Paperfolding and Catalan numbers

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Abstract[1]*:* This paper reproves a few results concerning paperfolding sequences using properties of Catalan numbers modulo 2. The guiding principle can be described by: Paperfolding = Catalan modulo 2 + signs given by 2−automatic sequences.

1 Main results

In this paper, a *continued fraction* is a formal expression

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \dots}} = b_0 + a_1 \Big/ \Big(b_1 + a_2 \Big/ \Big(b_2 + \dots \Big) \Big)$$

with $b_0, b_1, \dots, a_1, a_2, \dots \in \mathbb{C}[x]$ two sequences of complex rational functions. We denote such a continued fraction by

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \dots + \frac{a_k|}{|b_k} + \dots$$

and call it *convergent* if the coefficients of the (formal) Laurent-series expansions of its *partial convergents*

$$\frac{P_0}{Q_0} = \frac{b_0}{1},\ \frac{P_1}{Q_1} = \frac{b_0 b_1 + a_1}{b_1}, \dots, \frac{P_k}{Q_k} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \dots + \frac{a_k|}{|b_k}\ , \dots$$

are ultimately constant. The limit $\sum_{k \geq n} l_k x^k \in \mathbb{C}[[x]][x^{-1}]$ is then the obvious formal Laurent power series whose first coefficients agree with the Laurent series expansion of P_n/Q_n for n huge enough.

Let W_1 be the word $(-x_1)\ x_1$ of length 2 in the alphabet $\{x_1, -x_1\}$. For $k \geq 2$, we define recursively a word W_k of length $2(2^k - 1)$ over the alphabet $\{\pm x_1, \pm x_2, \dots, \pm x_k\}$ by considering the word

$$W_k = W_{k-1}\ x_k\ (-x_k)\ \overline{W_{k-1}}$$

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constructed by concatenation where $\overline{W_{k-1}}$ denotes the reverse of W_{k-1} obtained by reading W_{k-1} backwards. The first instances of the words W_k are

$$W_1 = (-x_1)\ x_1$$
$$W_2 = (-x_1)\ x_1\ x_2\ (-x_2)\ x_1\ (-x_1)$$
$$W_3 = (-x_1)\ x_1\ x_2\ (-x_2)\ x_1\ (-x_1)\ x_3\ (-x_3)\ (-x_1)\ x_1\ (-x_2)\ x_2\ x_1\ (-x_1)$$

and these words are initial subwords of a unique, well-defined infinite word $W_\infty = w_1\ w_2\ w_3, \ldots$ with letters $w_i \in \{\pm x_1, \pm x_2, \ldots\}$.

The following result has been known for some time, cf [6] or [13]. Similar or equivalent equivalent statements are for instance given in [9], Theorem 6.5.6 in [2]. It is closely related to paperfolding, see [2], [11], [5]. Related results are contained in [4], [12] and [13].

Theorem 1.1 *Consider a sequence $x_1, x_2, \ldots$ with values in $\mathbb{C}$. If the corresponding continued fraction*

$$\frac{1|}{|1} + \frac{w_1|}{|1} + \frac{w_2|}{|1} + \ldots + \frac{w_k|}{|1} + \ldots$$

associated with the sequence $W_\infty = w_1\ w_2\ \ldots$ defined by the letters of the infinite word W_∞ converges, then its limit is given by

$$1 + x_1 + x_1^2 x_2 + x_1^4 x_2^2 x_3 + x_1^8 x_2^4 x_3^2 x_4 + \ldots = 1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} x_i^{2^{j-i}} \ .$$

Remarks. (i) In fact, the proof will follow from the identity

$$\frac{P_{2(2^k-1)}}{Q_{2(2^k-1)}} = \frac{1|}{|1} + \frac{w_1|}{|1} + \frac{w_2|}{|1} + \ldots + \frac{w_{2(2^k-1)}|}{|1} = 1 + \sum_{j=1}^{k} \prod_{i=1}^{j} x_i^{2^{j-i}}$$

(cf. Lemma 6.5.5 in [2]) showing that the $2(2^k-1)-$th partial convergent of this continued fraction is a polynomial in $x_1, \ldots, x_k$.

(ii) It is of course possible to replace the field $\mathbb{C}$ by any other reasonable field or ring.

Examples. (1) Setting $x_i = x$ for all i and multiplying by x we get

$$\frac{x|}{|1} - \frac{x|}{|1} + \frac{x|}{|1} + \frac{x|}{|1} - \frac{x|}{|1} + \frac{x|}{|1} + \ldots$$

$$= x + x^2 + x^4 + x^8 + \ldots = \sum_{k=0}^{\infty} x^{2^k} \ .$$

The sequence of signs $w_1, w_2, \ldots \in \{\pm 1\}$ given by

$$\sum_{k=0}^{\infty} x^{2^k} = \frac{x|}{|1} + w_1 \frac{x|}{|1} + w_2 \frac{x|}{|1} + w_3 \frac{x|}{|1} + \ldots$$

can be recursively defined by

$$\begin{aligned} w_{4i+1} &= -w_{4i+2} = (-1)^{i+1} \ , \\ w_{8i+3} &= -w_{8i+4} = (-1)^{i} \ , \\ w_{8i+7} &= -w_{8i+8} = w_{4i+3} \end{aligned}$$

and is 2−automatic (cf. [2]).

(2) Setting $x_i = x^{1+3^{i-1}}$ and multiplying by x we get

$$\frac{x|}{|1} - \frac{x^2|}{|1} + \frac{x^2|}{|1} + \frac{x^4|}{|1} - \frac{x^4|}{|1} + \frac{x^2|}{|1} + \dots$$

$$= x + x^3 + x^9 + x^{27} + \dots = \sum_{k=0}^{\infty} x^{3^k} \ .$$

(3) Setting $x_i = x^{1+(i-1)i!}$ and multiplying by x we get

$$\frac{x|}{|1} - \frac{x|}{|1} + \frac{x|}{|1} + \frac{x^3|}{|1} - \frac{x^3|}{|1} + \frac{x|}{|1} + \dots$$

$$= x + x^2 + x^6 + x^{24} + \dots = \sum_{k=1}^{\infty} x^{k!} \ .$$

Consider the sequence $\mu = (\mu_0, \mu_1, \dots) = (1, 1, 0, 1, 0, 0, 0, 1, \dots)$ defined by $x \sum_{i=0}^{\infty} \mu_i x^i = \sum_{k=0}^{\infty} x^{2^k}$. We associate to $\mu = (\mu_0, \mu_1, \dots)$ its *Hankel matrix* H whose (i,j)−th coefficient $h_{i,j} = \mu_{i+j},\ 0 \le i, j$ depends only on the sum $i+j$ of its indices and is given by $(i+j)$−th element of the sequence μ.

Consider the (2−automatic) sequence $s = (s_0, s_1, s_2, \dots) = 1\ 1\ (-1)\ 1\ (-1)\ (-1) \dots$ defined recursively by $s_0 = 1$ and

$$\begin{aligned} s_{2i} &= (-1)^i \ s_i \ , \\ s_{2i+1} &= s_i \ . \end{aligned}$$

It is easy to check that $s_n = (-1)^{B_0(n)}$ where $B_0(n)$ counts the number of bounded blocks of consecutive $0'$s of the binary integer n (or, equivalently, $B_0(n)$ equals the number of blocks 10 appearing in the binary expansion of the n). E.g, $720 = 2^9 + 2^7 + 2^6 + 2^4$ corresponds to the binary integer 1011010000. Hence $B_0(720) = 3$. This description shows that we have $s_{2^{n+1}+n} = -s_n$ for $n < 2^n$. Since we have $B_0(2^{n+1} + n) = 1 + B_0(n)$ for $n < 2^n$ and $B_0(2^{n+1} + n) = B_0(n)$ for $2^n \le n < 2^{n+1}$, the sequence $s_0, s_1, \dots$ can also be constructed by iterating the application

$$W_\alpha W_\omega \longmapsto W_\alpha W_\omega (-W_\alpha) W_\omega$$

where where W_α, W_ω are the first and the second half of the finite word $s_0 \dots s_{2^n-1}$.

Denote by D_s the infinite diagonal matrix with diagonal entries $s_0, s_1, s_2, \ldots$ and by D_a the infinite diagonal matrix with diagonal entries given by the alternating, $2-$periodic sequence $1, -1, 1, -1, \overline{1, -1}$.

