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Paperfolding and Catalan numbers

Roland Bacher

June 17, 2004

*Abstract*¹: This paper reproves a few results concerning paperfolding sequences using properties of Catalan numbers modulo 2. The guiding principle can be described by: Paperfolding = Catalan modulo 2 + signs given by 2-automatic sequences.

1 Main results

In this paper, a *continued fraction* is a formal expression

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \dots}} = b_0 + a_1 / \left(b_1 + a_2 / \left(b_2 + \dots \right) \right)$$

with $b_0, b_1, \dots, a_1, a_2, \dots \in \mathbb{C}[x]$ two sequences of complex rational functions. We denote such a continued fraction by

$$b_0 + \frac{a_1|}{|b_1|} + \frac{a_2|}{|b_2|} + \dots + \frac{a_k|}{|b_k|} + \dots$$

and call it *convergent* if the coefficients of the (formal) Laurent-series expansions of its *partial convergents*

$$\frac{P_0}{Q_0} = \frac{b_0}{1}, \frac{P_1}{Q_1} = \frac{b_0 b_1 + a_1}{b_1}, \dots, \frac{P_k}{Q_k} = b_0 + \frac{a_1|}{|b_1|} + \frac{a_2|}{|b_2|} + \dots + \frac{a_k|}{|b_k|}, \dots$$

are ultimately constant. The limit $\sum_{k \geq n} l_k x^k \in \mathbb{C}[[x]][x^{-1}]$ is then the obvious formal Laurent power series whose first coefficients agree with the Laurent series expansion of P_n/Q_n for n huge enough.

Let W_1 be the word $(-x_1) x_1$ of length 2 in the alphabet $\{x_1, -x_1\}$. For $k \geq 2$, we define recursively a word W_k of length $2(2^k - 1)$ over the alphabet $\{\pm x_1, \pm x_2, \dots, \pm x_k\}$ by considering the word

$$W_k = W_{k-1} x_k (-x_k) \overline{W_{k-1}}$$

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constructed by concatenation where $\overline{W_{k-1}}$ denotes the reverse of W_{k-1} obtained by reading W_{k-1} backwards. The first instances of the words W_k are

$$\begin{aligned} W_1 &= (-x_1) x_1 \\ W_2 &= (-x_1) x_1 x_2 (-x_2) x_1 (-x_1) \\ W_3 &= (-x_1) x_1 x_2 (-x_2) x_1 (-x_1) x_3 (-x_3) (-x_1) x_1 (-x_2) x_2 x_1 (-x_1) \end{aligned}$$

and these words are initial subwords of a unique, well-defined infinite word $W_\infty = w_1 w_2 w_3, \dots$ with letters $w_i \in \{\pm x_1, \pm x_2, \dots\}$.

The following result has been known for some time, cf [6] or [13]. Similar or equivalent equivalent statements are for instance given in [9], Theorem 6.5.6 in [2]. It is closely related to paperfolding, see [2], [11], [5]. Related results are contained in [4], [12] and [13].

Theorem 1.1 *Consider a sequence x_1, x_2, \dots with values in \mathbb{C} . If the corresponding continued fraction*

$$\frac{1}{|1} + \frac{w_1|}{|1} + \frac{w_2|}{|1} + \dots + \frac{w_k|}{|1} + \dots$$

associated with the sequence $W_\infty = w_1 w_2 \dots$ defined by the letters of the infinite word W_∞ converges, then its limit is given by

$$1 + x_1 + x_1^2 x_2 + x_1^4 x_2^2 x_3 + x_1^8 x_2^4 x_3^2 x_4 + \dots = 1 + \sum_{j=1}^{\infty} \prod_{i=1}^j x_i^{2^{j-i}} .$$

Remarks. (i) In fact, the proof will follow from the identity

$$\frac{P_{2(2^k-1)}}{Q_{2(2^k-1)}} = \frac{1}{|1} + \frac{w_1|}{|1} + \frac{w_2|}{|1} + \dots + \frac{w_{2(2^k-1)}|}{|1} = 1 + \sum_{j=1}^k \prod_{i=1}^j x_i^{2^{j-i}}$$

(cf. Lemma 6.5.5 in [2]) showing that the $2(2^k - 1)$ -th partial convergent of this continued fraction is a polynomial in x_1, \dots, x_k .

(ii) It is of course possible to replace the field \mathbb{C} by any other reasonable field or ring.

Examples. (1) Setting $x_i = x$ for all i and multiplying by x we get

$$\begin{aligned} & \frac{x}{|1} - \frac{x}{|1} + \frac{x}{|1} + \frac{x}{|1} - \frac{x}{|1} + \frac{x}{|1} + \dots \\ &= x + x^2 + x^4 + x^8 + \dots = \sum_{k=0}^{\infty} x^{2^k} . \end{aligned}$$

The sequence of signs $w_1, w_2, \dots \in \{\pm 1\}$ given by

$$\sum_{k=0}^{\infty} x^{2^k} = \frac{x}{|1} + w_1 \frac{x}{|1} + w_2 \frac{x}{|1} + w_3 \frac{x}{|1} + \dots$$

can be recursively defined by

$$\begin{aligned} w_{4i+1} &= -w_{4i+2} = (-1)^{i+1}, \\ w_{8i+3} &= -w_{8i+4} = (-1)^i, \\ w_{8i+7} &= -w_{8i+8} = w_{4i+3} \end{aligned}$$

and is 2-automatic (cf. [2]).

(2) Setting $x_i = x^{1+3^{i-1}}$ and multiplying by x we get

$$\begin{aligned} \frac{x}{|1} - \frac{x^2}{|1} + \frac{x^2}{|1} + \frac{x^4}{|1} - \frac{x^4}{|1} + \frac{x^2}{|1} + \dots \\ = x + x^3 + x^9 + x^{27} + \dots = \sum_{k=0}^{\infty} x^{3^k}. \end{aligned}$$

(3) Setting $x_i = x^{1+(i-1)!}$ and multiplying by x we get

$$\begin{aligned} \frac{x}{|1} - \frac{x}{|1} + \frac{x}{|1} + \frac{x^3}{|1} - \frac{x^3}{|1} + \frac{x}{|1} + \dots \\ = x + x^2 + x^6 + x^{24} + \dots = \sum_{k=1}^{\infty} x^{k!}. \end{aligned}$$

Consider the sequence $\mu = (\mu_0, \mu_1, \dots) = (1, 1, 0, 1, 0, 0, 0, 1, \dots)$ defined by $x \sum_{i=0}^{\infty} \mu_i x^i = \sum_{k=0}^{\infty} x^{2^k}$. We associate to $\mu = (\mu_0, \mu_1, \dots)$ its *Hankel matrix* H whose (i, j) -th coefficient $h_{i,j} = \mu_{i+j}$, $0 \leq i, j$ depends only on the sum $i+j$ of its indices and is given by $(i+j)$ -th element of the sequence μ .

Consider the (2-automatic) sequence $s = (s_0, s_1, s_2, \dots) = 1 \ 1 \ (-1) \ 1 \ (-1) \ (-1) \ \dots$ defined recursively by $s_0 = 1$ and

$$\begin{aligned} s_{2i} &= (-1)^i s_i, \\ s_{2i+1} &= s_i. \end{aligned}$$

It is easy to check that $s_n = (-1)^{B_0(n)}$ where $B_0(n)$ counts the number of bounded blocks of consecutive 0's of the binary integer n (or, equivalently, $B_0(n)$ equals the number of blocks 10 appearing in the binary expansion of the n). E.g, $720 = 2^9 + 2^7 + 2^6 + 2^4$ corresponds to the binary integer 1011010000. Hence $B_0(720) = 3$. This description shows that we have $s_{2^{n+1}+n} = -s_n$ for $n < 2^n$. Since we have $B_0(2^{n+1} + n) = 1 + B_0(n)$ for $n < 2^n$ and $B_0(2^{n+1} + n) = B_0(n)$ for $2^n \leq n < 2^{n+1}$, the sequence s_0, s_1, \dots can also be constructed by iterating the application

$$W_\alpha W_\omega \mapsto W_\alpha W_\omega (-W_\alpha) W_\omega$$

where where W_α, W_ω are the first and the second half of the finite word $s_0 \dots s_{2^n-1}$.

