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Asymptotic Kinetic Theory of Magnetized Plasmas: Quasi-Particle Concept

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Abstract.

The asymptotic kinetic theory of magnetized plasmas is elaborated within the context of general statistical approach and asymptotic methods [1-5], developed by M. Krylov and M. Bohol'ubov, for linear and non-linear dynamic systems with a rapidly rotating phase [4]. The quasi-particles are introduced already on the microscopic level. Asymptotic expansions enable to close the description for slow processes, and to relate consistently particles and guiding centres to quasi-particles. The kinetic equation for quasi-particles is derived. It makes a basis for the reduced description of slow collective phenomena in the medium. The kinetic equation for quasi-particles takes into account self-consistent interaction fields, quasi-particle collisions and collective-fluctuation-induced relaxation of quasi-particle distribution function. The relationships between the distribution functions for particles, guiding centres and quasi-particles are derived taking into account fluctuations, which can be especially important in turbulent states. In this way macroscopic (statistical) particle properties can be obtained from those of quasi-particles in the general case of non-equilibrium media.

Key words: magnetized plasma; asymptotic methods; quasi-particles; kinetic equations; fluctuations and turbulence

Classification: 52.25.Dg; 52.35.Ra

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1 Introduction

The slow time evolution of nonlinear dynamic systems is frequently of concern. Then the reduction of the original description by means of asymptotic methods, which employ explicitly the separation of interesting time scales, is facilitating for both analytical and computational studies. This has been appreciated for a long time [1-7].

A peculiarity of the plasma as the system of many charged particles with long-range interaction between them is in its collective behaviour [8]. In non-equilibrium or turbulent plasmas the collective degrees of freedom are excited, and nonlinear effects become important. Diagnostics progress enables one to determine the characteristics of such plasmas in many more detail. Also, modern computers, especially those with parallel architecture, and advanced numerical simulation techniques make it possible to study complicated realistic situations.

Experimental observations and numerical simulations of space and fusion plasmas reveal their very complex behaviour, with low-frequency (with respect to the ion cyclotron frequency) oscillations.

Such oscillations and nonlinear interaction between them are possibly related to anomalous transport phenomena in the magnetized plasma. This makes one especially interested in the slow plasma motion, and the fundamental knowledge of the latter [9]. Experiments and simulations require strong theoretical support to set guidelines for the measurement interpretation, and developing benchmark experimental and simulation models. Comprehensive understanding of kinetic and nonlinear phenomena in plasmas under natural or laboratory conditions, involving low-frequency fluctuations as a ubiquitous feature, is indeed a great challenge in theory, experiments and simulations.

Even with the rapidly increasing power of computers, many important problems in plasma physics cannot be solved with conventional simulation techniques, for example particle-in-cell simulation. The latter model treats plasmas on the most basic level, using Maxwell's equations and Newton's law directly. Problems such as modeling a Tokamak for fusion energy research or modeling the magnetosphere for space plasma physics, have so large a range of space and time scales that the calculation is prohibitive on any foreseeable computer. These so called Grand Challenge problems require a reduced description if they are to be solved, where only some of the physical processes are included, hopefully the important ones.

Therefore, the general asymptotic and statistical methods become of fundamental importance for developing the reduced theory based on a self-consistent quasi-particle concept, and for providing a justification for quasi-particles introduced in the computational plasma studies.

Our objective is to derive from the first principles the Vlasov-type kinetic equation

for the quasi-particles in the magnetized plasma taking into account both mean self-consistent fields and plasma fluctuations.

In Sec. 2, basic quantities and equations are introduced. In Sec. 3, the closure of the quasi-particle description is considered. In Sec. 4 the adiabatic approximation is introduced, and the main features of the first-order theory and physical meaning of renormalizations are explained. Quasi-particle properties in the first-order adiabatic approximation are presented in Sec. 5. Sec. 6 deals with quasi-particle kinetics and fluctuations. Sec. 7 makes a summary.

2 Particles and Quasi-Particles

In this section, microscopic densities in the particle, guiding-centre and quasi-particle phase spaces are introduced, and the equations that they satisfy are formulated.

