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Dynamical Behavior of 2D Viscous-vortices and Formation of Vortex Crystals

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Vortex as a 2-dimensional (2D) coherent structure is of common interest in both self-organization and turbulent transport in plasmas. Much attention has been paid to inviscid vortices such as a drift vortex so far. We have observed the plasma hole in a cylindrical plasma with magnetic field, and identified it as a Burgers vortex, which is inherent to viscous fluids. The observation of viscous-vortex suggests that the viscosity of a plasma is not negligibly small and bears a key role in vortex formation.

The essential difference between an inviscid vortex and a viscous-vortex is the existence of radial flow, by which the viscous vortices can interact with a different manner from that of inviscid vortices. Starting from the fluid equation, we derived the equation of motion for “point viscous-vortices” and numerically examined the dynamical behavior to compare with that of point vortices. For a system of two viscous-vortices with the same sign of vorticity, they attract each other and coalesce into one as time elapses, while two point-vortices rotate each other and never coalesce into one. For a systems of N vortices $N \geq 3$, we obtained vortex crystals (or vortex lattices), which have much longer lifetime compared with the decay time due to viscosity.

1. Experimental Observation of a Viscous-Vortex

We have observed a density hole in a cylindrical plasma with magnetic field [1]. The end view image is shown in Fig.1. The central dark region in the figure indicates the density hole, the sizes of which are 6 cm in diameter and more than 100 cm in axial length. The density of hole region is about one tenth of that in the ambient plasma, and the width of transition layer between the hole and ambient plasma is about 1.2 cm, which corresponds to several ion Larmor radii.

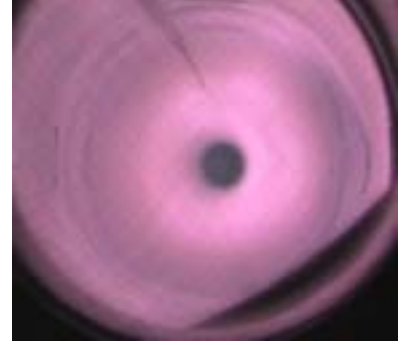


Fig.1 End view image of a plasma hole.

The flow velocity field of the plasma hole exhibits a monopole vortical structure with a sink in its center (see Fig.2).

When the direction of magnetic field is inversed, the azimuthal velocity also changes its direction of rotation, indicating that the azimuthal rotation is due to $\mathbf{E} \times \mathbf{B}$ drift. The remarkable characteristic of this vortex is the existence of radial flow, which remains unchanged under the field inversion. It is found that the radial flow is a $\mathbf{F} \times \mathbf{B}$ drift, \mathbf{F} being the viscous force due to shear in the azimuthal rotation. The plasma hole is identified as a Burgers vortex (viscous-vortex) [2, 3] and this suggests that the viscosity of a plasma is not negligibly small and bears a key role in vortex formation.

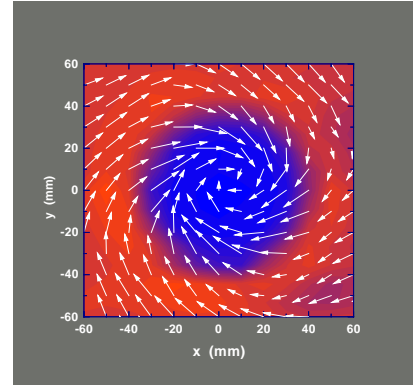


Fig.2 Velocity Field

When we consider the interaction of viscous-vortices, the existence of radial flow is of essential importance since it causes an attractive interaction between vortices. Consequently, the dynamical behavior of a system of viscous-vortices is expected to be qualitatively different from that of point vortices[4-6].