Denote by D_s the infinite diagonal matrix with diagonal entries s_0, s_1, s_2, \dots and by D_a the infinite diagonal matrix with diagonal entries given by the alternating, 2-periodic sequence $1, -1, 1, -1, \overline{1}, \overline{-1}$.

We introduce the infinite unipotent lower triangular matrix L with coefficients $l_{i,j} \in \{0, 1\}$ for $0 \leq i, j$ given by

$$l_{i,j} \equiv \binom{2i}{i-j} - \binom{2i}{i-j-1} \equiv \binom{2i}{i-j} + \binom{2i}{i-j-1} \equiv \binom{2i+1}{i-j} \pmod{2}$$

and denote by $L(k)$ the $k \times k$ submatrix with coefficients $l_{i,j}$, $0 \leq i, j < k$ of L . The matrix $L(8)$ for instance is given by

$$L(8) = \begin{pmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 0 & 1 & 1 & & & & & \\ 1 & 1 & 1 & 1 & & & & \\ 0 & 0 & 0 & 1 & 1 & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & & \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Theorem 1.2 (i) *We have*

$$H = D_s L D_a L^t D_s.$$

(ii) *(BA0BAB-construction for L) We have for $n \geq 0$*

$$L(2^{n+2}) = \begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix}$$

where A and B are the obvious submatrices of size $2^n \times 2^n$ in $L(2^{n+1})$.

Remarks. (i) Assertion (ii) of Theorem 1.2 yields an iterative construction for L by starting with $L(2) = \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$.

(ii) Conjugating H with an appropriate diagonal ± 1 -matrix, we can get the LU-decomposition of any Hankel matrix associated with a sequence having ordinary generating function $x^{-1} \left(\sum_{k=0}^{\infty} \epsilon_k x^{2^k} \right)$, $\epsilon_0 = 1, \epsilon_1, \epsilon_2, \dots \in \{\pm 1\}$. More precisely, the appropriate conjugating diagonal matrix has coefficients $d_{n,n} = (-1)^{\sum_{j=0}^n \nu_j \epsilon_{j+1}}$ for $0 \leq n = \sum_{j=0}^n \nu_j 2^j$ a binary integer. The case $\epsilon_0 = -1$ can be handled similarly by replacing the diagonal matrix D_a with its opposite $-D_a$.

The matrix $P = (D_s L D_s)^{-1} = D_s L^{-1} D_s$ with coefficients $p_{i,j}$, $0 \leq i, j$ encodes the formal orthogonal polynomials $Q_0, Q_1, \dots, Q_n = \sum_{k=0}^n p_{n,k} x^k$ with moments μ_0, μ_1, \dots given by $\sum_{i=0}^{\infty} \mu_i x^{i+1} = \sum_{k=0}^{\infty} x^{2^k}$.

In order to describe the matrix P we introduce a unipotent lower triangular matrix M with coefficients $m_{i,j} \in \{0, 1\}$, $0 \leq i, j$ by setting

$$m_{i,j} = \binom{i+j}{2j} \pmod{2}.$$

As above (with the matrix L) we denote by $M(k)$ the $k \times k$ submatrix with coefficients $m_{i,j}$, $0 \leq i, j < k$ of M .

Theorem 1.3 (i) *We have*

$$L D_a M = D_a$$

(where D_a denotes the diagonal matrix with 2-periodic diagonal coefficients $1, -1, 1, -1, \dots$).

(ii) (BA0BAB-construction for M) *We have have for $n \geq 0$*

$$M(2^{n+2}) = \begin{pmatrix} A' & & & & \\ B' & A' & & & \\ A' & B' & A' & & \\ B' & 0 & B' & A' & \end{pmatrix}$$

where A' and B' are the obvious submatrices of size $2^n \times 2^n$ in $M(2^{n+1})$.

The matrix $P = D_s D_a M D_a D_s = (D_s L D_s)^{-1}$ encoding the coefficients of the orthogonal polynomials with moments μ_0, μ_1, \dots has thus all its coefficients in $\{0, \pm 1\}$, see also [1] and [8].

The (formal) orthogonal polynomials Q_0, Q_1, \dots satisfy a classical three-term recursion relation with coefficients given by the continued Jacobi fraction expansion of $\sum \mu_i x^i = \frac{1}{x} \left(\sum_{k=0}^{\infty} x^{2k} \right)$. This expansion is described by the following result.

Theorem 1.4 *We have*

$$\begin{aligned} \sum_{k=0}^{\infty} x^{2k} &= \frac{x|}{|1-d_1x} + \frac{x^2|}{|1-d_2x} + \frac{x^2|}{|1-d_3x} + \dots \\ &= \frac{x|}{|1-x} + \frac{x^2|}{|1+2x} + \frac{x^2|}{|1} + \frac{x^2|}{|1} + \frac{x^2|}{|1-2x} + \dots \end{aligned}$$

where $d_n = \frac{s_n - s_{n-2}}{s_{n-1}}$ with $s_{-1} = 0$, $s_0 = 1$ and $s_{2i+1} = (-1)^i s_{2i} = s_i$ is the recursively defined sequence introduced previously (extended by $s_{-1} = 0$).

The formal orthogonal polynomials with moments μ_0, μ_1, \dots are thus recursively defined by $Q_0 = 1$, $Q_1 = x + 1$ and

$$Q_{n+1} = (x - d_n)Q_n + Q_{n-1} \text{ for } n \geq 2.$$

The rest of the paper is organised as follows.

Section 2 contains a proof of Theorem 1.1.

Section 3 recalls a few definitions and facts concerning Hankel matrices, formal orthogonal polynomials etc.

Section 4 is a little digression about Catalan numbers. All objects appearing in this paper are essentially obtained by considering the corresponding objects associated with Catalan numbers and by reducing them modulo 2 using sign conventions prescribed by recursively defined sequences (which are 2-automatic).

Section 5 recalls two classical and useful results on the reduction modulo 2 (or modulo a prime number) of binomial coefficients.

Section 6 is devoted to the proof of Theorem 1.2.

Section 7 contains a proof of Theorem 1.3 and a related amusing result.

Section 8 contains formulae for the matrix products $M L$ and $L M$.

Section 9 contains a proof of Theorem 1.4.

Section 10 displays the LU -decomposition of the Hankel matrix \tilde{H} associated with the sequences $1, 0, 1, 0, 0, 0, 1, 0, \dots$ obtained by shifting the sequence $1, 1, 0, 1, 0, 0, 0, 1, \dots$ associated with powers of 2 by one (or equivalently, by removing the first term).

Section 11 contains a uniqueness result.

2 Proof of Theorem 1.1

More or less equivalent reformulations of Theorem 1.1 can be found in several places. A few are [9], [11] and [2]. We give her a computational proof: Symmetry-arguments of continued fractions are replaced by polynomial identities.

Given a continued fraction

$$b_0 + \frac{a_1|}{|b_1|} + \frac{a_2|}{|b_2|} + \dots$$

it is classical (and easy, cf. for instance pages 4,5 of [10]) to show that the product

$$\begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & a_2 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_k \\ 1 & b_k \end{pmatrix}$$

equals

$$\begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix}$$

where

$$\frac{P_k}{Q_k} = b_0 + \frac{a_1|}{|b_1|} + \dots + \frac{a_k|}{|b_k|}$$

denotes the k -th partial convergent.