We introduce the infinite unipotent lower triangular matrix L with coefficients $l_{i,j} \in \{0, 1\}$ for $0 \leq i, j$ given by

$$l_{i,j} \equiv \binom{2i}{i-j} - \binom{2i}{i-j-1} \equiv \binom{2i}{i-j} + \binom{2i}{i-j-1} \equiv \binom{2i+1}{i-j} \pmod 2$$

and denote by $L(k)$ the $k \times k$ submatrix with coefficients $l_{i,j},\ 0 \leq i, j < k$ of L. The matrix $L(8)$ for instance is given by

$$L(8) = \begin{pmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 0 & 1 & 1 & & & & & \\ 1 & 1 & 1 & 1 & & & & \\ 0 & 0 & 0 & 1 & 1 & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & & \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 1.2 *(i) We have*

$$H = D_s\ L\ D_a\ L^t\ D_s\ .$$

(ii) (BA0BAB−construction for L) We have for $n \geq 0$

$$L(2^{n+2}) = \begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix}$$

where A and B are the obvious submatrices of size $2^n \times 2^n$ *in* $L(2^{n+1})$.

Remarks. (i) Assertion (ii) of Theorem 1.2 yields an iterative construction for L by starting with $L(2) = \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$.

(ii) Conjugating H with an appropriate diagonal $\pm 1-$matrix, we can get the the LU-decomposition of any Hankel matrix associated with a sequence having ordinary generating function $x^{-1}\left(\sum_{k=0}^{\infty} \epsilon_k x^{2^k}\right),\ \epsilon_0 = 1, \epsilon_1, \epsilon_2, \ldots \in \{\pm 1\}$. More precisely, the appropriate conjugating diagonal matrix has coefficients $d_{n,n} = (-1)^{\sum_{j=0} \nu_j \epsilon_{j+1}}$ for $0 \leq n = \sum_{j=0} \nu_j 2^j$ a binary integer. The case $\epsilon_0 = -1$ can be handled similarly by replacing the diagonal matrix D_a with its opposite $-D_a$.

The matrix $P = (D_s\ L\ D_s)^{-1} = D_s\ L^{-1}\ D_s$ with coefficients $p_{i,j},\ 0 \leq i, j$ encodes the formal orthogonal polynomials $Q_0, Q_1, \ldots, Q_n = \sum_{k=0}^{n} p_{n,k} x^k$ with moments $\mu_0, \mu_1, \ldots$ given by $\sum_{i=0}^{\infty} \mu_i x^{i+1} = \sum_{k=0}^{\infty} x^{2^k}$.

In order to describe the matrix P we introduce a unipotent lower triangular matrix M with coefficients $m_{i,j} \in \{0,1\},\ 0 \le i,j$ by setting

$$m_{i,j} = \binom{i+j}{2j} \pmod 2 \ .$$

As above (with the matrix L) we denote by $M(k)$ the $k \times k$ submatrix with coefficients $m_{i,j},\ 0 \le i,j < k$ of M.

Theorem 1.3 *(i) We have*

$$L\ D_a\ M = D_a$$

(where D_a denotes the diagonal matrix with $2-$periodic diagonal coefficients $1,-1,1,-1,\ldots$).

(ii) ($BA0BAB-$construction for M) We have have for $n \ge 0$

$$M(2^{n+2}) = \begin{pmatrix} A' & & & \\ B' & A' & & \\ A' & B' & A' & \\ B' & 0 & B' & A' \end{pmatrix}$$

where A' and B' are the obvious submatrices of size $2^n \times 2^n$ in $M(2^{n+1})$.

The matrix $P = D_s\ D_a\ M\ D_a\ D_s = (D_s\ L\ D_s)^{-1}$ encoding the coefficients of the orthogonal polynomials with moments $\mu_0, \mu_1, \ldots$ has thus all its coefficients in $\{0, \pm 1\}$, see also [1] and [8].

The (formal) orthogonal polynomials $Q_0, Q_1, \ldots$ satisfy a classical three-term recursion relation with coefficients given by the continued Jacobi fraction expansion of $\sum \mu_i\ x^i = \frac{1}{x}\left(\sum_{k=0}^{\infty} x^{2^k}\right)$. This expansion is described by the following result.

Theorem 1.4 *We have*

$$\begin{aligned} \sum_{k=0}^{\infty} x^{2^k} &= \tfrac{x|}{|1-d_1x} + \tfrac{x^2|}{|1-d_2x} + \tfrac{x^2|}{|1-d_3x} + \ldots \\ &= \tfrac{x|}{|1-x} + \tfrac{x^2|}{|1+2x} + \tfrac{x^2|}{|1} + \tfrac{x^2|}{|1} + \tfrac{x^2|}{|1-2x} + \ldots \end{aligned}$$

where $d_n = \frac{s_n - s_{n-2}}{s_{n-1}}$ with $s_{-1} = 0,\ s_0 = 1$ and $s_{2i+1} = (-1)^i s_{2i} = s_i$ is the recursively defined sequence introduced previously (extended by $s_{-1} = 0$).

The formal orthogonal polynomials with moments $\mu_0, \mu_1, \ldots$ are thus recursively defined by $Q_0 = 1$, $Q_1 = x + 1$ and

$$Q_{n+1} = (x - d_n)Q_n + Q_{n-1} \text{ for } n \ge 2 \ .$$

The rest of the paper is organised as follows.

Section 2 contains a proof of Theorem 1.1.

Section 3 recalls a few definitions and facts concerning Hankel matrices, formal orthogonal polynomials etc.

Section 4 is a little digression about Catalan numbers. All objects appearing in this paper are essentially obtained by considering the corresponding objects associated with Catalan numbers and by reducing them modulo 2 using sign conventions prescribed by recursively defined sequences (which are 2−automatic).

Section 5 recalls two classical and useful results on the reduction modulo 2 (or modulo a prime number) of binomial coefficients.

Section 6 is devoted to the proof of Theorem 1.2.

Section 7 contains a proof of Theorem 1.3 and a related amusing result.

Section 8 contains formulae for the matrix products $M\ L$ and $L\ M$.

Section 9 contains a proof of Theorem 1.4.

Section 10 displays the $LU-$decomposition of the Hankel matrix $\tilde{H}$ associated with the sequences $1, 0, 1, 0, 0, 0, 1, 0, \ldots$ obtained by shifting the sequence $1, 1, 0, 1, 0, 0, 0, 1, \ldots$ associated with powers of 2 by one (or equivalently, by removing the first term).

Section 11 contains a uniqueness result.

2 Proof of Theorem 1.1

More or less equivalent reformulations of Theorem 1.1 can be found in several places. A few are [9], [11] and [2]. We give her a computational proof: Symmetry-arguments of continued fractions are replaced by polynomial identities.

Given a continued fraction

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots$$

it is classical (and easy, cf. for instance pages 4,5 of [10]) to show that the product

$$\begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & a_2 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_k \\ 1 & b_k \end{pmatrix}$$

equals

$$\begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix}$$

where

$$\frac{P_k}{Q_k} = b_0 + \frac{a_1|}{|b_1} + \ldots + \frac{a_k|}{|b_k}$$

denotes the $k-$th partial convergent.

Given a finite word $U = u_1\ u_2\ \ldots\ u_l$ we consider the product

$$M(U) = \begin{pmatrix} 0 & u_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & u_2 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & u_l \\ 1 & 1 \end{pmatrix} .$$

Defining

$$\begin{array}{rcl} X_k & = & 1 + \sum_{j=1}^{k} \prod_{i=1}^{j} x_i^{2^{j-i}} \\ \tilde{X}_k & = & 1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} x_i^{2^{j-i}} - \prod_{i=1}^{k} x_i^{2^{k-i}} \end{array}$$

(example: $X_3 = 1 + x_1 + x_1^2 x_2 + x_1^4 x_2^2 x_3$ and $\tilde{X}_3 = 1 + x_1 + x_1^2 x_2 - x_1^4 x_2^2 x_3$) one checks easily the identities

$$X_n + x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n) = X_{n+1}$$

and

$$X_n - x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n) = \tilde{X}_{n+1} .$$

Lemma 2.1 *We have for all* $n \geq 1$

$$M(W_n) = \begin{pmatrix} \tilde{X}_n - X_{n-1}^2 & 1 - X_n \\ X_{n-1}^2 & X_n \end{pmatrix}$$

and

$$M(\overline{W_n}) = \begin{pmatrix} X_n - X_{n-1}^2 & 1 - \tilde{X}_n \\ X_{n-1}^2 & \tilde{X}_n \end{pmatrix} .$$

Proof. The computations

$$\begin{array}{rl} & M(W_1) = M((-x_1)\ x_1) = \begin{pmatrix} 0 & -x_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & x_1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -x_1 & -x_1 \\ 1 & 1 + x_1 \end{pmatrix} \\ = & \begin{pmatrix} (1 - x_1) - 1^2 & 1 - (1 + x_1) \\ 1^2 & 1 + x_1 \end{pmatrix} \end{array}$$

and

$$M(\overline{W_1}) = M(x_1\ (-x_1)) = \begin{pmatrix} 0 & x_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -x_1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} (1 + x_1) - 1^2 & 1 - (1 - x_1) \\ 1^2 & 1 - x_1 \end{pmatrix}$$

prove Lemma 2.1 for $n = 1$.

