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# A large deviation principle for the Yang-Mills measure

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## Abstract

We prove the first mathematical result relating the Yang-Mills measure on a compact surface and the Yang-Mills energy. We show that, at the small volume limit, the scaled Yang-Mills measures satisfy a large deviation principle with the Yang-Mills energy as rate function. This gives some rigorous content to the informal description of the Yang-Mills measure as the Gibbs measure of the Yang-Mills energy.

We make the assumption that the structure group is simply connected and plan to treat the general case in a forthcoming publication.

## Introduction

The Yang-Mills measure is a probability measure on the space of connections on a principal bundle over some manifold. Heuristically, it is the Gibbs measure corresponding to the Yang-Mills energy on the space of connections. In the case of a compact two-dimensional base manifold and compact structure group, this measure has been constructed by several authors [3, 14, 6, 9], using different techniques. However, in every construction so far, the expression of the Yang-Mills energy has been used only at an informal level, as a guide, but not incorporated as such in the construction. This is not more surprising than the fact that one can and usually does construct Brownian motion without referring to the Sobolev  $H^1$  norm.

So the situation now is the following : we have three constructions of essentially the same object, which are validated on the physical level by the fact that some characterizing properties of this object, namely some finite-dimensional distributions, are what physicists expect them to be - of course, physicists themselves derive their predictions from heuristical manipulations on the Yang-Mills energy.

The point of this paper is to relate at a mathematical level the Yang-Mills measure and the Yang-Mills energy. The result itself should surprise no one but it gives some rigorous content to the heuristic expression of the Yang-Mills measure. Hopefully, the proofs illustrate how it is possible to work with the Yang-Mills measure as it has been constructed in [9], using some of the tools developed there.

Our investigations of the large deviations of the Yang-Mills measure led naturally to the study of a space of  $H^1$  connections, which is a necessary completion of the space of smooth

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connections in this context. In a series of appendices, we show that the usual objects associated with a connection behave well in the  $H^1$  case, we give a simple proof of Fine's energy inequality [5], which is also extended to the  $H^1$  case. We also study the problem of minimizing the Yang-Mills action subject to constraints – imposed on the holonomies associated to an embedded graph. We construct explicit minimizers which are  $H^1$ , indeed Lipschitz, but are not typically smooth.

Let us give a very informal description of the result. The Yang-Mills measure can be expressed as

$$dP_T(\omega) = \frac{1}{Z} e^{-\frac{1}{2T}S(\omega)} \mathcal{D}\omega,$$

where  $S$  is the Yang-Mills energy and  $T$  a positive parameter. The large deviation result for the family of measures  $(P_T)_{T>0}$  says roughly that, if  $A$  is a nice set of connections, then

$$T \log P_T(A) \xrightarrow{T \rightarrow 0} -\frac{1}{2} \inf_{\omega \in A} S(\omega).$$

In this paper, we restrict our attention to the case of a simply connected structure group. The general compact case will be treated separately. This extension should involve some more geometry but no more analysis than the present case.

## 1 The large deviation principle

### 1.1 The Yang-Mills measure

Let  $M$  be a compact oriented<sup>1</sup> surface without boundary. Let  $G$  be a compact connected, simply connected, Lie group. Any principal bundle on  $M$  with simply connected structure group is trivial, so let us consider  $P = M \times G$ . Endow  $M$  with a surface measure  $\sigma$ , that is, a measure which is equivalent to Lebesgue measure in every coordinate chart.

By a parametrized embedded path on  $M$  we mean an embedding of  $[0, 1]$  into  $M$  which is, near each of its endpoints, in some chart, an interval of a coordinate axis. A parametrized path is by definition a finite concatenation of parametrized embedded paths. A path is defined as an equivalence class of embedded paths modulo reparametrization by increasing piecewise diffeomorphisms of  $[0, 1]$  with bounded derivatives. We denote by  $PM$  the class of paths on  $M$ .

The topology on  $PM$  is defined as follows. Put a Riemannian metric on  $M$ . Define the distance between two paths  $c$  and  $c'$  as

$$d_\ell(c, c') = d_\infty(c, c') + |\ell(c') - \ell(c)|,$$

where  $\ell$  denotes the length and  $d_\infty$  is the infimum over all parametrizations of the usual uniform distance. Different metrics give rise to equivalent distances, so there is an intrinsic topology of the convergence in length. We denote by  $\xrightarrow{\ell}$  the corresponding convergence.

We will sometimes use a stronger notion of convergence, saying that  $c_n \rightarrow c$  with fixed endpoints if  $c_n \xrightarrow{\ell} c$  and  $c_n$  and  $c$  share the same starting points and the same finishing points.

For any subset  $J \subset PM$ , let  $\mathcal{M}(J, G)$  be the set of multiplicative  $G$ -valued functions on  $J$ . By multiplicative, we mean the following : whenever two paths  $c_1$  and  $c_2$  of  $J$  can be concatenated

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<sup>1</sup>In this paper, we assume that an orientation of  $M$  has been chosen once for all.

to form  $c_1c_2 \in J$ , one has for every  $f \in \mathcal{M}(J, G)$  the relation  $f(c_1c_2) = f(c_2)f(c_1)$ . In other words,  $f$  is a morphism of groupoids, with multiplication reversed on  $G$ . The group  $\mathcal{F}(M, G)$  of all  $G$ -valued functions on  $M$  acts on  $\mathcal{M}(J, G)$  according to the following rule, which mimics the action of a gauge transformation on the holonomy induced by a connection : if  $j \in \mathcal{F}(M, G)$ ,  $f \in \mathcal{M}(J, G)$  and  $c \in J$  with endpoints  $c(0)$  and  $c(1)$ , then

$$(j \cdot f)(c) = j(c(1))^{-1}f(c)j(c(0)). \quad (1)$$

Endow  $\mathcal{M}(PM, G)$  with the cylinder  $\sigma$ -field  $\mathcal{C}$ , that generated by the canonical process  $(H_c : c \in PM)$ , defined by  $H_c(f) = f(c)$  for every  $f \in \mathcal{M}(PM, G)$ . The proper  $\sigma$ -field to work with would be that consisting of those events in  $\mathcal{C}$  that are invariant under the action of  $\mathcal{F}(M, G)$ . However, the canonical process is not measurable with respect to this smaller  $\sigma$ -field and we prefer to stick to a slightly redundant setting. So, in this paper, the Yang-Mills measure will be a probability measure on  $(\mathcal{M}(PM, G), \mathcal{C})$ .

In order to describe Yang-Mills measure, we need to explain quickly its discretization. A more detailed account of what follows can be found in [9], sections 1.1 to 1.4.

By a graph on  $M$  we mean a finite set of injective embedded paths on  $M$  which intersect, if at all, only at their endpoints. These endpoints are called vertices of the graph. The faces of a graph are the connected components of the space obtained from  $M$  by removing all the edges. Finally, a path in a graph is any path that can be constructed by concatenation of some edges of this graph and their reversals. We assume that the faces of a graph are diffeomorphic to disks. Actually, we will also assume that the boundary of each face is a simple curve in  $M$ , that is, a simple loop. In particular, this prevents a face from lying on both sides of its boundary. Finally, we assume that, at each vertex of the graph, all edges start in different directions. In other words, the boundary of a face can have corners but no cusp.

Let  $\Gamma = \{e_1, \dots, e_r\}$  be a graph. We associate to  $\Gamma$  the configuration space  $G^\Gamma \simeq G^r$  of discrete connections on  $\Gamma$ . To each path  $c$  in  $\Gamma$  corresponds a mapping  $h_c : G^r \rightarrow G$  defined as follows : if  $c = e_{i_1}^{\epsilon_1} \dots e_{i_m}^{\epsilon_m}$ , where  $\epsilon_1, \dots, \epsilon_m = \pm 1$ , then  $h_c(g_1, \dots, g_r) = g_{i_m}^{\epsilon_m} \dots g_{i_1}^{\epsilon_1}$ . This mapping is the discrete holonomy along  $c$ .

Consider on  $G$  the bi-invariant Riemannian metric, normalized to have total volume equal to 1. Thus, the Riemannian volume and the Haar measure coincide on  $G$ . We denote them by  $dg$ . Let  $p$  be the heat kernel of  $G$ , that is, the function  $p : (0, \infty) \times G \rightarrow (0, \infty)$ ,  $(t, g) \mapsto p_t(g)$ , solution to the equation

$$\left( \frac{\Delta}{2} - \frac{\partial}{\partial t} \right) p = 0$$

and subject to the initial condition

$$\int_G f(g)p_t(g) dg \xrightarrow{t \rightarrow 0} f(1)$$

for every function  $f$  continuous on  $G$ , where 1 is the unit of  $G$ .

For each face  $F$  of  $\Gamma$ , the boundary of  $F$  is an ill-defined path  $\partial F$  in  $\Gamma$ , because it depends on the choice of an origin<sup>2</sup>. However, the function  $p_{\sigma(F)}(h_{\partial F})$  is well-defined because of the nice invariance property  $p(xy) = p(yx)$ .

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<sup>2</sup>The orientation of  $\partial F$  is determined by that of  $M$ .

Fix a positive real number  $T$ . The discrete Yang-Mills measure  $P_T^\Gamma$  is the Borel probability measure on  $G^r$  defined by

$$dP_T^\Gamma = \frac{1}{Z} \prod_{F \text{ face}} p_{T\sigma(F)}(h_{\partial F}) dg^{\otimes r}, \quad (2)$$

where the product runs over all faces of  $\Gamma$  and  $Z$  is the normalization constant.

The measure  $P_T^\Gamma$  is gauge-invariant, in the sense that it is invariant under the action of  $\mathcal{F}(M, G)$  on  $\mathcal{M}(\Gamma, G) = G^\Gamma$ .

Finally, let  $d$  denote the Riemannian distance on  $G$ . We say that a sequence  $(H_n)_{n \geq 1}$  of  $G$ -valued random variables converges in probability to  $H$  if, for all  $\epsilon > 0$ ,  $\mathbb{P}(d(H_n, H) > \epsilon) \xrightarrow{n \rightarrow \infty} 0$ .

The Yang-Mills measure is defined by the following result ([9], theorem 2.72).

**Theorem 1.1** *Let  $T > 0$  be a positive real number. There exists a unique probability measure  $P_T$  on  $(\mathcal{M}(PM, G), \mathcal{C})$  under which the canonical process  $(H_c)_{c \in PM}$  satisfies the two following properties :*

1. *For all graphs  $\Gamma = \{e_1, \dots, e_r\}$  on  $M$ ,  $(H_{e_1}, \dots, H_{e_r})$  has the law  $P_T^\Gamma$  under  $P_T$ .*
2. *For all sequences  $(c_n)_{n \geq 1}$  of paths converging in length to a path  $c$  with fixed endpoints, one has  $H_{c_n} \xrightarrow[n \rightarrow \infty]{P} H_c$ .*

The measure  $P_T$  is invariant under the action of  $\mathcal{F}(M, G)$  on  $\mathcal{M}(PM, G)$ .

## 1.2 The Yang-Mills energy

Let  $\mathfrak{g}$  denote the Lie algebra of  $G$ , with bracket  $[\cdot, \cdot]$ . It is endowed with the scalar product corresponding to the bi-invariant metric on  $G$  with total volume equal to 1. A connection on  $P = M \times G$  is a  $\mathfrak{g}$ -valued 1-form on  $M$  with some regularity. For  $k$  integer or  $k = \infty$ ,  $p \in [1, +\infty]$ , and  $s \in \mathbb{R}$ , let  $C^k \mathcal{A}$ ,  $L^p \mathcal{A}$ ,  $W^{k,p} \mathcal{A}$  and  $H^s \mathcal{A}$  denote respectively the spaces of connections of class  $C^k$ ,  $L^p$ ,  $W^{k,p}$  and  $H^s$ , where  $H^s$  stands for the Sobolev  $W^{s,2}$  space. Similarly, a gauge transformation on  $P$  is a  $G$ -valued function on  $M$  and we define the spaces  $C^k \mathcal{J}$ ,  $L^p \mathcal{J} \dots$  of gauge transformations of different regularities.

These spaces are intrinsically defined on  $M$ , but it requires some structure to fix their norms. We fix once for all a Riemannian metric on  $M$  whose Riemannian volume is equal to  $\sigma$ . This will be useful for other purposes than just fixing norms, but one should keep in mind that all our results are independent of this choice.

If  $\omega$  is a smooth connection, its curvature is a smooth  $\mathfrak{g}$ -valued 2-form  $\Omega$ . Since an orientation is chosen on  $M$ , the surface measure  $\sigma$  can be identified with a smooth 2-form, that we still denote by  $\sigma$ . Then,  $\Omega$  can be expressed uniquely as  $\Omega = \tilde{\Omega} \sigma$ , where  $\tilde{\Omega}$  is a  $\mathfrak{g}$ -valued function on  $M$ . Then the Yang-Mills energy is defined as

$$S(\omega) = \int_M \|\tilde{\Omega}\|^2 d\sigma.$$

Locally, this can be written as follows. If, in some coordinates  $(x, y)$  on a domain  $U$  of  $M$ ,  $\omega = \omega_x dx + \omega_y dy$ , then  $\Omega = (\partial_x \omega_y - \partial_y \omega_x + [\omega_x, \omega_y]) dx \wedge dy$  and the corresponding contribution to  $S(\omega)$  is

$$S_U(\omega) = \int_U \|\partial_x \omega_y - \partial_y \omega_x + [\omega_x, \omega_y]\|^2 \left( \frac{dx dy}{d\sigma} \right)^2 d\sigma, \quad (3)$$

where  $\frac{d\sigma}{dx dy}$  is the Radon-Nikodym derivative of  $\sigma$  with respect to  $dx dy$ . It is clear from this expression that, if  $\sigma$  is multiplied by a positive real number  $T$ , then the Yang-Mills energy is multiplied by  $T^{-1}$ . Thus, the large deviation principle is related to the behaviour of the Yang-Mills measure when the area of  $M$  tends to zero.

It is also clear from (3) that the energy of a connection is controlled by the  $L^2$  norm of its derivatives and its  $L^4$  norm. It happens that both are controlled by the  $H^1$  norm. Actually, in two dimensions, one has a continuous embedding  $H^1 \subset L^p$  for each  $p \in [1, \infty)$ . Thus, the Yang-Mills energy is well-defined and continuous on  $H^1\mathcal{A}$ .

In order to define the rate function for the large deviation principle, we need to study the holonomy induced by Sobolev connections. For the moment, let us only state the main results. The proofs are given in the appendices.

### 1.3 Holonomy of Sobolev connections

Recall that the set  $PM$  is endowed with the topology of convergence in length.

**Proposition 1.2** *Let  $\omega$  be a connection in  $H^1\mathcal{A}$ . Then  $\omega$  determines an element of  $\mathcal{M}(PM, G)$ . Let  $f$  denote this element. For each  $c \in PM$ ,  $f(c)$  is the final value  $a_1$  of the solution to the differential equation*

$$\dot{a}_t a_t^{-1} = -\omega(\dot{c}(t)), \quad a_0 = 1, \quad (4)$$

where an arbitrary parametrization of  $c$  by the interval  $[0, 1]$  has been chosen. (Prop. A.1)

The function  $f : PM \rightarrow G$  is continuous with fixed endpoints : whenever  $(c_n)_n$  converges to  $c$  in length with fixed endpoints, one has  $f(c_n) \rightarrow f(c)$ . (Prop. A.4)

Moreover,  $f$  is of class  $H^1$  along every path. This means the following. For every  $c \in PM$  with a fixed parametrization and every  $t \in [0, 1]$ , call  $c^t$  the restriction of  $c$  to the interval  $[0, t]$ . Then, the function  $t \mapsto f(c^t)$  belongs to  $H^1([0, 1]; G)$ . (Prop. A.1)

Finally, given  $c \in PM$ , the holonomy of  $\omega \in H^1\mathcal{A}$  along  $c$  depends continuously on  $\omega$ . Moreover, it is sequentially weakly continuous with respect to  $\omega$  : if  $\omega_n \rightharpoonup \omega$  in  $H^1\mathcal{A}$ , then the holonomy of  $\omega_n$  along  $c$  converges to that of  $\omega$  along  $c$ . (Prop. A.2)

The following energy inequality proved by D. Fine [5] in the smooth case is still true for  $H^1$  connections and it will be very useful to us. Let  $\rho$  denote the function on  $G$  defined by  $\rho(x) = d(1, x)$ , where 1 is the unit of  $G$ .

**Proposition 1.3** *Let  $l$  be a simple loop on  $M$  which bounds a domain  $D$  diffeomorphic to a disk. Let  $\omega$  be an element of  $H^1\mathcal{A}$ . Let  $x$  be the holonomy of  $\omega$  along  $l$ . Then one has the inequality*

$$\rho(x)^2 \leq \sigma(D)S_D(\omega),$$

where  $S_D$  denotes the contribution of the domain  $D$  to the Yang-Mills energy of  $\omega$ . (Prop. A.3)

This result is so fundamental in this paper that we give a complete proof of it in the smooth case, in Appendix C. The extension to the  $H^1$  case is easy and it is proved as Proposition A.3.

