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Deformation quantization on a Hilbert space

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Summary. We study deformation quantization on an infinite-dimensional Hilbert space W endowed with its canonical Poisson structure. The standard example of the Moyal star-product is made explicit and it is shown that it is well defined on a subalgebra of $C^\infty(W)$. A classification of inequivalent deformation quantizations of exponential type, containing the Moyal and normal star-products, is also given.

1 Introduction

Deformation quantization provides an alternative formulation of Quantum Mechanics by interpreting quantization as a deformation of the commutative algebra of classical observables into a noncommutative algebra [1]. The quantum algebra is defined by a formal associative star-product \star_\hbar which encodes the algebraic structure of the set of observables.

Deformation quantization has been applied with increasing generality to several areas of mathematics and physics. Most of these applications deal with star-products on finite-dimensional manifolds. See [3] for a recent review.

It is natural to consider an extension of deformation quantization to infinite-dimensional manifolds as it appears to be a good setting where quantum field theory of nonlinear wave equations can be formulated (e.g. in the sense of I. Segal [11]). In the star-product approach, the first steps in that direction are given in [4, 5].

Recently, deformation quantization has become popular among field and string theorists. A generalization of Moyal star-product to infinite-dimensional spaces appears in several places in the literature. Let us just notice that the Witten star-product [12] appearing in string field theory is *heuristically* equivalent to an infinite-dimensional version of the Moyal star-product. A brute force generalization of Moyal star-product to field theory yields to some pathological and unpleasant features such as anomalies and breakdown of associativity. We think that it is worth writing down a mathematical study of the Moyal product in infinite dimension even if it is not an adequate product for field theory considerations.

In the finite-dimensional case, the existence of star-products on any (real) symplectic manifold has been established by DeWilde and Lecomte [2]. The general existence and classification problems for the deformation quantization of a Poisson manifold was solved by Kontsevich [9]. However, the very first problem that one faces when going over infinite-dimensional spaces, it to make sense of the star-product itself as a formal associative product. It contrasts with the finite-dimensional case where the deformation is defined on all of the smooth functions on the manifold.

This is by far too demanding in the infinite-dimensional case even when the Poisson structure is well-defined on all of the smooth functions (e.g. on Banach or Fréchet spaces). One should specify first an Abelian algebra of admissible functions which then can be deformed. For example, on $E = \mathcal{S} \times \mathcal{S}$, where \mathcal{S} is the Schwartz space on \mathbb{R}^n , endowed with its canonical Poisson structure, one cannot expect to write down a star-product defined on all holomorphic functions on E , but has to restrict it to some subalgebra. For example, in [5] it is shown, that for such a simple star-product as the normal star-product, it is defined on the subalgebra of holomorphic functions of a and \bar{a} (creation and annihilation ‘operators’) having semi-regular kernels. In [6], one can find a nice analysis for the normal star-product and the conditions on the kernel have been translated in terms of wave front set of the distributions.

After making precise what is a deformation quantization on a Hilbert space, we first present a study of Moyal product when the space-space is the direct sum of a Hilbert space with its dual. We identify a subalgebra of smooth functions, specified by conditions of Hilbert-Schmidt type on their derivatives, on which the Moyal product makes sense. We also define a family of star-products of exponential type, show that they are not all equivalent to each other and give the classification of their equivalence classes in terms of Hilbert-Schmidt operators.

2 Star-products on a Hilbert space

When infinite-dimensional spaces are involved, further conditions are needed to define a deformation quantization or a star-product. The algebra of functions on which the Poisson bracket and the star-product are defined should be specified along with the class of *admissible* cochains (especially when the issue of the equivalence of deformations is considered).

2.1 Notations

Let B be a Banach space over a field \mathbb{K} (\mathbb{R} or \mathbb{C}). The topological dual of B shall be denoted by B^* . The Banach space of bounded r -linear forms on B is denoted by $\mathcal{L}^r(B, \mathbb{K})$ and $\mathcal{L}_{\text{sym}}^r(B, \mathbb{K})$ is the subspace of $\mathcal{L}^r(B, \mathbb{K})$ consisting of bounded symmetric r -linear forms on B . We shall denote by $C^\infty(B, \mathbb{K})$ the space of \mathbb{K} -valued functions on B that are smooth in the Fréchet sense. The Fréchet derivative of $F \in C^\infty(W, \mathbb{K})$ is denoted by DF and it is a smooth map from B to $\mathcal{L}^1(B, \mathbb{K}) = B^*$, i.e., $DF \in C^\infty(B, B^*)$. For $F \in C^\infty(B, \mathbb{K})$, the higher derivative $D^{(r)}F$ belongs to $C^\infty(B, \mathcal{L}_{\text{sym}}^r(B, \mathbb{K}))$ and we shall use the following notation $D^{(r)}F(b).(b_1, \dots, b_r)$ for the r^{th} -derivative of F evaluated at $b \in B$ in the direction of $(b_1, \dots, b_r) \in B^r$.

Let W be an infinite-dimensional separable Hilbert space over a field \mathbb{K} . For notational reasons, as it will become clear later, it would be convenient for us to not identify W^* with W . For any orthonormal basis $\{e_i\}_{i \geq 1}$ in W and corresponding dual basis $\{e_i^*\}_{i \geq 1}$ in W^* , we shall denote the partial derivative of $F \in C^\infty(W, \mathbb{K})$ evaluated at w in the direction of e_i by $\partial_i F(w) \in \mathbb{K}$, i.e., $\partial_i F(w) = DF(w).e_i$. Since F is differentiable in the Fréchet sense, we have $DF(w) = \sum_{i \geq 1} \partial_i F(w)e_i^*$ and thus, for any $w \in W$, that $\sum_{i \geq 1} |\partial_i F(w)|^2 < \infty$.

2.2 Multidifferential operators

Let us first make precise what we call a Poisson structure on W . In the following, we will consider a map P that sends W into a space of (not necessarily bounded) bilinear forms on W^* and a subalgebra \mathcal{F} of $C^\infty(W, \mathbb{K})$. We define the subspace $\mathcal{D}_w^\mathcal{F} = \{DF(w) \mid F \in \mathcal{F}\}$ of W^* .

Definition 1. *Let W be a Hilbert space. Let \mathcal{F} be an Abelian subalgebra (for the pointwise product) of $C^\infty(W, \mathbb{K})$. A Poisson bracket on (W, \mathcal{F}) is a \mathbb{K} -bilinear map $\{\cdot, \cdot\}: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ such that:*

i) there exists a map P from W to the space of bilinear forms on W^ , so that the domain of $P(w)$ contains $\mathcal{D}_w^\mathcal{F} \times \mathcal{D}_w^\mathcal{F}$ and $\forall F, G \in \mathcal{F}$, $\{F, G\}(w) = P(w).(DF(w), DG(w))$ where $w \in W$.*

ii) $(\mathcal{F}, \{\cdot, \cdot\})$ is a Poisson algebra, i.e., skew-symmetry, Leibniz rule, and Jacobi identity are satisfied.

The triple $(W, \mathcal{F}, \{\cdot, \cdot\})$ is called a Poisson space.

Let us give an example where $P(w)$ is an unbounded bilinear form on W^* .