Given a finite word $U = u_1 u_2 \dots u_l$ we consider the product

$$M(U) = \begin{pmatrix} 0 & u_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & u_2 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & u_l \\ 1 & 1 \end{pmatrix}.$$

Defining

$$\begin{aligned} X_k &= 1 + \sum_{j=1}^k \prod_{i=1}^j x_i^{2^{j-i}} \\ \tilde{X}_k &= 1 + \sum_{j=1}^{k-1} \prod_{i=1}^j x_i^{2^{j-i}} - \prod_{i=1}^k x_i^{2^{k-i}} \end{aligned}$$

(example: $X_3 = 1 + x_1 + x_1^2 x_2 + x_1^4 x_2^2 x_3$ and $\tilde{X}_3 = 1 + x_1 + x_1^2 x_2 - x_1^4 x_2^2 x_3$)
one checks easily the identities

$$X_n + x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n) = X_{n+1}$$

and

$$X_n - x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n) = \tilde{X}_{n+1}.$$

Lemma 2.1 *We have for all $n \geq 1$*

$$M(W_n) = \begin{pmatrix} \tilde{X}_n - X_{n-1}^2 & 1 - X_n \\ X_{n-1}^2 & X_n \end{pmatrix}$$

and

$$M(\overline{W}_n) = \begin{pmatrix} X_n - X_{n-1}^2 & 1 - \tilde{X}_n \\ X_{n-1}^2 & \tilde{X}_n \end{pmatrix}.$$

Proof. The computations

$$\begin{aligned} M(W_1) &= M((-x_1) x_1) = \begin{pmatrix} 0 & -x_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & x_1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -x_1 & -x_1 \\ 1 & 1+x_1 \end{pmatrix} \\ &= \begin{pmatrix} (1-x_1) - 1^2 & 1 - (1+x_1) \\ 1^2 & 1+x_1 \end{pmatrix} \end{aligned}$$

and

$$M(\overline{W}_1) = M(x_1 (-x_1)) = \begin{pmatrix} 0 & x_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -x_1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} (1+x_1) - 1^2 & 1 - (1-x_1) \\ 1^2 & 1-x_1 \end{pmatrix}$$

prove Lemma 2.1 for $n = 1$.

For $n \geq 1$ we have

$$\begin{aligned} M(W_{n+1}) &= M(W_n x_{n+1} (-x_{n+1}) \overline{W}_n) \\ &= \begin{pmatrix} \tilde{X}_n - X_{n-1}^2 & 1 - X_n \\ X_{n-1}^2 & X_n \end{pmatrix} \begin{pmatrix} x_{n+1} & x_{n+1} \\ 1 & 1 - x_{n+1} \end{pmatrix} \begin{pmatrix} X_n - X_{n-1}^2 & 1 - \tilde{X}_n \\ X_{n-1}^2 & \tilde{X}_n \end{pmatrix} \\ &= \begin{pmatrix} (X_n - x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n)) - X_n^2 & 1 - (X_n + x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n)) \\ X_n^2 & X_n + x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{X}_{n+1} - X_n^2 & 1 - X_{n+1} \\ X_n^2 & X_{n+1} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
M(\overline{W_{n+1}}) &= M(W_n \begin{pmatrix} -x_{n+1} & x_{n+1} \\ x_{n+1} & \overline{W_n} \end{pmatrix}) \\
&= \begin{pmatrix} \tilde{X}_n - X_{n-1}^2 & 1 - X_n \\ X_{n-1}^2 & X_n \end{pmatrix} \begin{pmatrix} -x_{n+1} & -x_{n+1} \\ 1 & 1 + x_{n+1} \end{pmatrix} \begin{pmatrix} X_n - X_{n-1}^2 & 1 - \tilde{X}_n \\ X_{n-1}^2 & \tilde{X}_n \end{pmatrix} \\
&= \begin{pmatrix} (X_n + x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n)) - X_n^2 & 1 - (X_n - x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n)) \\ X_n^2 & X_n - x_{n+1}(X_{n-1}^2 - X_n \tilde{X}_n) \end{pmatrix} \\
&= \begin{pmatrix} X_{n+1} - X_n^2 & 1 - \tilde{X}_{n+1} \\ X_n^2 & \tilde{X}_{n+1} \end{pmatrix}
\end{aligned}$$

which proves Lemma 2.1 by induction on n . \square

Proof of Theorem 1.1. Lemma 2.1 shows that

$$\begin{aligned}
M(1 \ W_n) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{X}_n - X_{n-1}^2 & 1 - X_n \\ X_{n-1}^2 & X_n \end{pmatrix} \\
&= \begin{pmatrix} X_{n-1}^2 & X_n \\ \tilde{X}_n & 1 \end{pmatrix} .
\end{aligned}$$

The $2(2^n - 1)$ -th partial convergent of the continued fraction

$$\frac{1|}{|1} + \frac{w_1|}{|1} + \frac{w_2|}{|1} + \dots$$

defined by the word W_∞ equals thus

$$X_n = 1 + \sum_{j=1}^n \prod_{k=1}^j x_k^{2^{j-k}}$$

and the sequence of these partial convergents has limit

$$X_\infty = 1 + \sum_{j=1}^{\infty} \prod_{k=1}^j x_k^{2^{j-k}}$$

which is the statement of Theorem 1.1. \square

3 Formal orthogonal polynomials

The content of this section is classical and can for instance be found in [3] or [15].

We consider a formal power series

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k \in \mathbb{C}[[x]]$$

as a linear form on the complex vector space $\mathbb{C}[x]^*$ of polynomials by setting

$$l_g\left(\sum_{i=0}^n \alpha_i x^i\right) = \sum_{i=0}^n \alpha_i \mu_i .$$

We suppose that the n -th *Hankel matrix* $H(n)$ having coefficients $h_{i,j} = \mu_{i+j}$, $0 \leq i, j < n$ associated with $\mu = (\mu_0, \dots)$ is invertible for all n . Replacing such a sequence $\mu = (\mu_0, \mu_1, \dots)$ by $\frac{1}{\mu_0}\mu = (1, \frac{\mu_1}{\mu_0}, \dots)$ we may also suppose that $\mu_0 = 1$. We get a non-degenerate scalar product on $\mathbb{C}[x]$ by setting

$$\langle P, Q \rangle = l_g(PQ) .$$

Applying the familiar Gram-Schmitt orthogonalisation we obtain a sequence $Q_0 = 1$, $Q_1 = x - a_0$,

$$Q_{n+1} = (x - a_n)Q_n - b_n Q_{n-1}, \quad n \geq 1$$

of mutually orthogonal monic polynomials, called *formal orthogonal polynomials with moments* μ_0, μ_1, \dots

We encode the above recursion relation for the polynomials Q_0, Q_1, \dots by the *Stieltjes matrix*

$$S = \begin{pmatrix} a_0 & 1 & & & \\ b_1 & a_1 & 1 & & \\ & b_2 & a_2 & 1 & \\ & & b_3 & a_3 & 1 \\ & & & \ddots & \ddots & \ddots \end{pmatrix} .$$

The non-degeneracy condition on μ_0, μ_1, \dots shows that the (infinite) Hankel matrix H with coefficients $h_{i,j} = \mu_{i+j}$, $0 \leq i, j$ has an LU -decomposition $H = LU$ where L is lower triangular unipotent and $U = DL^t$ is upper triangular with D diagonal and invertible.

The diagonal entries of D are given by

$$d_{i,i} = \prod_{k=1}^i b_k$$

and we have thus

$$\det(H(n)) = \prod_{k=1}^{n-1} (b_k)^{n-k}$$

where $H(n)$ is the n -th Hankel matrix with coefficients $h_{i,j} = \mu_{i+j}$, $0 \leq i, j < n$. The i -th row vector

$$(m_{i,0}, m_{i,1}, \dots)$$

of $M = L^{-1}$ encodes the coefficients of the i -th orthogonal polynomial

$$Q_i = \sum_{k=0}^i m_{i,k} x^k .$$

The Stieltjes matrix S satisfies

$$LS = L_- = \begin{pmatrix} l_{1,0} & l_{1,1} & & \\ l_{2,0} & l_{2,1} & l_{2,2} & \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

where L_- is obtained by removing the first row of the unipotent lower triangular matrix L with coefficients $l_{i,j}$, $0 \leq i, j$ involved in the LU -decomposition of H .

The generating function $g = \sum_{k=0}^{\infty} \mu_k x^k$ can be expressed as a *continued fraction of Jacobi type*

$$g(x) = 1/(1 - a_0x - b_1x^2/(1 - a_1x - b_2x^2/(1 - a_2x - b_3x^2/(1 - \dots))))$$

where a_0, a_1, \dots and b_1, b_2, \dots are as above and are encoded in the Stieltjes matrix.

4 The Catalan numbers

The Catalan numbers $C_n = \binom{2n}{n} \frac{1}{n+1}$ are the coefficients of the algebraic generating function

$$\begin{aligned} c(x) &= \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots \\ &= \frac{1|}{|1} - \frac{x|}{|1} - \frac{x|}{|1} - \frac{x|}{|1} - \dots \\ &= \frac{1|}{|1-x} - \frac{x^2|}{|1-2x} - \frac{x^2|}{|1-2x} - \frac{x^2|}{|1-2x} - \dots \end{aligned}$$

and satisfies $c = 1 + x c^2$.