Gauge transformations of  $H^2\mathcal{J}$  act on  $H^1\mathcal{A}$  (see [16]), according to the rule

$$j \cdot \omega = j^{-1}\omega j - j^{-1}dj \quad (5)$$

and transform the holonomy according to the usual rule (see equation (1)). This makes sense, because  $H^2\mathcal{J} \subset C^0\mathcal{J}$ , so that  $H^2$  functions on  $M$  are defined pointwise.

**Proposition 1.4** *Let  $\omega$  and  $\omega'$  be two elements of  $H^1\mathcal{A}$ . Let  $f, f' \in \mathcal{M}(PM, G)$  be their respective holonomies. Assume that there exists  $j \in \mathcal{F}(M, G)$  such that  $j \cdot f = f'$ . Then  $j$  can be chosen in  $H^2(M; G)$  and such that  $j \cdot \omega = \omega'$ . In particular,  $\omega$  and  $\omega'$  have the same energy. (Prop. B.1)*

According to the last proposition, the mapping  $H^1\mathcal{A}/H^2\mathcal{J} \rightarrow \mathcal{M}(PM, G)/\mathcal{F}(M, G)$  is one-to-one. Let us say that a connection  $\omega$  and a function  $f : PM \rightarrow G$  are equivalent, and write  $\omega \sim f$ , if the class of  $\omega$  is sent on the class of  $f$  by this mapping. Observe that, if  $f \sim \omega$ , then  $S(\omega)$  depends only on  $f$ . It is now possible to define our rate function on  $\mathcal{M}(PM, G)$ .

**Definition 1.5** *Let  $I_{YM}$  be the function on  $\mathcal{M}(PM, G)$  defined by*

$$I_{YM}(f) = \begin{cases} \frac{1}{2}S(\omega) & \text{if there exists } \omega \in H^1\mathcal{A} \text{ s.t. } f \sim \omega, \\ +\infty & \text{otherwise.} \end{cases}$$

#### 1.4 Statement of the main result and idea of the proof

**Theorem 1.6** *The family of measures  $(P_T)_{T>0}$  on  $(\mathcal{M}(PM, G), \mathcal{C})$  satisfies a large deviation principle with good rate function  $I_{YM}$  as  $T$  tends to 0.*

Let us explain what this theorem means. The space  $\mathcal{M}(PM, G)$  is a compact topological space, as a closed subset of  $G^{PM}$ . It is also endowed with the  $\sigma$ -field  $\mathcal{C}^3$ . We are given a family of probability measures  $(P_T)_{T>0}$  on  $(\mathcal{M}(PM, G), \mathcal{C})$  and a function  $I_{YM} : \mathcal{M}(PM, G) \rightarrow [0, \infty)$ .

The theorem asserts first that  $I$  is a lower semi-continuous function with compact level sets, that is, for every  $\alpha \geq 0$ , the set  $\{f \in \mathcal{M}(PM, G) : I_{YM}(f) \leq \alpha\}$  is a compact subset of  $\mathcal{M}(PM, G)$ . Then, it asserts that, for every measurable subset  $A \in \mathcal{C}$ , with interior  $A^\circ$  and closure  $\bar{A}$ , one has the following equalities :

$$-\inf_{f \in A^\circ} I_{YM}(f) = \varliminf_{T \rightarrow 0} T \log P_T(A) \leq \overline{\varliminf}_{T \rightarrow 0} T \log P_T(A) = -\inf_{f \in \bar{A}} I_{YM}(f).$$

The reader unfamiliar with large deviations results may find [2] a very useful reference.

In order to prove this theorem, we are going to follow a route close to the one followed to construct the measure in [9]. The starting point is the classical short-time estimate of the heat kernel on a compact Riemannian manifold. An LDP for the discrete Yang-Mills measure follows. Just as in the construction of the measure, it is not enough to take the projective limit of these discrete LDP's : we must first obtain an LDP for the finite dimensional marginals of the holonomy process and this involves approximating arbitrary families of paths by families staying in graphs. Exponential approximation results allow us to do this. Then, Dawson-Gärtner's theorem produces the LDP for the whole process.

However, the existence of an LDP at each step of this procedure is not enough. We must identify the rate function in terms of the Yang-Mills action. This involves for example constructing a connection with minimal action for a given holonomy along the edges of a graph, or giving an expression of the rate function in terms of holonomy. We will make extensive use of the results of K. Uhlenbeck [16], the most important of which we restate, for the convenience of the reader, in the form that we actually use.

<sup>3</sup>Observe that  $\mathcal{C}$  is smaller than the Borel  $\sigma$ -field of  $\mathcal{M}(PM, G)$ , because  $PM$  is uncountable.

**Theorem 1.7 (K. Uhlenbeck [16])** *Let  $(\omega_n)_{n \geq 1}$  be a sequence in  $H^1 \mathcal{A}$  such that  $S(\omega_n)$  is bounded. Then there exists a subsequence  $(\omega_{n_k})_{k \geq 1}$ , a sequence  $(j_k)_{k \geq 1}$  in  $H^2 \mathcal{J}$  and an element  $\omega$  of  $H^1 \mathcal{A}$  such that*

1.  $j_k \cdot \omega_{n_k} \rightharpoonup \omega$  in  $H^1 \mathcal{A}$ ,
2.  $S(\omega) \leq \underline{\lim}_{k \rightarrow \infty} S(\omega_{n_k})$ .

This theorem is very powerful. It says that controlling the Yang-Mills energy is almost the same as controlling the  $H^1$  norm. This is a serious hint at the importance of the space  $H^1 \mathcal{A}$  in Yang-Mills theory.

## 2 Large deviations for finite-dimensional marginals

### 2.1 LD for the discrete Yang-Mills measure

The fundamental estimate is the following. The prototype of this result was proved by Varadhan [18, 17]. For the form given here the reader may consult [12] or [10].

**Theorem 2.1** *Let  $p_t(\cdot, \cdot)$  be the heat kernel on a compact Riemannian manifold. Then, uniformly for all  $x, y \in M$ , one has*

$$\lim_{t \rightarrow 0} -2t \log p_t(x, y) = d(x, y)^2.$$

From this theorem, we deduce directly the LDP for the discrete Yang-Mills measure on a graph.

**Proposition 2.2** *Let  $\Gamma$  be a graph on  $M$ . The family of measures  $(P_T^\Gamma)_{T > 0}$  on  $G^\Gamma$  satisfies an LDP with rate function*

$$I_\Gamma(g) = \frac{1}{2} \sum_{F \text{ face}} \frac{\rho(h_{\partial F}(g))^2}{\sigma(F)}.$$

*Proof* – By definition of  $P_T^\Gamma$  (see (2)) and Theorem 2.1, one has, as  $T \rightarrow 0$ ,

$$dP_T^\Gamma(g) = \frac{1}{Z_T^\Gamma} e^{-\frac{1}{T}(I_\Gamma(g) + o(1))} dg,$$

where  $o(1)$  is uniform on  $G^\Gamma$ .

The constant  $Z_T^\Gamma$  does not depend on  $\Gamma$  and it is explicitly given by

$$Z_T^\Gamma = \int_{G^{2q}} p_T([x_1, y_1] \dots [x_q, y_q]) dx_1 dy_1 \dots dx_q dy_q,$$

where  $q$  is the genus of  $M$  and  $[x, y] = xyx^{-1}y^{-1}$  (see [9], section 5.4.1). In particular,  $Z_T^\Gamma$  is less than or equal to  $\|p_T\|_\infty = O(T^{-\frac{\dim G}{2}})$ , the last estimate coming for example from [11]. Thus,  $T \log Z_T^\Gamma \rightarrow 0$  as  $T \rightarrow 0$ . The LDP follows easily.  $\square$



We need to identify this rate function in terms of the Yang-Mills energy. It is known, by the work of A. Sengupta and D. Fine [15, 5], that a connection on a disk of area  $T$  with holonomy  $x$  around the boundary has an energy at least equal to  $\rho(x)^2/T$ . Hence, given  $g$  in  $G^\Gamma$ , any connection which has  $g$  as its holonomy along the edges of  $\Gamma$  must have an energy at least equal to  $I_\Gamma(g)$ . We show that this is the actual infimum and that it is attained by some  $H^1$  connection.

Before that, let us define a relation between connections and functions on subsets of  $PM$ . Let  $J$  be any subset of  $PM$ , for example, the set of edges of a graph, or a set of loops. According to Proposition 1.2, any connection in  $H^1\mathcal{A}$  determines an element of  $\mathcal{M}(J, G)$ . Recall that the group  $\mathcal{F}(M, G)$  acts on  $\mathcal{M}(J, G)$ . Similarly, if  $K$  is a subset of  $PM$  containing  $J$ , any function of  $\mathcal{M}(K, G)$  determines by restriction a function of  $\mathcal{M}(J, G)$ . We can even do better. For every subset  $K$  of  $PM$ , let  $K^*$  the sub-groupoid of  $PM$  generated by  $K$ , that is, the set of paths that can be constructed by concatenating paths of  $K$ . There is a natural one-to-one correspondence between  $\mathcal{M}(K, G)$  and  $\mathcal{M}(K^*, G)$ . Hence, if  $K$  is a subset of  $PM$  such that  $J \subset K^*$ , then any function of  $\mathcal{M}(K, G)$  determines a function of  $\mathcal{M}(J, G)$ . The main example of this situation is the following :  $K$  is the set of edges of a graph and  $J$  is a set of paths in this graph.

**Definition 2.3** *Two connections of  $H^1\mathcal{A}$ , or two multiplicative  $G$ -valued functions defined on subsets of  $PM$  generating sub-groupoids containing  $J$ , or one such connection and one such function are said to agree up to gauge transformation on  $J$ , or simply to agree on  $J$ , if they induce the same class of  $\mathcal{M}(J, G)/\mathcal{F}(M, G)$ . We denote this relation by  $\sim_J$ .*

Observe that the relation  $\sim_{PM}$  is just the relation  $\sim$  that we have used in the definition of the rate function  $I_{YM}$  (Definition 1.5).

**Proposition 2.4** *Let  $\Gamma$  be a graph on  $M$ . Let  $g$  be an element of  $G^\Gamma$ . Then*

$$I_\Gamma(g) = \frac{1}{2} \inf\{S(\omega) : \omega \in H^1\mathcal{A}, \omega \sim_\Gamma g\}.$$

*Moreover, this infimum is attained by an element of  $H^1\mathcal{A}$ .*

This result will be our main tool, when combined with Uhlenbeck's compactness theorem, to identify the rate functions we will encounter in terms of the Yang-Mills energy.

In few words, one can describe a minimizer as follows : its curvature is given by  $\tilde{\Omega} = j^{-1}Xj$ , where  $j$  is a well-chosen gauge transformation and  $X$  a  $\mathfrak{g}$ -valued function on  $M$  constant on each face of  $\Gamma$ . The value of  $X$  on a face  $F$  is an element of minimal norm in  $\mathfrak{g}$  such that  $\exp(\sigma(F)X) = h_{\partial F}(g)$ , which makes sense once an origin has been chosen as starting point for the path  $\partial F$ .

Let us review quickly the main steps of the proof before actually getting into it. Let us pick a graph  $\Gamma$  and an element  $g$  in  $G^\Gamma$ . Fine's energy inequality (Prop. 1.3) implies immediately that  $2I_\Gamma(g) \leq \inf_{\omega \sim_\Gamma g} S(\omega)$ . In order to prove the other inequality, we show that there exists an  $H^1$  connection with holonomy  $g$  and energy  $2I_\Gamma(g)$ . For this, we construct first the minimizer on each face, because critical points of the energy on a disk with fixed holonomy along the boundary are easy to describe : they have constant curvature, up to gauge transformation. Then, we glue the faces together. To do this, we make an appropriate gauge transformation in each face so as to make every connection coincide on the boundary of the disk where it is defined with a given smooth connection which has the right holonomy. We show that the gauge transformation can be chosen in  $H^2\mathcal{J}$  on each face and this implies that the resulting connection belongs to  $H^1\mathcal{A}$ .

Notice that the fact that the infimum of the energy is attained is a consequence of Uhlenbeck's theorem. However, we don't use this in the proof.

**Lemma 2.5** *There exists a connection  $\eta$  in  $C^\infty\mathcal{A}$  such that  $\eta \sim_\Gamma g$  and  $\eta$  vanishes in the neighbourhood of every vertex of  $\Gamma$ .*

*Proof* – Let  $x$  be an element of  $G$  and  $X \in \mathfrak{g}$  be such that  $\exp X = x$ . Take  $\epsilon > 0$  and consider the tube  $[0, 1] \times ] - \epsilon, \epsilon[$ . Let  $u : [0, 1] \rightarrow [0, 1]$  be a smooth function such that  $u(x) = 0$  for  $x \leq 1/3$  and  $u(x) = 1$  for  $x \geq 2/3$ . Let  $\psi : ] - \epsilon, \epsilon[ \rightarrow [0, 1]$  be such that  $\psi(y) = 0$  for  $|y| \geq 2\epsilon/3$  and  $\psi(y) = 1$  for  $|y| \leq \epsilon/3$ . Then the connection  $\omega(x, y) = X\psi(y)du(x)$  has compact support in the interior of the tube and holonomy  $x$  along the segment  $[0, 1] \times \{0\}$ .

Let us consider now the graph  $\Gamma$ . Replace each edge by its restriction to a proper subinterval containing neither of its endpoints. The resulting paths on  $M$  are disjoint and admit disjoint tubular neighbourhoods. In each tube, the preceding construction of a  $\mathfrak{g}$ -valued 1-form with the appropriate holonomy can be done and the resulting locally defined forms can be smoothly interpolated by 0. The result is a connection with the desired properties.  $\square$

In the following statement, we focus on a single face  $F$  of  $\Gamma$ . We take this face to be a closed subset of  $M$  and denote its interior by  $U$ . Let us choose a vertex  $o(F)$  on the boundary of  $F$  and let  $\partial F$  be the loop based at  $o(F)$  going once around  $F$  in the positive direction. Let  $x$  denote  $h_{\partial F}(g)$ . Finally, let  $\mathcal{A}(D)$  denote the set of connections on a domain  $D$  contained in  $M$  and  $S_D$  the Yang-Mills energy on this domain. The following lemma is the main technical tool for the proof of Proposition 2.4.

**Lemma 2.6** *Let  $\eta$  be as in Lemma 2.5. There exists on  $F$  a connection  $\omega$  such that*

1.  $\omega \in H^1\mathcal{A}(U)$ .
2.  $S_F(\omega) = \rho(x)^2/\sigma(F)$ .
3.  $\omega = \eta$  on  $\partial F$ , in all directions.

*Proof* – For the sake of clarity, we divide this proof in several shorter steps.

• Step 1 : *there exists a connection  $\omega_1$  in  $C^\infty\mathcal{A}(F)$  with holonomy  $x$  along  $\partial F$ , and with energy  $\rho(x)^2/\sigma(F)$ .*

Take  $X$  in  $\mathfrak{g}$  of minimal norm such that  $\exp(X) = x$ . We have  $\|X\| = \rho(x)$ . Let  $\lambda$  be a 1-form on a neighbourhood of  $F$  such that  $d\lambda = \sigma$ . Then take  $\omega_1 = \frac{1}{\sigma(F)}X\lambda$ .

• Step 2 : *there exists a gauge transformation  $i$  smooth in a neighbourhood of  $F$  such that  $i \cdot \omega_1$  vanishes along  $\partial F$  in a neighbourhood of every vertex of  $F$ .*

Let  $v$  be a vertex of  $F$ . In a neighbourhood of  $v$ , there exists a system of polar coordinates such that the two outgoing edges are distinct radii. Let  $R > 0$  be such that the geodesic ball  $B(v, R)$  is contained in this neighbourhood. The radial gauge for  $\omega_1$  at  $v$  in  $B(v, R)$  is the gauge transformation  $i_v$  such that  $i_v(v) = 1$  and  $i_v \cdot \omega_1$  vanishes in radial directions in  $B(v, R)$ . Hence, it is defined as follows :  $i_v(r, \theta) = \text{hol}(\omega_1, \zeta_{r, \theta})$ , where  $\zeta_{r, \theta}$  denotes the segment joining  $v$  to the point of polar coordinates  $(r, \theta)$ . It is a smooth gauge transformation in  $B(v, R)$ . Let  $\kappa_v$  be the smooth logarithm of  $i_v$  such that  $\kappa_v(v) = 0$ . By multiplying  $\kappa_v$  by a smooth function on  $B(v, R)$  identically equal to 1 in  $B(v, R/3)$  and to 0 outside  $B(v, 2R/3)$ , we get the logarithm of a new smooth transformation  $\tilde{i}_v$ . It has the property that  $\tilde{i}_v \cdot \omega_1$  vanishes along  $\partial F$  in a neighbourhood of  $v$  and it is equal to 1 outside  $B(v, 2R/3)$ .

If we perform this construction around each vertex of  $F$ , in neighbourhoods small enough to contain only one vertex each, we may construct a gauge transformation  $i$  equal to  $i_v$  in a neighbourhood of  $v$  for each vertex  $v$  and equal to 1 outside these neighbourhoods. It is smooth and has the desired property.

Set  $\omega_2 = i \cdot \omega_1$ .