Example 1. Consider a real Hilbert space W with orthonormal basis $\{e_i\}_{i \geq 0}$. We will realize the following subalgebra of the Witt algebra:

$$[L_m, L_n] = (m - n)L_{m+n}, \quad m, n \geq 0,$$

by functions on W . For $w \in W$, let $\phi_i(w) = \langle e_i^*, w \rangle$, $i \geq 0$ be the coordinate functions. The algebra \mathcal{F} generated by the family of functions $\{\phi_i\}_{i \geq 0}$ is an Abelian subalgebra of $C^\infty(W, \mathbb{R})$ consisting of polynomial functions in a finite number of variables. The following expression:

$$\{F, G\}(w) = \sum_{m, n \geq 0} (m - n)\phi_{m+n}(w)\partial_m F(w)\partial_n G(w), \quad F, G \in \mathcal{F}, w \in W,$$

defines a Poisson bracket on (W, \mathcal{F}) . Indeed the right-hand side is a finite sum and is a function in \mathcal{F} , and we have:

$$\{\phi_i, \phi_j\} = (i - j)\phi_{i+j},$$

from which Jacobi identity follows. The special case when $j = 0$ gives:

$$\{\phi_i, \phi_0\}(w) = P(w).(D\phi_i(w), D\phi_0(w)) = P(w).(e_i^*, e_0^*) = i\phi_i(w).$$

By choosing an appropriate w (e.g. $w = \sum_{i \geq 1} i^{-3/4} e_i$), then $P(w).(e_i^*, e_0^*) = i\phi_i(w)$ can become as large as desired by varying i . This shows that the bilinear form $P(w)$ cannot be bounded.

The generalization of Def. 1 to multidifferential operators on W is straightforward. Again, given an Abelian subalgebra \mathcal{F} of $C^\infty(W, \mathbb{K})$, we define the following subspace of $\mathcal{L}_{\text{sym}}^r(W, \mathbb{K})$:

$$\mathcal{D}_w^\mathcal{F}(r) = \{D^{(r)}F(w) \mid F \in \mathcal{F}\}.$$

Definition 2. Let W be a Hilbert space. Let \mathcal{F} be an Abelian subalgebra, for the pointwise product, of $C^\infty(W, \mathbb{K})$. Let $r \geq 1$, an r -differential operator \mathbf{A} on (W, \mathcal{F}) is an r -linear map $\mathbf{A}: \mathcal{F}^r \rightarrow \mathcal{F}$ such that:

i) for $(n_1, \dots, n_r) \in \mathbb{N}^r$, there exists a map $a^{(n_1, \dots, n_r)}$ from W to a space of (not necessarily bounded) r -linear forms on $\mathcal{L}_{\text{sym}}^{n_1}(W, \mathbb{K}) \times \dots \times \mathcal{L}_{\text{sym}}^{n_r}(W, \mathbb{K})$, i.e.,

$$a^{(n_1, \dots, n_r)}(w): \mathcal{D}_w^{(n_1, \dots, n_r)} \subset \mathcal{L}_{\text{sym}}^{n_1}(W, \mathbb{K}) \times \dots \times \mathcal{L}_{\text{sym}}^{n_r}(W, \mathbb{K}) \rightarrow \mathbb{K},$$

so that the domain $\mathcal{D}_w^{(n_1, \dots, n_r)}$ of $a^{(n_1, \dots, n_r)}(w)$ contains $\mathcal{D}_w^{\mathcal{F}}(n_1) \times \dots \times \mathcal{D}_w^{\mathcal{F}}(n_r)$ and $a^{(n_1, \dots, n_r)}$ is 0 except for finitely many (n_1, \dots, n_r) ;

ii) for any $F_1, \dots, F_r \in \mathcal{F}$ and $w \in W$, we have

$$\mathbf{A}(F_1, \dots, F_r)(w) = \sum_{n_1, \dots, n_r \geq 0} a^{(n_1, \dots, n_r)}(w) \cdot (D^{(n_1)} F_1(w), \dots, D^{(n_r)} F_r(w)).$$

Notice that Poisson brackets as defined above are special cases of bidifferential operators in the sense of Def. 2 with $P = a^{(1,1)}$.

2.3 Deformation quantization on W

We now have all the ingredients to define what is meant by deformation quantization of a Poisson space $(W, \mathcal{F}, \{\cdot, \cdot\})$ when W is a Hilbert space.

Definition 3. Let W be a Hilbert space and $(W, \mathcal{F}, \{\cdot, \cdot\})$ be a Poisson space. A star-product on $(W, \mathcal{F}, \{\cdot, \cdot\})$ is a $\mathbb{K}[[\hbar]]$ -bilinear product $\star_\hbar: \mathcal{F}[[\hbar]] \times \mathcal{F}[[\hbar]] \rightarrow \mathcal{F}[[\hbar]]$ given by $F \star_\hbar G = \sum_{r \geq 0} \hbar^r C_r(F, G)$ for $F, G \in \mathcal{F}$ and extended by $\mathbb{K}[[\hbar]]$ -bilinearity to $\mathcal{F}[[\hbar]]$, and satisfying for any $F, G, H \in \mathcal{F}$:

i) $C_0(F, G) = FG$,

ii) $C_1(F, G) - C_1(G, F) = 2\{F, G\}$,

iii) for $r \geq 1$, $C_r: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ are bidifferential operators in the sense of Def. 2, vanishing on constants,

iv) $F \star_\hbar (G \star_\hbar H) = (F \star_\hbar G) \star_\hbar H$.

The triple $(W, \mathcal{F}[[\hbar]], \star_\hbar)$ is called a deformation quantization of the Poisson space $(W, \mathcal{F}, \{\cdot, \cdot\})$.

We also have a notion of equivalence of deformations adapted to our context:

Definition 4. Two deformation quantizations $(W, \mathcal{F}[[\hbar]], \star_\hbar^1)$ and $(W, \mathcal{F}[[\hbar]], \star_\hbar^2)$ of the same Poisson space $(W, \mathcal{F}, \{\cdot, \cdot\})$ are said to be equivalent if there exists a $\mathbb{K}[[\hbar]]$ -linear map $T: \mathcal{F}[[\hbar]] \rightarrow \mathcal{F}[[\hbar]]$ expressed as a formal series $T = \text{Id}_{\mathcal{F}} + \sum_{r \geq 1} \hbar^r T_r$ satisfying:

i) $T_r: \mathcal{F} \rightarrow \mathcal{F}$, $r \geq 1$, are differential operators in the sense of Def. 2, vanishing on constants,

ii) $T(F) \star_\hbar^1 T(G) = T(F \star_\hbar^2 G)$, $\forall F, G \in \mathcal{F}$.

3 Moyal product on a Hilbert space

We present an infinite-dimensional version of the Moyal product defined on a class of smooth functions specified by a Hilbert-Schmidt type of conditions on their derivatives.

3.1 Poisson structure

Let \mathcal{H} be an infinite-dimensional separable Hilbert space. We consider the phase-space $W = \mathcal{H} \oplus \mathcal{H}^*$ endowed with its canonical strong symplectic structure $\omega((x_1, \eta_1), (x_2, \eta_2)) = \eta_1(x_2) - \eta_2(x_1)$, where $x_1, x_2 \in \mathcal{H}$ and $\eta_1, \eta_2 \in \mathcal{H}^*$.