We have $g(x) \equiv c(x) \pmod{2}$ where $g(x) = \sum_{j=0}^{\infty} x^{2^j-1}$, cf. the last lines of [14]. A different proof can be given by remarking that c_n counts the number of planar binary rooted trees with $n+1$ leaves. Call two such trees equivalent if they are equivalent as abstract (non-planar) rooted trees. The cardinality of each equivalence class is then a power of two. An equivalence class containing only one element is given by a rooted 2-regular trees having 2^n leaves which are all at the same distance n from the root.

Many interesting mathematical objects associated with $g(x)$ are in fact obtained by reducing the corresponding objects of $c(x)$ modulo 2. Miraculously, all this works over \mathbb{Z} by choosing suitable signs given by a few recursively-defined sequences (which are in fact always 2-automatic, see [2] for a definition).

Consider the unipotent lower triangular matrices

$$L = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 5 & 9 & 5 & 1 & & \\ 14 & 28 & 20 & 7 & 1 & \\ 42 & 90 & 75 & 35 & 9 & 1 \end{pmatrix}$$

and

$$\tilde{L} = \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ 132 & 165 & 10 & 44 & 10 & 1 \end{pmatrix}$$

with coefficients $l_{i,j} = \binom{2i}{i-j} - \binom{2i}{i-j-1}$, respectively $\tilde{l}_{i,j} = \binom{2i+1}{i-j} - \binom{2i+1}{i-j-1}$ for $0 \leq i, j$ obtained by taking all even (respectively odd) lines (starting with line number zero) of the so-called *Catalan-triangle*

$$\begin{array}{cccccc} 1 & & & & & \\ & 1 & & & & \\ 1 & & 1 & & & \\ & 2 & & 1 & & \\ 2 & & 3 & & 1 & \\ & 5 & & 4 & & 1 \\ 5 & & 9 & & 5 & & 1 \\ & 14 & & 14 & & 6 & & 1 \\ 14 & & 28 & & 20 & & 7 & & 1 \end{array}$$

The matrix L is associated with the LU -decomposition of the Hankel matrix $H = L L^t$ having coefficients $h_{i,j} = C_{i+j} = \binom{2(i+j)}{i+j} / (i+j+1)$, $0 \leq i, j$ the Catalan numbers. The matrix \tilde{L} yields the LU -decomposition of the Hankel matrix $\tilde{H} = \tilde{L} \tilde{L}^t$ having coefficients $\tilde{h}_{i,j} = C_{i+j+1} = \binom{2(i+j+1)}{i+j+1} / (i+j+2)$, $0 \leq i, j$ associated with the shifted Catalan sequence $1, 2, 5, 14, 42, \dots$

The inverse matrices L^{-1} and \tilde{L}^{-1} are given by $D_a M D_a$ and $D_a \tilde{M} D_a$ where D_a is the diagonal matrix with 2-periodic alternating diagonal coefficients $1, -1, 1, -1, \dots$ and where M and \tilde{M} have coefficients $m_{i,j} = \binom{i+j}{2j}$ and $\tilde{m}_{i,j} = \binom{i+j+1}{2j+1}$ for $0 \leq i, j$.

Let us mention that the products $P = LM$ and $\tilde{P} = \tilde{L} \tilde{M}$ have non-zero coefficients $p_{i,j} = \frac{(2i)! j!}{i! (2j)! (i-j)!}$ and $\tilde{p}_{i,j} = 4^{i-j} \binom{i}{j}$ for $0 \leq j < i$. They can

also be given by

$$P = L M = \exp\left(\begin{pmatrix} 0 & & & & \\ 2 & 0 & & & \\ & 6 & 0 & & \\ & & 10 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}\right)$$

and

$$\tilde{P} = \tilde{L} \tilde{M} = \exp\left(\begin{pmatrix} 0 & & & & \\ 4 & 0 & & & \\ & 8 & 0 & & \\ & & 12 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}\right).$$

Computations suggest that the products $M L$ and $\tilde{M} \tilde{L}$ have also nice logarithms:

$$M L = \exp\left(\begin{pmatrix} 0 & & & & & & \\ 2 & 0 & & & & & \\ 0 & 6 & 0 & & & & \\ 2 & 0 & 10 & 0 & & & \\ 0 & 6 & 0 & 14 & 0 & & \\ 2 & 0 & 10 & 0 & 18 & 0 & \\ 0 & 6 & 0 & 14 & 0 & 22 & 0 \\ \vdots & \vdots & & & & & \ddots & \ddots \end{pmatrix}\right)$$

and

$$\tilde{M} \tilde{L} = \exp\left(\begin{pmatrix} 0 & & & & & & \\ 4 & 0 & & & & & \\ 0 & 8 & 0 & & & & \\ 4 & 0 & 12 & 0 & & & \\ 0 & 8 & 0 & 16 & 0 & & \\ 4 & 0 & 12 & 0 & 20 & 0 & \\ \vdots & \vdots & & & & & \ddots & \ddots \end{pmatrix}\right).$$

5 Binomial coefficients modulo 2

Chercher ref. (Concrete Maths? Graham, Knuth, Patashnik)

Using the Frobenius automorphism we have

$$(1+x)^{\sum_i \nu_i 2^i} \equiv \prod_i (1+x^{2^i})^{\nu_i} \pmod{2}$$

which shows the formula

$$\binom{n}{k} \equiv \prod_i \binom{\nu_i}{\kappa_i} \pmod{2}$$

where $n = \sum_{i \geq 0} \nu_i 2^i$, $\nu_i \in \{0, 1\}$ and $k = \sum_{i \geq 0} \kappa_i 2^i$, $\kappa_i \in \{0, 1\}$ are two natural binary integers. Kummer showed that the 2-valuation v_2 of $\binom{n}{k}$ (defined by $2^{v_2} | \binom{n}{k}$ and $2^{v_2+1} \nmid \binom{n}{k}$) equals the number of carry-overs when adding the two natural numbers k and $n - k$, written in base 2 (cf. [16]).

As an application, one (re)proves easily that $C_n = \binom{2n}{n} \frac{1}{n+1} \equiv 1 \pmod{2}$ if and only if $n+1 \in \{2^k\}_{k \in \mathbb{N}}$ is a power of two. Indeed, $\binom{2n}{n} \frac{1}{n+1} \equiv \frac{(2n+1)!}{n! n+1!} = \binom{2n+1}{n} \pmod{2}$ and addition of the binary integers n and $(n+1)$ needs always a carry except if $n = \sum_{i=0}^{k-1} 2^i$ and $n+1 = 2^k$.

6 Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2 which describes the LU -decomposition of the Hankel matrix H with coefficients $h_{i,j} = \mu_{i+j}$, $0 \leq i, j$ where

$$\sum_{k=0}^{\infty} \mu_k x^k = \sum_{j=0}^{\infty} x^{2^j-1} = 1 + x + x^3 + x^7 + x^{15} + \dots$$

We give in fact 3/2 proofs: We first prove assertion (i) by using the definition $l_{i,j} \equiv \binom{2i+1}{i-j} \pmod{2}$ of L .

We prove then assertion (ii) showing that the matrix L satisfies the $BA0BAB$ -construction.

Finally, we reprove assertion (i) using the recursive $BA0BAB$ -structure of L .

Proof of assertion (i) in Theorem 1.2. We denote by $\overline{\binom{n}{k}} \in \{0, 1\}$ the reduction $\pmod{2}$ with value in $\{0, 1\}$ of the binomial coefficient $\binom{n}{k}$. The main ingredient of the proof is the fact that

$$\overline{\binom{2n+1}{k}} = \overline{\binom{n}{\lfloor k/2 \rfloor}}$$

where $\lfloor k/2 \rfloor = k/2$ if k is even and $\lfloor k/2 \rfloor = (k-1)/2$ if k is odd.