2. Dynamics of Viscous-vortices

Fluid motion of a plasma in a constant magnetic field is described by the equation of continuity and the equation of motion

$$\frac{\partial V}{\partial t} + \nabla n V = 0 \quad (1)$$

$$\frac{\partial V}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{e}{M} \nabla \phi + \mathbf{V} \times \boldsymbol{\Omega} - \frac{\nabla p}{Mn} + \nu \Delta \mathbf{V} \quad (2)$$

The vorticity equation is given by taking rotation of above equation,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{V} \cdot \nabla \boldsymbol{\omega} = (\boldsymbol{\omega} + \boldsymbol{\Omega}) \cdot \nabla \mathbf{V} - (\boldsymbol{\omega} + \boldsymbol{\Omega}) \nabla \cdot \mathbf{V} + \nu \nabla^2 \boldsymbol{\omega} \quad (3)$$

where Ω is the ion cyclotron frequency and

$$\boldsymbol{\omega} = \nabla \times \mathbf{V} \quad (4)$$

When the magnetic field is in the z direction, a cylindrically symmetric stationary solution can be obtained,

$$\mathbf{V}_\perp = v \nabla \ln \hat{\omega}(r) + \frac{\mathbf{z} \times \mathbf{r}}{r^2} \int_0^r r \hat{\omega}(r) dr - \frac{\Omega}{2} \mathbf{z} \times \mathbf{r}, \quad \hat{\omega} = \omega + \Omega \quad (5)$$

For two dimensional vortices, the vorticity is generally concentrated at the center and is decreased in magnitude with the distance from the center. It means that the vorticity may be expressed as

$$\hat{\omega} = \frac{\hat{\omega}_0}{(1 + \beta r^2)^2} \quad (6)$$

where $\hat{\omega}_0$ is the strength of the vorticity at the center. The form of eq.(6) is particularly plausible since it gives a Burgers vortex near the center of axis. Then from eq.(5) we have for the radial velocity and the azimuthal velocity as

$$\begin{aligned} V_r &= -\frac{\alpha r}{1 + \beta r^2}, \quad \alpha = 4\beta v \\ V_\theta &= \frac{\hat{\omega}_0}{2} \frac{r}{1 + \beta r^2} - \frac{\Omega}{2} \end{aligned} \quad (7)$$

The vorticity distribution with respect to radius r are given in the following figure for the case

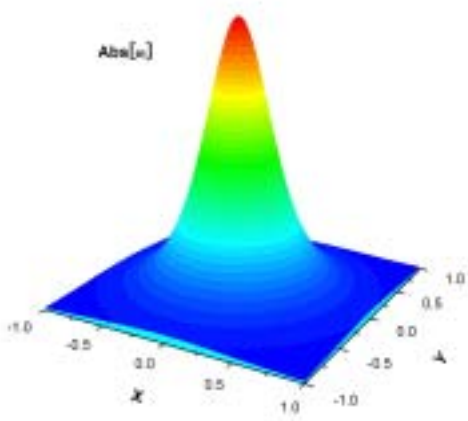


Fig.3 Vorticity distribution

$\beta = 5, v = 0.05$ The vorticity is localized in the center, which is due to the balance between inward transport and outward diffusion of vorticity (Fig.3). According to eq.(7), the vortex exhibits a rigid rotation in the near axis region, and in the far field region the azimuthal velocity is inversely proportional to the radius r , which is same as point vortices. Note that the radial velocity in the far field region is different from that of Burger's vortex,

which diverges with the radius r .

Now we consider a system consisting of N viscous-vortices in the two dimensional system as

$$\hat{\omega} = \sum \kappa_i(t) F_i(r - r_i(t)), \quad i = 1, \dots, N \quad (8)$$

where F_i is a localized function around $\mathbf{r} = \mathbf{r}_i(t)$ at a time t . Substituting the above expression into the vorticity equation in two dimension, and using

$$\mathbf{V}_\perp = \sum_i \mathbf{U}_i(\mathbf{r} - \mathbf{r}_i) - \frac{\Omega}{2} \mathbf{z} \times \mathbf{r} \quad (9)$$

we have

$$\sum_i \left[\nabla \cdot \left\{ F_i \left(-\frac{d\mathbf{r}_i}{dt} + \sum_j \mathbf{U}_j - \frac{\Omega}{2} \mathbf{z} \times \mathbf{r} \right) \right\} + \left(\frac{d\kappa_i}{dt} F_i - \nu \kappa_i \nabla^2 F_i \right) \right] = 0 \quad (10)$$

which is solved together with

$$\sum_i [(\nabla \times \mathbf{U}_i)_z - \kappa_i F_i] = 0 \quad (11)$$

We have with eq.(5)

$$\mathbf{U}_i = \nu \kappa(t) \nabla \ln F_i + \kappa(t) \frac{\mathbf{z} \times \mathbf{r}}{r^2} \int_0^r r F_i(\mathbf{r}) dr \quad (12)$$