Let $F: W \rightarrow \mathbb{C}$ be a C^∞ function (in the Fréchet sense). We shall denote by $D_1F(x, \eta)$ (resp. $D_2F(x, \eta)$) the first (resp. second) partial Fréchet derivative of F evaluated at point $(x, \eta) \in W$. With the identification $\mathcal{H}^{**} \sim \mathcal{H}$ we have $D_1F(x, \eta) \in \mathcal{H}^*$ and $D_2F(x, \eta) \in \mathcal{H}$. Let $\langle \cdot, \cdot \rangle: \mathcal{H}^* \times \mathcal{H} \rightarrow \mathbb{K}$ be the canonical pairing between \mathcal{H} and \mathcal{H}^* .

With these notations, the bracket associated with the canonical symplectic structure on W takes the form:

$$\{F, G\}(x, \eta) = \langle D_1F(x, \eta), D_2G(x, \eta) \rangle - \langle D_1G(x, \eta), D_2F(x, \eta) \rangle, \quad (1)$$

where $F, G \in C^\infty(W, \mathbb{K})$.

Proposition 1. *The space W endowed with the bracket (1) is an infinite-dimensional Poisson space or, equivalently, $(C^\infty(W, \mathbb{K}), \{\cdot, \cdot\})$ is a Poisson algebra.*

Proof. One has only to check that the map $(x, \eta) \mapsto \{F, G\}(x, \eta)$ belongs to $C^\infty(W, \mathbb{K})$ for any $F, G \in C^\infty(W, \mathbb{K})$. Then Leibniz property and Jacobi identity will follow. For $F, G \in C^\infty(W, \mathbb{K})$, the maps $(x, \eta) \mapsto (D_1F(x, \eta), D_2G(x, \eta))$ and $(\xi, y) \mapsto \langle \xi, y \rangle$ belong to $C^\infty(W, \mathcal{H}^* \times \mathcal{H})$ and $C^\infty(\mathcal{H}^* \times \mathcal{H}, \mathbb{K})$, respectively. The map $(x, \eta) \mapsto \{F, G\}(x, \eta)$, as composition of C^∞ maps, is therefore in $C^\infty(W, \mathbb{K})$. \square

For any orthonormal basis $\{e_i\}_{i \geq 1}$ in \mathcal{H} and dual basis $\{e_i^*\}_{i \geq 1}$ in \mathcal{H}^* , the complex number $\partial_i F(x, \eta)$ shall denote the partial derivative of F evaluated at (x, η) in the direction of e_i , i.e. $\partial_i F(x, \eta) = DF(x, \eta).(e_i, 0) = D_1F(x, \eta).e_i$, and, similarly, $\partial_{i^*} F(x, \eta) = DF(x, \eta).(0, e_i^*) = D_2F(x, \eta).e_i^*$ is the partial derivative in the direction of e_i^* . Notice that i^* should not be considered as a different index from i when sums are involved, it is merely a mnemonic notation to distinguish partial derivatives in \mathcal{H} and in \mathcal{H}^* .

For $F \in C^\infty(W, \mathbb{K})$, we have for any $(x, \eta) \in W$ that $\sum_{i \geq 1} |\partial_i F(x, \eta)|^2 < \infty$ and $\sum_{i \geq 1} |\partial_{i^*} F(x, \eta)|^2 < \infty$, hence the Poisson bracket (1) admits an equivalent form in terms of an absolutely convergent series:

$$\{F, G\}(x, \eta) = \sum_{i \geq 1} (\partial_i F(x, \eta) \partial_{i^*} G(x, \eta) - \partial_i G(x, \eta) \partial_{i^*} F(x, \eta)). \quad (2)$$

3.2 Functions of Hilbert-Schmidt type

We now define a subalgebra of $C^\infty(W, \mathbb{K})$ suited for our discussion. Let us start with some definitions and notations.

For any $F \in C^\infty(W, \mathbb{K})$ and $(x, \eta) \in W$, the higher derivatives

$$D^{(r)}F(x, \eta): W \times \cdots \times W \rightarrow \mathbb{K}, \quad r \geq 1,$$

are bounded symmetric r -linear maps and partial derivatives of F will be denoted $D_{\alpha_1 \dots \alpha_r}^{(r)}F(x, \eta)$ where $\alpha_1, \dots, \alpha_r$ are taking values 1 or 2. Let us introduce:

$$\mathcal{H}^{(\alpha)} = \begin{cases} \mathcal{H}, & \text{if } \alpha = 1; \\ \mathcal{H}^*, & \text{if } \alpha = 2. \end{cases} \quad \alpha^\flat = \begin{cases} 2, & \text{if } \alpha = 1; \\ 1, & \text{if } \alpha = 2. \end{cases} \quad i^{(\alpha)} = \begin{cases} i, & \text{if } \alpha = 1; \\ i^*, & \text{if } \alpha = 2. \end{cases} \quad (3)$$

Also i^\sharp will stand for either i or i^* . With these notations, partial derivatives of F are bounded r -linear maps:

$$D_{\alpha_1 \dots \alpha_r}^{(r)}F(x, \eta): \mathcal{H}^{(\alpha_1)} \times \cdots \times \mathcal{H}^{(\alpha_r)} \rightarrow \mathbb{K}.$$

It is convenient to introduce new symbols such as ∂_{ij^*k} for higher partial derivatives, e.g., $\partial_{ij^*k}F(x, \eta) \in \mathbb{K}$ stands for $D^{(3)}F(x, \eta) \cdot ((e_i, 0), (0, e_j^*), (e_k, 0)) = D_{121}^{(3)}F(x, \eta) \cdot (e_i, e_j^*, e_k)$, where $\{e_i\}_{i \geq 1}$ (resp. $\{e_i^*\}_{i \geq 1}$) is an orthonormal basis in \mathcal{H} (resp. \mathcal{H}^*).

Definition 5. Let $\{e_i\}_{i \geq 1}$ be an orthonormal basis in \mathcal{H} and $\{e_i^*\}_{i \geq 1}$ be the dual basis in \mathcal{H}^* . Functions of Hilbert-Schmidt type are functions F in $C^\infty(W, \mathbb{K})$ such that

$$\sum_{i_1, \dots, i_r \geq 1} |\partial_{i_1^\sharp \dots i_r^\sharp} F(x, \eta)|^2 < \infty, \quad \forall r \geq 1, \forall (x, \eta) \in W. \quad (4)$$

The sums involved have to be interpreted in the sense of summable families. By Schwarz lemma for partial derivatives, it should be understood that Eq. (4) represents $r + 1$ distinct sums corresponding to all of the choices $i^\sharp = i$ or i^* . The set of functions of Hilbert-Schmidt type on W will be denoted by \mathcal{F}_{HS} .

The definition above is independent of the choice of the orthonormal basis.

Remark 1. Let \mathbb{N}_* be the set of positive integers. For each $r \geq 1$, the set of families of elements $\{x_I\}_{I \in \mathbb{N}_*^r}$ in \mathbb{K} such that $\sum_{I \in \mathbb{N}_*^r} |x_I|^2 < \infty$ is the Hilbert space $\ell^2(\mathbb{N}_*^r)$ for the usual operations and inner product. Then condition (4) can be equivalently stated in the following way: $F \in C^\infty(W, \mathbb{K})$ is of Hilbert-Schmidt type if and only if, for any $r \geq 1$ and any $(x, \eta) \in W$, the 2^r families $\{\partial_{i_1^\sharp \dots i_r^\sharp} F(x, \eta)\}_{(i_1^\sharp, \dots, i_r^\sharp) \in \mathbb{N}_*^r}$ belong to $\ell^2(\mathbb{N}_*^r)$.