• Step 3 : *there exists a gauge transformation  $j$  smooth in a neighbourhood of  $F$  such that  $j \cdot \omega_2$  satisfies the properties 1 and 2 and such that  $j \cdot \omega_2$  and  $\eta$  coincide along  $\partial F$  in the tangential direction.*

Let  $c : [0, 1] \rightarrow M$  be a parametrization of  $\partial F$ . In particular,  $c(0) = o(F)$ . For each  $t \in [0, 1]$ , denote  $c^t$  the path obtained by restricting  $c$  to  $[0, t]$ . We define  $j$  on  $\partial F$  first, by defining  $j(c(t))$  for each  $t$ .

Choose  $j(c(0))$  such that  $j(c(0))^{-1} \text{hol}(\omega_2, c) j(c(0)) = \text{hol}(\eta, c)$  and define  $j$  along  $\partial F$  by setting, for each  $t \in [0, 1]$ ,  $j(c(t)) = \text{hol}(\omega_2, c^t) j(c(0)) \text{hol}(\eta, c^t)^{-1}$ . Since both  $\eta$  and  $\omega_2$  vanish along  $\partial F$  around the corners of  $F$ , the mappings  $t \mapsto \text{hol}(\eta, c^t)$  and  $t \mapsto \text{hol}(\omega_2, c^t)$  are smooth, and so is  $j$ . Since  $G$  is simply connected,  $j$  can be extended to a  $G$ -valued function smooth on a neighbourhood of  $F$ .

Set  $\omega_3 = j \cdot \omega_2$ .

• Step 4 : *there exists a gauge transformation  $k$  in  $H^2(U; G)$  such that  $k \cdot \omega_3$  satisfies the properties 1, 2 and 3.*

We will choose  $k$  so that  $k \cdot \omega_3$  has the properties of  $\omega_3$  from Step 3 and moreover agrees with  $\eta$  on  $\partial F$  in the transverse direction. Let us look for a logarithm of  $k$ , that is, a  $\mathfrak{g}$ -valued function  $\kappa \in H^2(U; \mathfrak{g})$ . Let us put  $\kappa = 0$  on  $\partial F$ . Then, the fact that  $k \cdot \omega_3$  and  $\eta$  coincide on  $\partial F$  is equivalent to the fact that  $d\kappa = \omega_3 - \eta$  holds on  $\partial F$ . Since  $\kappa = 0$  on  $\partial F$ , this equation holds already in tangential directions, so that it is enough to check it when evaluated in the normal direction. We are thus looking for a  $\mathfrak{g}$ -valued function  $\kappa$  subject to the boundary condition

$$\partial_\nu \kappa = (\omega_3 - \eta)(\partial_\nu), \quad (6)$$

where  $\partial_\nu$  denotes the derivative in the outward normal direction.

The right hand side is a function on the unit circle, smooth except at a finite number of points. At each such point, corresponding to a corner of  $\partial F$ ,  $\omega_3$  and  $\eta$  coincide in the directions of the two half-tangents to  $\partial F$  and thus coincide completely. Recall that there is, by definition, no cusp along the boundary of the face of a graph. So, the r.h.s of (6) vanishes at each singular point. Finally, the boundary condition is continuous and its derivative may jump at each corner, with finite limits on both sides of the jump. By Theorem 1.5.2.8 in [7], this ensures the existence of  $\kappa \in H^2(U; \mathfrak{g})$  satisfying (6).

Now,  $k = \exp(\kappa)$  satisfies the desired property :  $k \cdot \omega_3 = \eta$  on  $\partial F$ . The gauge transformation  $k$  belongs to  $H^2(U; G)$ . Hence, the connection  $\omega = k \cdot \omega_3$  belongs to  $H^1 \mathcal{A}(U)$  and it satisfies the properties 1, 2 and 3.  $\square$

*Proof of Proposition 2.4* – Let  $\Gamma$  be a graph on  $M$  and  $g$  an element of  $G^\Gamma$ . Let  $\omega \in H^1 \mathcal{A}$  be such that  $\omega \sim_\Gamma g$ . By the energy inequality of Proposition 1.3, we get  $S(\omega) \geq 2I_\Gamma(g)$ . Hence,  $I_\Gamma(g) \leq \inf_{\omega \sim_\Gamma g} S(\omega)$ .

To prove the other inequality, consider  $\eta$  given by Lemma 2.5. On each face  $F$  of  $\Gamma$ , take the connection given by Lemma 2.6 and denote it by  $\omega_F$ . The connections  $\omega_F$  glue together into an  $L^2$  connection  $\omega$  on  $M$ . We claim that  $\omega$  belongs to  $H^1 \mathcal{A}(M)$ .

However, by property 3 of Lemma 2.6, the connections  $\omega_F$  have the property that, whenever an edge  $e$  of  $\Gamma$  sits on the boundary of two faces  $F$  and  $F'$ , the traces of  $\omega_F$  and  $\omega_{F'}$  along  $e$  agree. This is exactly what is needed to ensure that the distributional gradient of  $\omega$  is just the sum of the gradients of the  $\omega_F$ , so that  $\omega$  actually belongs to  $H^1\mathcal{A}$ . Now,  $\omega$  agrees with  $g$  along  $\Gamma$  and its energy is exactly  $I_\Gamma(g)$ .  $\square$

Observe that the rate function  $I_\Gamma$  is invariant under the action of  $\mathcal{F}(M, G)$ , simply because the measures  $P_T^\Gamma$  are invariant under the same action.

This proof does not provide us with a concrete description of a minimizer. In particular, one may wonder how it behaves in a neighbourhood of the graph. In Appendix D, we give another proof of the existence of a minimizer, which is more constructive in the sense that it does not rely on an existence result like Thm 1.5.2.8 in [7].

## 2.2 LD for paths in a graph

By the contraction principle, we are now able to state an LDP for the law of the random holonomy along any finite set of paths in a graph.

**Proposition 2.7** *Let  $\Gamma$  be a graph. Let  $J = \{c_1, \dots, c_n\}$  be a finite set of paths in  $\Gamma$ . Then the laws of  $(H_{c_1}, \dots, H_{c_n})$  under  $P_T^\Gamma$ ,  $T > 0$ , satisfy an LDP on  $G^n$  with rate function*

$$I_J(g) = \frac{1}{2} \inf\{S(\omega) : \omega \in H^1\mathcal{A}, \omega \sim_J g\}.$$

*Proof* – The mapping  $h_J = (h_{c_1}, \dots, h_{c_n}) : G^\Gamma \longrightarrow G^J \simeq G^n$  is continuous. Hence, by contraction of the LDP on  $G^\Gamma$ , the laws of  $(H_{c_1}, \dots, H_{c_n})$  under  $P_T^\Gamma$  satisfy an LDP on  $G^J$  with rate function

$$\tilde{I}_J(g) = \inf\{I_\Gamma(k) : k \in G^\Gamma, h_J(k) = g\}.$$

First notice that  $h_J(k) = g$  implies  $g \in \mathcal{M}(J, G)$ . Hence, the support of  $I_J$  is contained in the closed subset  $\mathcal{M}(J, G)$  of  $G^J$ , which supports the laws of  $(H_{c_1}, \dots, H_{c_n})$  under  $P_T^\Gamma$ . So, the LDP holds on  $\mathcal{M}(J, G)$ .

Now, on one hand,  $h_J(k) = g$  implies  $k \sim_J g$ . On the other hand,  $k \sim_J g$  implies that there exists  $j \in \mathcal{F}(M, G)$  such that  $h_J(j \cdot k) = g$ . Since  $I_\Gamma$  is gauge-invariant, we have  $I_\Gamma(k) = I_\Gamma(j \cdot k)$ . Finally, we get the following expression for  $\tilde{I}_J$  :

$$\tilde{I}_J(g) = \inf\{I_\Gamma(k) : k \in G^\Gamma, k \sim_J g\} = \frac{1}{2} \inf\{S(\omega) : \exists k \in G^\Gamma, k \sim_J g, \omega \sim_\Gamma k\}.$$

It appears that  $\tilde{I}_J(g)$  is the infimum of  $S$  over a smaller set of connections than  $I_J(g)$ , so that  $\tilde{I}_J \geq I_J$ .

Now, take  $g \in G^J$  for which  $I_J(g) < \infty$ , that is, such that there exists an  $H^1$  connection which agrees with  $g$  on  $J$ . For every  $\alpha > 0$ , there exists a connection  $\omega_\alpha$  which agrees with  $g$  on  $J$  and such that  $S(\omega_\alpha) \leq I_J(g) + \alpha$ . This connection induces a certain  $k_\alpha \in G^\Gamma$  with which it agrees tautologically on  $\Gamma$ . Hence,  $\tilde{I}_J(g) \leq S(\omega_\alpha) \leq I_J(g) + \alpha$ . By letting  $\alpha$  tend to 0, we get  $\tilde{I}_J \leq I_J$ .  $\square$

For the last part of the proof, we could have argued that the infimum defining the function  $I_J$  is attained, as a consequence of Uhlenbeck's compactness theorem. Then, a minimizer induces a certain configuration in  $G^\Gamma$  with which it agrees and the inequality  $\tilde{I}_J \leq I_J$  follows.

### 2.3 LD for an arbitrary finite family of paths

The last step at the finite-dimensional level is to prove that Proposition 2.7 holds for an arbitrary finite subset  $J$  of  $PM$ . To do this, we use exponential approximation results.

Fix  $J = \{p_1, \dots, p_n\}$  an arbitrary finite subset of  $PM$ .

**Lemma 2.8** *There exist sequences  $(c_i^m)_{m \geq 1}$ ,  $i = 1 \dots n$ , of paths such that*

0. *For all  $m \geq 1$ , all  $i = 1 \dots n$ ,  $c_i^m$  and  $p_i$  have the same endpoints.*

1. *For all  $m \geq 1$ , there exists a graph  $\Gamma_m$  such that  $c_1^m, \dots, c_n^m$  are paths in  $\Gamma_m$ .*

2. *For all  $\delta > 0$ ,*

$$\lim_{m \rightarrow \infty} \overline{\lim}_{T \rightarrow 0} T \log P_T \left[ \max_{1 \leq i \leq n} d(H_{c_i^m}, H_{p_i}) > \delta \right] = -\infty.$$

3. *For all connections  $\omega \in H^1\mathcal{A}$ ,*

$$d(\text{hol}(\omega, c_i^m), \text{hol}(\omega, p_i)) \leq C[S(\omega)d_\ell(c_i^m, p_i)]^{\frac{1}{2}}$$

for some constant  $C$  independent of  $\omega$ .

*Proof* – Let us begin with the case where  $p_1, \dots, p_n$  are embedded paths. It is proved in [9], Section 2.5.3, that we can find sequences of piecewise geodesic paths  $(c_i^m)_{m \geq 1}$ ,  $i = 1 \dots n$ , converging in length to  $p_1, \dots, p_n$  with fixed endpoints, such that  $p_i^{-1}c_i^m$  bounds for each  $i$  a domain diffeomorphic to a disk, of arbitrarily small area. Let  $\sigma_i^m$  denote this area. Then,  $d(H_{p_i}, H_{c_i^m})$  has the same distribution as  $\rho = d(1, \cdot)$  under the heat kernel measure  $p_{T\sigma_i^m}$ . By the estimate of the heat kernel given for example in [19], Theorem V.4.3, we get, for some constant  $C$  depending only on  $G$  and for every  $\delta > 0$ ,

$$P_T[d(H_{c_i^m}, H_{p_i}) > \delta] \leq C(T\sigma_i^m)^{\frac{\dim G}{2}} e^{-\frac{\delta^2}{CT\sigma_i^m}}.$$

Hence,

$$\begin{aligned} \overline{\lim}_{T \rightarrow 0} T \log P_T \left[ \max_{1 \leq i \leq n} d(H_{c_i^m}, H_{p_i}) > \delta \right] &\leq \overline{\lim}_{T \rightarrow 0} T \log \sum_{i=1}^n P_T[d(H_{c_i^m}, H_{p_i}) > \delta] \\ &= \max_{1 \leq i \leq n} \overline{\lim}_{T \rightarrow 0} T \log P_T[d(H_{c_i^m}, H_{p_i}) > \delta] \\ &\leq -\min_{1 \leq i \leq n} \frac{\delta^2}{C\sigma_i^m}. \end{aligned}$$

Since the sequences  $(\sigma_i^m)_m$  converge to 0, the limit as  $m$  tends to infinity of this expression is equal to  $-\infty$ .

Let  $\omega$  be a connection of  $H^1\mathcal{A}$ . By the energy inequality (Proposition 1.3),

$$d(\text{hol}(\omega, c_i^m), \text{hol}(\omega, p_i))^2 = \rho(\text{hol}(\omega, p_i^{-1}c_i^m))^2 \leq \sigma_i^m S(\omega).$$

Now, the domain bounded by  $p_i^{-1}c_i^m$  is contained in a tube around  $p_i$  of width  $d_\infty(p_i, c_i^m) \leq d_\ell(p_i, c_i^m)$ . Thus, there exists a constant  $K$ , depending on the paths  $p_1, \dots, p_n$ , such that  $\sigma_i^m \leq Kd_\ell(p_i, c_i^m)$ . Finally, we get

$$d(\text{hol}(\omega, c_i^m), \text{hol}(\omega, p_i))^2 \leq Kd_\ell(p_i, c_i^m)S(\omega)$$

and property 3 holds.

If the paths  $p_1, \dots, p_n$  are only piecewise embedded, we can cut them into embedded paths  $q_1, \dots, q_r$  and apply the arguments above to this new set of paths. We find sequences  $(d_j^m)_{m \geq 1}$ ,  $j = 1 \dots r$ . Let us assume that  $\ell(d_j^m) \geq \ell(q_j)$  for all  $j$  and  $m$ . Assume now that  $p_1 = q_1 \dots q_s$ , where  $1 \leq s \leq r$ . Then, for all  $m \geq 1$ ,

$$d(H_{p_1}, H_{d_1^m \dots d_s^m}) \leq \sum_{j=1}^s d(H_{q_j}, H_{d_j^m}).$$

Set  $c_1^m = d_1^m \dots d_s^m$  and define the others  $c_i^m$  in a similar fashion. If  $N$  is the largest number of  $q_j$ 's that it is necessary to concatenate in order to get a  $p_i$ , we have

$$P_T \left[ \max_{1 \leq i \leq n} d(H_{c_i^m}, H_{p_i}) > \delta \right] \leq P_T \left[ \max_{1 \leq j \leq r} d(H_{d_j^m}, H_{q_j}) > \frac{\delta}{N} \right]$$

and property 2 follows.

To prove property 3, let us treat the case  $i = 1$ . By applying the special case of property 3 that we have proved above to  $q_1, \dots, q_s$ , we find a constant  $K'$  such that

$$\begin{aligned} d(\text{hol}(\omega, c_1^m), \text{hol}(\omega, p_1))^2 &\leq \left[ \sum_{j=1}^s d(\text{hol}(\omega, d_j^m), \text{hol}(\omega, q_j)) \right]^2 \\ &\leq K' S(\omega) s \sum_{j=1}^s d_\ell(d_j^m, q_j). \end{aligned}$$

Since  $\ell(q_j) \leq \ell(d_j^m)$ , we have for each  $j$  the inequality  $|\ell(d_j^m) - \ell(q_j)| \leq |\ell(c_i^m) - \ell(p_i)|$  and the last term is bounded above by  $K' s^2 S(\omega) d_\ell(c_1^m, p_1)$ . Finally,  $s \leq N$  and property 3 follows.  $\square$

For each  $m \geq 1$ , denote by  $J_m$  the set  $\{c_1^m, \dots, c_n^m\}$ .

**Proposition 2.9** *Let  $J = \{p_1, \dots, p_n\}$  be any finite subset of  $PM$ . The laws of  $(H_{p_1}, \dots, H_{p_n})$  under  $P_T$ ,  $T > 0$ , satisfy an LDP on  $\mathcal{M}(J, G)$  with rate function*

$$I_J(g) = \frac{1}{2} \inf \{ S(\omega) : \omega \in H^1 \mathcal{A}, \omega \sim_J g \}.$$

*Proof* – Let  $(c_i^m)_{m \geq 1}$ ,  $i = 1 \dots n$  be given by the preceding lemma. By a standard result on exponential approximations (see [2], Theorem 4.2.16), property 2 of Lemma 2.8 ensures that the laws of  $(H_{p_1}, \dots, H_{p_n})$  under  $P_T$ ,  $T > 0$  satisfy an LDP on  $G^J$  with rate function

$$\hat{I}_J(g) = \frac{1}{2} \sup_{\delta > 0} \lim_{m \rightarrow \infty} \inf_{h \in B(g, \delta)} \inf_{\omega \sim_{J_m} h} S(\omega).$$

Here,  $B(g, \delta)$  denotes the open ball of radius  $\delta$  around  $g$  in  $G^J$ . The proof is completed by the next lemma, after noticing that the domain of  $I_J$  is contained in the closed subset  $\mathcal{M}(J, G)$  of  $G^J$ .  $\square$

**Lemma 2.10** *Keep the preceding notation. Then,*

$$\sup_{\delta > 0} \liminf_{m \rightarrow \infty} \inf_{h \in B(g, \delta)} \inf_{\omega \sim_{J_m} h} S(\omega) = \inf_{\omega \sim_J g} S(\omega).$$

The proof of this lemma involves the two technical facts about connections that we mentioned earlier, namely that an  $H^1$  connection induces a continuous holonomy, and Uhlenbeck's weak compactness theorem.