We compute first the coefficient $r_{2i,2j}$ of the product $R = D_s L D_a L^t D_s$ (recall that L has coefficients $l_{i,j} = \overline{\binom{2i+1}{i-j}}$):

$$\begin{aligned} r_{2i,2j} &= s_{2i} s_{2j} \sum_k (-1)^k \overline{\binom{4i+1}{2i-k} \binom{4j+1}{2j-k}} \\ &= s_{2i} s_{2j} \left(\sum_{k=0} \overline{\binom{4i+1}{2i-2k} \binom{4j+1}{2j-2k}} - \sum_{k=1} \overline{\binom{4i+1}{2i-2k+1} \binom{4j+1}{2j-2k+1}} \right) \\ &= s_{2i} s_{2j} \left(\sum_{k=0} \overline{\binom{2i}{i-k} \binom{2j}{j-k}} - \sum_{k=1} \overline{\binom{2i}{i-k} \binom{2j}{j-k}} \right) \\ &= s_{2i} s_{2j} \overline{\binom{2i}{i} \binom{2j}{j}} = \begin{cases} 1 & \text{if } i = j = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, we get

$$\begin{aligned}
r_{2i+1,2j+1} &= s_{2i+1}s_{2j+1} \sum_k (-1)^k \overline{\binom{4i+3}{2i+1-k} \binom{4j+3}{2j+1-k}} \\
&= s_{2i+1}s_{2j+1} \left(\sum_{k=0} \overline{\binom{4i+3}{2i+1-2k} \binom{4j+3}{2j+1-2k}} - \sum_{k=0} \overline{\binom{4i+3}{2i-2k} \binom{4j+3}{2j-2k}} \right) \\
&= s_{2i+1}s_{2j+1} \left(\sum_{k=0} \overline{\binom{2i+1}{i-k} \binom{2j+1}{j-k}} - \sum_{k=0} \overline{\binom{2i+1}{i-k} \binom{2j+1}{j-k}} \right) = 0
\end{aligned}$$

For $r_{2i,2j+1} = r_{2j+1,2i}$ we have

$$\begin{aligned}
r_{2i,2j+1} &= s_{2i}s_{2j+1} \left(\sum_k \overline{\binom{4i+1}{2i-2k} \binom{4j+3}{2j+1-2k}} - \sum_k \overline{\binom{4i+1}{2i-2k-1} \binom{4j+3}{2j-2k}} \right) \\
&= s_{2i}s_{2j+1} \left(\sum_k \overline{\binom{2i}{i-k} \binom{2j+1}{j-k}} - \sum_k \overline{\binom{2i}{i-k-1} \binom{2j+1}{j-k}} \right) \\
&= s_{2i}s_{2j+1} \left(\sum_k \overline{\left(\binom{2i}{i-k} + \binom{2i}{i-k-1} \right) \binom{2j+1}{j-k}} - 2 \sum_k \overline{\binom{2i}{i-k-1} \binom{2j+1}{j-k}} \right) \\
&= s_{2i}s_{2j+1} \left(\sum_k \overline{\binom{2i+1}{i-k} \binom{2j+1}{j-k}} - 2 \sum_k \overline{\binom{2i}{i-k-1} \binom{2j+1}{j-k}} \right).
\end{aligned}$$

Using parity arguments and the recursion $s_{2i} = (-1)^i s_i$, $s_{2j+1} = s_j$ we get for i even

$$\begin{aligned}
r_{2i,2j+1} &= s_{2i}s_{2j+1} \left(\sum_k (-1)^k \overline{\binom{2i+1}{i-k} \binom{2j+1}{j-k}} + \right. \\
&\quad \left. + 2 \sum_k \overline{\left(\binom{2i+1}{i-2k-1} - \binom{2i}{i-2k-2} \right) \binom{2j+1}{j-2k-1}} \right) \\
&= s_i s_j \left(\sum_k (-1)^k \overline{\binom{2i+1}{i-k} \binom{2j+1}{j-k}} + 0 \right) = r_{i,j}.
\end{aligned}$$

Similarly, for i odd we have

$$\begin{aligned}
r_{2i,2j+1} &= -s_{2i}s_{2j+1} \left(\sum_k (-1)^k \overline{\binom{2i+1}{i-k} \binom{2j+1}{j-k}} + \right. \\
&\quad \left. - 2 \sum_k \overline{\left(\binom{2i+1}{i-2k} - \binom{2i}{i-2k-1} \right) \binom{2j+1}{j-2k}} \right) \\
&= s_i s_j \left(\sum_k (-1)^k \overline{\binom{2i+1}{i-k} \binom{2j+1}{j-k}} + 0 \right) = r_{i,j}.
\end{aligned}$$

Finally, the above computations together with induction on $i+j$ show that $r_{i,j} = 0$ except if $i+j = 2^k - 1$ for some $k \in \mathbb{N}$. In this case we get $r_{i,j} = 1$ which proves the result. \square

Proof of assertion (ii) in Theorem 1.2. Assume that $L(2^{n+1})$ is given by $\begin{pmatrix} A & \\ B & A \end{pmatrix}$. (This holds clearly for $n = 0$.)

We have for $0 \leq i, j < 2^{n+1}$

$$\begin{pmatrix} 2(i+2^{n+1})+1 & \\ (i+2^{n+1})-(j+2^{n+1}) & \end{pmatrix} = \begin{pmatrix} 2i+2^{n+2}+1 & \\ i-j & \end{pmatrix} \equiv \begin{pmatrix} 2i+1 & \\ i-j & \end{pmatrix} \pmod{2}$$

since $2i+1 < 2^{n+2}$. This shows already that $L(2^{n+2})$ is of the form $\begin{pmatrix} \tilde{A} & \\ \tilde{B} & \tilde{A} \end{pmatrix}$ with $\tilde{A} = L(2^{n+1})$.

Consider now a binomial coefficient of the form

$$\binom{2(2^{n+1} + i) + 1}{(2^{n+1} + i) - j} = \binom{2^{n+2} + 2i + 1}{2^{n+1} + i - j}$$

with $0 \leq i, j < 2^{n+1}$ which is odd. This implies $i + j + 1 \geq 2^{n+1}$ since otherwise $(2^{n+1} + i) - j < (2^{n+1} + i) + j + 1 < 2^{n+2} < 2^{n+2} + 2i + 1$ and there must therefore be a carry when adding the binary integers $((2^{n+1} + i) - j)$ and $((2^{n+1} + i) + j + 1)$.

Two such binary integers $((2^{n+1} + i) - j)$ and $((2^{n+1} + i) + j + 1)$ add thus without carry if and only if the binary integers $(2^{n+2} + 2^{n+1} + i - j)$ and $(2^{n+1} + i + j + 1 - 2^{n+2})$ add without carry. This shows the equality

$$\binom{2(2^{n+1} + i) + 1}{(2^{n+1} + i) - j} \equiv \binom{2(2^{n+1} + i) + 1}{(2^{n+1} + i) - (2^{n+2} - 1 - j)}$$

for $0 \leq i, j < 2^{n+1}$. Geometrically, this amounts to the fact that the block \tilde{B} is the vertical mirror of the block \tilde{A} and this symmetry is preserved by the *BA0BAB*-construction. \square

BA0BAB-proof of assertion (i). We have

$$\begin{aligned} & \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and conjugation by

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \text{ yields } \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The proof is now by induction using the *BA0BAB*-construction of L . Writing $\tilde{A} = D_a A^t$ and $\tilde{B} = D_a B^t$ (with D_a denoting the obvious submatrix of finite size of the previously infinite matrix D_a) we have

$$\begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} & 0 & \tilde{B} \\ & \tilde{A} & \tilde{B} & \tilde{A} \\ & & \tilde{A} & \tilde{B} \\ & & & \tilde{A} \end{pmatrix}$$

$$= \begin{pmatrix} A\tilde{A} & A\tilde{B} & 0 & A\tilde{B} \\ B\tilde{A} & B\tilde{B} + A\tilde{A} & A\tilde{B} & B\tilde{B} + A\tilde{A} \\ 0 & B\tilde{A} & B\tilde{B} + A\tilde{A} & B\tilde{A} + A\tilde{B} \\ B\tilde{A} & B\tilde{B} + A\tilde{A} & A\tilde{B} + B\tilde{A} & 2B\tilde{B} + 2A\tilde{A} \end{pmatrix}.$$

We have $B\tilde{B} + A\tilde{A} = 0$ by induction. The result is now correct up to signs if $A\tilde{B} = -B\tilde{A}$.