Remark 2. The set \mathcal{F}_{HS} does not contain all of the (continuous) polynomials on W . For example, the polynomial $P(y, \xi) = \langle \xi, y \rangle$ is not in \mathcal{F}_{HS} as $\sum_{i, j \geq 1} |\partial_{ij^*} P(x, \eta)|^2 = \sum_{i, j \geq 1} \delta_{ij} = \infty$. In a quantum field theory context, the polynomial P corresponds to a free Hamiltonian in the holomorphic representation.

Proposition 2. *The set of functions of Hilbert-Schmidt type is an Abelian subalgebra of $C^\infty(W, \mathbb{K})$ for the pointwise product of functions.*

Proof. Let $a, b \in \mathbb{K}$ and $F, G \in \mathcal{F}_{HS}$. It is clear from Remark 1 that $aF + bG$ is in \mathcal{F}_{HS} . The product FG belongs to \mathcal{F}_{HS} as a consequence of the Leibniz rule for the derivatives and from: if $\{x_I\}_{I \in \mathbb{N}_*^r} \in \ell^2(\mathbb{N}_*^r)$ and $\{y_J\}_{J \in \mathbb{N}_*^s} \in \ell^2(\mathbb{N}_*^s)$, then $\{x_I y_J\}_{I \times J \in \mathbb{N}_*^{r+s}} \in \ell^2(\mathbb{N}_*^{r+s})$. \square

Moreover the Poisson bracket (1) restricts to \mathcal{F}_{HS} and we have:

Proposition 3. *$(W, \mathcal{F}_{HS}, \{\cdot, \cdot\})$ is a Poisson space.*

Proof. Let F and G be in \mathcal{F}_{HS} . According to the proof of Prop. 1, the map $\Phi: (x, \eta) \mapsto \langle D_1 F(x, \eta), D_2 G(x, \eta) \rangle$ is in $C^\infty(W, \mathbb{K})$ and splits as follows:

$$\Phi: W \xrightarrow{\Psi_1} \mathcal{H}^* \times \mathcal{H} \xrightarrow{\Psi_2} \mathbb{K}$$

$$(x, \eta) \mapsto (D_1 F(x, \eta), D_2 G(x, \eta)) \mapsto \langle D_1 F(x, \eta), D_2 G(x, \eta) \rangle,$$

where both Ψ_1 and Ψ_2 are C^∞ maps. We only need to check that Φ is of Hilbert-Schmidt type.

By applying the chain rule to $\Phi = \Psi_2 \circ \Psi_1$, it is easy to see that we can freely interchange partial derivatives with the sum sign and we get that the partial derivatives of Φ is a finite sum of terms of the form:

$$a_{j_1^\# \dots j_r^\#, k_1^\# \dots k_s^\#} \equiv \sum_{i \geq 1} \partial_{i j_1^\# \dots j_r^\#} F(x, \eta) \partial_{i k_1^\# \dots k_s^\#} G(x, \eta). \quad (5)$$

The Cauchy-Schwarz inequality implies that the family $\{a_{j_1^\# \dots j_r^\#, k_1^\# \dots k_s^\#}\}$ is in $\ell^2(\mathbb{N}_*^{r+s})$ and thus Φ belongs to \mathcal{F}_{HS} . Hence \mathcal{F}_{HS} is closed under the Poisson bracket. \square

3.3 Moyal star-product on W

We are now in position to define the Moyal star-product on W as an associative product on $\mathcal{F}_{HS}[[\hbar]]$.

For $F, G \in \mathcal{F}_{HS}$, $(x, \eta) \in W$, $r \geq 1$, $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r$ equal to 1 or 2, and with the notations introduced previously, let us define:

$$\langle \langle D_{\alpha_1 \dots \alpha_r}^{(r)} F, D_{\beta_1 \dots \beta_r}^{(r)} G \rangle \rangle(x, \eta) = \sum_{i_1, \dots, i_r \geq 1} \partial_{i_1^{(\alpha_1)} \dots i_r^{(\alpha_r)}} F(x, \eta) \partial_{i_1^{(\beta_1)} \dots i_r^{(\beta_r)}} G(x, \eta). \quad (6)$$

Remark 3. The preceding definition does not depend on the choice of the orthonormal basis in \mathcal{H} and the series is absolutely convergent as a consequence of the Cauchy-Schwarz inequality.

Let Λ be the canonical symplectic 2×2 -matrix with $\Lambda^{12} = +1$. As in the finite-dimensional case, the powers of the Poisson bracket (1) are defined as:

$$C_r(F, G) = \sum_{\alpha_1, \dots, \alpha_r=1,2} \sum_{\beta_1, \dots, \beta_r=1,2} \Lambda^{\alpha_1 \beta_1} \dots \Lambda^{\alpha_r \beta_r} \langle \langle D_{\alpha_1 \dots \alpha_r}^{(r)} F, D_{\beta_1 \dots \beta_r}^{(r)} G \rangle \rangle. \quad (7)$$

The next Proposition shows that the C_r are bidifferential operators in the sense of Def. 2 and they close on \mathcal{F}_{HS} . We shall use a specific version of the Hilbert tensor product \otimes between Hilbert spaces (see e.g. Sect. 2.6 in [8]). Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ be Hilbert spaces with orthonormal bases $\{e_i^{(1)}\}_{i \geq 1}, \dots, \{e_i^{(r)}\}_{i \geq 1}$. There exists a Hilbert space $\mathcal{T} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_r$ and a bounded r -linear map $\Psi: (x_1, \dots, x_r) \mapsto x_1 \otimes \dots \otimes x_r$ from $\mathcal{H}_1 \times \dots \times \mathcal{H}_r$ to $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_r$ satisfying:

$$\sum_{i_1, \dots, i_r \geq 1} |\langle \Psi(e_{i_1}^{(1)}, \dots, e_{i_r}^{(r)}), x \rangle|^2 < \infty, \quad \forall x \in \mathcal{T},$$

such that for any bounded r -linear form: $\Xi: \mathcal{H}_1 \times \dots \times \mathcal{H}_r \rightarrow \mathbb{K}$ satisfying:

$$\sum_{i_1, \dots, i_r \geq 1} |\Xi(e_{i_1}^{(1)}, \dots, e_{i_r}^{(r)})|^2 < \infty,$$

there exists a unique bounded linear form L on \mathcal{T} so that $\Xi = L \circ \Psi$. This universal property allows to identify Ξ to an element of \mathcal{T}^* .