*Proof* – For each  $m \geq 1$  and  $\delta > 0$ , define the set

$$O_{m, \delta} = \{\omega \in H^1 \mathcal{A} : \exists h \in B(g, \delta), \omega \sim_{J_m} h\}.$$

Since the holonomy along a given path depends continuously on the connection in  $H^1 \mathcal{A}$  (see Proposition 1.2) these are open subsets of  $H^1 \mathcal{A}$ .

Now, let  $\omega$  be a connection in  $H^1 \mathcal{A}$  such that  $\omega \sim_J g$ . It induces a holonomy which depends continuously on the paths, with fixed endpoints (see Proposition 1.2). Hence, for every  $\delta > 0$ ,  $\omega$  belongs to  $O_{m, \delta}$  for  $m$  large enough.

Choose  $\alpha > 0$  and  $\omega_0$  in  $H^1 \mathcal{A}$  such that  $\omega_0 \sim_J g$  and such that  $S(\omega_0) \leq \inf\{S(\omega) : \omega \sim_J g\} + \alpha$ . Choose  $\delta > 0$ . By the observation above,

$$\liminf_{m \rightarrow \infty} \inf_{O_{m, \delta}} S \leq S(\omega_0).$$

By letting  $\alpha$ , then  $\delta$  tend to 0, we get the inequality  $\hat{I}_J \leq I_J$ .

Assume that this inequality is strict. Then, for some  $\alpha > 0$ , for all  $\delta > 0$ , one would have

$$\liminf_{m \rightarrow \infty} \inf_{O_{m, \delta}} S \leq \inf_{\omega \sim_J g} S(\omega) - \alpha = s - \alpha,$$

where we set  $s = \inf_{\omega \sim_J g} S(\omega)$ . Hence, we may construct a sequence  $(\omega_{m_k})$  with  $\omega_{m_k} \in O_{m_k, \delta}$  and  $\sup_k S(\omega_{m_k}) \leq s - \alpha/2$ . From this sequence with bounded energy, we can, by Uhlenbeck's theorem, extract a subsequence gauge-equivalent to a weakly convergent sequence of connections. Thus, there exists a subsequence  $(\omega_r)_{r \geq 1}$  of  $(\omega_{m_k})$ , a sequence  $(j_r)_{r \geq 1}$  in  $H^2 \mathcal{J}$  and  $\omega \in H^1 \mathcal{A}$  such that  $j_r \cdot \omega_r \rightarrow \omega$  and  $S(\omega) \leq s - \alpha/2$ . Let  $(N_r)_{r \geq 1}$  denote an increasing sequence such that  $\omega_r \in O_{N_r, \delta}$ .

For every path  $c$ , the holonomy along  $c$  of  $j_r \cdot \omega_r$  converges to that of  $\omega$  (see Proposition 1.2). In particular,  $\text{hol}(j_r \cdot \omega_r, p_i) \rightarrow \text{hol}(\omega, p_i)$  for each  $i$ . Moreover, since  $S(\omega_r)$  is bounded, by property 3 of Lemma 2.8, the distance  $d(\text{hol}(j_r \cdot \omega_r, c_i^{N_r}), \text{hol}(j_r \cdot \omega_r, p_i))$  tends to 0. Hence,

$$\text{hol}(j_r \cdot \omega_r, c_i^{N_r}) \xrightarrow[r \rightarrow \infty]{} \text{hol}(\omega, p_i) \quad , \quad i = 1 \dots n.$$

Hence,  $\omega \in O_{\infty, 2\delta}$ , where we take the convention  $J_\infty = J$ .

For every  $\delta > 0$ , we are thus able to construct a connection  $\omega$  in  $O_{\infty, 2\delta}$  such that  $S(\omega) \leq s - \alpha/2$ . For each  $n \geq 1$ , let us do this construction with  $\delta = 1/2n$ . This produces a sequence  $(\omega_n)$ , from which we may again extract a subsequence gauge-equivalent to a weakly  $H^1$  convergent one, with limit  $\omega_*$ . This limit satisfies both  $S(\omega_*) \leq s - \alpha/2$  and  $\omega_* \sim_J g$ . This contradicts the definition of  $s$ .  $\square$

### 3 Large deviations for the random holonomy process

As explained in [9], Section 2.10.2, the probability space  $(\mathcal{M}(PM, G), \mathcal{C}, P_T)$  is the projective limit of the spaces  $(\mathcal{M}(J, G), \mathcal{C}_J, P_T^J)$ , where  $J$  spans the set of finite subsets of  $PM$  and  $P_T^J$  denotes the distribution of the holonomy along the paths of  $J$  under  $P_T$ . A straightforward application of Dawson-Gärtner's theorem (see [2], Thm. 4.6.1 for instance) gives the following result. The topology on  $\mathcal{M}(PM, G)$  is that of pointwise convergence.

**Proposition 3.1** *The measures  $(P_T)_{T>0}$  satisfy on  $\mathcal{M}(PM, G)$  a LDP with good rate function*

$$\tilde{I}_{YM}(f) = \frac{1}{2} \sup_{J \subset PM, \#J < \infty} \inf_{\omega \sim_J f} S(\omega).$$

It is part of the theorem that the function  $\tilde{I}_{YM}$  is lower semi-continuous on  $\mathcal{M}(PM, G)$ . This means that the level sets  $\{f : \tilde{I}_{YM}(f) \leq \alpha\}$  are closed. That it is a *good* rate function means that these level sets are compact. This is automatically true here, because  $\mathcal{M}(PM, G)$  itself is a compact space, as a closed subspace of  $\mathcal{F}(PM, G) = G^{PM}$ .

This section is devoted to the proof of the fact that  $\tilde{I}_{YM} = I_{YM}$ .

**Lemma 3.2** *Let  $f \in \mathcal{M}(PM, G)$  be such that  $\tilde{I}_{YM}(f) < \infty$ . Then  $f$  is continuous with fixed endpoints.*

*Proof* – Let  $l$  be a simple loop on  $M$  bounding a domain  $D$  diffeomorphic to a disk. Let  $\omega$  be a connection in  $H^1\mathcal{A}$  such that  $\omega$  agrees with  $f$  on  $l$  and  $S(\omega) \leq \tilde{I}_{YM}(f) + 1$ . By the energy inequality (see Proposition 1.3),  $\rho(f(\partial D))^2 \leq \sigma(D)(\tilde{I}_{YM}(f) + 1)$ . This implies that  $f$  satisfies the assumptions of Proposition A.5. Hence, it is continuous with fixed endpoints.  $\square$

**Lemma 3.3** *Let  $f \in \mathcal{M}(PM, G)$  be such that  $\tilde{I}_{YM}(f) < \infty$ . Let  $m \in M$  be a fixed point. Then there exists  $\omega \in H^1\mathcal{A}$ , such that  $S(\omega) \leq \tilde{I}_{YM}(f)$  and  $\omega \sim_{\Pi} f$  for some dense subset  $\Pi$  of  $L_m M$ , the set of loops based at  $m$ .*

*Proof* – There exists in  $L_m M$  a countable dense subset of piecewise geodesic loops. For example, take a countable dense subset of  $M$  containing  $m$  and consider the set of piecewise geodesic loops based at  $m$  and joining a finite number of these points.

These piecewise geodesic loops can be enumerated as a sequence  $(\zeta_n)_{n \geq 1}$ . For every  $n \geq 1$ , there exists a graph  $\Gamma_n$  such that  $\zeta_1, \dots, \zeta_n$  are paths in  $\Gamma_n$ .

For each  $n$ , there exists by Proposition 2.4 a connection  $\omega_n \in H^1\mathcal{A}$  such that  $\omega_n \sim_{\Gamma_n} f$  and  $S(\omega_n) = I_{\Gamma_n}(f) \leq \tilde{I}_{YM}(f)$ .

Uhlenbeck's compactness theorem allows us to extract a weakly convergent subsequence of connections, up to gauge transformations. The weak limit of this subsequence has an energy no greater than  $\tilde{I}_{YM}(f)$ . Let  $h$  denote the holonomy that it induces. Then  $h$  agrees with  $f$  on  $\zeta_1, \dots, \zeta_n$  for every  $n$ . Hence, for each  $n \geq 1$ ,  $K_n = \{x \in G : x^{-1}h(\zeta_i)x = f(\zeta_i), i = 1 \dots n\}$  is non-empty. The sequence  $(K_n)$  is thus a non-increasing sequence of non-empty compact subsets of  $G$  and its limit is non-empty. Hence,  $h$  and  $f$  agree on  $\Pi$  itself, a dense subset of  $L_m M$ .  $\square$



**Lemma 3.4** *Let  $f \in \mathcal{M}(PM, G)$  be such that  $\tilde{I}_{YM}(f) < \infty$ . Then there exists  $\omega \in H^1\mathcal{A}$ , such that  $\omega \sim f$  and  $S(\omega) \leq \tilde{I}_{YM}(f)$ .*

*Proof* – It suffices to combine the two previous results : both  $f$  and  $\omega$  determine holonomies continuous with fixed endpoints, which agree on a dense subset of  $L_mM$ . By acting on  $f$  by a gauge transformation at the point  $m$  if necessary, we may assume that  $f(l) = \text{hol}(\omega, l)$  for every loop of  $\Pi$ . By continuity, this equality extends to every loop based at  $m$ . Finally, two elements of  $\mathcal{M}(PM, G)$  which agree on  $L_mM$  agree<sup>4</sup> on  $PM$  (see for example [13]).  $\square$

**Lemma 3.5** *Let  $f \in \mathcal{M}(PM, G)$  be such that there exists  $\omega \in H^1\mathcal{A}$  such that  $\omega \sim f$ . Then  $S(\omega) \leq \tilde{I}_{YM}(f)$ .*

*Proof* – The hypothesis implies that  $\tilde{I}_{YM}(f) < \infty$ . Hence, by Lemma 3.4,  $f$  is equivalent to an  $H^1$  connection  $\omega'$  such that  $S(\omega') \leq \tilde{I}_{YM}(f)$ . Now,  $\omega \sim \omega'$  and, by Proposition 1.4,  $S(\omega) = S(\omega')$ .  $\square$

**Proposition 3.6** *The two functions  $\tilde{I}_{YM}$  and  $I_{YM}$  are equal on  $\mathcal{M}(PM, G)$ .*

*Proof* – Take  $f \in \mathcal{M}(PM, G)$ . Assume first that  $f$  is equivalent to some  $H^1$  connection  $\omega$ . Then,  $\omega \sim_J f$  for every  $J \subset PM$ , so that  $\tilde{I}_{YM}(f) \leq I_{YM}(f)$ . On the other hand,  $I_{YM}(f) = S(\omega)$  and, by Lemma 3.5,  $\tilde{I}_{YM}(f) \geq S(\omega)$ . Hence,  $\tilde{I}_{YM}$  and  $I_{YM}$  take the same value on  $f$ .

Assume now that there is no  $H^1$  connection equivalent to  $f$ , that is,  $I_{YM}(f) = +\infty$ . By Lemma 3.4, this implies  $\tilde{I}_{YM}(f) = +\infty$ .  $\square$

This concludes the proof of the main theorem. The appendices are devoted to the proof of the technical results on  $H^1$  connections that were admitted in Section 1.3.

## A Holonomy of Sobolev connections – Proofs

### A.1 Existence of the holonomy

When one considers smooth objects, connections and paths can be paired to produce elements of the structure group, via the parallel transport. We show in this appendix that this pairing extends to the case of  $H^1$  connections, into a map

$$H^1\mathcal{A} \times PM \longrightarrow G,$$

which is continuous in each variable separately, in a sense that will be made more precise later.

Functions of class  $H^1$  on a two-dimensional domain are defined almost-everywhere, however it is a classical fact (see [1]) that they admit a trace along every regular enough one-dimensional subdomain. This allows us to consider the trace of an  $H^1$  connection along a path on  $M$ .

Let  $c : [0, 1] \longrightarrow M$  be a path with a fixed parametrization. If  $\omega \in C^\infty\mathcal{A}$ , then  $c^*\omega$  is a piecewise smooth 1-form on  $[0, 1]$  with values in  $\mathfrak{g}$ . For  $p \geq 1$ , its  $L^p$  norm is defined by

$$\|c^*\omega\|_{L^p}^p = \int_0^1 \|c^*\omega(\partial_t)\|^p dt = \int_0^1 \|\omega(\dot{c}_t)\|^p dt.$$

---

<sup>4</sup>Remember that *agree* means *are equal up to gauge transformations*, see Definition 2.3.

Notice that, for  $p = 1$ , this norm is independent of the parametrization of  $c$ .

Let  $\omega$  be an element of  $H^1\mathcal{A}$ . The trace of  $\omega$  along  $c$  can be defined as follows. If  $(\omega_n)_{n \geq 1}$  is a sequence of smooth connections converging to  $\omega$  in  $H^1\mathcal{A}$ , then the sequence  $(c^*\omega_n)_{n \geq 1}$  converges in  $L^2$  norm to a limit which depends only on  $\omega$  and is denoted by  $c^*\omega$ . Moreover, the mapping  $c^* : H^1\mathcal{A} \longrightarrow L^2\Omega^1([0, 1]) \otimes \mathfrak{g}$  is continuous<sup>5</sup>.

Let  $\omega$  be an element of  $H^1\mathcal{A}$  and  $c$  a parametrized path in  $PM$ . We identify the 1-form  $c^*\omega$  with the square-integrable  $\mathfrak{g}$ -valued function  $t \mapsto c^*\omega(\partial_t) = \omega(\dot{c}_t)$ . In order to define the holonomy of  $\omega$  along  $c$ , we need to solve the following differential equation in  $G$  :

$$\begin{cases} h_0 = 1 \\ \dot{h}_t h_t^{-1} = -(c^*\omega)_t. \end{cases} \quad (7)$$

The solution of this differential equation should belong to the space

$$H^1_\bullet([0, 1]; G) = \{a \in H^1([0, 1]; G) : a_0 = 1\}.$$

Although this space is no vector space, let us call  $H^1$  norm the function  $\|\cdot\|_{H^1_\bullet}$  defined on it by

$$\|a\|_{H^1_\bullet}^2 = \int_0^1 \|\dot{a}_t\|^2 dt.$$

**Proposition A.1** *Let  $\alpha$  be an element of  $L^2([0, 1]; \mathfrak{g})$ . There exists a unique  $a \in H^1_\bullet([0, 1]; G)$  such that*

$$\dot{a}_t a_t^{-1} = -\alpha_t \text{ a.e. } t \in [0, 1]. \quad (8)$$

*Through any faithful representation of  $G$  in  $GL_N(\mathbb{C})$ , the function  $a$  is the sum of the following absolutely convergent series in  $H^1([0, 1]; \mathbb{M}_N(\mathbb{C}))$*

$$a_t = 1 + \sum_{n \geq 1} (-1)^n \int_{0 \leq t_1 \leq \dots \leq t_n \leq t} \alpha_{t_n} \dots \alpha_{t_1} dt_1 \dots dt_n. \quad (9)$$

*Let  $D : L^2([0, 1]; \mathfrak{g}) \longrightarrow H^1_\bullet([0, 1]; G)$  denote the map which assigns to each  $\alpha$  the unique a solution to (8). Then  $D$  is a norm-preserving homeomorphism.*

*Proof* – If  $a$  and  $b$  are two solutions of (8), then one checks easily that the derivative of  $a^{-1}b$  is identically zero. Hence, the solution to (8), if it exists in  $H^1$ , is unique in that space.

Let us choose a faithful representation of  $G$  in  $GL_N(\mathbb{C})$  for some  $N$ . We identify  $G$  with its image under this representation and  $\mathfrak{g}$  with a subspace of  $\mathbb{M}_N(\mathbb{C})$ . Let us endow  $\mathbb{M}_N(\mathbb{C})$  with a norm  $\|\cdot\|$  which coincides with a  $G$ -invariant scalar product on  $\mathfrak{g}$ . Finally, let  $C$  denote a positive constant such that, for every  $b$  and  $c$  in  $\mathbb{M}_N(\mathbb{C})$ , one has  $\|bc\| \leq C \|b\| \|c\|$ .

For  $n \geq 1$ , set  $I_n(t) = \int_{0 \leq t_1 \leq \dots \leq t_n \leq t} \alpha_{t_n} \dots \alpha_{t_1} dt_1 \dots dt_n$ . It is an absolutely continuous function of  $t$ , with a.e. derivative

$$\dot{I}_n(t) = \alpha_t \int_{0 \leq t_1 \leq \dots \leq t_{n-1} \leq t} \alpha_{t_{n-1}} \dots \alpha_{t_1} dt_1 \dots dt_{n-1}.$$

---

<sup>5</sup>This mapping is in fact even defined and continuous on  $H^{1/2}\mathcal{A}$ . We will use this stronger property in the sequel.

Hence,

$$\| \dot{I}_n(t) \| \leq \frac{\| \alpha_t \|}{(n-1)!} \left( C \int_0^t \| \alpha_s \| ds \right)^{n-1}.$$

Let us consider all functions involved as taking their values in  $\mathbb{M}_N(\mathbb{C})$ . We can compute their  $L^2$  norms using the norm  $\| \cdot \|$  on  $\mathbb{M}_N(\mathbb{C})$ . We get in particular the inequality

$$\| \dot{I}_n \|_{L^2} \leq \| \alpha \|_{L^2} \frac{1}{(n-1)!} \left( C \int_0^1 \| \alpha_s \| ds \right)^{n-1}. \quad (10)$$

It follows that the series (9) converges absolutely in  $H^1$ .