Microscopic density F in particle phase spaces. Let us consider a system of non-relativistic charged particles (which can contain just one particle) of charge q and mass m in an external stationary and uniform magnetic field $\vec{B} \equiv B\vec{b}$, and additional force fields producing particle acceleration \vec{a} . The latter can include self-consistent microscopic fields created by particles themselves and externally imposed fields. The particle trajectory $\{ \vec{r}_p(t), \vec{v}_p(t) \}$ in phase space $\{ \vec{r}, \vec{v} \}$ is determined from the following equations:

$$\frac{d\vec{r}_p(t)}{dt} = \vec{v}_p(t), \quad \frac{d\vec{v}_p(t)}{dt} = \vec{v}_p(t) \times \vec{\Omega} + \vec{a}(\vec{r}_p(t), \vec{v}_p(t), t), \quad (1)$$

with $\vec{\Omega} \equiv q\vec{B}/mc$, $\Omega \equiv qB/mc$ the cyclotron frequency.

The microscopic density in particle phase space,

$$F(\vec{r}, \vec{v}, t) = \sum \delta(\vec{r} - \vec{r}_p(t)) \delta(\vec{v} - \vec{v}_p(t)), \quad (2)$$

with the summation over all particles of the same charge and mass, satisfies a continuity equation:

$$(\partial_t + \vec{v} \cdot \vec{\nabla}) F + \partial_{\vec{v}} \cdot \{ [\vec{a}(\vec{r}, \vec{v}, t) + \vec{v} \times \vec{\Omega}] F \} = 0. \quad (3)$$

Microscopic density F' in guiding-centre phase spaces. In order to allow for finite-Larmor-radius effects, associated with the non-uniformity of acceleration field \vec{a} at scales comparable to particle Larmor radii, one usually exploits the guiding-centre transformation from the particle \vec{r} to the guiding-centre position \vec{R} :

$$\vec{r} = \vec{R} + \vec{b} \times \vec{v} / \Omega. \quad (4)$$

The equations of guiding-centre motion have the standard form of a dynamic system with rapidly rotating phase as introduced by Boholiubov and Zubarev [4, 5], when the cylindrical coordinates in the particle velocity space are used, and for any guiding-centre transformation which coincides with (4) to the leading order of the asymptotic theory. It is convenient to specify such a transformation later, and introduce formally guiding-centre variables $\{\vec{R}, v_{\parallel}, \epsilon_{\perp}, \alpha\} \equiv \{\vec{x}, \alpha\}$ with respect to the triad of orthogonal unity vectors \vec{b} , \vec{N} , and $\vec{\beta}$, where the label \perp (\parallel) is for a vector component across(along) \vec{b} :

$$\vec{r} = \vec{r}(\vec{R}, \vec{v}) . \quad (5)$$

$$v_{\parallel} = \vec{b} \cdot \vec{v} , \quad \vec{v}_{\perp} = \vec{v} - v_{\parallel} \vec{b} = v_{\perp} [\cos\alpha \vec{N} + \sin\alpha \vec{\beta}] , \quad \epsilon_{\perp} \equiv v_{\perp}^2/2$$

For any function $Q(\vec{r}, \vec{v}, t)$ of particle variables,

$$\tilde{Q} \equiv \tilde{Q}(\vec{R}, v_{\parallel}, \epsilon_{\perp}, \alpha, t) = Q(\vec{r}, \vec{v}, t) , \quad (6)$$

after \vec{r} and \vec{v} are expressed in guiding-centre variables.

The equations of guiding-centre motion follow from (1), (5):

$$\begin{aligned} \frac{d\vec{R}_c}{dt} &= \vec{V}(\vec{x}, \alpha, t) , \quad \frac{dv_{\parallel c}}{dt} = \vec{b} \cdot \tilde{\vec{a}} \equiv \tilde{a}_{\parallel} , \quad \frac{d\epsilon_{\perp c}}{dt} = \vec{v}_{\perp} \cdot \tilde{\vec{a}} \equiv \tilde{a}_{\epsilon_{\perp}} , \\ \frac{d\alpha_c}{dt} &= -\Omega - \nu , \quad \nu \equiv -\frac{1}{v_{\perp}^2} (\vec{b} \times \vec{v}_{\perp}) \cdot \tilde{\vec{a}} , \end{aligned} \quad (7)$$

with ν the shift in the gyration frequency, \tilde{a}_{\parallel} ($\tilde{a}_{\epsilon_{\perp}}$) the acceleration in v_{\parallel} (ϵ_{\perp}). The latter quantities are determined by the acceleration projections in the moving local reference frame:

$$\vec{b} , \quad \hat{v} \equiv \vec{v}_{\perp}/v_{\perp} , \quad \hat{\alpha} \equiv \vec{b} \times \hat{v} .$$