Proposition 4. *For $F, G \in \mathcal{F}_{HS}$ and $r \geq 1$, the map $(x, \eta) \mapsto C_r(F, G)(x, \eta)$ belongs to the space of functions of Hilbert-Schmidt type \mathcal{F}_{HS} .*

Proof. Each term in the finite sum (7) is of the form

$$\langle \langle D_{\alpha_1 \dots \alpha_r}^{(r)} F(x, \eta), D_{\alpha_1^b \dots \alpha_r^b}^{(r)} G(x, \eta) \rangle \rangle, \quad (8)$$

where $\alpha_1, \dots, \alpha_r = 1$ or 2 . From the definition of \mathcal{F}_{HS} , expression (8) is well defined for any $F, G \in \mathcal{F}_{HS}$ and thus defines a function on W :

$$\Phi: (x, \eta) \mapsto \langle \langle D_{\alpha_1 \dots \alpha_r}^{(r)} F(x, \eta), D_{\alpha_1^b \dots \alpha_r^b}^{(r)} G(x, \eta) \rangle \rangle.$$

The case $r = 1$ has been already proved in Prop. 3. For $r \geq 2$, we need to slightly modify the argument used in the proof of Prop. 3 since the bilinear map $\langle \langle \cdot, \cdot \rangle \rangle$ defined by (6) is not a bounded bilinear form on the product of Banach spaces:

$$\mathcal{L}^r(\mathcal{H}^{(\alpha_1)}, \dots, \mathcal{H}^{(\alpha_r)}; \mathbb{K}) \times \mathcal{L}^r(\mathcal{H}^{(\alpha_1^b)}, \dots, \mathcal{H}^{(\alpha_r^b)}; \mathbb{K}). \quad (9)$$

In order to show that Φ is in $C^\infty(W, \mathbb{K})$, we shall use the universal property of the Hilbert tensor product \otimes mentioned above. For $F, G \in \mathcal{F}_{HS}$, we can consider the bounded r -linear maps $D_{\alpha_1 \dots \alpha_r}^{(r)} F(x, \eta)$ and $D_{\alpha_1^b \dots \alpha_r^b}^{(r)} G(x, \eta)$ as bounded linear forms on $\mathcal{H}^{(\alpha)} \equiv \mathcal{H}^{(\alpha_1)} \otimes \dots \otimes \mathcal{H}^{(\alpha_r)}$ and $\mathcal{H}^{(\alpha^b)} \equiv \mathcal{H}^{(\alpha_1^b)} \otimes \dots \otimes \mathcal{H}^{(\alpha_r^b)}$, respectively. Then the unbounded bilinear map $\langle \langle \cdot, \cdot \rangle \rangle$ on the product of spaces (9) restricts to the natural pairing of $\mathcal{H}^{(\alpha^b)} \sim \mathcal{H}^{(\alpha)*}$ and $\mathcal{H}^{(\alpha)}$ which is a smooth map. This shows that Ψ belongs to $C^\infty(W, \mathbb{K})$.

An argument similar to the one used in the proof of Prop. 3 shows that the partial derivatives of Φ involve a finite sum of terms of the form:

$$\sum_{i_1, \dots, i_r \geq 1} \partial_{i_1^{(\alpha_1)} \dots i_r^{(\alpha_r)} j_1^\# \dots j_r^\#} F(x, \eta) \partial_{i_1^{(\alpha_1^b)} \dots i_r^{(\alpha_r^b)} k_1^\# \dots k_r^\#} G(x, \eta),$$

where we have used the notations introduced at the beginning of Subsection 3.2. A direct application of the Cauchy-Schwarz inequality gives:

$$\begin{aligned} & \sum_{j_1, \dots, j_a \geq 1} \sum_{k_1, \dots, k_b \geq 1} \left| \sum_{i_1, \dots, i_r \geq 1} \partial_{i_1^{(\alpha_1)} \dots i_r^{(\alpha_r)}} j_1^{\#} \dots j_a^{\#} F \partial_{i_1^{(\alpha'_1)} \dots i_r^{(\alpha'_r)}} k_1^{\#} \dots k_b^{\#} G \right|^2 \leq \\ & \sum_{i_1, \dots, i_r, j_1, \dots, j_a \geq 1} \left| \partial_{i_1^{(\alpha_1)} \dots i_r^{(\alpha_r)}} j_1^{\#} \dots j_a^{\#} F \right|^2 \sum_{i_1, \dots, i_r, k_1, \dots, k_b \geq 1} \left| \partial_{i_1^{(\alpha'_1)} \dots i_r^{(\alpha'_r)}} k_1^{\#} \dots k_b^{\#} G \right|^2 < \infty. \end{aligned}$$

This shows that Φ is of Hilbert-Schmidt type and hence $(x, \eta) \mapsto C_r(F, G)(x, \eta)$ belongs to \mathcal{F}_{HS} . \square

We summarize all the previous facts in the following:

Theorem 1. *Let the C_r 's be given by (7), then the formula*

$$F \star_{\hbar}^M G = FG + \sum_{r \geq 1} \frac{\hbar^r}{r!} C_r(F, G), \quad (10)$$

defines an associative product on $\mathcal{F}_{HS}[[\hbar]]$ and hence a deformation quantization $(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^M)$ of the Poisson space $(W, \mathcal{F}_{HS}, \{ \cdot, \cdot \})$.

Proof. That $\star_{\hbar}^M : \mathcal{F}_{HS}[[\hbar]] \times \mathcal{F}_{HS}[[\hbar]] \rightarrow \mathcal{F}_{HS}[[\hbar]]$ is a bilinear map is a direct consequence of Prop. 4 and Prop. 2. Associativity follows from the same combinatorics used in the finite-dimensional case and the fact that the derivatives distribute in the pairing $\langle \langle \cdot, \cdot \rangle \rangle$ defining the C_r 's. \square

4 On the equivalence of deformation quantizations on W

We end this article by a discussion on the issue of equivalence of star-products on W . In the finite-dimensional case (i.e. on \mathbb{R}^{2n} endowed with its canonical Poisson bracket), it is well known that all star-products are equivalent to each other. The situation we are dealing with here, although it is a direct generalization of the flat finite-dimensional case, allows inequivalent deformation quantizations. We will illustrate this fact on a family of star-products of exponential type containing the important case of the normal star-product.

4.1 The Hochschild complex

The space of functions of Hilbert-Schmidt type \mathcal{F}_{HS} being an associative algebra over \mathbb{K} , we can consider the Hochschild complex $C^\bullet(\mathcal{F}_{HS}, \mathcal{F}_{HS})$ and its cohomology $H^\bullet(\mathcal{F}_{HS}, \mathcal{F}_{HS})$.

One has first to specify a class of cochains that would define the Hochschild complex. Here the cochains are simply r -differential operators in the sense of Def. 2 which vanishes on constants. The case where $\mathcal{F} = \mathcal{F}_{HS}$ in Def. 2 allows a more precise description of the r -differential operators. Consider an r -differential operator defined by:

$$\mathbf{A}(F_1, \dots, F_r)(w) = \sum_{n_1, \dots, n_r \geq 0} a^{(n_1, \dots, n_r)}(w) \cdot (D^{(n_1)} F_1(w), \dots, D^{(n_r)} F_r(w)),$$

where $F_1, \dots, F_r \in \mathcal{F}_{HS}$ and $w = (x, \eta) \in W$. For $F \in \mathcal{F}_{HS}$, the higher derivative $D^{(m)} F(x, \eta)$ defines an element of the m^{th} tensor power of W (here we identify W^* with W). For a fixed $w = (x, \eta) \in W$ and $m \geq 0$, the linear map $F \mapsto D^{(m)} F(x, \eta)$ from \mathcal{F}_{HS} to $W^{\otimes m}$ is onto and we can look at the restriction of the r -linear form

$$a^{(n_1, \dots, n_r)}(w): \mathcal{D}_w^{(n_1, \dots, n_r)} \subset \mathcal{L}_{\text{sym}}^{n_1}(W, \mathbb{K}) \times \dots \times \mathcal{L}_{\text{sym}}^{n_r}(W, \mathbb{K}) \rightarrow \mathbb{K},$$

to the product $W^{\otimes n_1} \times \dots \times W^{\otimes n_r}$ as a bounded r -linear form:

$$\tilde{a}^{(n_1, \dots, n_r)}(w): W^{\otimes n_1} \times \dots \times W^{\otimes n_r} \rightarrow \mathbb{K},$$

such that $w \mapsto \tilde{a}^{(n_1, \dots, n_r)}(w)$ is a smooth map from W to $\mathcal{L}^r(W^{\otimes n_1} \times \dots \times W^{\otimes n_r}, \mathbb{K})$.