For $n \geq 1$ we have

$$\begin{array}{rl} & M(W_{n+1}) = M(W_n\ x_{n+1}\ (-x_{n+1})\ \overline{W_n}) \\ = & \begin{pmatrix} \tilde{X}_n - X_{n-1}^2 & 1 - X_n \\ X_{n-1}^2 & X_n \end{pmatrix} \begin{pmatrix} x_{n+1} & x_{n+1} \\ 1 & 1 - x_{n+1} \end{pmatrix} \begin{pmatrix} X_n - X_{n-1}^2 & 1 - \tilde{X}_n \\ X_{n-1}^2 & \tilde{X}_n \end{pmatrix} \\ = & \begin{pmatrix} (X_n - x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n)) - X_n^2 & 1 - (X_n + x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n)) \\ X_n^2 & X_n + x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n) \end{pmatrix} \\ = & \begin{pmatrix} \tilde{X}_{n+1} - X_n^2 & 1 - X_{n+1} \\ X_n^2 & X_{n+1} \end{pmatrix} \end{array}$$

and

$$\begin{aligned}
& M(\overline{W_{n+1}}) = M(W_n\ (-x_{n+1})\ x_{n+1}\ \overline{W_n}) \\
= & \begin{pmatrix} \tilde{X}_n - X_{n-1}^2 & 1 - X_n \\ X_{n-1}^2 & X_n \end{pmatrix} \begin{pmatrix} -x_{n+1} & -x_{n+1} \\ 1 & 1 + x_{n+1} \end{pmatrix} \begin{pmatrix} X_n - X_{n-1}^2 & 1 - \tilde{X}_n \\ X_{n-1}^2 & \tilde{X}_n \end{pmatrix} \\
= & \begin{pmatrix} (X_n + x_{n+1}(X_{n-1}^2 - X_n\tilde{X}_n)) - X_n^2 & 1 - (X_n - x_{n+1}(X_{n-1}^2 - X_n\tilde{X}_n)) \\ X_n^2 & X_n - x_{n+1}(X_{n-1}^2 - X_n\tilde{X}_n) \end{pmatrix} \\
= & \begin{pmatrix} X_{n+1} - X_n^2 & 1 - \tilde{X}_{n+1} \\ X_n^2 & \tilde{X}_{n+1} \end{pmatrix}
\end{aligned}$$

which proves Lemma 2.1 by induction on n. □

Proof of Theorem 1.1. Lemma 2.1 shows that

$$\begin{aligned}
M(1\ W_n) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{X}_n - X_{n-1}^2 & 1 - X_n \\ X_{n-1}^2 & X_n \end{pmatrix} \\
&= \begin{pmatrix} X_{n-1}^2 & X_n \\ \tilde{X}_n & 1 \end{pmatrix} .
\end{aligned}$$

The $2(2^n - 1)-$th partial convergent of the continued fraction

$$\frac{1|}{|1} + \frac{w_1|}{|1} + \frac{w_2|}{|1} + \dots$$

defined by the word W_∞ equals thus

$$X_n = 1 + \sum_{j=1}^{n} \prod_{k=1}^{j} x_k^{2^{j-k}}$$

and the sequence of these partial convergents has limit

$$X_\infty = 1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} x_k^{2^{j-k}}$$

which is the statement of Theorem 1.1. □

3 Formal orthogonal polynomials

The content of this section is classical and can for instance be found in [3] or [15].

We consider a formal power series

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k \in \mathbb{C}[[x]]$$

as a linear form on the complex vector space $\mathbb{C}[x]^*$ of polynomials by setting

$$l_g(\sum_{i=0}^{n} \alpha_i x^i) = \sum_{i=0}^{n} \alpha_i \mu_i \ .$$

We suppose that the $n-$th *Hankel matrix* $H(n)$ having coefficients $h_{i,j} = \mu_{i+j},\ 0 \leq i,j < n$ associated with $\mu = (\mu_0, \ldots)$ is invertible for all n. Replacing such a sequence $\mu = (\mu_0, \mu_1, \ldots)$ by $\frac{1}{\mu_0}\mu = (1, \frac{\mu_1}{\mu_0}, \ldots)$ we may also suppose that $\mu_0 = 1$. We get a non-degenerate scalar product on $\mathbb{C}[x]$ by setting

$$\langle P, Q \rangle = l_g(PQ)\ .$$

Applying the familiar Gram-Schmitt orthogonalisation we obtain a sequence $Q_0 = 1,\ Q_1 = x - a_0$,

$$Q_{n+1} = (x - a_n)Q_n - b_n Q_{n-1},\ n \geq 1$$

of mutually orthogonal monic polynomials, called *formal orthogonal polynomials with moments* $\mu_0, \mu_1, \ldots$.

We encode the above recursion relation for the polynomials $Q_0, Q_1, \ldots$ by the *Stieltjes matrix*

$$S = \begin{pmatrix} a_0 & 1 & & & & \\ b_1 & a_1 & 1 & & & \\ & b_2 & a_2 & 1 & & \\ & & b_3 & a_3 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}\ .$$

The non-degeneracy condition on $\mu_0, \mu_1, \ldots$ shows that the (infinite) Hankel matrix H with coefficients $h_{i,j} = \mu_{i+j},\ 0 \leq i,j$ has an $LU-$decomposition $H = LU$ where L is lower triangular unipotent and $U = DL^t$ is upper triangular with D diagonal and invertible.

The diagonal entries of D are given by

$$d_{i,i} = \prod_{k=1}^{i} b_k$$

and we have thus

$$\det(H(n)) = \prod_{k=1}^{n-1} (b_k)^{n-k}$$

where $H(n)$ is the $n-$th Hankel matrix with coefficients $h_{i,j} = \mu_{i+j},\ 0 \leq i,j < n$. The $i-$th row vector

$$(m_{i,0}, m_{i,1}, \ldots)$$

of $M = L^{-1}$ encodes the coefficients of the $i-$th orthogonal polynomial

$$Q_i = \sum_{k=0}^{i} m_{i,k} x^k\ .$$

The Stieltjes matrix S satisfies

$$LS = L_- = \begin{pmatrix} l_{1,0} & l_{1,1} & & \\ l_{2,0} & l_{2,1} & l_{2,2} & \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

where L_- is obtained by removing the first row of the unipotent lower triangular matrix L with coefficients $l_{i,j},\ 0 \leq i, j$ involved in the $LU-$decomposition of H.

The generating function $g = \sum_{k=0}^{\infty} \mu_k x^k$ can be expressed as a *continued fraction of Jacobi type*

$$g(x) = 1/(1 - a_0 x - b_1 x^2/(1 - a_1 x - b_2 x^2/(1 - a_2 x - b_3 x^2/(1 - \dots))))$$

where $a_0, a_1, \dots$ and $b_1, b_2, \dots$ are as above and are encoded in the Stieltjes matrix.

4 The Catalan numbers

The Catalan numbers $C_n = \binom{2n}{n}\frac{1}{n+1}$ are the coefficients of the algebraic generating function

$$\begin{array}{rcccl} c(x) & = & \sum_{n=0}^{\infty} C_n x^n & = & \frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots \\ & & & = & \dfrac{1|}{|1} - \dfrac{x|}{|1} - \dfrac{x|}{|1} - \dfrac{x|}{|1} - \dots \\ & & & = & \dfrac{1|}{|1-x} - \dfrac{x^2|}{|1-2x} - \dfrac{x^2|}{|1-2x} - \dfrac{x^2|}{|1-2x} - \dots \end{array}$$

and satisfies $c = 1 + x\ c^2$.

