Finite Larmor Radius Effects and Quantum Chaos

Robert Dewar, Ben B.F. Mcmillan

Finite Larmor Radius Effects and Quantum Chaos

R L Dewar†* and B F McMillan†‡

† Department of Theoretical Physics and Plasma Research Laboratory, Research School of Physical Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia

‡ Present address: School of Physics, University of Sydney, Sydney NSW 2006, Australia

Abstract. Informed by insights from quantum chaos theory, three approaches for estimating finite-Larmor-radius (FLR) stabilization of ideal magnetohydrodynamic (MHD) instabilities are discussed: phase space criteria based on the Weyl formula, "pseudo-FLR" modifications of MHD to suppress large perpendicular wavenumbers, and inclusion of drift corrections to the dispersion relation.

PACS numbers: 52.35.Bj,05.45.Mt

1. Introduction

It is well known [1] that quantum chaos theory has applications in classical wave physics, such as in microwave cavities and in acoustics, but its potential for providing new insights for characterizing plasma waves in complex geometries has been little explored.

One signature of quantum chaos is the failure of semiclassical quantization due to the occurrence of chaotic ray paths in the Wentzel–Kramers–Brillouin (WKB) approximation. Another is the appearance of level repulsion in the statistics of the separation between nearest-neighbour eigenvalues. Preliminary results [2] on the analysis of spectral datasets of ideal magnetohydrodynamic (MHD) interchange growth rates for a Mercier-unstable stellarator case, studied using the CAS3D code, indicate evidence of level repulsion. However a careful analysis of the spectrum of ideal interchange modes in a cylindrical approximation [3] reveals non-generic behaviour of the spectral statistics due to the peculiarities of ideal MHD, such as the importance of number-theoretic effects on the distribution of rational surfaces in the rotational transform profile. Thus caution must be applied in applying conventional quantum chaos theory.

Another barrier to application of quantum chaos theory is that the addition of more realistic physics than is included in ideal MHD, such as resistivity [4], makes the eigenvalue problem non-Hermitian and the spectrum complex. This makes statistical analysis of the spectrum difficult.

* E-mail: robert.dewar@anu.edu.au

In this paper we discuss three approaches to finding a compromise approach that retains the simplicity of ideal MHD, yet allows an estimate of the effect of finite-Larmor-radius (FLR) effects in a non-dissipative way

The crudest approach is based on phase-space volume (Weyl formula [5, 3]) considerations. A slightly more sophisticated approach [2] is to make a minimal modification to ideal MHD in order to suppress short perpendicular wavelengths in a natural way, which we call *pseudo-FLR regularization*. The third way [6] is to include a drift correction in the dispersion relation.

Although our goal is the analysis of three-dimensional equilibria, much insight can be found by a careful study of the cylindrical limit. We study a plasma in which the Suydam criterion [7] for the stability of interchange modes is violated, so the number of unstable modes tends to infinity as the short-wavelength cutoff tends to zero. The eigenvalue equation for a reduced MHD model (effectively cylindrical) of a stellarator is presented in Sec. 2.

In Sec. 3 we discuss pseudo-FLR regularization effected by solving the eigenvalue problem in a restricted domain in radial Fourier space, such that $k_\perp \rho_* < 1$. In Sec. 4 we discuss the Weyl formula when the poloidal mode number m is restricted to be less than or equal to a cutoff, $m_{\max}$. Choosing $m_{\max}$ on the basis of FLR considerations leads to a stabilization criterion. In Sec. 5 we briefly discuss another form of FLR stabilization obtained by adding a drift correction to the dispersion relation.

2. Cylindrical model eigenvalue equation

We use the model of [3], nondimensionalizing by measuring the radius r in units of the minor radius of the plasma column, a, and the time t in units of the poloidal Alfvén time $\tau_{\mathrm{A}} = R_0\sqrt{\mu_0\rho}/B_0$, where B_0 is the toroidal magnetic field and μ_0 is the permeability of free space. Thus the frequency ω and growth-rate γ are in units of τ_{A}^{-1}. Defining $\lambda \equiv \omega^2 \equiv -\gamma^2$ we seek the spectrum of λ-values satisfying the scalar generalized eigenvalue equation

$$L\varphi = \lambda M\varphi \tag{1}$$

under the boundary conditions $\varphi(0) = 0$ at the magnetic axis and $\varphi(1) = 0$, appropriate to a perfectly conducting wall at the plasma edge. The operators L and M given below are Hermitian under the inner product defined, for arbitrary functions f and g satisfying the boundary conditions, by

$$\langle f, g\rangle \equiv \int_0^1 f^*(r)g(r)\, r\, dr\ . \tag{2}$$

The weight factor r in the inner product is a Jacobian factor coming from $d^3x = rdrd\theta dz$.