The Leibniz rule for the derivatives of a product can be written here for $F, G \in \mathcal{F}_{HS}$ as:

$$\begin{aligned} D^{(m)}(FG)(w) \cdot (w_1, \dots, w_m) \\ = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sum_{k=0}^{k=m} \binom{m}{k} (D^{(k)} F(w) \otimes D^{(m-k)} G(w)) (w_{\sigma_1} \otimes \dots \otimes w_{\sigma_m}), \end{aligned}$$

where \mathfrak{S}_m is the symmetric group of degree m .

Let $\mathbf{A}(F) = \sum_{m \geq 0} \tilde{a}^{(m)}(w) \cdot (D^{(m)} F(w))$ be a differential operator with $\tilde{a}^{(m)} \in C^\infty(W, \mathcal{L}(W^{\otimes m}, \mathbb{K}))$, then it follows from the above form for the Leibniz rule that for $F, G \in \mathcal{F}_{HS}$, $\mathbf{A}(FG)$ can be written as a finite sum

$$\sum_{m_1, m_2 \geq 0} \tilde{b}^{(m_1, m_2)}(w) \cdot (D^{(m_1)} F(w), D^{(m_2)} G(w))$$

for some $\tilde{b}^{(m_1, m_2)} \in C^\infty(W, \mathcal{L}^2(W^{\otimes m_1} \times W^{\otimes m_2}, \mathbb{K}))$, and thus $(F, G) \mapsto \mathbf{A}(FG)$ is a bidifferential operator. The generalization to r -differential operators of this fact is straightforward.

The Hochschild complex consists of multidifferential operators vanishing on constants, i.e., $C^\bullet(\mathcal{F}_{HS}, \mathcal{F}_{HS}) = \oplus_{k \geq 0} C^k(\mathcal{F}_{HS}, \mathcal{F}_{HS})$ where, for $k \geq 1$, the space of k -cochains is:

$$\begin{aligned} C^k(\mathcal{F}_{HS}, \mathcal{F}_{HS}) \\ = \{\mathbf{A}: \mathcal{F}_{HS}^k \rightarrow \mathcal{F}_{HS} \mid \mathbf{A} \text{ is a } k\text{-differential operator vanishing on } \mathbb{K}\}. \end{aligned}$$

The differential of a k -cochain \mathbf{A} is the $(k+1)$ -linear map $\delta \mathbf{A}$ given by:

$$\begin{aligned} \delta \mathbf{A}(F_0, \dots, F_k) &= F_0 \mathbf{A}(F_1, \dots, F_k) - \mathbf{A}(F_0 F_1, F_2, \dots, F_k) + \dots \\ &+ (-1)^k \mathbf{A}(F_0, F_1, \dots, F_{k-1} F_k) + (-1)^{k+1} \mathbf{A}(F_0, \dots, F_{k-1}) F_k. \end{aligned} \quad (11)$$

satisfies $\delta^2 = 0$, and according to the discussion above $\delta \mathbf{A}$ is a $(k+1)$ -differential operator, vanishing on constants whenever \mathbf{A} does. Thus $\delta \mathbf{A}$ is a cochain that belongs to $C^{k+1}(\mathcal{F}_{HS}, \mathcal{F}_{HS})$, hence we indeed have a complex.

A k -cochain \mathbf{A} is a k -cocycle if $\delta \mathbf{A} = 0$. We denote by $Z^k(\mathcal{F}_{HS}, \mathcal{F}_{HS})$ the space of k -cocycles and by $B^k(\mathcal{F}_{HS}, \mathcal{F}_{HS})$ the space of those k -cocycles which are coboundaries. The k^{th} Hochschild cohomology space of \mathcal{F}_{HS} valued in \mathcal{F}_{HS} is defined as the quotient $H^k(\mathcal{F}_{HS}, \mathcal{F}_{HS}) = Z^k(\mathcal{F}_{HS}, \mathcal{F}_{HS})/B^k(\mathcal{F}_{HS}, \mathcal{F}_{HS})$.

4.2 Star-products of the exponential type

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on \mathcal{H} and $\mathcal{B}_2(\mathcal{H})$, the two-sided $*$ -ideal of Hilbert-Schmidt operators on \mathcal{H} . We shall describe a family of deformation quantizations $\{(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^A)\}_{A \in \mathcal{B}(H)}$. Each star-product \star_{\hbar}^A where $A \in \mathcal{B}(H)$ shall be the exponential of a Hochschild 2-cocycle, with the Moyal star-product corresponding to the case $A = 0$. It will turn out that the set of equivalence classes of star-products of this type is parameterized by $\mathcal{B}(\mathcal{H})/\mathcal{B}_2(\mathcal{H})$.

Let $A \in \mathcal{B}(\mathcal{H})$. For $F, G \in \mathcal{F}_{HS}$, the map

$$(x, \eta) \mapsto E_A(F, G)(x, \eta) \equiv \langle D_1 F(x, \eta), AD_2 G(x, \eta) \rangle + \langle D_1 G(x, \eta), AD_2 F(x, \eta) \rangle \quad (12)$$

defines a smooth function on W and is symmetric in F and G . Moreover we have:

Proposition 5. *The bilinear map $(F, G) \mapsto E_A(F, G)$ is a Hochschild 2-cocycle.*

Proof. It is clear that $\delta E_A = 0$ and, since E_A vanishes on constants, it is sufficient to check that E_A is a bidifferential operator on (W, \mathcal{F}_{HS}) , i.e., that the smooth function $(x, \eta) \mapsto E_A(F, G)(x, \eta)$ is of Hilbert-Schmidt type for any $F, G \in \mathcal{F}_{HS}$. Let $\{e_i\}_{i \geq 1}$ be an orthonormal basis of \mathcal{H} and $\{e_i^*\}_{i \geq 1}$ the dual basis. Consider the basis $\{f_i\}_{i \geq 1}$ in $W = \mathcal{H} \oplus \mathcal{H}^*$ defined by $f_i = (e_{\frac{i+1}{2}}, 0)$ if i is odd, and $f_i = (0, e_{\frac{i}{2}}^*)$ if i is even. To show that $E_A(F, G)$ is in \mathcal{F}_{HS} is equivalent to show that

$$\sum_{i_1, \dots, i_n \geq 1} |D^{(n)} E_A(F, G)(x, \eta) \cdot (f_{i_1}, \dots, f_{i_n})|^2 < \infty. \quad (13)$$

holds for any $n \geq 1$ and $w = (x, \eta) \in W$.