We have $g(x) \equiv c(s) \pmod 2$ where $g(x) = \sum_{j=0}^{\infty} x^{2^j-1}$, cf. the last lines of [14]. A different proof can be given by remarking that c_n counts the number of planar binary rooted trees with $n+1$ leaves. Call two such trees equivalent if they are equivalent as abstract (non-planar) rooted trees. The cardinality of each equivalence class is then a power of two. An equivalence classe containing only one element is given by a rooted $2-$regular trees having 2^n leaves which are all at the same distance n from the root.

Many interesting mathematical objects associated with $g(x)$ are in fact obtained by reducing the corresponding objects of $c(x)$ modulo 2. Miraculously, all this works over $\mathbb{Z}$ by choosing suitable signs given by a few recursively-defined sequences (which are in fact always $2-$automatic, see [2] for a definition).

Consider the unipotent lower triangular matrices

$$L = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 5 & 9 & 5 & 1 & & \\ 14 & 28 & 20 & 7 & 1 & \\ 42 & 90 & 75 & 35 & 9 & 1 \end{pmatrix}$$

and

$$\tilde{L} = \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ 132 & 165 & 10 & 44 & 10 & 1 \end{pmatrix}$$

with coefficients $l_{i,j} = \binom{2i}{i-j} - \binom{2i}{i-j-1}$, respectively $\tilde{l}_{i,j} = \binom{2i+1}{i-j} - \binom{2i+1}{i-j-1}$ for $0 \leq i, j$ obtained by taking all even (respectively odd) lines (starting with line number zero) of the so-called *Catalan-triangle*

$$\begin{array}{ccccccccccc} 1 & & & & & & & & & & \\ & 1 & & & & & & & & & \\ 1 & & 1 & & & & & & & & \\ & 2 & & 1 & & & & & & & \\ 2 & & 3 & & 1 & & & & & & \\ & 5 & & 4 & & 1 & & & & & \\ 5 & & 9 & & 5 & & 1 & & & & \\ & 14 & & 14 & & 6 & & 1 & & & \\ 14 & & 28 & & 20 & & 7 & & 1 & & \end{array}$$

The matrix L is associated with the $LU-$decomposition of the Hankel matrix $H = L\ L^t$ having coefficients $h_{i,j} = C_{i+j} = \binom{2(i+j)}{i+j}/(i+j+1),\ 0 \leq i, j$ the Catalan numbers. The matrix $\tilde{L}$ yields the $LU-$decomposition of the Hankel matrix $\tilde{H} = \tilde{L}\ \tilde{L}^t$ having coefficients $\tilde{h}_{i,j} = C_{i+j+1} = \binom{2(i+j+1)}{i+j+1}/(i+j+2),\ 0 \leq i, j$ associated with the shifted Catalan sequence $1, 2, 5, 14, 42, \ldots$.

The inverse matrices L^{-1} and $\tilde{L}^{-1}$ are given by $D_a\ M\ D_a$ and $D_a\ \tilde{M}\ D_a$ where D_a is the diagonal matrix with 2-periodic alternating diagonal coefficients $1, -1, 1, -1$, and where M and $\tilde{M}$ have coefficients $m_{i,j} = \binom{i+j}{2j}$ and $\tilde{m}_{i,j} = \binom{i+j+1}{2j+1}$ for $0 \leq i, j$.

Let us mention that the products $P = LM$ and $\tilde{P} = \tilde{L}\ \tilde{M}$ have non-zero coefficients $p_{i,j} = \frac{(2i)!\ j!}{i!\ (2j)!\ (i-j)!}$ and $\tilde{p}_{i,j} = 4^{(i-j)}\binom{i}{j}$ for $0 \leq j < i$. They can

also be given by

$$P = L\ M = \exp\left(\begin{pmatrix} 0 & & & & & \\ 2 & 0 & & & & \\ & 6 & 0 & & & \\ & & 10 & 0 & & \\ & & & \ddots & \ddots & \end{pmatrix}\right)$$

and

$$\tilde{P} = \tilde{L}\ \tilde{M} = \exp\left(\begin{pmatrix} 0 & & & & & \\ 4 & 0 & & & & \\ & 8 & 0 & & & \\ & & 12 & 0 & & \\ & & & \ddots & \ddots & \end{pmatrix}\right).$$

Computations suggest that the products $M\ L$ and $\tilde{M}\ \tilde{L}$ have also nice logarithms:

$$M\ L = \exp\left(\begin{pmatrix} 0 & & & & & & & \\ 2 & 0 & & & & & & \\ 0 & 6 & 0 & & & & & \\ 2 & 0 & 10 & 0 & & & & \\ 0 & 6 & 0 & 14 & 0 & & & \\ 2 & 0 & 10 & 0 & 18 & 0 & & \\ 0 & 6 & 0 & 14 & 0 & 22 & 0 & \\ \vdots & \vdots & & & & & \ddots & \ddots \end{pmatrix}\right)$$

and

$$\tilde{M}\ \tilde{L} = \exp\left(\begin{pmatrix} 0 & & & & & & \\ 4 & 0 & & & & & \\ 0 & 8 & 0 & & & & \\ 4 & 0 & 12 & 0 & & & \\ 0 & 8 & 0 & 16 & 0 & & \\ 4 & 0 & 12 & 0 & 20 & 0 & \\ \vdots & \vdots & & & & \ddots & \ddots \end{pmatrix}\right).$$

5 Binomial coefficients modulo 2

Chercher ref. (Concrete Maths? Graham, Knuth, Patashnik)

Using the Frobenius automorphism we have

$$(1+x)^{\sum_i \nu_i 2^i} \equiv \prod_i (1+x^{2^i})^{\nu_i} \pmod 2$$

which shows the formula

$$\binom{n}{k} \equiv \prod_i \binom{\nu_i}{\kappa_i} \pmod 2$$

where $n = \sum_{i\geq 0} \nu_i\ 2^i,\ \nu_i \in \{0,1\}$ and $k = \sum_{i\geq 0} \kappa_i\ 2^i,\ \kappa_i \in \{0,1\}$ are two natural binary integers. Kummer showed that the $2-$valuation v_2 of $\binom{n}{k}$ (defined by $2^{v_2} | \binom{n}{k}$ and $2^{v_2+1} \not| \binom{n}{k}$) equals the number of carry-overs when adding the two natural numbers k and $n-k$, written in base 2 (cf. [16]).

As an application, one (re)proves easily that $C_n = \binom{2n}{n}\frac{1}{n+1} \equiv 1 \pmod 2$ if and only if $n+1 \in \{2^k\}_{k\in\mathbb{N}}$ is a power of two. Indeed, $\binom{2n}{n}\frac{1}{n+1} \equiv \frac{(2n+1)!}{n!\ 'n+1)!} = \binom{2n+1}{n} \pmod 2$ and addition of the binary integers n and $(n+1)$ needs always a carry except if $n = \sum_{i=0}^{k-1} 2^k$ and $n+1 = 2^k$.

6 Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2 which describes the $LU-$decomposition of the Hankel matrix H with coefficients $h_{i,j} = \mu_{i+j},\ 0 \leq i,j$ where

$$\sum_{k=0}^{\infty} \mu_i x^i = \sum_{j=0}^{\infty} x^{2^j-1} = 1 + x + x^3 + x^7 + x^{15} + \ldots.$$

We give in fact 3/2 proofs: We first prove assertion (i) by using the definition $l_{i,j} \equiv \binom{2i+1}{i-j} \pmod 2$ of L.

We prove then assertion (ii) showing that the matrix L satisfies the $BA0BAB-$construction.

Finally, we reprove assertion (i) using the recursive $BA0BAB-$structure of L.

Proof of assertion (i) in Theorem 1.2. We denote by $\overline{\binom{n}{k}} \in \{0,1\}$ the reduction $\pmod 2$ with value in $\{0,1\}$ of the binomial coefficient $\binom{n}{k}$. The main ingredient of the proof is the fact that

$$\overline{\binom{2n+1}{k}} = \overline{\binom{n}{\lfloor k/2 \rfloor}}$$

where $\lfloor k/2 \rfloor = k/2$ if k is even and $\lfloor k/2 \rfloor = (k-1)/2$ if k is odd.