The operator L is given by

$$\begin{aligned} L \equiv & -\frac{1}{r}\frac{d}{dr}(n - m\iota\!\!\!-)^2 r\frac{d}{dr} \\ & + \frac{m^2}{r^2}\left[(n - m\iota\!\!\!-)^2 - D_{\mathrm{S}} + \frac{\ddot{\iota\!\!\!-}}{m}(n - m\iota\!\!\!-)\right]\ , \end{aligned} \tag{3}$$

where the Suydam stability parameter $D_{\rm S}$ is

$$D_{\rm S} \equiv -\frac{\beta_0}{2\epsilon^2} p'(r)\Omega'(r) \;, \tag{4}$$

with $\epsilon \equiv a/R_0 \ll 1$ the inverse aspect ratio, $p(r)$ the plasma pressure normalized to unity at $r = 0$, $\beta_0 \equiv 2\mu_0 p_0/B_0^2$ the ratio of plasma pressure to magnetic pressure at the magnetic axis, and Ω' the average field line curvature. Here

$$\Omega \equiv \epsilon^2 N \left(r^2 \iota\!\!\!- + 2 \int r \iota\!\!\!- dr \right) \tag{5}$$

where the rotational transform is produced by helical current windings making $N \gg 1$ turns as ζ goes from 0 to 2π, $\Omega'(r)$ giving the averaged field-line curvature. (Note that ϵ cancels out in $D_{\rm S}$.) We use the notation $\dot{f} \equiv r f'(r)$ for an arbitrary function f, so $\dot{\iota\!\!\!-} \equiv r d\iota/dr$ is a measure of the magnetic shear and $\ddot{\iota\!\!\!-}$ measures the variation of the shear with radius. The term Ω is a measure of the "magnetic hill" [8] that allows pressure energy to be released by interchanging field lines, thus driving the interchange instability.

The criterion for interchange instability is $G > 1/4$, where

$$G \equiv \frac{D_{\rm S}}{\dot{\iota\!\!\!-}^2} \;. \tag{6}$$

The operator arising from the inertial term in the equation of motion,

$$M \equiv -\nabla_\perp^2 = -\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} + \frac{m^2}{r^2} \;, \tag{7}$$

is easily seen to be positive definite under the inner product Eq. (2).

3. Pseudo-FLR regularization

Here we seek only a *minimal* modification of Eq. (1) that has some physical basis but makes as little change to ideal MHD as possible. To preserve the Hermitian nature of ideal MHD we cannot use the drift correction used for estimating FLR stabilization of interchange modes by Kulsrud [9]. However it is possible to effect a pseudo-FLR regularization of ideal MHD by restricting $\mathbf{k}_\perp$ to a disk of radius less than or of the order of the inverse ion Larmor radius. In our nondimensionalized, large-aspect ratio model this implies

$$(k_\theta^2 + k_r^2)^{1/2} \rho_* \le \alpha_* \;, \tag{8}$$

where k_r and k_θ are the radial and poloidal components of the wavevector, respectively, ρ_* is the ion Larmor radius (at a typical energy) in units of the minor radius, and α_* is a constant of order unity, taken to be exactly 1 in this paper.

In drift-wave turbulence studies the ion energy used in defining ρ_* is taken to be the electron temperature, here assumed independent of r. It is consistent with gyrokinetic calculations of electromagnetic drift instabilities to take $\alpha_* \approx 1$, [10]. Henceforth we take $\alpha_* \equiv 1$. Typical physical values for ρ_* range between 10^{-1} for a small laboratory experiment to 10^{-4} for a projected fusion reactor.