The chain rule applied to

$$\Phi: W \xrightarrow{\Psi_1} \mathcal{H}^* \times \mathcal{H} \xrightarrow{\Psi_2} \mathbb{K}$$

$$(x, \eta) \mapsto (D_1 F(x, \eta), D_2 G(x, \eta)) \mapsto \langle D_1 F(x, \eta), AD_2 G(x, \eta) \rangle,$$

shows that the derivatives of E_A distributes in the pairing $\langle \cdot, \cdot \rangle$.

The n^{th} derivative of $E_A(F, G)$ in the direction of $(f_{i_1}, \dots, f_{i_n})$ is a finite sum of terms of the form:

$$\langle D^{(r)} D_1 F(x, \eta) \cdot (f_{k_1}, \dots, f_{k_r}), AD^{(s)} D_2 G(x, \eta) \cdot (f_{l_1}, \dots, f_{l_s}) \rangle, \quad (14)$$

where $r + s = n$ and $(k_1, \dots, k_r, l_1, \dots, l_s)$ is a permutation of (i_1, \dots, i_n) , and similar terms with F and G inverted. It is worth noting that $D^{(r)} D_1 F(x, \eta) \cdot (f_{k_1}, \dots, f_{k_r})$ is the element of \mathcal{H}^* defined by the bounded linear form

$$h \mapsto D^{(r+1)} F(x, \eta) \cdot ((h, 0), f_{k_1}, \dots, f_{k_r})$$

on \mathcal{H} and, similarly, $D^{(s)} D_2 G(x, \eta) \cdot (f_{l_1}, \dots, f_{l_s})$ is the element of \mathcal{H} defined by the bounded linear form on \mathcal{H}^* :

$$\xi \mapsto D^{(s+1)} G(x, \eta) \cdot ((0, \xi), f_{l_1}, \dots, f_{l_s}).$$

The modulus squared of (14) is bounded by the constant $\|A\|^2 \alpha_{k_1 \dots k_r} \beta_{l_1 \dots l_s}$, where

$$\begin{aligned}\alpha_{k_1 \dots k_r} &= \sum_{i \geq 1} |D^{(r+1)} F(x, \eta) \cdot ((e_i, 0), f_{k_1}, \dots, f_{k_r})|^2, \\ \beta_{l_1 \dots l_s} &= \sum_{j \geq 1} |D^{(s+1)} G(x, \eta) \cdot ((0, e_j^*), f_{l_1}, \dots, f_{l_s})|^2.\end{aligned}$$

Since $F, G \in \mathcal{F}_{HS}$, we have $\sum_{k_1, \dots, k_r \geq 1} \alpha_{k_1 \dots k_r} < \infty$ and $\sum_{l_1, \dots, l_s \geq 1} \beta_{l_1 \dots l_s} < \infty$, from which inequality (13) follows. \square

For any $A \in \mathcal{B}(\mathcal{H})$, let us define:

$$\begin{aligned}C_1^A(F, G) &= \{F, G\} + E_A(F, G) \\ &= \langle D_1 F, (A + \mathbb{I}) D_2 G \rangle + \langle D_1 G, (A - \mathbb{I}) D_2 F \rangle,\end{aligned}\tag{15}$$

where \mathbb{I} is the identity operator on \mathcal{H} . Plainly, C_1^A is a 2-cocycle with constant coefficients, and one can define a star-product by taking the exponential of C_1^A :

$$F \star_h^A G = \exp(\hbar C_1^A)(F, G) = FG + \sum_{r \geq 1} \frac{\hbar^r}{r!} C_r^A(F, G),$$

where $C_r^A = (C_1^A)^r$ in the sense of bidifferential operators. This formula defines an associative product on $\mathcal{F}_{HS}[[\hbar]]$, and we get a family of deformation quantizations $\{(W, \mathcal{F}_{HS}[[\hbar]], \star_h^A)\}_{A \in \mathcal{B}(H)}$ of $(W, \mathcal{F}_{HS}, \{\cdot, \cdot\})$. This family of star-products is easily described by their symbols. Let us consider the following family of smooth functions on W :

$$\Phi_{y, \xi}(x, \eta) = \exp(\langle \eta, y \rangle + \langle \xi, x \rangle), \quad x, y \in \mathcal{H}, \quad \eta, \xi \in \mathcal{H}^*.\tag{16}$$

The $\Phi_{y, \xi}$'s belong to \mathcal{F}_{HS} and from (15) we deduce that:

$$C_r^A(\Phi_{y, \xi}, \Phi_{y', \xi'}) = (\langle \xi, (A + \mathbb{I})y' \rangle + \langle \xi', (A - \mathbb{I})y \rangle)^r \Phi_{y+y', \xi+\xi'},$$

and consequently:

$$\Phi_{y, \xi} \star_h^A \Phi_{y', \xi'} = \exp(\hbar(\langle \xi, (A + \mathbb{I})y' \rangle + \langle \xi', (A - \mathbb{I})y \rangle)) \Phi_{y+y', \xi+\xi'}.\tag{17}$$

Example 2. Set $A = \mathbb{I}$ in (15), then $C_1^{\mathbb{I}}(F, G) = 2\langle D_1 F, D_2 G \rangle$ and the corresponding star-product reads

$$F \star_h^{\mathbb{I}} G = FG + \sum_{r \geq 1} \frac{(2\hbar)^r}{r!} \langle D_{1 \dots 1}^{(r)} F, D_{2 \dots 2}^{(r)} G \rangle.\tag{18}$$

It is the well-known normal star-product (or Wick or standard depending on the interpretation of the variables $(x, \eta) \in W$). The cochains defining the normal star-product $C_r^{\mathbb{I}}$ correspond to only one term in the sum defining the r^{th} cochain of the Moyal star-product, namely the term corresponding to $\alpha_1 = \dots = \alpha_r = 1$ and $\beta_1 = \dots = \beta_r = 2$ in the sum (7). One would expect that conditions (4) defining the functions of Hilbert-Schmidt type are not all needed in order to make sense of the normal star-product and would guess that this product can be defined on a wider class of functions. Actually, the normal star-product defines a deformation quantization on a larger space of functions (containing the free Hamiltonian) that we shall describe in a forthcoming paper.