Let us check that  $a$  is a solution of (8). For this, define, for  $N \leq 1$ ,  $a_t^N$  as the partial sum up to order  $N$  of the series (9). One checks easily that  $\dot{a}_t^N = -\alpha_t a_t^{N-1}$ . By letting  $N$  tend to infinity, we find  $\dot{a}_t = -\alpha_t a_t$ . There remains to check that  $a$  takes its values in  $G$ .

Now let  $f$  be any smooth function on  $\mathbb{M}_N(\mathbb{C})$  which vanishes on the group  $G$ . Replacing  $f$  by  $x \mapsto \int_G f(gx) dg$ , we may assume that  $f$  is left  $G$ -invariant, that is, that  $f(gx) = f(x)$  for all  $x \in \mathbb{M}_N(\mathbb{C})$  and  $g \in G$ . Differentiating this relation with respect to  $g$ , we find  $d_x f(\alpha x) = 0$  for every  $x \in \mathbb{M}_N(\mathbb{C})$  and  $\alpha \in \mathfrak{g}$ . Now,  $f(a_0) = 0$  and

$$\frac{d}{dt} \Big|_{t=t_0} f(a_t) = d_{a_{t_0}} f(\dot{a}_{t_0}) = -d_{a_{t_0}} f(\alpha_{t_0} a_{t_0}) = 0.$$

This equation holds a.e. on  $[0, 1]$ , but since  $t \mapsto f(a_t)$  is absolutely continuous, it implies that  $f(a_t)$  vanishes identically.

By taking for  $f$  the square of the distance to  $G$  in  $\mathcal{M}_n(\mathcal{C})$ , we get the result.  $\square$

It is clear now that, given a path  $c$ , the mapping  $H^1 \mathcal{A} \rightarrow G$ ,  $\omega \mapsto \text{hol}(\omega, c)$  is continuous. However, more can be said.

**Proposition A.2** *Let  $(\omega_n)_{n \geq 1}$  be a sequence in  $H^1 \mathcal{A}$  which converges weakly to  $\omega \in H^1 \mathcal{A}$ . Then, for every  $c \in PM$ , one has the convergence*

$$\text{hol}(\omega_n, c) \xrightarrow[n \rightarrow \infty]{} \text{hol}(\omega, c).$$

*Proof* – This result relies on the fact that a connection needs only be of class  $H^{\frac{1}{2}}$  in order to admit a  $L^2$  trace along a path. Hence, we have two continuous linear mappings

$$H^1 \mathcal{A} \xhookrightarrow{i} H^{\frac{1}{2}} \mathcal{A} \xrightarrow{c^*} L^2 \Omega^1([0, 1]) \otimes \mathfrak{g},$$

the first one being the inclusion. The point is that  $i$  is compact and transforms a weak convergence into a strong one : the result follows immediately.  $\square$

## A.2 Energy inequality and continuity of the holonomy

Let  $l$  be a simple loop in  $M$  which bounds a domain  $D$  diffeomorphic to an open disk. Let  $\omega$  be a smooth connection on  $M$ . D. Fine has proved in [5] the following inequality :

$$\rho(\text{hol}(\omega, l))^2 \leq \sigma(D) S_D(\omega), \quad (11)$$

where  $S_D(\omega)$  is the contribution of the restriction of  $\omega$  to  $D$  to the Yang-Mills energy of  $\omega$ , that is,

$$S_D(\omega) = \int_D \|\tilde{\Omega}\|^2 d\sigma.$$

**Proposition A.3** *Inequality (11) still holds when  $\omega$  belongs to  $H^1\mathcal{A}$ .*

*Proof* – Let us approximate  $\omega$  in  $H^1\mathcal{A}$  by a sequence of smooth connections. Both the Yang-Mills energy and the holonomy along  $l$  are continuous functionals on  $H^1\mathcal{A}$ , so that inequality (11) passes to the limit.  $\square$

This inequality is essential. For example, it suffices to imply the following continuity result.

**Proposition A.4** *Let  $\omega \in H^1\mathcal{A}$  be fixed. The mapping  $PM \rightarrow G$ ,  $c \mapsto \text{hol}(\omega, c)$  is continuous with fixed endpoints. This means that, whenever  $c_n$  tends to  $c$  in length with fixed endpoints, one has  $\text{hol}(\omega, c_n) \rightarrow \text{hol}(\omega, c)$ .*

We prefer to state and prove the following slightly more general result.

**Proposition A.5** *Let  $f$  be an element of  $\mathcal{M}(PM, G)$ . Assume that there exists a constant  $K$  such that, for every domain  $D$  diffeomorphic to an open disk and bounded by a simple loop  $\partial D$ , one has the inequality*

$$\rho(f(\partial D))^2 \leq K\sigma(D). \quad (12)$$

*Then  $f$  is continuous with fixed endpoints : whenever  $(c_n)_{n \geq 1}$  is a sequence of paths with fixed endpoints converging in length to  $c$ ,  $f(c_n)$  converges to  $f(c)$ .*

*Proof* – The arguments for this proof are spread in the sections 2.4 to 2.6 of [9], although this result was not stated there. We give here a full sketch of proof and refer the reader to [9] for the details.

Let  $f \in \mathcal{M}(PM, G)$  satisfy (12). The first step is to use an isoperimetric inequality on  $M$  ([9], Prop. 2.15), which holds locally, to deduce the existence of a new constant, still denoted by  $K$ , such that, for  $l$  a simple loop of sufficiently small length  $\ell(l)$ ,  $\rho(l) \leq K\ell(l)$ .

Since  $f$  is multiplicative, this inequality can be extended to short loops with finite self-intersection, defined in [9], Def. 2.11. This involves lasso decomposition (Prop. 2.14) and holds in particular for piecewise geodesic loops. Hence, for every short enough piecewise geodesic loop  $\zeta$ , one still has  $\rho(f(\zeta)) \leq K\ell(\zeta)$ .

Now, let  $c$  be an embedded path. It is possible to find a sequence of piecewise geodesic paths  $(\alpha_n)_{n \geq 1}$  converging to  $c$  with fixed endpoints and such that, for all  $n$ ,  $\alpha_n^{-1}c$  is a simple loop bounding a domain diffeomorphic to a disk, whose area tends to zero as  $n$  tends to infinity. Hence,  $f(\alpha_n) \rightarrow f(c)$ . This is explained in section 2.5.3.

Then, the arguments of the proofs of Lemma 2.36 and Prop. 2.35 show that, whenever  $(\zeta_n)$  is a sequence of piecewise geodesic paths converging to  $c$  with fixed endpoints,  $f(\zeta_n)$  tends to  $f(c)$ . This is the main step of the proof. It involves cutting each  $\zeta_n$  in three parts, two short loops based at the endpoints of  $c$  and one path with the same endpoints as  $c$  and staying in a tubular neighbourhood of  $c$ . The two short loops do not contribute asymptotically to  $f(\zeta_n)$  and the path in the tubular neighbourhood is shown to have a value under  $f$  close to that of  $c$ , by comparing it with an appropriate term in the first approximating sequence  $(\alpha_n)$ .

After extending the result to the case of a piecewise embedded path  $c$  (Section 2.6.4), one concludes easily that  $f$  is continuous with fixed endpoints, see Prop. 2.42.  $\square$

The same arguments imply the following result.

**Proposition A.6** *Let  $\alpha$  be a smooth 1-form on a domain  $D$ . Let  $c$  be a path in  $D$  and  $(c_n)_{n \geq 1}$  a sequence of path converging in length to  $c$  with fixed endpoints. Assume that  $c$  and all the  $c_n$ 's are contained in some compact subdomain of  $D$ . Then  $\int_{c_n} \alpha$  converges as  $n$  tends to infinity to  $\int_c \alpha$ .*

*Proof* – Inside the compact subdomain where  $c$  and the  $c_n$ 's stay,  $\alpha$  and  $d\alpha$  are bounded. Hence, for any loop  $l$  bounding a domain  $\Delta$  diffeomorphic to a disk,

$$\left| \int_l \alpha \right| \leq K \sigma(\Delta) \leq K' \sigma(\Delta)^{\frac{1}{2}}$$

for some constants  $K$  and  $K'$ . From there on, the proof goes just as the previous one.  $\square$

## B Gauge equivalence of $H^1$ connections

Gauge transformations of class  $H^2$  act on  $H^1\mathcal{A}$  and modify the holonomy according to (1). We address here the following question : is it true that two connections which determine equivalent elements of  $\mathcal{M}(PM, G)$  differ by a gauge transformation of class  $H^2$  ? The answer is positive.

**Proposition B.1** *Let  $\omega$  and  $\omega'$  be two connections of  $H^1\mathcal{A}$ . Assume that the functions  $c \mapsto \text{hol}(\omega, c)$  and  $c \mapsto \text{hol}(\omega', c)$  of  $\mathcal{M}(PM, G)$  are equivalent, that is, belong to the same orbit under  $\mathcal{F}(M, G)$ . Then there exists  $j \in H^2\mathcal{J}$  such that  $\omega' = j \cdot \omega$ . In particular,  $\omega$  and  $\omega'$  have the same energy.*

Let  $\omega$  and  $\omega'$  be as above. Let  $f$  and  $f'$  denote their respective holonomies. We proceed as follows. First, we find a gauge transformation  $j$  which transforms  $f$  into  $f'$ . Then, we show that  $j$  is of class  $H^2$ . Hence,  $j \cdot \omega$  and  $\omega'$  induce the same holonomy. We conclude that they are equal.

Let us begin with the last point.

**Lemma B.2** *Let  $\omega$  and  $\omega'$  be two connections of  $H^1\mathcal{A}$  which induce the same holonomy. Then  $\omega = \omega'$ .*

*Proof* – In local coordinates  $(x, y)$ , where  $\omega = \omega_x dx + \omega_y dy$  and  $\omega' = \omega'_x dx + \omega'_y dy$ , the hypothesis implies that the traces of  $\omega_x$  and  $\omega'_x$  coincide along every horizontal segment and those of  $\omega_y$  and  $\omega'_y$  along every vertical segment. Now, an  $H^1$  function whose trace vanishes identically along every horizontal segment is identically zero. Indeed, it belongs to  $L^1$  and, by Fubini's theorem, its integral over any rectangle is zero.  $\square$

Let us fix two connections  $\omega$  and  $\omega'$  as in Proposition B.1, inducing holonomies  $f$  and  $f'$ . Let  $m$  be a point on  $M$ . Let  $j$  be a gauge transformation such that  $j \cdot f = f'$ . If  $c$  is a path starting from  $m$  and ending at some point  $p$ , the fact that  $j \cdot f = f'$  evaluated at  $c$  reads

$$j(p) = f(c)j(m)f'(c)^{-1}. \quad (13)$$

Hence, if  $d$  denotes the distance in  $G$ ,  $d(j(p), j(m)) \leq \rho(f(c)) + \rho(f'(c))$ .

**Proposition B.3** *Let  $\omega$  be an element of  $H^1\mathcal{A}$ , inducing the holonomy  $f \in \mathcal{M}(PM, G)$ . Let  $m$  be a point in  $M$ . There exists two constants  $R > 0$ ,  $K > 0$  such that, if  $\zeta$  is a geodesic starting at  $m$  of length less than  $R$ ,*

$$\rho(f(\zeta)) \leq K\sqrt{\ell(\zeta)}.$$

The square root plays no special role here. It could actually be improved into any power of  $\ell(\zeta)$  in the interval  $[1/2, 1)$  or even into  $\ell(\zeta) \log(\ell(\zeta))$ .

*Proof* – Let  $R$  be smaller than the radius of injectivity of  $M$ . Using normal coordinates in  $B(m, R)$ , we work on  $D(0, R)$ , on which we put polar coordinates  $(r, \theta)$ . Given  $r \in [0, R)$  and  $\theta \in [0, 2\pi)$ , let  $\zeta_{r,\theta}$  denote the segment joining 0 to  $(r, \theta)$ . Let us parametrize this segment by the radius on the interval  $[0, r]$ . Taking the trace along  $\zeta_{r,\theta}$  defines a mapping

$$\zeta_{r,\theta}^* : H^1\mathcal{A}(D(0, r)) \longrightarrow L^2\mathcal{A}([0, r]).$$

Then, relation (7) and a Cauchy-Schwarz inequality ensure that

$$\rho(f(\zeta_{r,\theta})) \leq \sqrt{r} \|\zeta_{r,\theta}^*\omega\|_{L^2([0,r])}, \quad (14)$$

where we have as before  $\|\zeta_{r,\theta}^*\omega\|_{L^2([0,r])}^2 = \int_0^r \|\omega(\dot{\zeta}_{r,\theta})\|^2 dt$ .

Hence, it is crucial to estimate the norm of  $\zeta_{r,\theta}^*$ . To do this, we realize  $\zeta_{r,\theta}^*$  as the composition of several linear mappings.

$$\begin{array}{ccc} H^1\mathcal{A}(D(0, R)) & \xrightarrow{\text{Rot}(\theta)^*} & H^1\mathcal{A}(D(0, R)) \\ & & \downarrow \\ & & H^1\mathcal{A}(D(0, r)) \xrightarrow{(r\text{Id})^*} H^1\mathcal{A}(D(0, 1)) \xrightarrow{\zeta_{1,0}^*} L^2\mathcal{A}([0, 1]) \xrightarrow{(r^{-1}\text{Id})^*} L^2\mathcal{A}([0, r]), \end{array}$$

where the vertical arrow is just the restriction. Let us review these linear mappings one after the other.

The mapping  $U(1) \longrightarrow \text{End}(H^1\mathcal{A}(D(0, R)))$  sending  $e^{i\theta}$  on  $\text{Rot}(\theta)^*$  is continuous. Hence, its image is bounded, and so is the norm of  $\text{Rot}(\theta)^*$ , uniformly in  $\theta$ .

The norm of the restriction is just 1.

The norm of  $\zeta_{1,0}^*$ , which is just taking the trace along the segment  $[0, 1]$ , is some finite constant.

What is of importance to us is what happens in the two dilatations. We leave the reader check the two following equalities :

$$\|(r\text{Id})^*\omega\|_{L^2(D(0,1))} = \|\omega\|_{L^2(D(0,r))}, \quad \|\nabla(r\text{Id})^*\omega\|_{L^2(D(0,1))} = r \|\nabla\omega\|_{L^2(D(0,r))}, \quad (15)$$

where  $\nabla\omega$  stands for the gradient of any component of  $\omega$ . The norms are understood with respect to the pullback of the metric on  $M$  by the exponential map. Since  $\omega$  belongs to  $L^4(D(0,1))$ , we have<sup>6</sup>  $\|\omega\|_{L^2(D(0,r))} \leq \sigma(D(0,r))^{1/4} \|\omega\|_{L^4(D(0,r))} \leq K\sqrt{r} \|\omega\|_{H^1(D(0,r))}$ . Finally, we get  $\|(r\text{Id})^*\| \leq K\sqrt{r}$  for some constant  $K$ .

A similar but simpler argument shows that  $\|(r^{-1}\text{Id})^*\| \leq K/\sqrt{r}$ . Putting all this together, we conclude that the norm of  $\zeta_{r,\theta}^*$  is bounded uniformly with respect to  $r$  and  $\theta$ . The result follows by applying (14).  $\square$

**Corollary B.4** *Let  $\omega, \omega'$  be two connections of  $H^1\mathcal{A}$ . Assume that they induce equivalent holonomies  $f$  and  $f'$ . Then there exists a continuous gauge transformation which sends  $f$  to  $f'$ .*

We prove now that  $j$  admits a weak derivative.

**Proposition B.5** *With the notation of the corollary above,  $j$  admits a weak differential equal to  $j\omega' - \omega j$ . In particular, it belongs to  $W^{1,4}\mathcal{J}$ . Moreover, if  $\omega$  and  $\omega'$  are smooth, then  $j$  is smooth.*

*Proof* – Let us prove this first on an open subset of  $M$  diffeomorphic to  $(-1,1)^2$ , with coordinates  $(x,y)$ . Let us focus on the partial derivative with respect to  $x$ . For each  $y \in (-1,1)$  and  $x \in [0,1)$ , consider the segment  $S_y(x) = [0,x] \times \{y\}$ . By (13), we have

$$j(x,y) = f(S_y(x))j(0,y)f'(S_y(x))^{-1}.$$

Hence, for fixed  $y$ , the map  $x \mapsto j(x,y)$  is the product of two  $H^1$  functions, so that it belongs to  $W^{1,1}$ . Let us compute its derivative :

$$\begin{aligned} \partial_x j(x,y) &= -\omega(\partial_x)f(S_y(x))j(0,y)f'(S_y(x))^{-1} + f(S_y(x))j(0,y)f'(S_y(x))^{-1}\omega'(\partial_x) \\ &= j(x,y)\omega'(\partial_x) - \omega(\partial_x)j(x,y). \end{aligned}$$

If  $\omega$  and  $\omega'$  are smooth, this computation and the similar one in the other direction imply that  $j$  too is smooth.

Otherwise,  $j$  is absolutely continuous along every horizontal segment and admits there an almost everywhere derivative which is the trace of the  $L^4$  function  $j\omega'(\partial_x) - \omega(\partial_x)j$ .