We compute first the coefficient $r_{2i,2j}$ of the product $R = D_s L D_a L^t D_s$ (recall that L has coefficients $l_{i,j} = \overline{\binom{2i+1}{i-j}}$):

$$\begin{aligned}
r_{2i,2j} &= s_{2i}s_{2j} \textstyle\sum_k (-1)^k \overline{\binom{4i+1}{2i-k}\binom{4j+1}{2j-k}} \\
&= s_{2i}s_{2j} \left(\textstyle\sum_{k=0} \overline{\binom{4i+1}{2i-2k}\binom{4j+1}{2j-2k}} - \sum_{k=1} \overline{\binom{4i+1}{2i-2k+1}\binom{4j+1}{2j-2k+1}} \right) \\
&= s_{2i}s_{2j} \left(\textstyle\sum_{k=0} \overline{\binom{2i}{i-k}\binom{2j}{j-k}} - \sum_{k=1} \overline{\binom{2i}{i-k}\binom{2j}{j-k}} \right) \\
&= s_{2i}s_{2j} \overline{\binom{2i}{i}\binom{2j}{j}} = \begin{cases} 1 & \text{if } i=j=0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Similarly, we get

$$\begin{aligned} r_{2i+1,2j+1} &= s_{2i+1}s_{2j+1}\textstyle\sum_k (-1)^k \overline{\binom{4i+3}{2i+1-k}\binom{4j+3}{2j+1-k}} \\ &= s_{2i+1}s_{2j+1}\left(\textstyle\sum_{k=0} \overline{\binom{4i+3}{2i+1-2k}\binom{4j+3}{2j+1-2k}} - \sum_{k=0} \overline{\binom{4i+3}{2i-2k}\binom{4j+3}{2j-2k}}\right) \\ &= s_{2i+1}s_{2j+1}\left(\textstyle\sum_{k=0} \overline{\binom{2i+1}{i-k}\binom{2j+1}{j-k}} - \sum_{k=0} \overline{\binom{2i+1}{i-k}\binom{2j+1}{j-k}}\right) = 0 \end{aligned}$$

For $r_{2i,2j+1} = r_{2j+1,2i}$ we have

$$\begin{aligned} r_{2i,2j+1} &= s_{2i}s_{2j+1}\left(\textstyle\sum_k \overline{\binom{4i+1}{2i-2k}\binom{4j+3}{2j+1-2k}} - \sum_k \overline{\binom{4i+1}{2i-2k-1}\binom{4j+3}{2j-2k}}\right) \\ &= s_{2i}s_{2j+1}\left(\textstyle\sum_k \overline{\binom{2i}{i-k}\binom{2j+1}{j-k}} - \sum_k \overline{\binom{2i}{i-k-1}\binom{2j+1}{j-k}}\right) \\ &= s_{2i}s_{2j+1}\left(\textstyle\sum_k \left(\overline{\binom{2i}{i-k}} + \overline{\binom{2i}{i-k-1}}\right)\overline{\binom{2j+1}{j-k}} - 2\sum_k \overline{\binom{2i}{i-k-1}\binom{2j+1}{j-k}}\right) \\ &= s_{2i}s_{2j+1}\left(\textstyle\sum_k \overline{\binom{2i+1}{i-k}\binom{2j+1}{j-k}} - 2\sum_k \overline{\binom{2i}{i-k-1}\binom{2j+1}{j-k}}\right) \ . \end{aligned}$$

Using parity arguments and the recursion $s_{2i} = (-1)^i s_i,\ s_{2j+1} = s_j$ we get for i even

$$\begin{aligned} r_{2i,2j+1} &= s_{2i}s_{2j+1}\Big(\textstyle\sum_k (-1)^k \overline{\binom{2i+1}{i-k}\binom{2j+1}{j-k}} + \\ &\qquad +2\sum_k \left(\overline{\binom{2i+1}{i-2k-1}} - \overline{\binom{2i}{i-2k-2}}\right)\overline{\binom{2j+1}{j-2k-1}}\Big) \\ &= s_i s_j\left(\textstyle\sum_k (-1)^k \overline{\binom{2i+1}{i-k}\binom{2j+1}{j-k}} + 0\right) = r_{i,j} \ . \end{aligned}$$

Similarly, for i odd we have

$$\begin{aligned} r_{2i,2j+1} &= -s_{2i}s_{2j+1}\Big(\textstyle\sum_k (-1)^k \overline{\binom{2i+1}{i-k}\binom{2j+1}{j-k}} + \\ &\qquad -2\sum_k \left(\overline{\binom{2i+1}{i-2k}} - \overline{\binom{2i}{i-2k-1}}\right)\overline{\binom{2j+1}{j-2k}}\Big) \\ &= s_i s_j\left(\textstyle\sum_k (-1)^k \overline{\binom{2i+1}{i-k}\binom{2j+1}{j-k}} + 0\right) = r_{i,j} \ . \end{aligned}$$

Finally, the above computations together with induction on $i+j$ show that $r_{i,j} = 0$ except if $i+j = 2^k - 1$ for some $k \in \mathbb{N}$. In this case we get $r_{i,j} = 1$ which proves the result. □

Proof of assertion (ii) in Theorem 1.2. Assume that $L(2^{n+1})$ is given by $\begin{pmatrix} A & \\ B & A \end{pmatrix}$. (This holds clearly for $n = 0$.)

We have for $0 \le i, j < 2^{n+1}$

$$\binom{2(i+2^{n+1})+1}{(i+2^{n+1})-(j+2^{n+1})} = \binom{2i+2^{n+2}+1}{i-j} \equiv \binom{2i+1}{i-j} \pmod 2$$

since $2i+1 < 2^{n+2}$. This shows already that $L(2^{n+2})$ is of the form $\begin{pmatrix} \tilde{A} & \\ \tilde{B} & \tilde{A} \end{pmatrix}$ with $\tilde{A} = L(2^{n+1})$.

Consider now a binomial coefficient of the form

$$\binom{2(2^{n+1}+i)+1}{(2^{n+1}+i)-j} = \binom{2^{n+2}+2i+1}{2^{n+1}+i-j}$$

with $0 \leq i,j < 2^{n+1}$ which is odd. This implies $i+j+1 \geq 2^{n+1}$ since otherwise $(2^{n+1}+i)-j < (2^{n+1}+i)+j+1 < 2^{n+2} < 2^{n+2}+2i+1$ and there must therefore be a carry when adding the binary integers $((2^{n+1}+i)-j)$ and $((2^{n+1}+i)+j+1)$.

Two such binary integers $((2^{n+1}+i)-j)$ and $((2^{n+1}+i)+j+1)$ add thus without carry if and only if the binary integers $(2^{n+2}+2^{n+1}+i-j)$ and $(2^{n+1}+i+j+1-2^{n+2})$ add without carry. This shows the equality

$$\binom{2(2^{n+1}+i)+1}{(2^{n+1}+i)-j} \equiv \binom{2(2^{n+1}+i)+1}{(2^{n+1}+i)-(2^{n+2}-1-j)}$$

for $0 \leq i,j < 2^{n+1}$. Geometrically, this amounts to the fact that the block $\tilde{B}$ is the vertical mirror of the block $\tilde{A}$ and this symmetry is preserved by the $BA0BAB-$construction. □

$BA0BAB-$**proof of assertion (i).** We have

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and conjugation by

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \text{ yields } \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

The proof is now by induction using the $BA0BAB-$construction of L. Writing $\tilde{A} = D_a\ A^t$ and $\tilde{B} = D_a\ B^t$ (with D_a denoting the obvious submatrix of finite size of the previously infinite matrix D_a) we have

$$\begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} & 0 & \tilde{B} \\ & \tilde{A} & \tilde{B} & \tilde{A} \\ & & \tilde{A} & \tilde{B} \\ & & & \tilde{A} \end{pmatrix}$$

$$= \begin{pmatrix} A\tilde{A} & A\tilde{B} & 0 & A\tilde{B} \\ B\tilde{A} & B\tilde{B}+A\tilde{A} & A\tilde{B} & B\tilde{B}+A\tilde{A} \\ 0 & B\tilde{A} & B\tilde{B}+A\tilde{A} & B\tilde{A}+A\tilde{B} \\ B\tilde{A} & B\tilde{B}+A\tilde{A} & A\tilde{B}+B\tilde{A} & 2B\tilde{B}+2A\tilde{A} \end{pmatrix}.$$

We have $B\tilde{B}+A\tilde{A}=0$ by induction. The result is now correct up to signs if $A\tilde{B}=-B\tilde{A}$.