Here we may take an equilibrium solution for $F_i(\mathbf{r})$ as

$$F_i(\mathbf{r} - \mathbf{r}_i(t)) = \frac{1}{[1 + \beta_i(\mathbf{r} - \mathbf{r}_i(t))^2]^2} \quad (13)$$

which is localized around $\mathbf{r} = \mathbf{r}_i(t)$. Noting that the first term of the left hand side of eq.(10) represents the characteristics of the vorticity equation, we may put

$$\begin{aligned} \frac{d\mathbf{r}_i}{dt} &= \sum_{j \neq i} \mathbf{U}_j(\mathbf{r}_i - \mathbf{r}_j(t)) - \frac{\Omega}{2} \mathbf{z} \times \mathbf{r}_i(t) \\ \frac{d\kappa_i}{dt} &= -2\alpha_i \kappa_i(t), \quad (\alpha_i = 4\beta_i \nu) \end{aligned} \quad (14)$$

where

$$\mathbf{U}_j(\mathbf{r}_i - \mathbf{r}_j) = -\frac{\alpha_j(\mathbf{r}_i - \mathbf{r}_j)}{1 + \beta_j(\mathbf{r}_i - \mathbf{r}_j)^2} + \frac{\kappa_j}{2} \cdot \frac{\mathbf{z} \times (\mathbf{r}_i - \mathbf{r}_j)}{1 + \beta_j(\mathbf{r}_i - \mathbf{r}_j)^2} \quad (15)$$

In general when N vortices are apart from each other ($\sim R$) they approach with a converging rate $\alpha/\beta R$ and at the same time the vorticity strength is subject to damping due to viscosity.

Once they come together close enough to the characteristic length $(1/\beta_j)^{0.5}$ the converging rate diminishes linearly to zero while κ_j decays exponentially. Thus in viscous plasmas, vortices are eventually to converge and be dissipated. An interesting thing is that although in a system of Euler vortices which interact through a logarithmic potential the velocity field becomes singular when two vortices approach each other, in a viscous system the velocity field vanishes linearly with the separation distance go to zero and therefore the vortices can merge into a single vortex without singularity. This implies that in viscous plasmas, the number of vortices can change in the course of dynamical evolution while in the Euler system the number of vortices are conserved and the merging process never be studied.

There is a constant of motion for the above equation when the size and strength of all the vortices

are the same

$$\frac{d}{dt} \sum_i \mathbf{r}_i(t) = -\frac{\Omega}{2} \mathbf{z} \times \sum_i \mathbf{r}_i(t) \quad (16)$$

which shows the center of mass is conserved under the replacement of \mathbf{r}_i by $\mathbf{r}_i \exp[-i\Omega t]$. Furthermore in this case the vortices finally collapse into the origin as

$$\frac{d}{dt} \sum_i \mathbf{r}_i(t)^2 = -\sum_{i,j} \frac{\alpha(\mathbf{r}_i(t) - \mathbf{r}_j(t))^2}{1 + \beta(\mathbf{r}_i(t) - \mathbf{r}_j(t))^2} \leq 0 \quad (17)$$

Equation (14) with eq.(15) is rewritten by putting $z_j = x_j + iy_j$ as

$$\begin{aligned} \frac{dz_j}{dt} = & \sum_{\ell \neq j} \left[-\frac{\alpha_\ell(z_j - z_\ell)}{1 + \beta_j |z_j - z_\ell|^2} \right. \\ & \left. + i \frac{\kappa_\ell(0) e^{-2\alpha_\ell t}}{2} \cdot \frac{z_j - z_\ell}{1 + \beta_j |z_j - z_\ell|^2} \right] - i \frac{\Omega}{2} z_j \end{aligned} \quad (18)$$