We conjugate now this last matrix by the diagonal matrix with diagonal entries s_0, \dots, s_{2^n-1} which we can write using blocks of size 2^n in the form:

$$\begin{pmatrix} D_\alpha & & & \\ & D_\omega & & \\ & & -D_\alpha & \\ & & & D_\omega \end{pmatrix}.$$

We get (after simplification using induction and the assumption $A\tilde{B} = -B\tilde{A}$)

$$\begin{pmatrix} D_\alpha A\tilde{A}D_\alpha & D_\alpha A\tilde{B}D_\omega & 0 & D_\alpha A\tilde{B}D_\omega \\ D_\omega B\tilde{A}D_\alpha & 0 & -D_\omega A\tilde{B}D_\alpha & \\ 0 & -D_\alpha B\tilde{A}D_\omega & & \\ D_\omega B\tilde{A}D_\alpha & & & \end{pmatrix}.$$

Supposing the identity $A\tilde{B} = -B\tilde{A}$ we get

$$-D_\alpha B\tilde{A}D_\omega = D_\alpha A\tilde{B}D_\omega \text{ and } -D_\omega A\tilde{B}D_\alpha = D_\omega B\tilde{A}D_\alpha$$

which have the correct form (one's on the antidiagonal and zeros everywhere else) by induction.

We have yet to prove that $A\tilde{B} = -B\tilde{A}$. Writing $A = \begin{pmatrix} a & \\ b & a \end{pmatrix}$, $B = \begin{pmatrix} & b \\ b & a \end{pmatrix}$ and $\tilde{A} = D_a A^t$, $\tilde{B} = D_a B^t$ we have

$$A\tilde{B} = \begin{pmatrix} a & \\ b & a \end{pmatrix} \begin{pmatrix} \tilde{b} & \\ \tilde{b} & \tilde{a} \end{pmatrix} = \begin{pmatrix} 0 & a\tilde{b} \\ a\tilde{b} & 0 \end{pmatrix}$$

and

$$B\tilde{A} = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{a} \end{pmatrix} = \begin{pmatrix} 0 & b\tilde{a} \\ b\tilde{a} & 0 \end{pmatrix}$$

which proves $A\tilde{B} = -B\tilde{A}$ since $a\tilde{b} = -b\tilde{a}$ by induction. \square

7 Proof of Theorem 1.3

Denote by D_e the diagonal matrix with diagonal coefficients $1, 0, 1, 0, 1, 0, \dots, \overline{1, 0}$, and by $D_o = 1 - D_e$ the diagonal matrix with coefficients $0, 1, 0, 1, 0, 1, \dots, \overline{0, 1}$. We have thus $1 = D_e + D_o$ and $D_a = D_e - D_o$. The main ingredient for proving Theorem 1.3 is the following proposition which is also of independent interest.

Proposition 7.1 *We have*

$$M D_e L = A + D_e, \quad M D_o L = A + D_o$$

where A is the strictly lower triangular matrix with coefficients $a_{i,j} = 1$ if $i > j$ and $a_{i,j} = 0$ otherwise.

Proof. We denote by $\alpha_{i,j} = (M D_e L)_{i,j}$ respectively $\beta_{i,j} = (M D_o L)_{i,j}$ the corresponding coefficients of both products. The proof is by induction on $i + j$. Obviously, $\alpha_{i,j} = \beta_{i,j} = 0$ for $i < j$. The proof for $i = j$ is obvious since M and L are lower triangular unipotent matrices. Consider now for $i > j$

$$\alpha_{i,j} = \sum_k \overline{\binom{i+2k}{4k}} \overline{\binom{4k+1}{2k-j}}.$$

(recall that $m_{i,k} = \overline{\binom{i+k}{2k}} \equiv \binom{i+k}{2k} \pmod{2}$ and $l_{k,j} = \overline{\binom{2k+1}{k-j}} \equiv \binom{2k+1}{k-j} \pmod{2}$).

Since

$$\overline{\binom{2i+2k}{4k}} = \overline{\binom{2i+1+2k}{4k}} \text{ and } \overline{\binom{4k}{2k-2j}} = \overline{\binom{4k+1}{2k-2j}} = \overline{\binom{4k+1}{2k-(2j-1)}}$$

we have

$$\begin{aligned} \alpha_{i,j} &= \sum_k \overline{\binom{\lfloor i/2 \rfloor + k}{2k}} \overline{\binom{2k}{k - \lfloor (j+1)/2 \rfloor}} \\ &= \sum_{k, k \equiv \lfloor (j+1)/2 \rfloor \pmod{2}} \overline{\binom{\lfloor i/2 \rfloor + k}{2k}} \overline{\binom{2k+1}{k - \lfloor (j+1)/2 \rfloor}} \\ &= \begin{cases} \alpha_{\lfloor i/2 \rfloor, \lfloor (j+1)/2 \rfloor} & \text{if } \lfloor (j+1)/2 \rfloor \equiv 0 \pmod{2}, \\ \beta_{\lfloor i/2 \rfloor, \lfloor (j+1)/2 \rfloor} & \text{if } \lfloor (j+1)/2 \rfloor \equiv 1 \pmod{2}. \end{cases} \end{aligned}$$

We have similarly

$$\begin{aligned} \beta_{i,j} &= \sum_k \overline{\binom{i+2k+1}{4k+2}} \overline{\binom{4k+3}{2k+1-j}} \\ &= \sum_k \overline{\binom{2\lfloor (i-1)/2 \rfloor + 1 + 2k+1}{4k+2}} \overline{\binom{4k+3}{2k+1-2\lfloor j/2 \rfloor}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k, k \equiv \lfloor (i-1)/2 \rfloor \pmod{2}} \overline{\binom{\lfloor (i-1)/2 \rfloor + k + 1}{2k+1} \binom{2k+1}{k - \lfloor j/2 \rfloor}} \\
&= \sum_{k, k \equiv \lfloor (i-1)/2 \rfloor \pmod{2}} \overline{\binom{\lfloor (i+1)/2 \rfloor + k}{2k} \binom{2k+1}{k - \lfloor j/2 \rfloor}} \\
&= \begin{cases} \alpha_{\lfloor (i+1)/2 \rfloor, \lfloor j/2 \rfloor} & \text{if } \lfloor (i-1)/2 \rfloor \equiv 0 \pmod{2}, \\ \beta_{\lfloor (i+1)/2 \rfloor, \lfloor j/2 \rfloor} & \text{if } \lfloor (i-1)/2 \rfloor \equiv 1 \pmod{2}. \end{cases}
\end{aligned}$$

except if $\lfloor (i-1)/2 \rfloor = \lfloor j/2 \rfloor = s$ where

$$\sum_{k, k \equiv s \pmod{2}} \overline{\binom{s+k+1}{2k+1} \binom{2k+1}{k-s}} = 1$$

since only $k = s$ yields a non-zero contribution. This ends the proof. \square

Proof of assertion (i) in Theorem 1.3 Using the notations of Proposition 7.1 we have

$$M D_a L = (M D_e L) - (M D_o L) = (A + D_e) - (A + D_o) = D_a$$

which shows that $D_a M D_a = L^{-1}$ and ends the proof. \square

Proof of assertion (ii) in Theorem 1.3 We have

$$m_{i,j} = \overline{\binom{i+j}{2j}} = \overline{\binom{i+j+2^{n+2}}{2j+2^{n+2}}} = m_{i+2^{n+1}, j+2^{n+1}}$$

for $i, j < 2^{n+1}$. This shows (together with induction) that the lower right corner of $M(2^{n+2})$ is given by $\begin{pmatrix} A' & \\ B' & A' \end{pmatrix}$ and has thus the correct form.

In order to prove the recurrence formula for the lower left corner, we remark that B' is by induction the horizontal mirror of A' . We have thus to show that $m_{i+2^{n+1}, j} = m_{2^{n+1}-1-i, j}$ or equivalently that

$$\binom{2^{n+1} + i + j}{2j} \equiv \binom{2^{n+1} - 1 - i + j}{2j} \pmod{2}$$

for $0 \leq i, j < 2^{n+1}$.