Any vector \vec{A} can be decomposed in this frame:

$$\vec{A} = A_{\parallel} \vec{b} + A_{v_{\perp}} \hat{v} + A_{\alpha} \hat{\alpha} , \quad \partial_{\vec{v}} = \partial_{v_{\parallel}} \vec{b} + \partial_{\epsilon_{\perp}} v_{\perp} \hat{v} + \frac{1}{v_{\perp}} \partial_{\alpha} \hat{\alpha} .$$

All the quantities on the right-hand sides of the equations (7) are evaluated along the trajectory in the guiding-centre phase space $\{\vec{x}, \alpha\}$, while the guiding-centre velocity can be found after transformation (5) is specified. For the transformation (4): $\vec{V} = v_{\parallel} \vec{b} + \tilde{\vec{a}} \times \vec{b}/\Omega$.

The microscopic density in the guiding-centre phase space is introduced like in the particle one,

$$F' \equiv F'(\vec{x}, \alpha, t) = \sum \delta(\vec{x} - \vec{x}_c(t)) \sum_n \delta(\alpha - \alpha_c(t) + 2\pi n) , \quad (8)$$

with the summation over all guiding centres. This density satisfies also a continuity equation but in its proper phase space:

$$\partial_t F' + \partial_{\vec{x}} \cdot (\vec{X} F') - \partial_\alpha [(\Omega + \nu) F'] = 0. \quad (9)$$

Here, $\vec{X} \equiv \{\vec{V}, \tilde{a}_\parallel, \tilde{a}_{\epsilon_\perp}\}$ is a generalized velocity corresponding to the phase-space variable $\vec{x} \equiv \{\vec{R}, v_\parallel, \epsilon_\perp\}$. The infinite sum over n in (8) provides the periodicity of the quantity of interest with respect to the gyrophase α (gyroangle).

Microscopic density G in quasi-particle phase spaces. The exact microscopic density G in the quasi-particle (reduced) phase space \vec{x} is introduced in relation to the gyrophase-independent part of F' :

$$G \equiv 2\pi \int d\alpha F'. \quad (10)$$

Such a quantity also satisfies a continuity equation, which follows from (9) after integration over the gyrophase α . The equation obtained in this way is not closed, as it contains other harmonics of F' as unknown quantities. The harmonics are introduced by means of Fourier series,

$$F' = \frac{1}{2\pi} G + \sum_{n \neq 0} e^{-i\alpha n} F_n, \quad F_0 \equiv \frac{1}{2\pi} G. \quad (11)$$

The infinite set of harmonics provides complete information on the gyrophase dependence within the context of exact description. For slow time evolution, one can achieve closure after relating the harmonics of F' to G by means of an asymptotic expansion in a small parameter. In this way the quasi-particle description is constructed formally up to arbitrary orders.

It is important to notice that the notion of guiding centre introduced above differs from the one usually encountered in the guiding-centre plasma models and drift-kinetic theories elaborated on their basis. In such models, e.g. [10, 11], the guiding-centre is introduced after averaging over the gyrophase, like the quasi-particle in the present theory.

Let us consider the continuity equation for G in the reduced phase space $\vec{x} \equiv \{\vec{R}, v_\parallel, \epsilon_\perp\}$:

$$\partial_t G + \partial_{\vec{x}} \cdot (\vec{X}_0 G) = S, \quad S \equiv -2\pi \sum_{n \neq 0} \partial_{\vec{x}} \cdot (\vec{X}_n F_n^*), \quad (12)$$

with the effects of other harmonics represented by a source term S on the right-hand side. Any quantity $Q(\vec{x}, v_\parallel, v_\perp, \alpha, t)$ of guiding-centre variables is represented by the Fourier series,

$$Q = \sum_n e^{-i\alpha n} Q_n, \quad Q_n \equiv \{Q\}_n = \frac{1}{2\pi} \int_0^{2\pi} d\alpha Q e^{i\alpha n}. \quad (13)$$

The harmonics $F_n(\vec{x}, t)$ are governed by an interconnected chain of equations

$$(\partial_t + in\Omega)F_n + \sum_{n_1 + n_2 = n} [\partial_{\vec{x}} \cdot (\vec{X}_{n_1} F_{n_2}) + in\nu_{n_1} F_{n_2}] = 0. \quad (14)$$

