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A short note about Morozov's formula

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Abstract

The purpose of this short note, is to rewrite Morozov's formula for correlation functions over the unitary group, in a much simpler form, involving the computation of a single determinant.

1 Introduction

The main result of this paper is given in 2.13.

In [1], A. Morozov proposed a formula for correlation functions of unitary matrices with Itzykson-Zuber's type measure:

$$\langle |U_{ji}|^2 \rangle_{U(N)} := \frac{1}{I(X, Y)} \int_{U(N)} dU |U_{ji}|^2 e^{\text{tr}(XU^\dagger YU)} \quad (1.1)$$

where dU is the Haar measure over the unitary group $U(N)$ (appropriately normalized, so that 1.4 below holds with prefactor 1), and X and Y are two given diagonal complex matrices:

$$X = \text{diag}(x_1, \dots, x_N) \quad , \quad Y = \text{diag}(y_1, \dots, y_N) \quad (1.2)$$

and the normalization factor $I(X, Y)$ is the so-called Harish-Chandra-Itzykson-Zuber integral:

$$I(X, Y) := \int_{U(N)} dU e^{\text{tr}(XU^\dagger YU)} \quad (1.3)$$

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It is well-known [4, 5] that:

$$I(X, Y) = \frac{\det E}{\Delta(X)\Delta(Y)} \quad (1.4)$$

where E is the matrix with entries:

$$E_{ij} := e^{x_i y_j} \quad (1.5)$$

and where $\Delta(X)$ and $\Delta(Y)$ are the Vandermonde determinants:

$$\Delta(X) := \prod_{i < j} (x_i - x_j) \quad , \quad \Delta(Y) := \prod_{i < j} (y_i - y_j) . \quad (1.6)$$

Here we shall assume that $\det E \neq 0$ and $\Delta(X) \neq 0$ and $\Delta(Y) \neq 0$.

Morozov's formula was proven in [2], from Shatashvili's formula [3]. Morozov's formula was originally written as follows, for any arbitrary sequences of complex numbers a_i, b_j one has:

$$(1.7) \quad \sum_{i,j=1}^N a_i b_j \int_{U(N)} dU |U_{ji}|^2 e^{\text{tr}(XU^+YU)} = \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho \in S_N} \epsilon(\rho) e^{\sum x_\ell y_{\rho(\ell)}}$$

$$\sum_{n=0}^{N-1} (-1)^n \sum_{i_1 < i_2 < \dots < i_{n+1}} \frac{\det \begin{pmatrix} 1 & \dots & 1 \\ x_{i_1} & \dots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n-1} & \dots & x_{i_{n+1}}^{n-1} \\ a_{i_1} & \dots & a_{i_{n+1}} \end{pmatrix} \det \begin{pmatrix} 1 & \dots & 1 \\ y_{\rho(i_1)} & \dots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^{n-1} & \dots & y_{\rho(i_{n+1})}^{n-1} \\ b_{\rho(i_1)} & \dots & b_{\rho(i_{n+1})} \end{pmatrix}}{\det \begin{pmatrix} 1 & \dots & 1 \\ x_{i_1} & \dots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^n & \dots & x_{i_{n+1}}^n \end{pmatrix} \det \begin{pmatrix} 1 & \dots & 1 \\ y_{\rho(i_1)} & \dots & y_{\rho(i_{n+1})} \\ \vdots & & \vdots \\ y_{\rho(i_1)}^n & \dots & y_{\rho(i_{n+1})}^n \end{pmatrix}} ,$$

where S_N is the symmetric group of rank N .

2 Rewriting Morozov's formula

The purpose of this short note is to rewrite this formula in a much simpler form, in a way very similar to what was done in [2].

For any two complex numbers x and y (such that $|x| > \max |x_i|$ and $|y| > \max |y_i|$), consider the choice:

$$a_i = \frac{1}{x - x_i} = \sum_{r=0}^{\infty} \frac{x_i^r}{x^{r+1}} \quad , \quad b_i = \frac{1}{y - y_i} = \sum_{s=0}^{\infty} \frac{y_i^s}{y^{s+1}} \quad (2.8)$$

Introduce the Schur polynomials (corresponding to hook diagrams):

$$\begin{aligned}
S_r(x_{i_1}, \dots, x_{i_{n+1}}) &:= \frac{\det \begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ x_{i_1}^r & \cdots & x_{i_{n+1}}^r \end{pmatrix}}{\det \begin{pmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & & \vdots \\ x_{i_1}^n & \cdots & x_{i_{n+1}}^n \end{pmatrix}} \\
&= \sum_{a_1 \leq a_2 \leq \dots \leq a_{r-n}} \prod_{k=1}^{r-n} x_{i_{a_k}} = \sum_{j_1 + \dots + j_{n+1} = r-n} x_{i_1}^{j_1} \cdots x_{i_{n+1}}^{j_{n+1}}. \quad (2.9)
\end{aligned}$$