At this stage, a natural question arises: are the deformation quantizations $\{(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^A)\}_{A \in \mathcal{B}(H)}$ equivalent to each other? The answer is given in the:

Proposition 6. *Let $A \in \mathcal{B}(H)$. The Hochschild 2-cocycle E_A defined by (12) is a coboundary if and only if A is a Hilbert-Schmidt operator.*

Proof. Let A be a Hilbert-Schmidt operator on \mathcal{H} . The map from W to \mathbb{K} defined by $(x, \eta) \mapsto \langle \eta, Ax \rangle$ is of Hilbert-Schmidt type and therefore there exists a bounded linear form $\tilde{A}: \mathcal{H}^* \otimes \mathcal{H} \rightarrow \mathbb{K}$ so that $\langle \eta, Ax \rangle = \langle \tilde{A}, \eta \otimes x \rangle$. The 2-cocycle E_A can then be written as:

$$E_A(F, G) = \langle \tilde{A}, D_1 F \otimes D_2 G + D_1 G \otimes D_2 F \rangle, \quad F, G \in \mathcal{F}_{HS}.$$

For $F \in \mathcal{F}_{HS}$, the mixed derivative $D_{12}^{(2)} F$ belongs to $C^\infty(W, \mathcal{L}(\mathcal{H} \otimes \mathcal{H}^*, \mathbb{K})) \sim C^\infty(W, \mathcal{H}^* \otimes \mathcal{H})$ and $T_A(F)(x, \eta) = -\langle \tilde{A}, D_{12}^{(2)} F(x, \eta) \rangle$ defines a differential operator on (W, \mathcal{F}_{HS}) vanishing on constants. For any two functions in \mathcal{F}_{HS} , the Leibniz rule reads:

$$D_{12}^{(2)}(FG) = FD_{12}^{(2)}(G) + GD_{12}^{(2)}(F) + D_1 F \otimes D_2 G + D_1 G \otimes D_2 F,$$

it belongs to $C^\infty(W, \mathcal{H}^* \otimes \mathcal{H})$ and a simple computation gives $\delta T_A = E_A$, therefore E_A is a coboundary if A is in the Hilbert-Schmidt class.

Conversely, if E_A is a coboundary, there exists a differential operator S on (W, \mathcal{F}_{HS}) vanishing on constants, so that $E_A = \delta S$. If a term of degree one occurs in S (a derivation) it can be subtracted without changing δS , hence we can assume that S has the form:

$$S(F)(x, \eta) = \sum_{m \geq 2} a^{(m)}(x, \eta) \cdot (D^{(m)} F(x, \eta)),$$

where $a^{(m)} \in C^\infty(W, \mathcal{L}(W^{\otimes m}, \mathbb{K}))$ and only finitely many of them are nonzero. By computing δS and using $E_A = \delta S$, we find that only the term of degree 2 contributes:

$$\begin{aligned} & \langle D_1 F(w) AD_2 G(w) \rangle + \langle D_1 G(w), AD_2 F(w) \rangle \\ &= -a^{(2)}(w) \cdot (DF(w) \otimes DG(w) + DG(w) \otimes DF(w)), \quad w = (x, \eta) \in W. \end{aligned}$$

If we evaluate the equality above on $F(x, \eta) = \langle \xi, x \rangle$, $\xi \in \mathcal{H}^*$, and $G(x, \eta) = \langle \eta, y \rangle$, $y \in \mathcal{H}$, we find (with a slight abuse of notations):

$$\langle \xi, Ay \rangle = -a^{(2)}(w) \cdot (\xi \otimes y), \quad \forall \xi \in \mathcal{H}^*, \forall y \in \mathcal{H},$$

from which follows that $A \in \mathcal{B}(H)$ is a Hilbert-Schmidt operator on \mathcal{H} , as $a^{(2)}(w)$ is a bounded linear form on $\mathcal{H}^* \otimes \mathcal{H}$. \square

As an immediate consequence of Prop. (6), we deduce a classification result for deformation quantizations of exponential type $\{(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^A)\}_{A \in \mathcal{B}(H)}$.

Theorem 2. *Let $A, B \in \mathcal{B}(H)$. Two deformation quantizations $(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^A)$ and $(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^B)$ are equivalent if and only if $A - B$ is in the Hilbert-Schmidt class. Consequently, the set of equivalence classes of $\{(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^A)\}_{A \in \mathcal{B}(H)}$ is parameterized by $\mathcal{B}(H)/\mathcal{B}_2(H)$.*

Proof. Suppose that \star_{\hbar}^A and \star_{\hbar}^B are equivalent, i.e., there exists a formal series of differential operators vanishing on constants: $T = \text{Id}_{\mathcal{F}_{HS}} + \sum_{r \geq 1} \hbar^r T_r$, so that $T(F \star_{\hbar}^A G) = TF \star_{\hbar}^B TG$. Then it follows that $C_1^A = C_1^B + \delta T_1$ and, from the definitions (12) and (15) of E_A and C_1^A , we have $E_{A-B} = E_A - E_B = \delta T_1$, showing that E_{A-B} is a coboundary and hence $A - B$ is a Hilbert-Schmidt operator on \mathcal{H} .

Conversely, if $S \equiv A - B$ is a Hilbert-Schmidt operator, it defines a bounded linear form \tilde{S} on $\mathcal{H}^* \otimes \mathcal{H}$ and a differential operator $T_1(F) = -\langle \tilde{S}, D_{12}^{(2)} F \rangle$ on (W, \mathcal{F}_{HS}) (cf. the proof of Prop. 6). Since the star-products \star_{\hbar}^A and \star_{\hbar}^B are defined by constant coefficient bidifferential operators, it is sufficient to establish the equivalence at the level of symbols. Now define the formal series of differential operators $T = \exp(\hbar T_1)$. Its symbol is given by $T(\Phi_{y,\xi}) = \exp(\hbar \langle \xi, (B - A)y \rangle) \Phi_{y,\xi}$, where $\Phi_{y,\xi}$ has been defined in (16). Using the symbol (17) associated to a star-product, we find:

$$T(\Phi_{y,\xi} \star_{\hbar}^A \Phi_{y',\xi'}) = T \Phi_{y,\xi} \star_{\hbar}^B T \Phi_{y',\xi'}, \quad y, y' \in \mathcal{H}, \xi, \xi' \in \mathcal{H}^*.$$

Therefore the deformation quantizations $(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^A)$ and $(W, \mathcal{F}_{HS}[[\hbar]], \star_{\hbar}^B)$ are equivalent. \square

The Moyal and normal star-products correspond to $A = 0$ and $A = \mathbb{I}$ in (15), respectively. Since the identity operator on the infinite-dimensional Hilbert space \mathcal{H} is not in the Hilbert-Schmidt class, we have:

Corollary 1. *The Moyal and normal star-products are not equivalent deformations on $(W, \mathcal{F}_{HS}[[\hbar]])$.*

Remark 4. One can generalize the class of exponential type of star-products by allowing formal series with coefficients in $\mathcal{B}(\mathcal{H})$ in (15) or (17). The set of equivalence classes would then be $(\mathcal{B}(\mathcal{H})/\mathcal{B}_2(\mathcal{H}))[[\hbar]]$. Since we did not show that any star-product on $(W, \mathcal{F}_{HS}, \{\cdot, \cdot\})$ is equivalent to a star-product of the exponential type, $(\mathcal{B}(\mathcal{H})/\mathcal{B}_2(\mathcal{H}))[[\hbar]]$ can only be considered as a *lower bound* for the classification space of all the star-products on $(W, \mathcal{F}_{HS}, \{\cdot, \cdot\})$.

Remark 5. Recall that all the star-products on \mathbb{R}^{2n} endowed with its canonical Poisson structure are equivalent to each other. This fact should be put in relation with Von Neumann's uniqueness theorem on the irreducible (continuous) representations of Weyl systems associated to the canonical commutation relations (CCR) $\{q_i, p_j\} = \delta_{ij}$. The inequivalent representations of Weyl systems associated to the infinite dimensional CCR have been described long time ago by Gårding and Wightman [7], and Segal [10]. Here the existence of inequivalent deformation quantizations conveys the idea that there should be a close link between the set of equivalence classes of star-products and representations of Weyl systems associated to the CCR. It might turn out that they actually are identical.

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