Consider

$$\frac{(2^{n+1} + i + j)(2^{n+1} + i + j - 1) \cdots (2^{n+1} + i - j + 1)}{(2j)!}$$

for $0 \leq i, j < 2^{n+1}$. Since all terms of the numerator are $< 2^{n+2}$ we have

$$\begin{aligned}
\binom{2^{n+1} + i + j}{2j} &\equiv \frac{(2^{n+1} + i + j - 2^{n+2}) \cdots (2^{n+1} + i - j + 1 - 2^{n+2})}{(2j)!} \pmod{2} \\
&\equiv \frac{(2^{n+1} - i - j)(2^{n+1} - i - j + 1) \cdots (2^{n+1} - i - j - 1)}{(2j)!} \pmod{2} \\
&\equiv \binom{2^{n+1} - 1 - i + j}{2j} \pmod{2}
\end{aligned}$$

which ends the proof. \square

BA0BAB–proof of assertion (i) in Theorem 1.3 Computing $L(4) D_a M(4)$ (with D_a denoting a diagonal matrix with alternating entries $1, -1, 1, -1, \dots$ of the correct size) we have

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 0 & 1 & 1 \end{pmatrix} = D_a .$$

Writing

$$\begin{pmatrix} \tilde{A}' & & & \\ \tilde{B}' & \tilde{A}' & & \\ \tilde{A}' & \tilde{B}' & \tilde{A}' & \\ \tilde{B}' & 0 & \tilde{B}' & \tilde{A}' \end{pmatrix} = D_a \begin{pmatrix} A' & & & \\ B' & A' & & \\ A' & B' & A' & \\ B' & 0 & B' & A' \end{pmatrix}$$

we have

$$\begin{aligned} & \begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix} \begin{pmatrix} \tilde{A}' & & & \\ \tilde{B}' & \tilde{A}' & & \\ \tilde{A}' & \tilde{B}' & \tilde{A}' & \\ \tilde{B}' & 0 & \tilde{B}' & \tilde{A}' \end{pmatrix} \\ &= \begin{pmatrix} A\tilde{A}' & & & \\ B\tilde{A}' + A\tilde{B}' & A\tilde{A}' & & \\ B\tilde{B}' + A\tilde{A}' & B\tilde{A}' + A\tilde{B}' & A\tilde{A}' & \\ 2B\tilde{A}' + 2A\tilde{B}' & A\tilde{A}' + B\tilde{B}' & B\tilde{A}' + A\tilde{B}' & A\tilde{A}' \end{pmatrix} \\ &= \begin{pmatrix} D_a & & & \\ 0 & D_a & & \\ B\tilde{B}' + A\tilde{A}' & 0 & D_a & \\ 0 & A\tilde{A}' + B\tilde{B}' & 0 & D_a \end{pmatrix} \end{aligned}$$

by induction.

Since we have by induction $A\tilde{A}' = D_a$, it will be enough to show that $B\tilde{B}' = -D_a$. Writing

$$B = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \text{ and } \tilde{B}' = \begin{pmatrix} \tilde{a}' & \tilde{b}' \\ \tilde{b}' & 0 \end{pmatrix}$$

we get

$$B\tilde{B}' = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \begin{pmatrix} \tilde{a}' & \tilde{b}' \\ \tilde{b}' & 0 \end{pmatrix} = \begin{pmatrix} b\tilde{b}' & 0 \\ b\tilde{a}' + a\tilde{b}' & b\tilde{b}' \end{pmatrix} .$$

Since $b\tilde{a}' + a\tilde{b}' = 0$ by the computation above and $b\tilde{b}' = -a\tilde{a}'$ by induction, we get the result. \square

8 ML and LM

As before, we denote by L and M the lower triangular unipotent matrices with coefficients $l_{i,j}, m_{i,j} \in \{0, 1\}$ defined by

$$l_{i,j} \equiv \binom{2i+1}{i-j} \pmod{2}, \quad m_{i,j} \equiv \binom{i+j}{2j} \pmod{2}$$

for $0 \leq i, j$.

Proposition 8.1 *The matrix $R = M L$ has coefficients*

$$r_{i,j} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ 2 & \text{if } i > j \end{cases}$$

Proposition 8.2 *The product LM yields a matrix which can recursively be constructed by iterating*

$$\begin{pmatrix} A & & & \\ B & A & & \\ & & & \\ & & & \end{pmatrix} \mapsto \begin{pmatrix} A & & & \\ B & A & & \\ 2A & B & A & \\ 2B & 2A & B & A \end{pmatrix}$$

starting with $A = 1$ and $B = 2$.

Assertion (i) of Theorem 1.3 implies of course easily the identities

$$(M L)^{-1} = D_a M L D_a \text{ and } (L M)^{-1} = D_a L M D_a .$$

Proof of Proposition 8.1. This follows immediately from Proposition 7.1 since $ML = (M D_e L) + (M D_o L)$. \square

Proof of Proposition 8.2. Using the $BA0BAB$ -construction of L and M we have

$$\begin{aligned} & \begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix} \begin{pmatrix} A' & & & \\ B' & A' & & \\ A' & B' & A' & \\ B' & 0 & B' & A' \end{pmatrix} \\ &= \begin{pmatrix} AA' & & & \\ BA' + AB' & AA' & & \\ BB' + AA' & BA' + AB' & AA' & \\ 2BA' + 2AB' & AA' + BB' & BA' + AB' & AA' \end{pmatrix} \end{aligned}$$

Computing

$$AA' = \begin{pmatrix} a & \\ b & a \end{pmatrix} \begin{pmatrix} a' & \\ b' & a' \end{pmatrix} = \begin{pmatrix} aa' & \\ ba' + ab' & aa' \end{pmatrix}$$

and

$$BB' = \begin{pmatrix} 0 & b \\ b & a \end{pmatrix} \begin{pmatrix} a' & b' \\ b' & 0 \end{pmatrix} = \begin{pmatrix} bb' & \\ ba' + ab' & bb' \end{pmatrix}$$

shows $AA' = BB'$ by induction. This finishes the proof by induction. \square

9 Proof of Theorem 1.4

We prove Theorem 1.4 by showing that the $n \times n$ submatrix formed by the first n rows and columns of the Stieltjes matrix associated with $\sum_{k=0}^{\infty} x^{2^k}$ has determinant s_n for all n .

Proof of Theorem 1.4 Let H and L be as in Theorem 1.2. We denote by L_- the infinite matrix obtained by deleting the first row of L . Denote by C the matrix with coefficients $c_{i,j} = 1$ if $(i-j) \in \{-1, 0\}$ and $c_{i,j} = 0$ otherwise (for $0 \leq i, j$).

Elementary properties of the Catalan triangle show now that $L = L_- C$. This implies that the finite matrix $L_-(n)$ formed by the first n rows and columns of L_- has determinant 1. Denoting by $D_{s_{+1}}(n)$ the $n \times n$ diagonal matrix with entries s_1, \dots, s_n we see that the finite Stieltjes matrix $S(n) = D_s(n) (L(n))^{-1} D_s(n) D_{s_{+1}}(n) L_-(n) D_s(n)$ associated with H has determinant $s_n \in \pm 1$.

Easy computations show that $\det(H(2)) = -1$ and $\det(H(2^k)) = 1$ for $0 \leq k \neq 1$. We have also $\det(H(2^k + a)) = (-1)^a \det(H(2^k - a))$ for $0 \leq a < 2^k$. This shows that $\det(H(n)) = (-1)^{\binom{n}{2}}$ and implies that the Stieltjes matrix associated with H is of the form

$$S(n) = \begin{pmatrix} 1 & 1 & & & & \\ -1 & d_2 & 1 & & & \\ & -1 & d_3 & 1 & & \\ & & -1 & d_4 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We have thus

$$s_n = \det(S(n)) = d_n \det(S(n-1)) + \det(S(n-2)) = d_n s_{n-1} + s_{n-2}$$

for $n > 1$ which shows

$$d_n = \frac{s_n - s_{n-2}}{s_{n-1}} \in \{0, \pm 2\}, \quad n > 1$$

and proves the result. \square

10 The shifted Hankel matrix

We consider the shifted Hankel matrix \tilde{H} with coefficients $\tilde{h}_{i,j} \in \{0, 1\}$, $0 \leq i, j$ given by $\tilde{h}_{i,j} = 1$ if $i + j + 2 = 2^k$ for some natural integer $k \geq 1$ and $\tilde{h}_{i,j} = 0$ otherwise. We describe the LU -decomposition of \tilde{H} . The associated Stieltjes matrix can be recovered from Theorem 1.1 by setting $x_1 = x_2 = x_3 = \dots = x^2$.