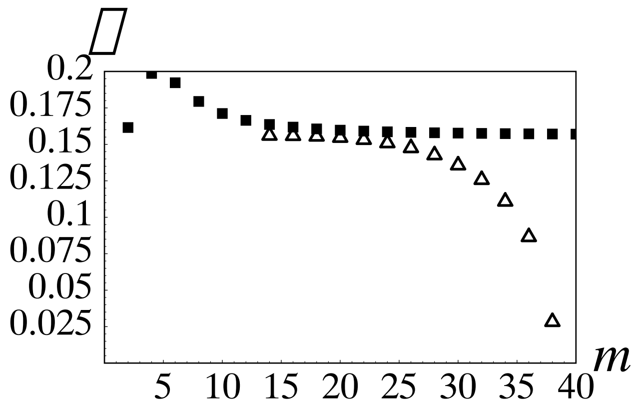

Figure 1. Exact, unregularized eigenvalues (squares) and high-m, pseudo-FLR regularized eigenvalues (triangles) in the case $\mu = 1/2$, radial mode number $l = 0$ with the short-wavelength cutoff parameter $\rho_* = 0.01$. The profiles are the same as studied in [3].

To apply Eq. (8) precisely we need to relate k_r and k_θ to the eigenvalue problem. We see that $k_\theta = m/r$. For defining k_r our first approach is to use Fourier transformation in the radial direction. This is only practical in the large-m limit, when modes are localized near the resonant surfaces $r = r_\mu$ where $n - m\iota\!\!\!-(r_\mu) = 0$, so we restrict discussion of the Fourier method to leading order in the $1/m$ expansion.

We use the stretched radial coordinate $x \equiv m(r-r_\mu)/r_\mu$. Then we define $k_\theta \equiv m/r_\mu$ and $k_r \equiv m\kappa/r_\mu$, where κ is the Fourier-space independent variable conjugate to x,

$$\varphi(x) = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi}\, \varphi_\kappa e^{i\kappa x} \ . \tag{9}$$

With the substitutions $d/dx \mapsto i\kappa$, $x \mapsto id/d\kappa$, and using the fact that $\kappa d/d\kappa$ and $(d/d\kappa)\kappa \equiv 1 + \kappa d/d\kappa$ commute, we have

$$\left[-\frac{d}{d\kappa}(1+\kappa^2)\frac{d}{d\kappa} + \Gamma^2(1+\kappa^2) - G\right]\varphi_\kappa = 0 \ , \tag{10}$$

where G is defined in Eq. (6) and $\Gamma \equiv \gamma/\dot{\iota}(r_\mu)$, both evaluated at $r = r_\mu$. The transformation $\kappa = \sinh\eta$ then leads back to the Schrödinger-like eigenvalue equation studied in [3],

$$\frac{d^2\psi}{d\eta^2} + [G - \tfrac{1}{4} - \tfrac{1}{4}\mathrm{sech}^2\,\eta - \Gamma^2\cosh^2\eta]\psi = 0 \ , \tag{11}$$

with η now to be interpreted as a distorted Fourier-space independent variable, rather than as a real-space coordinate!

Equation (8) implies that Eq. (10) is to be solved on the domain $-\kappa_{\max} \le \kappa \le \kappa_{\max}$ where

$$\kappa_{\max}(\mu) \equiv \left[\left(\frac{r_\mu}{m\rho_*}\right)^2 - 1\right]^{1/2} \ . \tag{12}$$

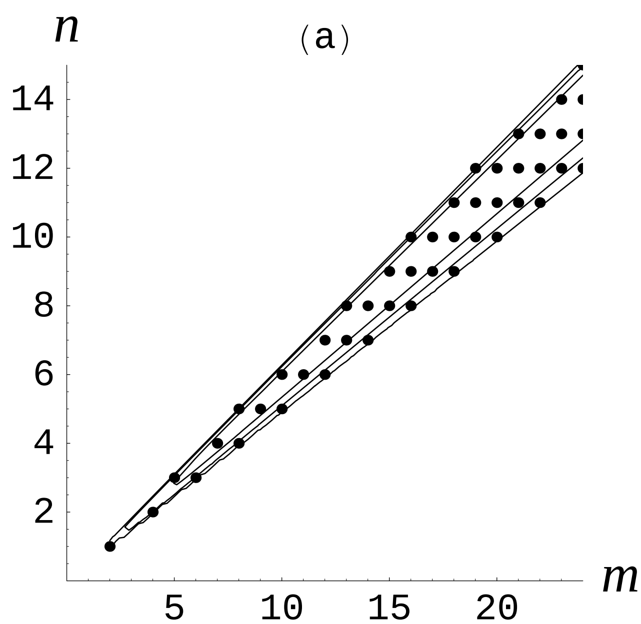

Figure 2. Lattice of quantum numbers on which the part of the spectrum between the "ground state" and the threshold for the entry of the $l = 1$ mode is defined, and the contours of constant eigenvalue (λ or γ), which are seen to be unbounded, asymptoting at large m to straight lines passing through the origin.