It is necessary to specify initial conditions. Given the initial particle phase space position $\{\vec{r}(t_0), \vec{v}(t_0)\}$, the guiding-centre position $\{\vec{x}_c(t_0), \alpha_c(t_0)\}$ is found by virtue of (5), while the initial quasi-particle position is $\vec{x}_c(t_0)$, $G(\vec{x}, t_0) = \sum \delta(\vec{x} - \vec{x}_c(t_0))$. The summation is over all quasi-particles, with their number being equal to the number of guiding centres and particles,

$$\int d\vec{x} G = \int d\vec{x} d\alpha F' = \int d\vec{r} d\vec{v} F.$$

When the number of particles is conserved the number of quasi-particles is also conserved. The conservation of quasi-particles has nothing to do with the specific structure of the equation (12) or its approximations. The exact equation for G (or any of its approximations) complies with the conservation properties of the original continuity equation. The structure of the latter in the case considered implies no local sources or sinks of particles and no factors producing random particle motion (at the very basic level of description no information is lost yet). These particle properties are not to be preserved in general.

Equation (12) is exact. If its right-hand side is treated as given, then the general solution is the sum of the solution G_h of the homogeneous equation

$$\partial_t G_h + \partial_{\vec{x}} \cdot (\vec{X}_0 G_h) = 0, \quad (15)$$

and the particular solution of the inhomogeneous equation. If the homogeneous solution is specified to take into account the initial condition, then it can be treated as a microscopic quasi-particle density, and

$$G_h(\vec{x}, t) = \sum \delta(\vec{x} - \vec{x}_q(t)), \quad \frac{d\vec{x}_q(t)}{dt} = \vec{X}_0(\vec{x}_q(t), t). \quad (16)$$

That is, the quasi-particle is point-wise. It is governed by the above equations of motion. The quasi-particle has initially the same position, parallel velocity and the magnitude of perpendicular velocity as the original guiding centre. Initial coordinates of the guiding centre and the particle are related by (5). In general, two close particles can be represented by two distant quasi-particles, and vice versa. Quasi-particle equations of motion look like equations (7) averaged in some sense over the gyrophase. However no approximations have been introduced yet.

3 Asymptotic Expansions

In this section, a formal expansion is introduced for the deviation of the guiding-centre density from the quasi-particle one, and a closed quasi-particle description is developed on its basis to arbitrary orders in a small parameter.

Following Boholiubov and Krylov, the closure can be achieved by means of a formal expansion in powers of a small parameter λ ($\omega/\Omega \sim \lambda$, where ω is a characteristic frequency and Ω is a cyclotron frequency):

$$F' = \frac{1}{2\pi} G + \sum_{j=1}^{\infty} F'^{(j)}, \quad (17)$$

where

$$F'^{(j)} = \sum_{n \neq 0} e^{-i\alpha n} F_n^{(j)}, \quad (18)$$

and

$$F_n = \sum_{j=1}^{\infty} F_n^{(j)} \sim \lambda G, \quad \lambda \ll 1. \quad (19)$$

Then the source term from (12) reappears as an expansion in increasing powers of λ :

$$S = \sum_{j=1}^{\infty} S^{(j)} \equiv -2\pi \sum_{n \neq 0} \partial_{\vec{x}} \cdot (\vec{X}_n^* F_n^{(j)}). \quad (20)$$

After the closure incomplete information on the gyrophase dependence becomes available only in the form of a probability distribution. The closure makes it possible to relate guiding-centre (particle) and quasi-particle characteristics. Equation (17) introduces a general relationship between the guiding-centres and the quasi-particles.

Combining (17) and (14), one derives an equation for the first iteration

$$[\hat{D}_0 + in(\Omega + \nu_0)] F_n^{(1)} = -\frac{1}{2\pi} [in\nu_n G + \partial_{\vec{x}} \cdot (\vec{X}_n G)], \quad (21)$$

as well as the recurrence relations for higher-order iterations:

$$\begin{aligned} & [\hat{D}_0 + in(\Omega + \nu_0)] F_n^{(j+1)} = \\ & = - \sum_{n_1 + n_2 = n, n_1 n_2 \neq 0} [in\nu_{n_1} F_{n_2}^{(j)} + \partial_{\vec{x}} \cdot (\vec{X}_{n_1} F_{n_2}^{(j)})]. \end{aligned} \quad (22)$$

Here for arbitrary $g(\vec{x}, t)$, $\hat{D}_0 g \equiv \partial_t g + \partial_{\vec{x}} \cdot (\vec{X}_0 g)$.

In the simplest approximation the influence of other harmonics on the space-time evolution of the quasi-particle density is completely disregarded:

$$\partial_t G + \partial_{\vec{x}} \cdot (\vec{X}_0 G) = 0, \quad F' = \frac{1}{2\pi} G. \quad (23)$$

The discussion of the quasi-particle nature and the derivation of the quasi-particle equations of motion presented in the previous section are also appropriate in this case.