We conjugate now this last matrix by the diagonal matrix with diagonal entries $s_0,\ldots,s_{2^{n+2}-1}$ which we can write using blocks of size 2^n in the form:

$$\begin{pmatrix} D_\alpha & & & \\ & D_\omega & & \\ & & -D_\alpha & \\ & & & D_\omega \end{pmatrix}.$$

We get (after simplification using induction and the assumption $A\tilde{B}=-B\tilde{A}$)

$$\begin{pmatrix} D_\alpha A\tilde{A}D_\alpha & D_\alpha A\tilde{B}D_\omega & 0 & D_\alpha A\tilde{B}D_\omega \\ D_\omega B\tilde{A}D_\alpha & 0 & -D_\omega A\tilde{B}D_\alpha & \\ 0 & -D_\alpha B\tilde{A}D_\omega & & \\ D_\omega B\tilde{A}D_\alpha & & & \end{pmatrix}.$$

Supposing the identity $A\tilde{B}=-B\tilde{A}$ we get

$$-D_\alpha B\tilde{A}D_\omega = D_\alpha A\tilde{B}D_\omega \text{ and } -D_\omega A\tilde{B}D_\alpha = D_\omega B\tilde{A}D_\alpha$$

which have the correct form (one's on the antidiagonal and zeros everywhere else) by induction.

We have yet to prove that $A\tilde{B}=-B\tilde{A}$. Writing $A=\begin{pmatrix} a & \\ b & a \end{pmatrix}$, $B=\begin{pmatrix} & b \\ b & a \end{pmatrix}$ and $\tilde{A}=D_a\ A^t, \tilde{B}=D_a\ B^t$ we have

$$A\tilde{B}=\begin{pmatrix} a & \\ b & a \end{pmatrix}\begin{pmatrix} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix}=\begin{pmatrix} 0 & a\tilde{b} \\ a\tilde{b} & 0 \end{pmatrix}$$

and

$$B\tilde{A}=\begin{pmatrix} 0 & b \\ b & a \end{pmatrix}\begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{a} \end{pmatrix}=\begin{pmatrix} 0 & b\tilde{a} \\ b\tilde{a} & 0 \end{pmatrix}$$

which proves $A\tilde{B}=-B\tilde{A}$ since $a\tilde{b}=-b\tilde{a}$ by induction. □

7 Proof of Theorem 1.3

Denote by D_e the diagonal matrix with diagonal coefficients $1,0,1,0,1,0,\ldots,\overline{1,0,}$ and by $D_o = 1 - D_e$ the diagonal matrix with coefficients $0,1,0,1,0,1,\ldots,\overline{0,1,}$. We have thus $1 = D_e + D_o$ and $D_a = D_e - D_o$. The main ingredient for proving Theorem 1.3 is the following proposition which is also of independent interest.

Proposition 7.1 *We have*

$$M\ D_e\ L = A + D_e, \qquad M\ D_o\ L = A + D_o$$

where A is the strictly lower triangular matrix with coefficients $a_{i,j} = 1$ if $i > j$ and $a_{i,j} = 0$ otherwise.

Proof. We denote by $\alpha_{i,j} = (M\ D_e\ L)_{i,j}$ respectively $\beta_{i,j} = (M\ D_o\ L)_{i,j}$ the corresponding coefficients of both products. The proof is by induction on $i+j$. Obviously, $\alpha_{i,j} = \beta_{i,j} = 0$ for $i < j$. The proof for $i = j$ is obvious since M and L are lower triangular unipotent matrices. Consider now for $i > j$

$$\alpha_{i,j} = \sum_k \overline{\binom{i+2k}{4k}\binom{4k+1}{2k-j}}\ .$$

(recall that $m_{i,k} = \overline{\binom{i+k}{2k}} \equiv \binom{i+k}{2k} \pmod 2$ and $l_{k,j} = \overline{\binom{2k+1}{k-j}} \equiv \binom{2k+1}{k-j} \pmod 2$).

Since

$$\overline{\binom{2i+2k}{4k}} = \overline{\binom{2i+1+2k}{4k}} \text{ and } \overline{\binom{4k}{2k-2j}} = \overline{\binom{4k+1}{2k-2j}} = \overline{\binom{4k+1}{2k-(2j-1)}}$$

we have

$$\alpha_{i,j} = \sum_k \overline{\binom{\lfloor i/2\rfloor + k}{2k}\binom{2k}{k-\lfloor (j+1)/2\rfloor}}$$

$$= \sum_{k,\ k\equiv\lfloor (j+1)/2\rfloor \pmod 2} \overline{\binom{\lfloor i/2\rfloor + k}{2k}\binom{2k+1}{k-\lfloor (j+1)/2\rfloor}}$$

$$= \begin{cases} \alpha_{\lfloor i/2\rfloor,\lfloor (j+1)/2\rfloor} & \text{if } \lfloor (j+1)/2\rfloor \equiv 0 \pmod 2\ , \\ \beta_{\lfloor i/2\rfloor,\lfloor (j+1)/2\rfloor} & \text{if } \lfloor (j+1)/2\rfloor \equiv 1 \pmod 2\ . \end{cases}$$

We have similarly

$$\beta_{i,j} = \sum_k \overline{\binom{i+2k+1}{4k+2}\binom{4k+3}{2k+1-j}}$$

$$= \sum \overline{\binom{2\lfloor (i-1)/2\rfloor + 1 + 2k + 1}{4k+2}\binom{4k+3}{2k+1-2\lfloor j/2\rfloor}}$$

$$= \sum_{k,\ k\equiv\lfloor(i-1)/2\rfloor \pmod 2} \overline{\binom{\lfloor(i-1)/2\rfloor+k+1}{2k+1}\binom{2k+1}{k-\lfloor j/2\rfloor}}$$

$$= \sum_{k,\ k\equiv\lfloor(i-1)/2\rfloor \pmod 2} \overline{\binom{\lfloor(i+1)/2\rfloor+k}{2k}\binom{2k+1}{k-\lfloor j/2\rfloor}}$$

$$= \begin{cases} \alpha_{\lfloor(i+1)/2\rfloor,\lfloor j/2\rfloor} & \text{if } \lfloor(i-1)/2\rfloor \equiv 0 \pmod 2\ , \\ \beta_{\lfloor(i+1)/2\rfloor,\lfloor j/2\rfloor} & \text{if } \lfloor(i-1)/2\rfloor \equiv 1 \pmod 2\ . \end{cases}$$

except if $\lfloor(i-1)/2\rfloor = \lfloor j/2\rfloor = s$ where

$$\sum_{k,\ k\equiv s \pmod 2} \overline{\binom{s+k+1}{2k+1}\binom{2k+1}{k-s}} = 1$$

since only $k = s$ yields a non-zero contribution. This ends the proof. □

Proof of assertion (i) in Theorem 1.3 Using the notations of Proposition 7.1 we have

$$M\ D_a\ L = (M\ D_e\ L) - (M\ D_o\ L) = (A + D_e) - (A + D_o) = D_a$$

which shows that $D_a\ M\ D_a = L^{-1}$ and ends the proof. □

Proof of assertion (ii) in Theorem 1.3 We have

$$m_{i,j} = \overline{\binom{i+j}{2j}} = \overline{\binom{i+j+2^{n+2}}{2j+2^{n+2}}} = m_{i+2^{n+1},j+2^{n+1}}$$

for $i, j < 2^{n+1}$. This shows (together with induction) that the lower right corner of $M(2^{n+2})$ is given by $\begin{pmatrix} A' & \\ B' & A' \end{pmatrix}$ and has thus the correct form.

In order to prove the recurrence formula for the lower left corner, we remark that B' is by induction the horizontal mirror of A'. We have thus to show that $m_{i+2^{n+1},j} = m_{2^{n+1}-1-i,j}$ or equivalently that

$$\binom{2^{n+1}+i+j}{2j} \equiv \binom{2^{n+1}-1-i+j}{2j} \pmod 2$$

for $0 \le i, j < 2^{n+1}$.

