delta_1. \end{aligned}$$

The vector bundle $\Lambda^p M$ decomposes naturally as

$$\Lambda^p M \cong \bigoplus_{i=0}^p \Lambda^{i,p-i} M,$$

where $\Lambda^{i,p-i} M \cong \Lambda^i M_1 \otimes \Lambda^{p-i} M_2$. Obviously, d_1 and δ_1 map $\Lambda^{i,p-i} M$ to $\Lambda^{i+1,p-i} M$ and $\Lambda^{i-1,p-i} M$ respectively, and d_2 and δ_2 map $\Lambda^{i,p-i} M$ to $\Lambda^{i,p-i+1} M$ and $\Lambda^{i,p-i-1} M$ respectively.

With respect to the above decomposition, every p -form can be written $u = u_0 + \dots + u_p$, where $u_i \in \Lambda^i M_1 \otimes \Lambda^{p-i} M_2$. For the remaining of this paper, u will denote a twistor p -form on M , where $1 \leq p \leq n + m - 1$. The twistor equation reads

$$\nabla_X u = \frac{1}{p+1} X \lrcorner (d_1 u + d_2 u) - \frac{1}{m+n-p+1} X \wedge (\delta_1 u + \delta_2 u), \quad \forall X \in TM. \quad (2)$$

By projection onto the different irreducible components of $\Lambda^p M$, (2) can be translated into the following two systems of equations:

$$\nabla_X u_k = \frac{1}{p+1} X \lrcorner (d_1 u_k + d_2 u_{k+1}) - \frac{1}{m+n-p+1} X \wedge (\delta_1 u_k + \delta_2 u_{k-1}), \quad \forall X \in TM_1, \quad (3)$$

and

$$\nabla_X u_k = \frac{1}{p+1} X \lrcorner (d_1 u_{k-1} + d_2 u_k) - \frac{1}{m+n-p+1} X \wedge (\delta_1 u_{k+1} + \delta_2 u_k), \quad \forall X \in TM_2. \quad (4)$$

Taking the wedge product with X in (3) and summing over an orthonormal basis of TM_1 yields $d_1u_k = \frac{k+1}{p+1}(d_1u_k + d_2u_{k+1})$, so

$$(p-k)d_1u_k = (k+1)d_2u_{k+1}, \quad (5)$$

and similarly, taking the interior product with X and summing over an orthonormal basis of TM_1 yields $\delta_1u_k = \frac{m-k+1}{m+n-p+1}(\delta_1u_k + \delta_2u_{k-1})$, thus

$$(n+k-p)\delta_1u_k = (m-k+1)\delta_2u_{k-1}. \quad (6)$$

Suppose first that p is strictly smaller than m and n . For $k < p$, (5) and (6) imply

$$\delta_1d_1u_k = \frac{k+1}{p-k}\delta_1d_2u_{k+1} = -\frac{k+1}{p-k}d_2\delta_1u_{k+1} = -\frac{(k+1)(m-k)}{(p-k)(n+k-p+1)}d_2\delta_2u_k. \quad (7)$$

Integrating over M yields $0 = d_1u_k = \delta_2u_k$, $\forall k < p$. Similarly one gets $0 = d_2u_k = \delta_1u_k$, $\forall k > 0$. Moreover, we have $0 = \delta_2u_p = \delta_1u_0$ (tautologically), so in particular $\delta_1u_k = \delta_2u_k = 0$, $\forall k$. From (3) and (4) we see that u_1, \dots, u_{p-1} are parallel forms on M , and u_0 and u_p are pull-backs of Killing forms on M_2 and M_1 respectively. In other words, u can be written in the following form:

$$u = u_{M_1} + u_{M_2} + \text{parallel form},$$

where u_{M_1} and u_{M_2} are Killing forms on M_1 and M_2 respectively.

If p is strictly larger than m and n , then the argument above applied to the Hodge dual $*u$ – which is again a twistor form on M by (1) – shows that u is equal to a sum

$$u = u_{M_1} \wedge \text{vol}_{M_2} + u_{M_2} \wedge \text{vol}_{M_1} + \text{parallel form},$$

where u_{M_1} and u_{M_2} are closed twistor forms on M_1 and M_2 and vol_{M_1} and vol_{M_2} denote the volume forms of M_1 and M_2 respectively.

It remains to treat the case where p is a number between m and n , and we suppose without loss of generality that $m \leq p \leq n$. Obviously $u_{m+1} = \dots = u_p = 0$. Now, for every $k \leq m-1$, we obtain as before, using (7) and integrating over M , that $0 = d_1u_k = \delta_2u_k$, $\forall k \leq m-1$ and similarly, $0 = d_2u_k = \delta_1u_k$, $1 \leq k \leq m$. As before, (3) and (4) show that u_1, \dots, u_{m-1} are parallel forms on M , u_0 is the pull-back of a Killing form on M_2 , and u_m can be written as $\text{vol}_{M_1} \wedge u_{M_2}$, where u_{M_2} is a closed twistor form on M_2 . This proves the theorem. □

As an application of this result, we have the following:

Corollary 2.2. *Let (M^n, g) be a compact simply connected Riemannian manifold. If M carries a conformal vector field which is not Killing, then $\text{Hol}(M) = \text{SO}_n$.*

Proof. Assume first that $(M, g) = (M_1, g_1) \times (M_2, g_2)$ is a Riemannian product with $\dim(M_1), \dim(M_2) \geq 1$. Then, taking into account that the isomorphism between 1-forms and vector fields defined by the Riemannian metric maps twistor forms to

conformal vector fields and Killing forms to Killing vector fields, Theorem 2.1 implies that every conformal vector field on M is a Killing vector field. Thus M is irreducible.

Assume next that $\text{Hol}(M) \neq \text{SO}_n$. From the Berger-Simons holonomy theorem, M is either an irreducible symmetric space (in particular Einstein), or its holonomy group is U_m , SU_m , Sp_k , $Sp_k \cdot Sp_1$, G_2 or $Spin_7$. In the first 3 cases the manifold is Kähler and in the last 3 it is Einstein. Now, a classical result by Obata states that a conformal vector field on a compact manifold M is already a Killing vector field if M is Kähler or if M is Einstein and not isometric to the round sphere.

The only possibility left is therefore $\text{Hol}(M) = \text{SO}_n$.

□

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