The guiding-centre density is no more the delta-function of the trajectory in guiding-centre phase space, and its interpretation as a probability distribution is natural. After the closure a part of information about the gyrophase, as well as about rapid variations of other coordinates, is lost. The reduced guiding-centre distribution F' contains but incomplete information on the gyrophase dependence. An expansion of phase-space point, which represents the guiding centre (or the particle), to a probability "circle" occurs.

Let us illustrate this point for the case of uniformly magnetized plasma. Within the original exact description, a microscopic particle density in the physical space $n(\vec{r}, t) \equiv \int d\vec{v} F = \delta(\vec{r} - \vec{r}_p(t))$. Meanwhile, taking into account the guiding-centre transformation in the usual form $\vec{R} = \vec{r} + \vec{v} \times \vec{b}/\Omega$, one obtains from (23):

$$n(\vec{r}, t) = \frac{1}{2\pi} \delta(r_{\parallel} - R_{\parallel q}(t)) \delta[(\vec{r}_{\perp} - \vec{R}_{\perp q}(t))^2 - r_L^2], \quad (24)$$

where $r_L \equiv v_{\perp q}(t)/\Omega$ is the Larmor radius. The latter equation enables one to imagine a probability circle of radius r_L moving in the physical space with its plane being perpendicular to \vec{B} , and its center being at the point $\vec{R}_q(t)$. The probability to find the particle at any point of this circle is the same, according to (24). This probability circle can be distorted due to the effects of harmonics. Eq.(24) predicts a "visually observable" motion of probability circle with the velocity \vec{V}_0 , while the particle velocity calculated from the reduced distribution in (23) is $v_{\parallel q}(t) \vec{b}$. In order to describe particle drift motion across the magnetic field higher-order approximations are needed.

4 First-Order Theory

The adiabatic approximation, the main features of the first-order theory, and the physical meaning of renormalizations are treated here.

We introduce the adiabatic approximation assuming that

$$(\partial_t + in\nu_0) F_n + \partial_{\vec{x}} \cdot (\vec{X}_0 F_n) \sim \lambda n\Omega F_n. \quad (25)$$

According to this approximation, the slow time evolution of distribution functions is considered only (as compared to the rapid particle gyration with the cyclotron frequency), and the forces applied to the particle are assumed to produce the guiding centre accelerations and drift velocities slowly evolving in time, and weak enough.

Combining (17) and (25) with (14), one finds the first-order iteration $F_n^{(1)}$ in terms of quasi-particle density:

$$F_n^{(1)} = \frac{i}{2\pi n\Omega} [in\nu_n G + \partial_{\vec{x}} \cdot (\vec{X}_n G)], \quad (26)$$

$$F'^{(1)} = - \frac{1}{2\pi\Omega} (\nu - \nu_0) G + \frac{1}{2\pi\Omega} \int^\alpha d\alpha \partial_{\vec{x}} \cdot [(\vec{X} - \vec{X}_0) G],$$

and the recurrence relations for higher-order iterations,

$$\begin{aligned} F_n^{(j+1)} &= \frac{i}{n\Omega} \{ (\partial_t + in\nu_0) F_n^{(j)} + \partial_{\vec{x}} \cdot (\vec{X}_0 F_n^{(j)}) + \\ &+ \sum_{n_1 + n_2 = n, n_1 n_2 \neq 0} [\partial_{\vec{x}} \cdot (\vec{X}_{n_1} F_{n_2}^{(j)}) + in\nu_{n_1} F_{n_2}^{(j)}] \}, \quad (27) \\ F'^{(j+1)} &= - \frac{1}{\Omega} (\nu F'^{(j)} - \{ \nu F'^{(j)} \}_0) + \\ &+ \frac{1}{\Omega} \int^\alpha d\alpha [\partial_t F'^{(j)} + \partial_{\vec{x}} \cdot (\vec{X} F'^{(j)} - \{ \vec{X} F'^{(j)} \}_0)]. \end{aligned}$$

In the first-order approximation, the source term is involved in the time evolution of quasi-particle density, and the relevant contribution of harmonics into the guiding-centre density becomes important:

$$F_n = F_n^{(1)}, \quad S = S^{(1)}, \quad F' = \frac{1}{2\pi} G + \sum_{n \neq 0} e^{-i\alpha n} F_n^{(1)}. \quad (28)$$