Consider

$$\frac{(2^{n+1}+i+j)(2^{n+1}+i+j-1)\cdots(2^{n+1}+i-j+1)}{(2j)!}$$

for $0 \le i, j < 2^{n+1}$. Since all terms of the numerator are $< 2^{n+2}$ we have

$$\begin{aligned} \binom{2^{n+1}+i+j}{2j} &\equiv \tfrac{(2^{n+1}+i+j-2^{n+2})\cdots(2^{n+1}+i-j+1-2^{n+2})}{(2j)!} \pmod 2 \\ &\equiv \tfrac{(2^{n+1}-i-j)(2^{n+1}-i-j+1)\cdots(2^{n+1}-i+j-1)}{(2j)!} \pmod 2 \\ &\equiv \tbinom{2^{n+1}-1-i+j}{2j} \pmod 2 \end{aligned}$$

which ends the proof. □

BA0BAB−**proof of assertion (i) in Theorem 1.3** Computing $L(4)\ D_a\ M(4)$ (with D_a denoting a diagonal matrix with alternating entries $1, -1, 1, -1, \ldots$ of the correct size) we have

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 0 & 1 & 1 \end{pmatrix} = D_a \ .$$

Writing

$$\begin{pmatrix} \tilde{A}' & & & \\ \tilde{B}' & \tilde{A}' & & \\ \tilde{A}' & \tilde{B}' & \tilde{A}' & \\ \tilde{B}' & 0 & \tilde{B}' & \tilde{A}' \end{pmatrix} = D_a \begin{pmatrix} A' & & & \\ B' & A' & & \\ A' & B' & A' & \\ B' & 0 & B' & A' \end{pmatrix}$$

we have

$$\begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix} \begin{pmatrix} \tilde{A}' & & & \\ \tilde{B}' & \tilde{A}' & & \\ \tilde{A}' & \tilde{B}' & \tilde{A}' & \\ \tilde{B}' & 0 & \tilde{B}' & \tilde{A}' \end{pmatrix}$$

$$= \begin{pmatrix} A\tilde{A}' & & & \\ B\tilde{A}' + A\tilde{B}' & A\tilde{A}' & & \\ B\tilde{B}' + A\tilde{A}' & B\tilde{A}' + A\tilde{B}' & A\tilde{A}' & \\ 2B\tilde{A}' + 2A\tilde{B}' & A\tilde{A}' + B\tilde{B}' & B\tilde{A}' + A\tilde{B}' & A\tilde{A}' \end{pmatrix}$$

$$= \begin{pmatrix} D_a & & & \\ 0 & D_a & & \\ B\tilde{B}' + A\tilde{A}' & 0 & D_a & \\ 0 & A\tilde{A}' + B\tilde{B}' & 0 & D_a \end{pmatrix}$$

by induction.

Since we have by induction $A\tilde{A}' = D_a$, it will be enough to show that $B\tilde{B}' = -D_a$. Writing

$$B = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \text{ and } \tilde{B}' = \begin{pmatrix} \tilde{a}' & \tilde{b}' \\ \tilde{b}' & 0 \end{pmatrix}$$

we get

$$B\tilde{B}' = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \begin{pmatrix} \tilde{a}' & \tilde{b}' \\ \tilde{b}' & 0 \end{pmatrix} = \begin{pmatrix} b\tilde{b}' & 0 \\ b\tilde{a}' + a\tilde{b}' & b\tilde{b}' \end{pmatrix} \ .$$

Since $b\tilde{a}' + a\tilde{b}' = 0$ by the computation above and $b\tilde{b}' = -a\tilde{a}'$ by induction, we get the result. □

8 ML and LM

As before, we denote by L and M the lower triangular unipotent matrices with coefficients $l_{i,j}, m_{i,j} \in \{0,1\}$ defined by

$$l_{i,j} \equiv \binom{2i+1}{i-j} \pmod 2, \; m_{i,j} \equiv \binom{i+j}{2j} \pmod 2$$

for $0 \leq i, j$.

Proposition 8.1 *The matrix $R = M\ L$ has coefficients*

$$r_{i,j} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ 2 & \text{if } i > j \end{cases}$$

Proposition 8.2 *The product LM yields a matrix which can recursively be constructed by iterating*

$$\begin{pmatrix} A & \\ B & A \end{pmatrix} \longmapsto \begin{pmatrix} A & & & \\ B & A & & \\ 2A & B & A & \\ 2B & 2A & B & A \end{pmatrix}$$

starting with $A = 1$ and $B = 2$.

Assertion (i) of Theorem 1.3 implies of course easily the identities

$$(M\ L)^{-1} = D_a\ M\ L\ D_a \text{ and } (L\ M)^{-1} = D_a\ L\ M\ D_a\ .$$

Proof of Proposition 8.1. This follows immediately from Proposition 7.1 since $ML = (M\ D_e\ L) + (M\ D_o\ L)$. □

Proof of Proposition 8.2. Using the $BA0BAB-$construction of L and M we have

$$\begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix} \begin{pmatrix} A' & & & \\ B' & A' & & \\ A' & B' & A' & \\ B' & 0 & B' & A' \end{pmatrix}$$

$$= \begin{pmatrix} AA' & & & \\ BA' + AB' & AA' & & \\ BB' + AA' & BA' + AB' & AA' & \\ 2BA' + 2AB' & AA' + BB' & BA' + AB' & AA' \end{pmatrix}$$

Computing

$$AA' = \begin{pmatrix} a & \\ b & a \end{pmatrix} \begin{pmatrix} a' & \\ b' & a' \end{pmatrix} = \begin{pmatrix} aa' & \\ ba' + ab' & aa' \end{pmatrix}$$

and

$$BB' = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \begin{pmatrix} a' & b' \\ b' & 0 \end{pmatrix} = \begin{pmatrix} bb' & \\ ba' + ab' & bb' \end{pmatrix}$$

shows $AA' = BB'$ by induction. This finishes the proof by induction. □

9 Proof of Theorem 1.4

We prove Theorem 1.4 by showing that the $n \times n$ submatrix formed by the first n rows and columns of the Stieltjes matrix associated with $\sum_{k=0}^{\infty} x^{2^k}$ has determinant s_n for all n.

Proof of Theorem 1.4 Let H and L be as in Theorem 1.2. We denote by L_- the infinite matrix obtained by deleting the first row of L. Denote by C the matrix with coefficients $c_{i,j} = 1$ if $(i-j) \in \{-1, 0\}$ and $c_{i,j} = 0$ otherwise (for $0 \leq i, j$).

Elementary properties of the Catalan triangle show now that $L = L_- \, C$. This implies that the finite matrix $L_-(n)$ formed by the first n rows and columns of L_- has determinant 1. Denoting by $D_{s_+1}(n)$ the $n \times n$ diagonal matrix with entries $s_1, \ldots, s_n$ we see that the finite Stieltjes matrix $S(n) = D_s(n) \; (L(n))^{-1} D_s(n) \; D_{s_+1}(n) \;\; L_-(n) \; D_s(n)$ associated with H has determinant $s_n \in \pm 1$.

Easy computations show that $\det(H(2)) = -1$ and $\det(H(2^k)) = 1$ for $0 \leq k \neq 1$. We have also $\det(H(2^k + a)) = (-1)^a \;\; \det(H(2^k - a))$ for $0 \leq a < 2^k$. This shows that $\det(H(n)) = (-1)^{\binom{n}{2}}$ and implies that the Stieltjes matrix associated with H is of the form

$$S(n) = \begin{pmatrix} 1 & 1 & & & & \\ -1 & d_2 & 1 & & & \\ & -1 & d_3 & 1 & & \\ & & -1 & d_4 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} .$$

We have thus

$$s_n = \det(S(n)) = d_n \det(S(n-1)) + \det(S(n-2)) = d_n s_{n-1} + s_{n-2}$$

for $n > 1$ which shows

$$d_n = \frac{s_n - s_{n-2}}{s_{n-1}} \in \{0, \pm 2\}, \;\; n > 1$$

and proves the result. □

10 The shifted Hankel matrix

We consider the shifted Hankel matrix $\tilde{H}$ with coefficients $\tilde{h}_{i,j} \in \{0, 1\}$, $0 \leq i, j$ given by $\tilde{h}_{i,j} = 1$ if $i + j + 2 = 2^k$ for some natural integer $k \geq 1$ and $\tilde{h}_{i,j} = 0$ otherwise. We describe the $LU-$decomposition of $\tilde{H}$. The associated Stieltjes matrix can be recovered from Theorem 1.1 by setting $x_1 = x_2 = x_3 = \ldots = x^2$.