Define lower triangular matrices \tilde{L} and \tilde{M} with coefficients $\tilde{l}_{i,j}, \tilde{m}_{i,j} \in \{0, 1\}$ given by $\tilde{l}_{i,j} \equiv \binom{2i+1}{i-j} - \binom{2i+1}{i-j-1} \equiv \binom{2i+2}{i-j} \pmod{2}$ and $\tilde{m}_{i,j} \equiv \binom{i+j+1}{2j+1} \pmod{2}$.

It easy to see that one has recursive formulae

$$\tilde{l}_{2i,2j} = l_{i,j} \equiv \binom{2i+1}{i-j} \pmod{2}, \quad \tilde{l}_{2i+1,2j+1} = \tilde{l}_{i,j}$$

and

$$\tilde{m}_{2i,2j} = m_{i,j} \equiv \binom{i+j}{2j} \pmod{2}, \quad \tilde{m}_{2i+1,2j+1} = \tilde{m}_{i,j}.$$

The products $\tilde{L} \tilde{M}$ and $\tilde{M} \tilde{L}$ satisfy analogous recursive formulae.

In order to state the main result of this section we need also two sign-sequences $\tilde{s}_0, \tilde{s}_1, \dots$ and \tilde{t}_0, \tilde{t}_1 with $\tilde{s}_i, \tilde{t}_i \in \{\pm 1\}$. The sequence \tilde{s}_i is recursively defined by

$$\tilde{s}_{2i} = (-1)^i \text{ and } \tilde{s}_{2i+1} = \tilde{s}_i$$

and is obtained by removing the initial term of the famous paperfolding sequence

$$1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, -1, -1, \dots$$

describing the peaks and valleys in a strip of paper which has iteratedly been folded (with all foldings executed in a similar way), see e.g. [1], [2] or [8].

The sequence \tilde{t}_i is defined by $\tilde{t}_0 = 1$ and

$$\tilde{t}_{2i+1} = \tilde{t}_i, \quad \tilde{t}_{4i} = (-1)^i \tilde{t}_{2i}, \quad \tilde{t}_{4i+2} = \tilde{t}_{2i}.$$

As always, we denote by $D_{\tilde{s}}$, respectively $D_{\tilde{t}}$ the diagonal matrices with diagonal entries $\tilde{s}_0, \tilde{s}_1, \dots$ respectively $\tilde{t}_0, \tilde{t}_1, \dots$.

Theorem 10.1 (i) *We have*

$$D_{\tilde{t}} \tilde{L} D_{\tilde{s}} \tilde{L}^t D_{\tilde{t}} = \tilde{H}.$$

(ii) *We have*

$$\tilde{L} D_{\tilde{s}} \tilde{M} = \tilde{M} D_{\tilde{s}} \tilde{L} = D_{\tilde{s}}.$$

Proof. Let $r_{i,j}$ be the coefficient (i, j) of the matrix product $D_{\tilde{t}} \tilde{L} D_{\tilde{s}} \tilde{L}^t D_{\tilde{t}}$. Since $\tilde{l}_{i,j} = 0$ if $i \not\equiv j \pmod{2}$ we have easily that $r_{i,j} = 0$ if $i \not\equiv j \pmod{2}$. For $r_{2i,2j}$ the computation reduces to assertion (i) of Theorem 1.2 since $\tilde{l}_{2i,2j} = \overline{\binom{4i+2}{2i-2j}} = \overline{\binom{2i+1}{i-j}}$ and $\tilde{t}_{2i} = s_i$.

For $r_{2i+1,2j+1}$ we get

$$\begin{aligned} & \tilde{s}_{2i+1} \tilde{s}_{2j+1} \sum_k \overline{\binom{4i+4}{2i+1-2k-1}} \tilde{t}_{2k+1} \overline{\binom{4j+4}{2j+1-2k-1}} \\ &= \tilde{s}_i \tilde{s}_j \sum_k \overline{\binom{2i+2}{i-k}} \tilde{t}_k \overline{\binom{2j+2}{j-k}} = r_{i,j} \end{aligned}$$

which proves assertion (i) by induction on $i + j$.

In order to prove assertion (ii), we denote by $r_{i,j}$ the coefficient (i, j) of the product $\tilde{L} D_{\tilde{s}} \tilde{M}$. For $i \not\equiv j \pmod{2}$ we have $\tilde{l}_{i,j} = \overline{\binom{2i+2}{i-j}} = \tilde{m}_{i,j} = \overline{\binom{i+j+1}{2j+1}} = 0$. This shows that $r_{i,j} = 0$ if $i \not\equiv j \pmod{2}$. For $r_{2i,2j}$ we have

$$\begin{aligned} & \sum_k \overline{\binom{4i+2}{2i-2k}} \tilde{s}_{2k} \overline{\binom{2k+2j+1}{4j+1}} \\ &= \sum_k \overline{\binom{2i+1}{i-k}} (-1)^k \overline{\binom{k+j}{2j}} = \begin{cases} 0 & \text{if } i \neq j \\ (-1)^i & \text{if } i = j \end{cases} \end{aligned}$$

by Theorem 1.3 which proves assertion (ii) in this case.

For $r_{2i+1,2j+1}$ we get

$$\begin{aligned} & \sum_k \overline{\binom{4i+4}{2i+1-2k-1}} \tilde{s}_{2k+1} \overline{\binom{2k+2j+3}{4j+3}} \\ &= \sum_k \overline{\binom{2i+2}{i-k}} \tilde{s}_k \overline{\binom{k+j+1}{2j+1}} = r_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ \tilde{s}_i = \tilde{s}_{2i+1} & \text{if } i = j \end{cases} \end{aligned}$$

which proves the result by induction on $i + j$. \square

10.1 BA0BAB–constructions for \tilde{L} and \tilde{M}

Denote by \tilde{L}_0 the infinite strictly lower triangular matrix obtained by adding a first row of zeros to \tilde{L} . We define also \tilde{M}_0 by adding a first row of zeros to \tilde{M} .

It can be shown that the matrix \tilde{L}_0 is obtained by iterating

$$\begin{pmatrix} A & & & \\ B & A & & \\ & B & A & \\ & & B & A \end{pmatrix} \mapsto \begin{pmatrix} A & & & \\ B & A & & \\ 0 & B & A & \\ B & A & B & A \end{pmatrix}$$

where one starts with with $\begin{pmatrix} A & & & \\ B & A & & \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Similarly, \tilde{M}_0 is obtained by iterating

$$\begin{pmatrix} A & & & \\ B & A & & \\ & A & B & A \\ & & B & 0 & B & A \end{pmatrix} \mapsto \begin{pmatrix} A & & & & & \\ B & A & & & & \\ A & B & A & & & \\ B & 0 & B & A & & \end{pmatrix}$$

starting with $\begin{pmatrix} A & & & & & \\ B & A & & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

11 Uniqueness

Sequences with generating function

$$\sum_{k=0}^{\infty} \epsilon_k x^{2^k}, \quad \epsilon_i \in \{\pm 1\}$$

are bijectively related to the set of all paperfolding sequences and can be characterized in terms of Hankel matrices as follows:

Proposition 11.1 *Let s_0, s_1, \dots be a sequence with $s_i \in \{\pm 1, 0\}$ such that $\det(H(n)), \det(\tilde{H}(n)) = \pm 1$ for all n where $H(n)$ and $\tilde{H}(n)$ have coefficients $h_{i,j} = s_{i+j}$ and $\tilde{h}_{i,j} = s_{i+j+1}$ for $0 \leq i, j < n$. Then $\sum_{k=0}^{\infty} s_k x^{k+1} = \sum_{k=0}^{\infty} \epsilon_k x^{2^k}$ for a suitable choice $\epsilon_0, \epsilon_1, \dots \in \{\pm 1\}$.*

Proof. Computing $\det(H(n)) \pmod{2}$ determines $s_{2n-2} \pmod{2}$ and computing $\det(\tilde{H}(n)) \pmod{2}$ determines $s_{2n-1} \pmod{2}$. This shows that $\sum_{k=0}^{\infty} s_k x^{k+1} \equiv \sum_{k=0}^{\infty} x^{2^k} \pmod{2}$ and the signs can of course be chosen arbitrarily. \square

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