In the adiabatic approximation, the expression (26) is valid, and the microscopic relationship between particles and quasi-particles takes the form

$$F' = \frac{1}{2\pi} G - \frac{1}{2\pi\Omega} (\nu - \nu_0) G + \frac{1}{2\pi\Omega} \int^\alpha d\alpha \partial_{\vec{x}} \cdot [(\vec{X} - \vec{X}_0) G]. \quad (29)$$

When the adiabatic approximation (25) is used to calculate the source term, one discovers that the structure of the microscopic equation for the quasi-particle density remains unchanged:

$$\partial_t G + \partial_{\vec{x}} \cdot (\vec{X}_r G) = 0. \quad (30)$$

Here, the contribution from $F_n^{(1)}$ into the source term is taken into account by renormalizing the quasi-particle velocity and accelerations. For any function $Q(\vec{x}, \alpha, t)$ of guiding-centre phase-space variables the renormalized value Q_r is

$$Q_r \equiv Q_r(\vec{x}, t) = Q_0 - \sum_{n \neq 0} (\nu_n + \frac{i}{n} \vec{X}_n \cdot \partial_{\vec{x}}) \frac{1}{\Omega} Q_n^*. \quad (31)$$

The structure of the continuity equation in the reduced phase space, and the initial conditions determine the nature of quasi-particles and their equations of motion.

However, the structure of the original continuity equation does not preclude in general the effects of random motion, or local sources and sinks for quasi-particles. Such effects can be involved within reduced theory approximations.

The solution to (30) taking account of initial conditions can be represented in the form of zero-order approximation:

$$G = \sum \delta(\vec{x} - \vec{x}_q(t)) , \quad \frac{d}{dt} \vec{x}_q(t) = \vec{X}_r(\vec{x}_q(t), t) , \quad (32)$$

which means point quasi-particles.

In order to elucidate the relation of renormalized (quasi-particle) quantities to the particle, let us take an arbitrary function $Q(\vec{r}, \vec{v}, t)$ of particle dynamical variables and derive a reduced expression for it. It is convenient to introduce the following notations:

$$\begin{aligned} \langle \dots \rangle_F &\equiv \int d\vec{r} d\vec{v} (\dots) F , & \langle \dots \rangle_{F'} &\equiv \int d\vec{x} d\alpha (\dots) F' , \\ \langle \dots \rangle_G &\equiv \int d\vec{x} (\dots) G , & d\vec{x} &\equiv d\vec{R} dv_{\parallel} d\epsilon_{\perp} . \end{aligned} \quad (33)$$

For arbitrary quantity $Q(\vec{r}, \vec{v}, t)$ of particle variables, we notice easily that

$$Q(t) \equiv Q(\vec{r}_p(t), \vec{v}_p(t), t) \equiv \langle Q \rangle_F = \langle \tilde{Q} \rangle_{F'} = \langle \tilde{Q}_0 \rangle_G + 2\pi \sum_{n \neq 0} \int d\vec{x} \tilde{Q}_n^* F_n$$

from where the reduced expression follows immediately:

$$\langle Q \rangle_F = \langle \tilde{Q}_0 \rangle_G + 2\pi \sum_{j=1}^{\infty} \sum_{n \neq 0} \int d\vec{x} \tilde{Q}_n^* F_n^{(j)} . \quad (34)$$

In the first-order adiabatic approximation, the term arising from $F_n^{(1)}$ is retained only in the infinite sum. With $F_n^{(1)}$ from (26),

$$\langle Q \rangle_F = \langle Q_r \rangle_G . \quad (35)$$

Clearly, $\langle Q \rangle_F = Q_r(\vec{x}_q(t), t)$ if the quasi-particle is point-wise.

The following exact relations hold:

$$\vec{R}_c(t) \equiv \langle \vec{R} \rangle_{F'} = \langle \vec{R} \rangle_G , \quad (36)$$

$$v_{\parallel c}(t) \equiv \langle v_{\parallel} \rangle_{F'} = \langle v_{\parallel} \rangle_G , \quad v_{\perp c}(t) \equiv \langle v_{\perp} \rangle_{F'} = \langle v_{\perp} \rangle_G$$

(and similar relations for any other functions of guiding-centre variables except the gyrophase). That is, the mean guiding-centre position is exactly the mean quasi-particle position, and in the reduced theory it is so to arbitrary orders in the small parameter. Within the adiabatic approximation, when the quasi-particle is point-wise, $\langle \vec{R} \rangle_{F'} = \vec{R}_q(t)$: the quasi-particle and mean guiding-centre positions coincide. Similar relationship holds for the parallel velocities and the magnitudes of perpendicular velocities of the guiding centre and the quasi-particle.