Define lower triangular matrices $\tilde{L}$ and $\tilde{M}$ with coefficients $\tilde{l}_{i,j},\ \tilde{m}_{i,j} \in \{0,1\}$ given by $\tilde{l}_{i,j} \equiv \binom{2i+1}{i-j} - \binom{2i+1}{i-j-1} \equiv \binom{2i+2}{i-j} \pmod 2$ and $\tilde{m}_{i,j} \equiv \binom{i+j+1}{2j+1} \pmod 2$.

It easy to see that one has recursive formulae

$$\tilde{l}_{2i,2j} = l_{i,j} \equiv \binom{2i+1}{i-j} \pmod 2, \quad \tilde{l}_{2i+1,2j+1} = \tilde{l}_{i,j}$$

and

$$\tilde{m}_{2i,2j} = m_{i,j} \equiv \binom{i+j}{2j} \pmod 2, \quad \tilde{m}_{2i+1,2j+1} = \tilde{m}_{i,j}\ .$$

The products $\tilde{L}\ \tilde{M}$ and $\tilde{M}\ \tilde{L}$ satisfy analogous recursive formulae.

In order to state the main result of this section we need also two sign-sequences $\tilde{s}_0, \tilde{s}_1, \ldots$ and $\tilde{t}_0, \tilde{t}_1$ with $\tilde{s}_i, \tilde{t}_i \in \{\pm 1\}$. The sequence $\tilde{s}_i$ is recursively defined by

$$\tilde{s}_{2i} = (-1)^i \text{ and } \tilde{s}_{2i+1} = \tilde{s}_i$$

and is obtained by removing the initial term of the famous paperfolding sequence

$$1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, -1, -1, \ldots$$

describing the peaks and valleys in a strip of paper which has iteratedly been folded (with all foldings executed in a similar way), see e.g. [1], [2] or [8].

The sequence $\tilde{t}_i$ is defined by $\tilde{t}_0 = 1$ and

$$\tilde{t}_{2i+1} = \tilde{t}_i,\ \tilde{t}_{4i} = (-1)^i \tilde{t}_{2i},\ \tilde{t}_{4i+2} = \tilde{t}_{2i}\ .$$

As always, we denote by $D_{\tilde{s}}$, respectively $D_{\tilde{t}}$ the diagonal matrices with diagonal entries $\tilde{s}_0, \tilde{s}_1, \ldots$ respectively $\tilde{t}_0, \tilde{t}_1, \ldots$.

Theorem 10.1 *(i) We have*

$$D_{\tilde{t}}\ \tilde{L}\ D_{\tilde{s}}\ \tilde{L}^t\ D_{\tilde{t}} = \tilde{H}\ .$$

(ii) We have

$$\tilde{L}\ D_{\tilde{s}}\ \tilde{M} = \tilde{M}\ D_{\tilde{s}}\ \tilde{L} = D_{\tilde{s}}\ .$$

Proof. Let $r_{i,j}$ be the coefficient (i,j) of the matrix product $D_{\tilde{t}}\ \tilde{L}\ D_{\tilde{s}}\ \tilde{L}^t\ D_{\tilde{t}}$. Since $\tilde{l}_{i,j} = 0$ if $i \not\equiv j \pmod 2$ we have easily that $r_{i,j} = 0$ if $i \not\equiv j \pmod 2$. For $r_{2i,2j}$ the computation reduces to assertion (i) of Theorem 1.2 since $\tilde{l}_{2i,2j} = \overline{\binom{4i+2}{2i-2j}} = \overline{\binom{2i+1}{i-j}}$ and $\tilde{t}_{2i} = s_i$.

For $r_{2i+1,2j+1}$ we get

$$\begin{aligned} & \tilde{s}_{2i+1}\tilde{s}_{2j+1} \textstyle\sum_k \overline{\binom{4i+4}{2i+1-2k-1}}\tilde{t}_{2k+1}\overline{\binom{4j+4}{2j+1-2k-1}} \\ = \ & \tilde{s}_i\tilde{s}_j \textstyle\sum_k \overline{\binom{2i+2}{i-k}}\tilde{t}_k\overline{\binom{2j+2}{j-k}} = r_{i,j} \end{aligned}$$

which proves assertion (i) by induction on $i+j$.

In order to prove assertion (ii), we denote by $r_{i,j}$ the coefficient (i,j) of the product $\tilde{L}\ D_{\tilde{s}}\ \tilde{M}$. For $i \not\equiv j \pmod 2$ we have $\tilde{l}_{i,j} = \overline{\binom{2i+2}{i-j}} = \tilde{m}_{i,j} = \overline{\binom{i+j+1}{2j+1}} = 0$. This shows that $r_{i,j} = 0$ if $i \not\equiv j \pmod 2$. For $r_{2i,2j}$ we have

$$\sum_k \overline{\binom{4i+2}{2i-2k}} \tilde{s}_{2k} \overline{\binom{2k+2j+1}{4j+1}}$$

$$= \sum_k \overline{\binom{2i+1}{i-k}} (-1)^k \overline{\binom{k+j}{2j}} = \begin{cases} 0 & \text{if } i \neq j \\ (-1)^i & \text{if } i = j \end{cases}$$

by Theorem 1.3 which proves assertion (ii) in this case.

For $r_{2i+1,2j+1}$ we get

$$\sum_k \overline{\binom{4i+4}{2i+1-2k-1}} \tilde{s}_{2k+1} \overline{\binom{2k+2j+3}{4j+3}}$$

$$= \sum_k \overline{\binom{2i+2}{i-k}} \tilde{s}_k \overline{\binom{k+j+1}{2j+1}} = r_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ \tilde{s}_i = \tilde{s}_{2i+1} & \text{if } i = j \end{cases}$$

which proves the result by induction on $i+j$. □

10.1 $BA0BAB-$constructions for $\tilde{L}$ and $\tilde{M}$

Denote by $\tilde{L}_0$ the infinite strictly lower triangular matrix obtained by adding a first row of zeros to $\tilde{L}$. We define also $\tilde{M}_0$ by adding a first row of zeros to $\tilde{M}$.

It can be shown that the matrix $\tilde{L}_0$ is obtained by iterating

$$\begin{pmatrix} A & \\ B & A \end{pmatrix} \longmapsto \begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix}$$

where one starts with with $\begin{pmatrix} A & \\ B & A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Similarly, $\tilde{M}_0$ is obtained by iterating

$$\begin{pmatrix} A & \\ B & A \end{pmatrix} \longmapsto \begin{pmatrix} A & & & \\ B & A & & \\ A & B & A & \\ B & 0 & B & A \end{pmatrix}$$

starting with $\begin{pmatrix} A & \\ B & A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

11 Uniqueness

Sequences with generating function

$$\sum_{k=0}^{\infty} \epsilon_k \; x^{2^k}, \;\; \epsilon_i \in \{\pm 1\}$$

are bijectively related to the set of all paperfolding sequences and can be characerized in terms of Hankel matrices as follows:

Proposition 11.1 *Let $s_0, s_1, \ldots$ be a sequence with $s_i \in \{\pm 1, 0\}$ such that $\det(H(n)), \det(\tilde{H}(n)) = \pm 1$ for all n where $H(n)$ and $\tilde{H}(n)$ have coefficients $h_{i,j} = s_{i+j}$ and $\tilde{h}_{i,j} = s_{i+j+1}$ for $0 \leq i, j < n$. Then $\sum_{k=0}^{\infty} s_k \; x^{k+1} = \sum_{k=0}^{\infty} \epsilon_k \; x^{2^k}$ for a suitable choice $\epsilon_0, \epsilon_1, \ldots \in \{\pm 1\}$.*

Proof. Computing $\det(H(n)) \pmod 2$ determines $s_{2n-2} \pmod 2$ and computing $\det(\tilde{H}(n)) \pmod 2$ determines $s_{2n-1} \pmod 2$. This shows that $\sum_{k=0}^{\infty} s_k \; x^{k+1} \equiv \sum_{k=0}^{\infty} x^{2^k} \pmod 2$ and the signs can of course be chosen arbitrarily. □

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