This exists provided $|m| < m_{\rm max}$, where

$$m_{\rm max}(\mu) \equiv r_\mu/\rho_* \; . \tag{13}$$

Analogously to quantum mechanical box-quantization we use Dirichlet boundary conditions at $\pm\kappa_{\rm max}$.

As illustrated in Fig. 1 this procedure removes the accumulation point at $m = \infty$, completely stabilizing the interchange modes before m reaches $m_{\rm max} = 50$ in the case shown. With this modification to the interchange dispersion relation the eigenvalue isocurves in the m, n plane no longer radiate out to infinity as in Fig. 2, but instead form closed curves just as for generic wave systems [11]. Thus we would expect that the spectral statistics would also be generic (i.e. Poisson in this separable case), and this explains the approximately exponential behaviour of the distribution of nearest neighour eigenvalue separations observed in [2].

4. The Weyl formula and FLR stabilization

For fixed radial mode number l and large $m_{\rm max}$ the number of eigenvalues $N_l(\mu)$ in an interval of the ratio n/m between μ and the value of n/m, μ_l, giving the maximum asymptotic growth rate is asymptotically equal to the area in the m, n plane of the triangle bounded by the lines $n = \mu m$, $n = \mu_l m$ and $m = m_{\rm max}$ (see Fig. 2). That is, $N_l(\mu) \sim \frac{1}{2}|\mu - \mu_l| m_{\rm max}^2$.

Since contours of constant λ (and γ) asymptote to lines of constant μ as $m \to \infty$ we can estimate the number of eigenvalues between two values of λ (or γ) by inverting the function $\lambda_{\mu,l}$ for μ and substituting this into the above expression for $N_l(\mu)$. The inverse is double-valued: $\mu = \mu_l^+(\lambda) > \mu_l$ and $\mu_l^-(\lambda) < \mu_l$. Then the number of eigenvalues $\leq \lambda$ is approximately

$$\overline{N}_l(\lambda) \equiv \tfrac{1}{2}[\mu^+(\lambda) - \mu^-(\lambda)]m_{\max}^2 \, . \tag{14}$$

This is the analog of the Weyl formula [12, p. 258] for the integral of the smoothed spectral density ("density of states"). The general argument is based on the idea that, when the number of eigenvalues is large, the number of eigenvalues below a given value λ depends asymptotically only on the phase-space volume enclosed by the isosurface $\lambda = \text{const}$ in the semiclassical $\mathbf{x}, \mathbf{k}$ phase space, irrespective of whether the rays are chaotic or regular. The corresponding formula for ballooning modes in general three-dimensional geometry was given in [5].

As shown in [2] the formula Eq. (14) gives good agreement with the coarse-grained distribution of a large dataset of eigenvalues, but of course does not capture the fine detail. Nevertheless we can use it to make a rough estimate of FLR stabilization by using the criterion that the system is stable to the mode in question if not even one mode with eigenvalue less than the marginal stability value $\lambda = 0$ can be supported,

$$\overline{N}_l(0) < 1 \, , \tag{15}$$

where $\overline{N}_l(\lambda)$ is computed from Eq. (14) with $m_{\max}$ taken to be $m_{\max} = 1/\rho_*$ [cf. Eq. (13)].

For determining the stability threshold we can restrict attention to the most unstable radial eigenmode, $l = 0$. Approximating $\lambda_{\mu,0} = \lambda_0 + \frac{1}{2}(\partial^2\lambda_{\mu_0,0}/\partial\mu_0^2)(\mu - \mu_0)^2$ we get $\mu_0^\pm(\lambda) = \mu_0 \pm \sqrt{2}(\lambda - \lambda_0)^{1/2}/(\partial^2\lambda_{\mu_0,0}/\partial\mu_0^2)^{1/2}$. Substituting in Eqs. (14) and (15) leads to the FLR stabilization criterion

$$\gamma_{\max} < \rho_*^4 \frac{\partial^2\gamma}{\partial\mu^2} \, , \tag{16}$$

where the partial derivatives are evaluated at the position of the maximum growth rate, $\gamma_{\max} \equiv (-\lambda_0)^{1/2}$.