Hence,  $j$  admits an  $L^4$  weak partial derivative with respect to  $x$ , namely  $j\omega' - \omega j$  evaluated on the vector field  $\partial_x$ . A similar statement holds in the other direction and  $j$  admits an  $L^4$  weak differential equal to  $j\omega' - \omega j$ .  $\square$

**Corollary B.6** *Still with the same notation,  $j$  belongs to  $H^2(M;G)$ .*

*Proof* – It suffices to show that  $j\omega' - \omega j$  belongs to  $H^1$ . First, notice that it belongs to  $L^4$ , because  $\omega$  and  $\omega'$  do and  $j$  is bounded. The differential in some direction of a component of this form reads  $(\partial j)\omega' - \omega(\partial j) + j(\partial\omega') - (\partial\omega)j$ . The first two terms belong to  $L^2$  as products of two terms in  $L^4$ . The last two terms also belong to  $L^2$  as products of a bounded term and an

<sup>6</sup>This argument can be improved :  $\omega$  belongs to  $L^p$  for every finite  $p$ . This is how the square root in the statement of Prop. B.3 could be replaced by a higher power. However, this would serve no purpose here.

$L^2$  one. □

*Proof of Proposition B.1* – Let  $\omega$  and  $\omega'$  be two  $H^1$  connections which induce equivalent holonomies  $f$  and  $f'$ . According to Corollary B.6, there exists a gauge transformation  $j$  in  $H^2\mathcal{J}$  such that  $j \cdot f = f'$ . On the other hand,  $j$  acts on  $\omega$  and  $j \cdot \omega$  has the same holonomy as  $\omega'$ . By Lemma B.2, we deduce that  $j \cdot \omega = \omega'$ . In particular,  $S(\omega) = S(\omega')$ . □

Let us conclude this appendix with a result that will be useful in the explicit construction of an  $H^1$  connection with minimal energy given the equivalence class of its holonomy along the edges of a graph (Appendix D).

**Proposition B.7** *Consider the rectangle  $R = (-1, 1)^2$ , the straight path  $S = (-1, 1) \times \{0\}$  in  $R$  and the upper and lower halves  $R_+ = (-1, 1) \times [0, 1]$ ,  $R_- = (-1, 1) \times (-1, 0]$ .*

*Consider two connections  $\omega$  and  $\omega'$  which are  $H^1$  on  $R$ , which coincide along  $S$  in the tangential direction and which have the same holonomy along any loop in  $R$  which is either contained in  $R_+$  or  $R_-$ .*

*Then there exists an  $H^2$  gauge transformation  $j : R \rightarrow G$  identically equal to 1 along  $S$  such that  $\omega' = j \cdot \omega$ .*

*Moreover, if  $\omega$  and  $\omega'$  are smooth on  $R - S$ , then  $j$  is also smooth on  $R - S$ .*

*Proof* – Let  $f$  and  $f'$  denote the holonomies induced by  $\omega$  and  $\omega'$  respectively. Let  $m$  be a point in  $R$ . Define  $j(m) = f(c)f'(c)^{-1}$ , where  $c$  is any path joining a point of  $S$  to  $m$  and staying in the half of  $R$  where  $m$  sits. The assumptions on  $\omega$  and  $\omega'$  imply that  $j(m)$  does not depend on the choice of  $c$ . Moreover,  $j(m) = 1$  whenever  $m \in S$ .

Now,  $j \cdot f$  and  $f'$  coincide along all paths contained in  $R_+$  and all paths contained in  $R_-$ , as well as along all paths obtained by concatenation of a finite number paths staying in  $R_+$  or  $R_-$ .

The proofs of statements B.2 to B.6 do not require more than that. The continuity of  $j$  can be checked separately in each half of  $R$ . Then, the fact that it belongs to  $W^{1,4}$  relies only on the equality of  $j \cdot f$  and  $f'$  along segments parallel to the axes. In the case where  $\omega$  and  $\omega'$  are smooth, we can also deduce the fact that  $j$  is smooth on  $R - S$ . The next step, the fact that  $j$  is  $H^2$ , comes from the fact that both  $\omega$  and  $\omega'$  are  $H^1$ . Finally, one deduces that  $j \cdot \omega = \omega'$  from the fact that they induce the same holonomy along segments parallel to the axes. □

## C A proof of the energy inequality

For the convenience of the reader, we give a proof of Fine's energy inequality. It consists basically in applying Cauchy-Schwarz inequality in the right context.

Let  $D$  be the closed unit disk in  $\mathbb{R}^2$  endowed with a smooth volume form  $\sigma$ . Pick a connection  $\omega$  which is smooth on  $D$ , that is, a smooth  $\mathfrak{g}$ -valued 1-form defined on a neighbourhood of  $D$ . For  $s, t \in [0, 1]$ , set

$$\gamma(s, t) = (1 - s + s \cos(2\pi t), s \sin(2\pi t)) \in \mathbb{R}^2.$$

For every  $s \in [0, 1]$ ,  $t \mapsto \gamma(s, t)$  is a smooth loop based at the point  $(1, 1)$ . The mapping  $\gamma$  realizes a smooth homotopy between the constant loop at  $(1, 1)$  and the boundary of  $D$  starting at  $(1, 1)$  with the usual orientation.



Let  $h(s, t)$  denote the holonomy of  $\omega$  along  $\gamma(s, \cdot)$  at time  $t$ , that is : for every  $s \in [0, 1]$ ,  $t \mapsto h(s, t)$  is the solution of  $h(s, 0) = 1$ ,  $(\partial_t h)(s, t)h(s, t)^{-1} = -\omega((\partial_t \gamma)(s, t))$ . By standard results on ordinary differential equations,  $h$  is a smooth mapping from  $[0, 1]^2$  to  $G$ . Its derivative with respect to  $s$  is given by the following proposition, which we prove at the end of this section.

Recall that the curvature  $\Omega$  of  $\omega$  is the  $\mathfrak{g}$ -valued 2-form defined by  $\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$ .

**Proposition C.1** *The following equality holds for all  $s \in [0, 1]$  :*

$$h(s, 1)^{-1}(\partial_s h)(s, 1) = \int_0^1 h(s, t)^{-1} \Omega((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t)) h(s, t) dt.$$

Let  $\sigma(D)$  denote the integral of  $\sigma$  over  $D$ .

**Corollary C.2 (Energy inequality)** *The following inequality holds :*

$$\rho(\text{hol}(\omega, \partial D))^2 \leq S_D(\omega) \sigma(D).$$

*Proof* – Since conjugation preserves the norm in  $\mathfrak{g}$ ,

$$\|(\partial_s h)(s, 1)\| \leq \int_0^1 \|\Omega((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t))\| dt.$$

Let us write  $\Omega = \tilde{\Omega}\sigma$ , where  $\tilde{\Omega}$  is a  $\mathfrak{g}$ -valued function on  $D$ . By applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left| \int_0^1 \|\Omega((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t))\| dt \right|^2 &= \left| \int_0^1 \|\tilde{\Omega}(\gamma(s, t))\| |\sigma((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t))| dt \right|^2 \\ &\leq \int_0^1 \|\tilde{\Omega}(\gamma(s, t))\|^2 |\sigma((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t))| dt \int_0^1 |\sigma((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t))| dt. \end{aligned}$$

Set  $A(r) = \int_0^r ds \int_0^1 |\sigma((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t))| dt$ . It is the area enclosed by the path  $t \mapsto \gamma(r, t)$ . The function  $A$  is a diffeomorphism of  $[0, 1]$  onto  $[0, \sigma(D)]$ . Reparametrize  $s \mapsto h(s, 1)$  by setting

$$k(u) = h(A^{-1}(\sigma(D)u), 1).$$

The path  $u \mapsto k(u)$  is a path in  $G$  from 1 to  $h(1, 1)$ . Hence,

$$\rho(h(1, 1))^2 \leq \ell(k)^2 = \left| \int_0^1 \|\dot{k}(u)\| du \right|^2 \leq \int_0^1 \|\dot{k}(u)\|^2 du.$$

The last term is the energy of the path  $k$ , which can be estimated as follows.

$$\begin{aligned} \int_0^1 \|\dot{k}(u)\|^2 du &= \int_0^1 \|(\partial_s h)(A^{-1}(\sigma(D)u), 1)\|^2 \frac{\sigma(D)^2}{A'(A^{-1}(\sigma(D)u))^2} du \\ &= \int_0^1 \|(\partial_s h)(s, 1)\|^2 \frac{\sigma(D)}{A'(s)} ds \\ &\leq \sigma(D) \int_0^1 \left( \int_0^1 \|\tilde{\Omega}(\gamma(s, t))\|^2 |\sigma((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t))| dt \right) ds \\ &= \sigma(D) \int_D \|\tilde{\Omega}\|^2 \sigma \\ &= \sigma(D) S_D(\omega). \end{aligned}$$

This implies the result because  $h(1, 1) = \text{hol}(\omega, \partial D)$ .  $\square$

*Proof of Proposition C.1* – For every  $s \in [0, 1]$ , consider the vector field  $X_s$  defined on  $[0, 1] \times G$  by

$$X_s(u, g) = (1, -\omega((\partial_t \gamma)(s, u))g).$$

Let  $(\Phi_s^t)_{t \geq 0}$  denote the flow of  $X_s$ . The group  $G$  acts on  $[0, 1] \times G$  by multiplication on the right on the second factor. Since  $X_s$  is  $G$ -invariant, its flow is also invariant. By definition of  $h(s, t)$ , one has

$$\Phi_s^t(u, g) = (u + t, h(s, u + t)h(s, u)^{-1}g). \quad (16)$$

In particular,  $\Phi_s^1(0, 1) = (1, h(s, 1))$ . Since  $X_s$  depends smoothly on  $s$ , a classical result (see [4], Thm. B.3 for example) asserts that

$$\frac{\partial}{\partial s} \Phi_s^1(0, 1) = \int_0^1 d_{\Phi_s^t(0, 1)} \Phi_s^{1-t} \left[ \frac{\partial X_s}{\partial s}(\Phi_s^t(0, 1)) \right] dt.$$

The term inside the brackets is equal to  $(0, -\frac{\partial}{\partial s} [\omega((\partial_t \gamma)(s, t))] h(s, t))$ . By differentiating (16), one gets

$$d_{\Phi_s^t(0, 1)} \Phi_s^{1-t} \left[ (0, -\frac{\partial}{\partial s} [\omega((\partial_t \gamma)(s, t))] h(s, t)) \right] = (0, h(s, 1)h(s, t)^{-1} \frac{\partial}{\partial s} [\omega((\partial_t \gamma)(s, t))] h(s, t)).$$

Hence,

$$h(s, 1)^{-1} (\partial_s h)(s, 1) = - \int_0^1 h(s, t)^{-1} \frac{\partial}{\partial s} [\omega((\partial_t \gamma)(s, t))] h(s, t) dt. \quad (17)$$

Since the vector fields  $\partial_t \gamma$  and  $\partial_s \gamma$  commute, one has at every point

$$-\frac{\partial}{\partial s} [\omega(\partial_t \gamma)] = d\omega(\partial_t \gamma, \partial_s \gamma) - \frac{\partial}{\partial t} [\omega(\partial_s \gamma)]. \quad (18)$$

Let us compute  $-\int_0^1 h(s, t)^{-1} \frac{\partial}{\partial t} [\omega(\partial_s \gamma)] h(s, t) dt$  by integration by parts. The boundary terms vanish, because the loops  $t \mapsto \gamma(s, t)$  share the same basepoint. There remains

$$- \int_0^1 h(s, t)^{-1} \frac{\partial}{\partial t} [\omega(\partial_s \gamma)] h(s, t) dt = \int_0^1 h(s, t)^{-1} [\omega(\partial_t \gamma), \omega(\partial_s \gamma)] h(s, t) dt, \quad (19)$$

where the last bracket is the Lie bracket of  $\mathfrak{g}$ . Combining (17), (18) and (19), we find what we want, that is,

$$h(s, 1)^{-1} (\partial_s h)(s, 1) = \int_0^1 h(s, t)^{-1} \Omega((\partial_t \gamma)(s, t), (\partial_s \gamma)(s, t)) h(s, t) dt. \quad \square$$

## D Explicit construction of a connection with minimal energy constrained by its holonomy along the edges of a graph

In this section, we present an alternative proof of Prop. 2.4. In fact, we prove the following result in a simpler but longer fashion than we did in Section 2.1.

**Proposition D.1** *Let  $\Gamma$  be a graph on  $M$ . Let  $g$  be an element of  $G^\Gamma$ . Then there exists an element  $\omega \in H^1\mathcal{A}$  such that  $S(\omega) = 2I_\Gamma(g)$  and  $\omega \sim_\Gamma g$ . In fact,  $\omega$  can even be chosen in  $W^{1,\infty}\mathcal{A}$ , hence continuous on  $M$ , and smooth outside  $\Gamma$ .*

We begin by proving the existence of a covering of  $M$  which is nicely adapted to  $\Gamma$ . Recall that  $M$  is endowed with a Riemannian metric. This metric allows us to define tubular neighbourhoods around embedded submanifolds of  $M$ , see for example [9], Section 2.5.2.

In what follows, we denote by  $V(\Gamma)$  the set of vertices of  $\Gamma$ . If  $e$  is an edge and  $v$  a vertex of  $\Gamma$ , we denote by  $e \rightarrow v$  the fact that  $e$  is incident to  $v$ .

**Lemma D.2** *There exist two real numbers  $R, L > 0$  such that the following holds :*

1. *The balls  $B(v, R)$  are diffeomorphic to disks and pairwise disjoint, where  $v$  spans  $V(\Gamma)$ . Moreover, an edge  $e$  meets the ball  $B(v, R)$  if and only if  $v$  is one of its endpoints and this intersection is connected.*
2. *Let  $v$  be a vertex. Let  $f_1, \dots, f_k$  be the edges sharing  $v$  as an endpoint. We assume that they are oriented as outgoing edges and that they are indexed in their natural cyclic order around  $v$ . It is possible to choose polar normal coordinates  $(r, \theta)$  in  $B(v, R)$  such that there exist  $k$  smooth functions  $\theta_1, \dots, \theta_k : [0, R) \rightarrow [0, 2\pi)$  and  $2k$  real numbers  $0 \leq \theta_1^- < \theta_1^+ < \dots < \theta_k^- < \theta_k^+ < 2\pi$ , such that, for all  $j = 1 \dots k$ ,  $(r, \theta_j(r))$  is an oriented parametrization of  $f_j$  inside  $B(v, R)$  and  $\theta_1^- < \inf_{[0, R)} \theta_j < \sup_{[0, R)} \theta_j < \theta_j^+$ . We call  $\{(r, \theta) : \theta_j^- < \theta < \theta_j^+\}$  the angular sector of  $f_j$  at  $v$ .*  
*For each edge  $e$ , let  $e^\circ$  denote the intersection of  $e$  with  $\bigcap_{v \in V(\Gamma)} B(v, R/2)^c$ . Let  $T_e$  denote the tubular neighbourhood of radius  $L$  around  $e^\circ$ .*
3. *The tube  $T_e$  exists for every edge  $e$  and any two distinct tubes are disjoint.*
4. *Let  $v$  be a vertex and  $e$  an edge. The tube  $T_e$  meets  $B(v, R)$  only if  $v$  is an endpoint of  $e$ .*
5. *Let  $e$  be an edge and  $v$  one of its endpoints. Then the open subset  $T_e \cap B(v, R)$  of  $M$  is diffeomorphic to  $(-1, 1)^2$  by a mapping which sends  $e$  to  $(-1, 1) \times \{0\}$ .*
6. *The last property holds with  $B(v, R)$  replaced by  $B(v, 3R/4)$ .*

*Proof* – Let  $R_i$  be the injectivity radius of  $M$ , so that any ball of radius smaller than  $R_i$  is diffeomorphic to a disk. Set  $R_1 = R_i \wedge \frac{1}{2} \inf_{v \neq w} d(v, w)$ , where the infimum is taken over all pairs of distinct vertices. The balls  $B(v, R), v \in V(\Gamma)$  are diffeomorphic to disks and pairwise disjoint as soon as  $R \leq R_1$ .

Let us choose in each ball  $B(v, R_1)$  a system of normal polar coordinates. This amounts to choosing the initial speed of the geodesic of equation  $\{\theta = 0\}$ . We choose it in such a way that no edge incident to  $v$  is tangent to this geodesic at  $v$ .

Let  $v$  be a vertex and  $e$  an edge incident to  $v$ . Since  $e$  is a segment of an embedded submanifold, it can be parametrized near  $v$  and inside  $B(v, R_1)$  as  $e(s) = (r(s), \theta(s))$ , where  $s \geq 0$  and  $e(0) = v$ . Then  $\dot{r}(0) > 0$ . Choose  $s_v(e) > 0$  such that  $\dot{r}(s) > 0$  for all  $s \in [0, s_v(e)]$ . Set  $R_2 = \inf\{r(s_v(e)) : v \in V(\Gamma), e \rightarrow v\}$ . Let  $e$  be an edge with endpoints  $\underline{e}$  and  $\bar{e}$ . Set  $\tilde{e} = e \cap (B(\underline{e}, R_2) \cup B(\bar{e}, R_2))^c$ , where  $e$  is seen as a subset of  $M$ . Finally, set  $R_3 = \frac{1}{2} \inf d(\tilde{e}, v)$ , where the infimum is taken over all edges and vertices. Observe that  $R_3 < R_2$ . Take  $R \leq R_3$ .