The last term of the right hand side is eliminated by the transformation $z \rightarrow z \exp[-i(\Omega/2)t]$. The velocity field acting on a certain vortex from the rest is different, by screening effects, from the Euler point vortex equation given as

$$\frac{dz_j}{dt} = i \sum_{\ell \neq j} \kappa_\ell \frac{z_j - z_\ell}{|z_j - z_\ell|^2} \quad (19)$$

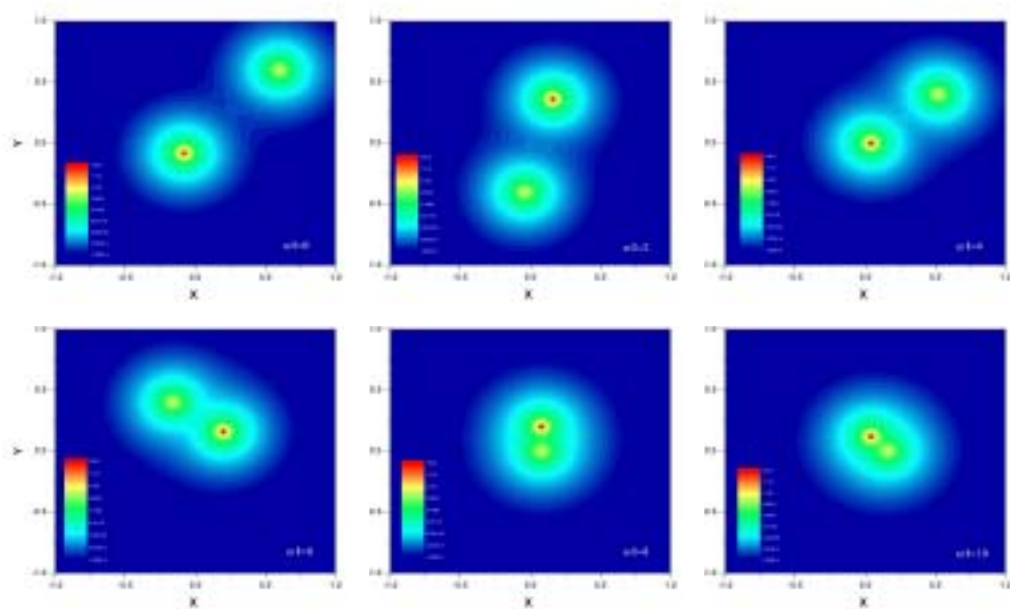


Fig.4 Vortex merging (2 vortices)

The snapshots of configurations are shown in Fig.4 for the case that $\kappa_1(0)=300$, $\kappa_2(0)=3$, $\beta_1=100$, $\beta_2=1$ $v=0.05$. The vortex with smaller vorticity rotates around the one with larger vorticity no matter how large they are. This shows a good agreement with the experimental observations in non-neutral

plasmas[7]. This figure (2 vortices merging) shows a clear contrast to the inviscid case ($\nu=0$) where vortices rotate around each other (see Fig. 5).

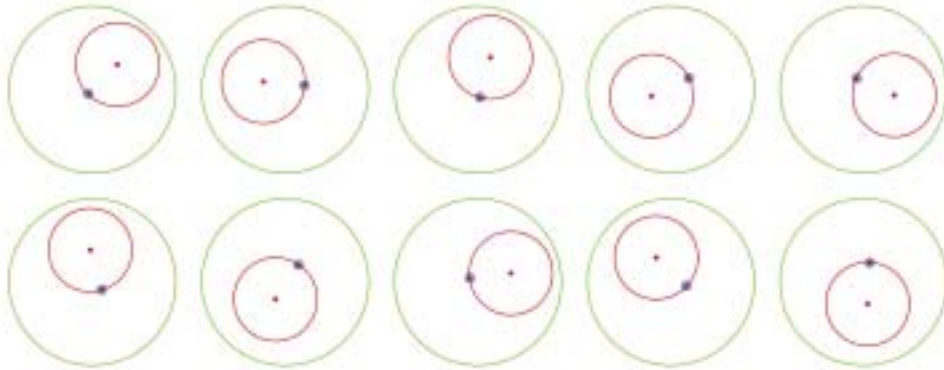


Fig. 5 Vortex dynamics for $\nu=0$ case

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