The formal generating function of these Schur polynomials is:

$$\sum_{r=0}^{\infty} \frac{1}{x^{r+1}} S_r(x_{i_1}, \dots, x_{i_{n+1}}) = \prod_{k=1}^{n+1} \frac{1}{x - x_{i_k}} \quad (2.10)$$

Inserting that into 1.7, one gets:

$$\begin{aligned}
I(X, Y) &\sum_{i,j} \frac{1}{x - x_i} \frac{1}{y - y_j} \langle |U_{ji}|^2 \rangle_{U(N)} \\
&= \int_{U(N)} dU \operatorname{tr} \left(\frac{1}{x - X} U \frac{1}{y - Y} U^\dagger \right) e^{\operatorname{tr}(XU^\dagger YU)} \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{r,s=0}^{\infty} \sum_{\rho \in S_N} \epsilon(\rho) e^{\sum x_\ell y_{\rho(\ell)}} \\
&\quad \sum_{n=0}^{N-1} (-1)^n \sum_{i_1 < i_2 < \dots < i_{n+1}} \frac{S_r(x_{i_1}, \dots, x_{i_{n+1}})}{x^{r+1}} \frac{S_s(y_{\rho(i_1)}, \dots, y_{\rho(i_{n+1})})}{y^{s+1}} \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho \in S_N} \epsilon(\rho) e^{\sum x_\ell y_{\rho(\ell)}} \sum_{n=0}^{N-1} (-1)^n \sum_{i_1 < i_2 < \dots < i_{n+1}} \prod_{k=1}^{n+1} \frac{1}{x - x_{i_k}} \prod_{l=1}^{n+1} \frac{1}{y - y_{\rho(i_l)}} \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho \in S_N} \epsilon(\rho) e^{\sum x_\ell y_{\rho(\ell)}} \left[1 - \prod_{i=1}^N \left(1 - \frac{1}{x - x_i} \frac{1}{y - y_{\rho(i)}} \right) \right] \\
&= \frac{1}{\Delta(X)\Delta(Y)} \sum_{\rho \in S_N} \epsilon(\rho) \left[\prod_{i=1}^N e^{x_i y_{\rho(i)}} - \prod_{i=1}^N \left(e^{x_i y_{\rho(i)}} - \frac{1}{x - x_i} e^{x_i y_{\rho(i)}} \frac{1}{y - y_{\rho(i)}} \right) \right] \\
&= \frac{1}{\Delta(X)\Delta(Y)} \left[\det(e^{x_i y_j}) - \det \left(e^{x_i y_j} - \frac{1}{x - x_i} e^{x_i y_j} \frac{1}{y - y_j} \right) \right] \\
&(2.11)
\end{aligned}$$

Thus, Morozov's formula can be rewritten:

$$\int_{U(N)} dU \operatorname{tr} \left(\frac{1}{x - X} U \frac{1}{y - Y} U^\dagger \right) e^{\operatorname{tr}(XU^\dagger YU)} = \frac{\det E - \det \left(E - \frac{1}{x-X} E \frac{1}{y-Y} \right)}{\Delta(X)\Delta(Y)} \quad (2.12)$$

or:

$$\left\langle \operatorname{tr} \left(\frac{1}{x-X} U \frac{1}{y-Y} U^\dagger \right) \right\rangle = 1 - \frac{\det \left(E - \frac{1}{x-X} E \frac{1}{y-Y} \right)}{\det E}$$

$$= 1 - \det \left(1 - \frac{1}{x-X} E \frac{1}{y-Y} E^{-1} \right)$$
(2.13)

3 Concluding remarks

From that expression of Morozov's formula, it is rather easy to recover any individual correlator by taking residues:

$$\left\langle U_{ij} U_{ji}^\dagger \right\rangle = \operatorname{Res}_{x \rightarrow x_i} \operatorname{Res}_{y \rightarrow x_j} \left\langle \operatorname{tr} \left(\frac{1}{x-X} U \frac{1}{y-Y} U^\dagger \right) \right\rangle$$
(3.14)

Notice also that 2.13 is very similar to what was found in [2] after integration over X and Y .

Notice that by expanding the determinant in 2.13 along its last column, one can find a recursion relation relating $U(N)$ to $U(N-1)$ integrals, which is equivalent to Shatashvili's approach [3].

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