Then the balls  $B(v, R)$  satisfy the first point of the lemma. Moreover, we can say that, if  $e$  is incident to  $v$ , it crosses the circle of radius  $R$  around  $v$  transversally.

Let us consider a vertex  $v$  and an edge  $e$  incident at  $v$ . Once again, because  $e$  is a segment of an embedded submanifold, the local parametrization  $(r(s), \theta(s))$  of  $e$  defined above is such that  $s \mapsto \theta(s)$  can be extended by continuity at  $s = 0$ .

Let us denote by  $f_1, \dots, f_k$  be the edges incident to  $v$ , given in their cyclic order around  $v$ , and  $(r_1, \theta_1), \dots, (r_k, \theta_k)$  their local parametrizations. We assume that the edges are indexed in such a way that  $0 < \theta_1(0) < \dots < \theta_k(0) < 2\pi$ . Set  $\delta = \theta_1(0) \wedge (2\pi - \theta_k(0)) \wedge \inf_{1 \leq i \leq k-1} |\theta_{i+1}(0) - \theta_i(0)|$ . Choose  $s'_v > 0$  such that, for all  $i = 1 \dots k$  and all  $s \in [0, s'_v]$ ,  $|\theta_i(s) - \theta_i(0)| < \delta/4$ . Finally, set  $R_4 = \inf r_i(s'_v)$ , where  $v$  runs over  $V(\Gamma)$  and  $r_i$  is the local parametrization of an edge incident to  $v$ . Then, any  $R$  such that  $0 < R \leq R_4$  satisfies the two first points of the lemma.

Let us choose  $R = R_4$ . This allows us to define  $e^\circ$  for each edge  $e$ .

For each edge  $e$ , let  $L_e$  be the largest width of a tube around  $e^\circ$ . Set  $L_1 = \inf_e L_e \wedge \frac{1}{2} \inf_{e \neq f} d(e^\circ, f^\circ)$ . Then any positive  $L$  smaller than  $L_1$  satisfies the third point. Now set  $L_2 = \frac{1}{2} \inf_{e, v} d(e^\circ, B(v, R))$ , where the infimum runs over all pairs  $(e, v)$  with  $e$  non incident to  $v$ . Any positive  $L$  smaller than  $L_3 = L_1 \wedge L_2$  satisfies the fourth point.

Let  $e$  be an edge. Let  $T_e$  be a tube around  $e^\circ$ , endowed with Fermi coordinates  $(t, s)$ , such that  $e^\circ$  is defined by the equation  $s = 0$ . Consider a curve  $\gamma$  which crosses  $e^\circ$  only once and transversally. Near its intersection point with  $e^\circ$ ,  $\gamma$  can be parametrized as  $(t(\tau), s(\tau))$ , with  $\tau = 0$  corresponding to the intersection point. Then,  $\dot{s}(0) \neq 0$ . Hence, there exists a positive number  $L'_e(\gamma)$  such that the portion of  $\gamma$  contained in a tube of width smaller than  $L'_e(\gamma)$  is the graph in Fermi coordinates of a smooth function  $s \mapsto T(s)$ , which crosses the boundary of the tube transversally. Then, the domain  $\{(s, t) : t \leq T(s)\}$  is diffeomorphic to a rectangle by a diffeomorphism which sends  $e^\circ$  to a segment parallel to an edge.

By applying for each edge  $e$  this argument to the circles of centers  $\underline{e}$  and  $\bar{e}$  and radii  $R$  and  $3R/4$ , we find a positive width  $L'_e$  such that  $L = L_3 \wedge \inf_e L'_e$  satisfies the fifth and sixth points.  $\square$

Let  $R$  and  $L$  be two positive numbers given by the lemma above. We are going to construct a minimizing connection on the balls  $B(v, R)$  first, then on the tubes  $T_e$  and finally extend it on the interior of the faces of  $\Gamma$ .

For each face  $F$  of  $\Gamma$ , choose a vertex  $o(F)$  on the boundary of  $F$  and call  $\partial F$  the loop based at  $o(F)$  going once around  $F$  with the positive orientation. If  $v$  is a vertex on  $F$  other than  $o(F)$ , denote by  $\partial F_{o \rightarrow v}$  the portion of  $\partial F$  going from  $o(F)$  to  $v$ . Let us decide that  $\partial F_{o \rightarrow o}$  is the path  $\partial F$  itself.

Recall that a configuration  $g = (g_e)_{e \in \Gamma} \in G^\Gamma$  is given. For each face  $F$  and each vertex  $v$  on the boundary of  $F$ , set  $x_F(v) = h_{\partial F_{o \rightarrow v}}(g)$ . We denote  $x_F(o(F)) = h_{\partial F}(g)$  simply by  $x_F$ .

For each  $F$  again, choose an element  $X_F \in \mathfrak{g}$  of minimal norm such that  $\exp(\sigma(F)X_F) = x_F$ . Observe for future use that  $\rho(x_F) = \sigma(F) \parallel X_F \parallel$ . Finally, for each  $F$  and  $v$  on the boundary of  $F$ , set  $X_F(v) = x_F(v)X_F x_F(v)^{-1}$ . For  $v = o(F)$ ,  $X_F(v) = \text{Ad}(x_F)X_F = X_F$ .

Recall that we consider  $\sigma$  sometimes as a measure and sometimes as a 2-form, the two points of view being equivalent since  $M$  is oriented.

Finally, from now on we will always consider faces of  $\Gamma$  as open subsets of  $M$ .

**Proposition D.3** *We keep the notation of the preceding lemma. Let  $v$  be a vertex of  $\Gamma$ . There exists on  $B(v, R)$  a  $W^{1, \infty}$  connection  $\omega_v$  such that the following holds :*

1.  $\omega_v$  vanishes in all directions on the portion contained in  $B(v, R)$  of any edge incident to  $v$ .
2. Let  $F$  be a face such that  $v$  sits on  $\partial F$ . Inside the face  $F$ ,  $\omega_v$  is smooth and it can be written as  $X_F(v)\lambda$ , where  $\lambda$  is a smooth real-valued 1-form such that  $d\lambda = \sigma$ . In particular, it has curvature  $X_F(v)\sigma$ .

*Proof* – Pick an integer  $i$  between 1 and  $k$ , where  $k$  is the number of edges incident to  $v$ . Set  $X_i = X_F(v)$ , where  $F$  is the face sitting between  $f_i$  and  $f_{i+1}$ , with the convention  $f_{k+1} = f_1$ . Define, for  $r \in [0, R)$ ,

$$\beta_i(r) = X_i \int_{\theta_i(r)}^{\theta_{i+1}(r)} \sigma(r, \theta) d\theta$$

and

$$\alpha_i(r) = \frac{2}{r^2} \int_0^r u \beta_i(u) du,$$

where  $\sigma(r, \theta)$  is the smooth positive function on  $B(v, R)$  such that  $\sigma = \sigma(r, \theta)rdrd\theta$ . Let  $\phi_i : [0, 2\pi) \rightarrow [0, \infty)$  be a smooth nonnegative function such that  $\int_0^{2\pi} \phi_i(\theta) d\theta = 1$  and such that  $\text{supp}(\phi_i) \subset (\theta_i^+, \theta_{i+1}^-)$ . Set

$$a(r, \theta) = \phi_i(\theta)\alpha_i(r), \quad \theta_i(r) \leq \theta < \theta_{i+1}(r),$$

and define a function  $b$  on  $B(v, R)$  by setting  $b(r, \theta_1(r)) = 0$  and

$$\frac{\partial b}{\partial \theta} = \sigma(r, \theta)X_i - \phi_i(\theta)\beta_i(r), \quad \theta_i(r) \leq \theta < \theta_{i+1}(r).$$

After doing this for each  $i = 1 \dots k$ , we have defined two functions  $a$  and  $b$  on  $B(v, R)$ . Finally, set

$$\omega_v = \frac{r^2}{2} a(r, \theta) d\theta - r b(r, \theta) dr.$$

We claim that  $\omega_v$  belongs to  $W^{1, \infty} \mathcal{A}(B(v, R))$ . To see this, let us write  $\omega_v$  in Cartesian coordinates corresponding to our choice of polar coordinates :

$$\omega_v = \left( -\frac{y}{2} a(r, \theta) - x b(r, \theta) \right) dx + \left( \frac{x}{2} a(r, \theta) - y b(r, \theta) \right) dy.$$

For each  $i = 1 \dots k$ , the function  $\beta_i$  is continuous and bounded on  $[0, R)$  and so is  $\alpha_i$ . Both are also smooth on  $(0, R)$ . Hence,  $a$  is bounded on  $B(v, R)$  and smooth except at  $v$ . Observe that  $\int_{\theta_i(r)}^{\theta_{i+1}(r)} \sigma(r, \theta)X_i - \phi_i(\theta)\beta_i(r) d\theta = 0$ , so that  $b$  is continuous and bounded on  $B(v, R) - \{v\}$ . It is also smooth in  $B(v, R)$  outside  $\Gamma$ .

So far, we have proved that the components of  $\omega_v$  belongs to  $L^\infty$ . Now, along any segment in  $B(v, R)$  parallel to one of the coordinate axes and which does not contain  $v$ ,  $a$  is smooth and  $b$  is continuous and piecewise smooth with bounded derivative. In particular, both are absolutely continuous and so are the components of  $\omega_v$ . Let us show that the almost-everywhere defined derivatives of these components are uniformly bounded on  $B(v, R)$ . For example,

$$\frac{\partial}{\partial x}(ya)(r, \theta) = \frac{2}{r^2} \int_0^r u \beta_i(u) du \left( -\frac{y^2}{r^2} \phi'(\theta) - \frac{2xy}{r^2} \phi(\theta) \right) + \frac{2xy}{r^2} \phi_i(\theta) \beta_i(r),$$

so that

$$\|\partial_x(ya)\|_{L^\infty} \leq 10 \sup_{1 \leq i \leq k} (\|\beta_i\|_\infty (\|\phi_i\|_\infty + \|\phi_i'\|_\infty)) < \infty.$$

A similar computation shows that the partial derivatives of  $xa, ya, xb, yb$  are all uniformly bounded on  $B(v, R)$ . Finally, the components of  $\omega_v$  are  $L^\infty$ , absolutely continuous along almost all segments parallel to one of the coordinate axes inside  $B(v, R)$ , with partial derivative belonging to  $L^\infty(B(v, R))$ . This implies that the components of  $\omega_v$  belong to  $W^{1,\infty}$ .

The fact that  $\omega_v$  vanishes along each edge starting at  $v$  comes from the fact that  $a$  vanishes in the angular sector of each edge and  $b(r, \theta_i(r))$  vanishes identically for each  $i = 1 \dots k$ .

Let  $F$  be a face incident to  $v$ . By construction,  $\omega_v$  is smooth inside  $F$  and it can be written as  $X_F(v)$  times a real-valued 1-form. It is readily checked that the differential of this form is  $\sigma$  itself :  $a$  and  $b$  have been designed for that purpose. The statement on the curvature of  $\omega_v$  follows immediately.  $\square$

The next step is to construct the minimizer in the neighbourhood of each edge.

**Proposition D.4** *We keep the notation of Lemma D.2. Let  $e$  be an edge of  $\Gamma$ . Let  $\underline{e}$  denote the starting point of  $e$  and  $\bar{e}$  its terminal point. There exists on  $T_e$  a  $W^{1,\infty}$  connection  $\omega_e$  such that the following holds :*

1.  $\omega_e$  vanishes in all directions on  $e$  inside  $T_e \cap (B(\underline{e}, R) \cup B(\bar{e}, R))$ .
2. The holonomy of  $\omega_e$  along  $e^\circ$  is  $h_e(g) = g_e$ .
3. Let  $F$  be one of the two faces bounded by  $e$ . Then, on  $T_e \cap F$ ,  $\omega_e$  is smooth and it is gauge-equivalent to  $X_F \lambda$ , where  $\lambda$  is a smooth real-valued 1-form such that  $d\lambda = \sigma$ . In particular, the curvature of  $\omega_e$  is of the form  $(j^{-1} X_F j) \sigma$  for some smooth function  $j : T_e \cap F \rightarrow G$ .
4. Let  $F$  be one of the two faces bounded by  $e$  and  $v$  be one of its endpoints. Then, on  $T_e \cap F \cap B(v, R)$ ,  $\omega_e$  has curvature  $X_F(v) \sigma$ .

*Proof* – We start by constructing a connection on  $T_e$  which satisfies the first and third points above and in fact vanishes tangentially along the whole  $e^\circ$ . Then, by an appropriate gauge transformation, we make it satisfy the two other points.

Let  $(x, y) \in [0, 1] \times (-L, L)$  be Fermi coordinates in  $T_e$ , such that  $e^\circ$  is the subset of  $T_e$  defined by the equation  $y = 0$  and oriented consistently with the  $x$ -axis. Let  $\sigma(x, y)$  be the smooth positive function on  $[0, 1] \times (-L, L)$  such that the equality  $\sigma = \sigma(x, y) dx dy$  holds.

Let  $F_+$  denote the face above  $e$ , that is, such that  $F_+ \cap \{y > 0\} \neq \emptyset$ . Similarly, let  $F_-$  denote the face below  $e$ . With our conventions,  $e$  is oriented as a piece of the boundary of  $F_+$ .

Set  $X_+ = X_{F_+}(\underline{e})$  and  $X_- = X_{F_-}(\underline{e})$ . Finally, set

$$\eta_e(x, y) = -(X_+ \mathbf{1}_{y \geq 0} + X_- \mathbf{1}_{y \leq 0}) \left( \int_0^y \sigma(x, t) dt \right) dx.$$

Since  $(x, y) \mapsto \sigma(x, y)$  is bounded on  $T_e$  as well as its derivatives,  $\eta_e$  is Lipschitzian, hence  $W^{1,\infty}$ . It is also smooth on  $T_e$  outside  $e$ .

The first and third points are satisfied by  $\eta_e$  and will still be satisfied after we act on it by a gauge transformation  $j : T_e \rightarrow G$  which is smooth on  $T_e$  and locally constant in  $T_e \cap (B(\underline{e}, R) \cup B(\bar{e}, R))$ .

Let  $s$  and  $t$  be two real numbers with  $0 < s < t < 1$  and such that the portion of  $e$  defined by the equation  $s \leq x \leq t$  does not meet  $B(\underline{e}, R) \cup B(\bar{e}, R)$ .

Let  $\psi : [0, 1] \rightarrow G$  be a smooth mapping such that  $\psi(x) = 1$  whenever  $x \leq s$  and  $\psi(x) = g_e^{-1}$  whenever  $x \geq t$ . Finally, set  $j(x, y) = \psi(x)$  and define

$$\omega_e = j \cdot \eta_e = j^{-1} \eta_e j + j^{-1} dj.$$

Since  $j$  is constant inside the balls around the endpoints of  $e$ ,  $\omega_e$  vanishes there tangentially along  $e$ . The third property is also still satisfied by  $\omega_e$ . Let us check that  $\omega_e$  satisfies the second and fourth properties.

Since the holonomy of  $\eta_e$  along  $e^\circ$  is 1, that of  $\omega_e$  is  $j(1, 0)^{-1} j(0, 0) = g_e$ . Then, the fourth point is satisfied by definition in the neighbourhood of the starting point of  $e$ . Near  $\bar{e}$ , in  $F_+$ ,  $\omega_e$  has curvature  $g_e X_{F_+}(\underline{e}) g_e^{-1}$ , that is,  $g_e h_{\partial F_+ \circ \rightarrow \underline{e}}(g) X_{F_+} h_{\partial F_+ \circ \rightarrow \underline{e}}^{-1}(g) g_e^{-1}$ . Now,  $g_e h_{\partial F_+ \circ \rightarrow \underline{e}}(g)$  is nothing but  $h_{\partial F_+ \circ \rightarrow \bar{e}}(g)$ , because  $e$  is oriented as a piece of the boundary of  $F_+$ . Hence,  $g_e X_{F_+}(\underline{e}) g_e^{-1} = X_{F_+}(\bar{e})$ . Taking orientation into account, the same argument works in  $F_-$  and guarantees that the fourth condition is also satisfied near the terminal point of  $e$ .  $\square$

Consider an edge  $e$  and one of its endpoints  $v$ . There is no reason why the connections  $\omega_v$  and  $\omega_e$  constructed in the two preceding propositions should coincide on  $B(v, R) \cap T_e$ . We make sure that they do by applying a suitable gauge transformation to  $\omega_e$ .

**Proposition D.5** *Let  $e$  be an edge of  $\Gamma$ . There exists an  $H^2$  gauge transformation  $j_e : T_e \rightarrow G$  such that  $j_e$  is identically equal to 1 along  $e^\circ$  and  $j \cdot \omega_e$  coincides with  $\omega_{\underline{e}}$  (resp.  $\omega_{\bar{e}}$ ) on  $T_e \cap B(\underline{e}, 3R/4)$  (resp.  $T_e \cap B(\bar{e}, 3R/4)$ ). Moreover,  $j_e$  is smooth on  $T_e$  outside  $e$ .*

*Proof* – By the fifth point of Lemma D.2, there exists a diffeomorphism from  $T_e \cap B(\underline{e}, R)$  to  $(-1, 1)^2$  which sends  $e$  to  $(-1, 1) \times \{0\}$ . Let us work on the rectangle  $R = (-1, 1)^2$  that we identify with  $T_e \cap B(\underline{e}, R)$ . We have two connections there, namely  $\omega_{\underline{e}}$  and  $\omega_e$ , which are  $H^1$ , being understood that the reference measure is  $\sigma$ .