5 Quasi-Particle Constants of Motion

Here the quasi-particle properties are studied in the case of potential acceleration which can correspond to the potential electric field.

Let us assume that the particle acceleration derives from a velocity-independent potential:

$$\vec{a} = -\vec{\nabla}\theta, \quad \theta \equiv \theta(\vec{r}, t), \quad (37)$$

and adopt the usual guiding-centre transformation (4), when the guiding-centre velocity is:

$$\vec{V} = v_{\parallel} \vec{b} + \frac{1}{\Omega} \tilde{a} \times \vec{b}.$$

Then, in the case of uniform magnetic fields, particle acceleration expressed in guiding-centre variables is governed by a potential as well, we denote this potential by $\tilde{\theta}$:

$$\tilde{a} = -\partial_{\vec{R}}\tilde{\theta}, \quad \partial_{\vec{R}} \cdot \vec{V} = 0. \quad (38)$$

Therefore, the flow in the guiding-centre configuration space is incompressible.

Other derivatives of the potential are also expressed in guiding-centre variables. Such derivatives produce a shift of the gyration frequency, and an acceleration in ϵ_{\perp} :

$$\nu = \Omega \partial_{\epsilon_{\perp}}\tilde{\theta}, \quad \tilde{a}_{\epsilon_{\perp}} = \Omega \partial_{\alpha}\tilde{\theta}, \quad \partial_{\alpha}\nu - \partial_{\epsilon_{\perp}}\tilde{a}_{\epsilon_{\perp}} = 0. \quad (39)$$

Therefore, the flow in the subspace $\{\epsilon_{\perp}, \alpha\}$ is also incompressible. These properties hold in the case of a non-uniformly magnetized plasma as well.

Thus, the flow in the guiding-centre phase space appears to be incompressible:

$$\partial_{\vec{x}} \cdot \vec{X} - \partial_{\alpha}\nu = 0, \quad (40)$$

as $\partial_{v_{\parallel}}\tilde{a}_{\parallel} = 0$. One proves a general *Theorem*: if the flow in the original phase space is incompressible, then the renormalized flow in the reduced (quasi-particle) phase space is also incompressible:

$$\partial_{\vec{x}} \cdot \vec{X} - \partial_{\alpha}\nu = 0 \quad \rightarrow \quad \partial_{\vec{x}} \cdot \vec{X}_r = 0. \quad (41)$$

Proof. Definition (31) takes this form in the case (40):

$$Q_r = Q_0 - \partial_{\vec{x}} \cdot \sum_{n \neq 0} \frac{i}{n\Omega} \vec{X}_n Q_n^*. \quad (42)$$

It is evident that $\partial_{\vec{x}} \cdot \vec{X}_0 = 0$, since the operator $\partial_{\vec{x}}$ is gyrophase-independent by definition. When $Q = \vec{X}$, the divergence of the second term on the right-hand side of (42) changes sign after n is replaced by $-n$, and therefore vanishes. This completes the proof.

In the case of potential acceleration,

$$Q_r = Q_0 - \sum_{n \neq 0} [\partial_{\epsilon_{\perp}} (\tilde{\theta}_n Q_n^*) + \frac{i}{n\Omega} (\tilde{a}_{\parallel} n \partial_{v_{\parallel}} + \vec{V}_n \cdot \partial_{\vec{R}}) Q_n^*], \quad (43)$$

with $\tilde{a}_{\epsilon_{\perp}n} = -in\Omega\tilde{\theta}_n$ from (39). In particular, the renormalized potential

$$\tilde{\theta}_r = \tilde{\theta}_0 - \sum_{n \neq 0} [\partial_{\epsilon_{\perp}} |\tilde{\theta}_n|^2 + \frac{i}{n\Omega^2} \tilde{a}_n \times \tilde{a}_n^* \cdot \vec{b}], \quad (44)$$

and the renormalized acceleration:

$$\tilde{a}_{\epsilon_{\perp}r} = \Omega \{ \partial_{\alpha} \tilde{\theta} \}_r = \Omega \sum_{n \neq 0} in [\partial_{\epsilon_{\perp}} |\tilde{\theta}_n|^2 + \frac{i}{n\Omega^2} \tilde{a}_n \times \tilde{a}_n^* \cdot \vec{b}].$$

As the terms in the infinite sum are odd functions of n ,

$$\tilde{a}_{\epsilon_{\perp}r} = 0. \quad (45)$$

This property has important consequences: one can introduce a stream potential and discover the first-order quasi-particle invariant.