5. Drift-corrected dispersion relation

Finite Larmor radius also affects stability through the introduction of drift corrections, which can be roughly estimated [13, 6] by changing the ideal-MHD dispersion relation $\omega^2 = \lambda_{l,m,n}$ to

$$\omega(\omega - \omega_*) = \lambda_{l,m,n} \, , \tag{17}$$

where ω_* is an appropriate drift frequency. Strictly (for high-m) this should be evaluated at the rational surface where $n = \iota m$, where the mode is localized, but as a global estimate for ω_* let us take $|\boldsymbol{\nabla} p| \sim p/a$. Then our nondimensionalized drift frequency may be estimated as $\omega_* \sim m\rho_*^2\omega_{\rm ci}\tau_{\rm A}$. The new factor $\omega_{\rm ci}\tau_{\rm A}$ shows this is an inherently different FLR effect from that discussed in the previous sections.

The dispersion relation Eq. (17) is a quadratic equation and can have complex roots, making semiclassical ray tracing difficult. However, if the drift effect is sufficiently strong it can stabilize the instabilities, making ω real. It is found [14] that this makes the three-dimensional ray dynamics much closer to integrable than found when regularizing using a sharp cutoff, as in [5].

6. Conclusion

The simple phase space criterion is the most practical way to estimate FLR stabilization from ideal-MHD calculations, either global eigenmode computations or localized ballooning equation estimates.

The pseudo-FLR method may lead to improvements in the design of global ideal-MHD eigenvalue codes that make the spectra more physical, and also makes MHD more generic in the quantum chaos sense, thus allowing the application of standard quantum chaos methods.

Finally, the drift-corrected dispersion relation method gives insight into the relation of the MHD spectrum to spectra calculated with kinetic codes.

Acknowledgments

We acknowledge support by the Australian Research Council.

References

[1] Stöckmann, H. J., *Quantum Chaos: An Introduction*, Cambridge University Press, Cambridge, 1999.

[2] Dewar, R. L., Nührenberg, C., and Tatsuno, T., J. Plasma Fusion Res. SERIES, in press. E-Print arXiv:physics/0409070 (2004), Proceedings of the 13th International Toki Conference, Toki, Japan, 9-12 December 2003.

[3] Dewar, R. L., Tatsuno, T., Yoshida, Z., Nührenberg, C., and McMillan, B. F., To be published in Phys. Rev. E (2004), E-Print arXiv:physics/0405095.

[4] McMillan, B. F., Dewar, R. L., and Storer, R. G., Plasma Phys. Control. Fusion **46** (2004) 1027, E-Print arXiv:physics/0405002.

[5] Dewar, R. L., Cuthbert, P., and Ball, R., Phys. Rev. Letters **86** (2001) 2321, E-Print arXiv:physics/0102065.

[6] Nevins, W. M. and Pearlstein, L. D., Phys. Fluids **31** (1988) 1988.

[7] Suydam, B. R., in *Proc. Second Int. Conf. on the Peaceful Uses of Atomic Energy*, volume 31, page 157, Geneva, 1958, United Nations.

[8] Wakatani, M., *Stellarator and Heliotron Devices*, Number 95 in The international series of monographs on physics, Oxford University Press, New York, 1998.

[9] Kulsrud, R. M., Phys. Fluids **6** (1963) 904.

[10] F. S. Jenko 2002, private communication.

[11] Berry, M. V. and Tabor, M., Proc. R. Soc. Lond. A **356** (1977) 375.

[12] Gutzwiller, M. C., *Chaos in Classical and Quantum Mechanics*, Number 1 in Interdisciplinary Applied Mathematics Series, Springer-Verlag, New York, 1990.

[13] Tang, W. M., Dewar, R. L., and Manickam, J., Nucl. Fusion **22** (1982) 1079.

[14] McMillan, B. F., *Global Wavemodes of Dissipative Fluid Models in Strongly Three Dimensional Plasmas: Calculation and Interpretation*, PhD thesis, The Australian National University, 2004.