Let  $F_+$  and  $F_-$  denote the faces of  $\Gamma$  corresponding to the upper and lower halves  $R_+ = (-1, 1) \times [0, 1)$  and  $R_- = (-1, 1) \times (-1, 0]$  of  $R$ . Then, on  $R_+$ , both 1-forms are of the form  $X_{F_+}(\underline{e})\lambda$ , where  $\lambda$  is a 1-form such that  $d\lambda = \sigma$ . A similar property holds on  $R_-$ .

Set  $X_+ = X_{F_+}(\underline{e})$  and  $X_- = X_{F_-}(\underline{e})$ . The holonomy of both connections along any loop  $l$  contained in  $R_+$  (resp. in  $R_-$ ) is equal to  $\exp(X_+ \int_l \lambda)$  (resp.  $\exp(X_- \int_l \lambda)$ ), which does not depend on  $\lambda$  such that  $d\lambda = \sigma$ , because we are working on a simply connected domain.

Finally, since  $\omega_{\underline{e}}$  and  $\omega_e$  both vanish along  $e$ , the hypotheses of Proposition B.7 are satisfied and we conclude that there exists an  $H^2$  gauge transformation  $j_{\underline{e}} : T_e \cap B(\underline{e}, R) \rightarrow G$ , smooth outside  $e$ , equal to 1 along  $e$  and which transforms  $\omega_e$  into  $\omega_{\underline{e}}$ .

We want to do the same construction at the other end of  $e$  and interpolate our two gauge transformations on the whole domain  $T_e$ . In order to interpolate correctly, let us consider inside  $R$  the image of  $T_e \cap B(\bar{e}, 3R/4)$ . Let  $t \in (-1, 1)$  be such that this image is contained in  $(-1, t) \times (-1, 1)$ . Let  $\psi : (-1, 1) \rightarrow [0, 1]$  be a smooth function such that  $\psi(x) = 1$  when  $x \leq t$  and  $\psi(x) = 0$  in a neighbourhood of 1. Let  $J_{\underline{e}}$  be the lift of  $j_{\underline{e}}$  to  $\mathfrak{g}$  which is equal to 0 along  $e$ . Define  $j_{\underline{e}}^0(x, y) = \exp(J_{\underline{e}}(x, y)\psi(x))$ . Then  $j_{\underline{e}}^0$  is  $H^2$  on  $R$ , identically equal to 1 along  $e$  and in a domain of the form  $(1 - \epsilon, 1) \times (-1, 1)$  and  $j_{\underline{e}}^0 \cdot \omega_e = \omega_{\underline{e}}$  inside  $B(\bar{e}, 3R/4)$ .

The same construction at the other end of  $e$  provides us with a gauge transformation  $j_e^0$  identically equal to 1 along  $e$ , such that  $j_e^0 \cdot \omega_e = \omega_{\bar{e}}$  on  $B(\bar{e}, 3R/4)$  and which can be smoothly interpolated by 1 outside  $T_e \cap B(\bar{e}, R)$ .

So, after interpolating  $j_{\underline{e}}^0$  and  $j_{\bar{e}}^0$  outside  $T_e \cap (B(\underline{e}, R) \cup B(\bar{e}, R))$  by 1, we get a gauge transformation with the desired properties.  $\square$

At this point, we have a collection of locally defined connections, one for each vertex, namely  $\omega_v$  in  $B(v, 3R/4)$  and one for each edge, namely  $j_e \cdot \omega_e$  in  $T_e$ . These connections agree pairwise whenever the domains where they are defined overlap. Hence, we have defined a connection  $\omega_\Gamma$  on the neighbourhood  $\bigcup_v B(v, 3R/4) \cup \bigcup_e T_e$  of  $\Gamma$ . We will denote this neighbourhood by  $U_\Gamma$ .

It remains to extend  $\omega_\Gamma$  inside the faces of  $\Gamma$ .

**Proposition D.6** *Let  $F$  be a face of  $\Gamma$ . Let  $\lambda_F$  be a 1-form on  $F$  such that  $d\lambda_F = \sigma$ . Set*

$$\omega_F = X_F \lambda_F.$$

*There exists a neighbourhood  $\tilde{U}_\Gamma$  of  $\Gamma$  contained in  $U_\Gamma$  and a smooth gauge transformation  $j_F : F \rightarrow G$  such that  $j_F \cdot \omega_F = \omega_\Gamma$  on  $F \cap \tilde{U}_\Gamma$ .*

*Proof* – First observe that both  $\omega_F$  and  $\omega_\Gamma$  are smooth inside  $F \cap U_\Gamma$ . This makes matters simpler : it suffices to show that, for some basepoint  $m$  in  $F \cap U_\Gamma$ ,  $\omega_\Gamma$  and  $\omega_F$  have the same holonomy along every loop based at  $m$ .

If  $l$  is a loop in  $F \cap U_\Gamma$ , the holonomy of  $\omega_F$  along  $l$  is  $\exp(X_F \int_l \lambda_F)$ . Assume for a moment that the holonomy of  $\omega_\Gamma$  is the same. Then, there exists a smooth gauge transformation  $j : F \cap U_\Gamma \rightarrow G$  such that  $j \cdot \omega_F = \omega_\Gamma$  where  $j$  is defined. In order to extend  $j$  to  $F$ , consider a neighbourhood  $\tilde{U}_\Gamma$  of  $\Gamma$ , such that  $\tilde{U}_\Gamma \subset\subset U_\Gamma$ . Then there exists a smooth function  $\psi : F \rightarrow [0, 1]$  such that  $\psi(m) = 1$  if  $m \in \tilde{U}_\Gamma$  and  $\psi(m) = 0$  if  $m \notin U_\Gamma$ . Let  $J$  be a logarithm of  $j$ , which exists because  $G$  is simply connected. Then,  $j_F = \exp(\psi J)$  is well-defined and smooth in  $F$  and satisfies the desired property.

In order to complete this proof, it remains to compute the holonomy of  $\omega_\Gamma$  inside  $F \cap U_\Gamma$ . This is what we are going to do now.  $\square$

Let  $\tilde{\Omega}_\Gamma : F \cap U_\Gamma \rightarrow \mathfrak{g}$  be the unique smooth function such that the curvature of  $\omega_\Gamma$  is  $\tilde{\Omega}_\Gamma \sigma$  on this domain. Let  $f_\Gamma$  denote the holonomy induced by  $\omega_\Gamma$ .

**Lemma D.7** *Let  $m$  be a point in  $F \cap U_\Gamma$ . Let  $c$  be a path starting from  $m$  and ending at some point  $p$ , not necessarily distinct from  $m$ . Then one has the relation*

$$\tilde{\Omega}_\Gamma(m) = f_\Gamma(c)^{-1} \tilde{\Omega}_\Gamma(p) f_\Gamma(c).$$

*Proof* – One of the most important features of the connection  $\omega_\Gamma$  on  $F \cap U_\Gamma$  is that it is locally gauge equivalent to a connection of the form  $X\lambda$ , where  $X$  is some vector of  $\mathfrak{g}$  and  $\lambda$  a real-valued 1-form such that  $d\lambda = \sigma$ . Indeed, it has been constructed in that way. It is even true that  $F \cap U_\Gamma$  can be covered by finitely many domains where  $\omega_\Gamma$  has this form.

Now, the relation which we want to prove is gauge-invariant : if it is satisfied by some connection  $\omega$  on a given domain, it is also satisfied by  $j \cdot \omega$  for all gauge transformations  $j$ . Indeed,  $\tilde{\Omega}(m)$  becomes  $j(m)^{-1} \tilde{\Omega}(m) j(m)$ ,  $\tilde{\Omega}(p)$  becomes  $j(p)^{-1} \tilde{\Omega}(p) j(p)$  and  $f(c)$  becomes  $j(p)^{-1} f(c) j(m)$ .



Also, a connection of the form  $X\lambda$  satisfies this relation because its curvature is  $X\sigma$  and its holonomy commutes to  $X$ .

Hence, the statement is true for a path  $c$  small enough to be contained in a domain where  $\omega_\Gamma$  has, up to gauge equivalence, the special form mentioned above. The general case follows easily by subdividing an arbitrary path into pieces each of which is contained one of the domains.  $\square$

**Proposition D.8** *Let  $m$  be a point in  $F \cap U_\Gamma$ . Let  $l$  be a smooth loop based at  $m$  and homotopic to a point in this domain. Then the holonomy of  $\omega_\Gamma$  along  $l$  is  $\exp(\tilde{\Omega}_\Gamma(m) \int_l \lambda_F)$ .*

*Proof* – Since  $l$  is smooth and homotopic to a point, there is a smooth homotopy  $H : (t, s) \mapsto l_s(t)$  from  $[0, 1]^2$  to  $F \cap U_\Gamma$  such that for each  $s$ ,  $l_s$  is a smooth loop based at  $m$ ,  $l_0$  is the constant loop and  $l_1$  is just  $l$ . Let us pull all our objects back by this smooth homotopy and work on  $[0, 1]^2$ . For each  $s \in [0, 1]$ , let  $h_s$  denote the holonomy of  $\omega_\Gamma$  along  $l_s$ , that is, the holonomy of  $H^*\omega_\Gamma$  along  $[0, 1] \times \{s\}$ . Then, according to [8], Thm. 2.2, and by the preceding lemma, we have

$$h_s^{-1} \frac{\partial h_s}{\partial s} = \int_0^1 \tilde{\Omega}_\Gamma(m) H^* \sigma_{(t,s)}(\partial_s, \partial_t) dt.$$

This implies

$$h_1 = \exp \left( \tilde{\Omega}_\Gamma(m) \int_0^1 \int_0^1 H^* \sigma_{(t,s)}(\partial_s, \partial_t) dt ds \right).$$

On the other hand,

$$\frac{\partial}{\partial s} \int_{l_s} \lambda_F = \int_0^1 \partial_s [(H^* \lambda_F)_{(t,s)}(\partial_t)] dt = \int_0^1 d(H^* \lambda_F)_{(t,s)}(\partial_s, \partial_t) dt = \int_0^1 H^* \sigma_{(t,s)}(\partial_s, \partial_t) dt,$$

since  $d_{(0,s)}H(\partial_s) = d_{(1,s)}H(\partial_s) = 0$  for all  $s$ . Hence,

$$h_1 = \exp \left( \tilde{\Omega}_\Gamma(m) \int_0^1 \frac{\partial}{\partial s} \int_{l_s} \lambda_F ds \right) = \exp \left( \tilde{\Omega}_\Gamma(m) \int_l \lambda_F \right).$$

The result is proved.  $\square$

Recall that, in  $F$  near the vertex  $o(F)$ , the curvature of  $\omega_\Gamma$  is equal to  $X_F\sigma$ . Let us choose a point  $m$  in  $F \cap B(o(F), 3R/4)$ . We have  $\tilde{\Omega}_\Gamma(m) = X_F$ .

Let  $\rho$  be the geodesic path from  $m$  to  $o(F)$  and let  $C$  denote the path  $\rho \partial F \rho^{-1}$ . This path generates the fundamental group  $\pi_1(F \cap U_\Gamma, m)$ .

**Lemma D.9** *The holonomy of  $\omega_\Gamma$  along  $C$  is  $x_F = \exp(\sigma(F)X_F)$ .*

*Proof* – On one hand, the holonomy of  $\omega_\Gamma$  along  $\partial F$  is  $x_F$ , by construction. On the other hand,  $\omega_\Gamma$  takes its values in  $\mathbb{R}X_F$  inside  $F \cap B(o(F), 3R/4)$  because it is equal to  $\omega_{o(F)}$  on this domain. Hence, the holonomy of  $\omega_\Gamma$  along  $\rho$  belongs to the one-parameter subgroup of  $G$  generated by  $X_F$  and in any case commutes to  $x_F$ . The result follows.  $\square$

**Proposition D.10** *Let  $l$  be a smooth loop in  $F \cap U_\Gamma$  based at  $m$ . Then the holonomy of  $\omega_\Gamma$  along  $l$  is equal to  $\exp(X_F \int_l \lambda_F)$ .*

*Proof* – Since the path  $C$  generates  $\pi_1(F \cap U_\Gamma)$ , there exists an integer  $r \in \mathbb{Z}$  such that the loop  $lC^r$  is homotopic to a constant loop. Now, let  $(C_n)_{n \geq 1}$  be a sequence of smooth loops based at  $m$ , homotopic to  $C$ , converging in length to  $C$  and such that  $lC_n^r$  is a smooth path for each  $n$ . Hence, by Prop. A.4, the holonomy of  $\omega_\Gamma$  along  $C_n$  converges to that along  $C$ , namely  $x_F$  by Lemma D.9. Let  $h$  denote the holonomy of  $\omega_\Gamma$  along  $l$ .

On one hand, the holonomy along  $lC^r$  is equal to  $x_F^r h$ . On the other hand, by Prop. D.8, the holonomy along  $lC_n^r$  is equal to  $\exp(X_F \int_{lC_n^r} \lambda_F) = \exp(r X_F \int_{C_n} \lambda_F) \exp(X_F \int_l \lambda_F)$  and converges to the holonomy along  $lC^r$ .

By Prop. A.6 and since  $C_n$  converges in length to  $C$ ,  $\int_{C_n} \lambda_F$  tends to  $\int_C \lambda_F$  as  $n$  tends to infinity, so that the holonomy along  $lC_n^r$  converges to  $x_F^r \exp(X_F \int_l \lambda_F)$ . Hence,  $h$  is equal to  $\exp(X_F \int_l \lambda_F)$ .  $\square$

This finishes the proof of Proposition D.6.

*Proof of Proposition D.1* – Let us put together the restriction of  $\omega_\Gamma$  to  $\tilde{U}_\Gamma$  and the connections  $j_F \cdot \omega_F$ , one for each face. We get a connection defined on  $M$  itself, that we denote simply by  $\omega$ .

The connection  $\omega$  is  $W^{1,\infty}$ , hence continuous on  $M$ , and it is smooth outside  $\Gamma$ . Its holonomy along each edge  $e$  of  $\Gamma$  is  $g_e$ . Inside any face  $F$  of  $\Gamma$ , its curvature is of the form  $j^{-1} X_F j$  for some smooth  $j : F \rightarrow G$ . Hence, the Yang-Mills energy of  $\omega$  is

$$S(\omega) = \sum_F \int_F \|X_F\|^2 d\sigma = \sum_F \frac{(\sigma(F) \|X_F\|)^2}{\sigma(F)} = \sum_F \frac{\rho(x_F)^2}{\sigma(F)} = \sum_F \frac{\rho(h_{\partial F}(g))^2}{\sigma(F)} = 2I_\Gamma(g).$$

Finally,  $\omega$  realizes the minimum of the Yang-Mills energy under the constraint  $\omega \sim_\Gamma g$ .  $\square$

## References

- [1] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
- [2] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*. Jones and Bartlett Publishers, Boston, MA, 1993.
- [3] Bruce K. Driver.  $YM_2$ : continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.*, 123(4):575–616, 1989.
- [4] J. J. Duistermaat and J. A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.
- [5] Dana Fine. Energy in Yang-Mills on a Riemann surface. *J. Math. Phys.*, 41(7):4387–4390, 2000.
- [6] Dana S. Fine. Quantum Yang-Mills on a Riemann surface. *Comm. Math. Phys.*, 140(2):321–338, 1991.
- [7] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

- [8] Leonard Gross. A Poincaré lemma for connection forms. *J. Funct. Anal.*, 63(1):1–46, 1985.
- [9] Thierry Lévy. Yang-Mills measure on compact surfaces. *Mem. Amer. Math. Soc.*, 166(790):xiv+122, 2003.
- [10] Peter Li and Shing-Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.
- [11] S. A. Molčanov. Diffusion processes, and Riemannian geometry. *Uspehi Mat. Nauk*, 30(1(181)):3–59, 1975.
- [12] James R. Norris. Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds. *Acta Math.*, 179(1):79–103, 1997.
- [13] Ambar Sengupta. The Yang-Mills measure for  $S^2$ . *J. Funct. Anal.*, 108(2):231–273, 1992.
- [14] Ambar Sengupta. Gauge theory on compact surfaces. *Mem. Amer. Math. Soc.*, 126(600):viii+85, 1997.
- [15] Ambar Sengupta. A Yang-Mills inequality for compact surfaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 1(1):1–16, 1998.
- [16] Karen K. Uhlenbeck. Connections with  $L^p$  bounds on curvature. *Comm. Math. Phys.*, 83(1):31–42, 1982.
- [17] S. R. S. Varadhan. Diffusion processes in a small time interval. *Comm. Pure Appl. Math.*, 20:659–685, 1967.
- [18] S. R. S. Varadhan. On the behavior of the fundamental solution of the heat equation with variable coefficients. *Comm. Pure Appl. Math.*, 20:431–455, 1967.
- [19] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.