Stream potential. Incompressibility of renormalized flow in the reduced phase space together with vanishing $a_{\epsilon_{\perp}}$ imply that the renormalized flow in the quasi-particle configuration space is incompressible. Therefore, the renormalized quasi-particle velocity can be represented by a stream function, and that is the potential of the renormalized acceleration over Ω :

$$\partial_{\vec{R}} \cdot \vec{V}_r = 0 \rightarrow \tilde{a}_r = -\partial_{\vec{R}} \tilde{\theta}_s. \quad (46)$$

The potential $\tilde{\theta}_s$ of the renormalized acceleration (henceforth called the stream potential) is found from (43) with $Q = \tilde{a}$:

$$\tilde{a}_r = -\partial_{\vec{R}} \tilde{\theta}_0 + \sum_{n \neq 0} [\partial_{\epsilon_{\perp}} (\tilde{\theta}_n \frac{\partial \tilde{\theta}_n^*}{\partial \vec{R}}) + \frac{i}{n\Omega^2} \frac{\partial \tilde{\theta}_n}{\partial \vec{R}} \times \vec{b} \cdot \partial_{\vec{R}} \frac{\partial \tilde{\theta}_n^*}{\partial \vec{R}}].$$

By replacing here n with $-n$, and adding the resulting expression to the original one, we find:

$$\tilde{\theta}_s = \tilde{\theta}_0 - \frac{1}{2} \sum_{n \neq 0} [\partial_{\epsilon_{\perp}} |\tilde{\theta}_n|^2 + \frac{i}{n\Omega^2} \tilde{a}_n \times \tilde{a}_n^* \cdot \vec{b}]. \quad (47)$$

Thus the stream potential differs from the renormalized one.

The summation over n can be carried out:

$$\tilde{\theta}_s = \tilde{\theta}_0 - \frac{1}{2} \partial_{\epsilon_{\perp}} (\{ \tilde{\theta}^2 \}_0 + \tilde{\theta}_0^2) + \frac{1}{2\Omega} \{ \tilde{a} \cdot \int^{\alpha} d\alpha (\vec{V} - \vec{V}_0) \}_0, \quad (48)$$

$$\tilde{\theta}_r = 2\tilde{\theta}_s - \tilde{\theta}_0 . \quad (49)$$

First-order quasi-particle invariant. It is well known that the magnitude of the perpendicular particle velocity (or the Larmor radius, or the kinetic energy of cross-field motion, or the magnetic moment) is a constant of motion in a uniform magnetic field, when no other forces are applied. According to (45) and (32),

$$\frac{d\epsilon_{\perp} q}{dt} = \tilde{a}_{\epsilon_{\perp} r} = 0 ,$$

that is ϵ_{\perp} is an invariant of quasi-particle motion in the first order allowing for strongly non-uniform (at scales comparable to particle Larmor radii) but slowly evolving in time potential force fields.

Therefore, the continuity equation for quasi-particles simplifies:

$$(\partial_t + \vec{V}_r \cdot \partial_{\vec{R}} + \tilde{a}_{\parallel} r \partial_{v_{\parallel}}) G = 0 , \quad (50)$$

$$\vec{V}_r = v_{\parallel} \vec{b} + \frac{1}{\Omega} \tilde{a}_r \times \vec{b} , \quad \tilde{a}_r = -\partial_{\vec{R}} \tilde{\theta}_s , \quad \tilde{a}_{\parallel} r = -\vec{b} \cdot \partial_{\vec{R}} \tilde{\theta}_s .$$

The latter equation takes into account explicitly the first-order quasi-particle invariant and the existence of the stream potential.

According to general relations between mean values,

$$K_{\perp} \equiv \left\langle \frac{v_{\perp}^2}{2} \right\rangle_F = \langle \epsilon_{\perp} \rangle_{F'} = \langle \epsilon_{\perp} \rangle_G . \quad (51)$$

Therefore the reduced kinetic energy K_{\perp} (per unit mass) of cross-field particle motion is the first-order adiabatic invariant, with regard to strongly non-uniform potential force fields besides stationary and uniform magnetic fields. This result must be reconsidered in the next orders.

Microscopic relationship between guiding-centres and quasi-particles. The microscopic relationship (29) between guiding-centres and quasi-particles simplifies in this case

$$F' = \frac{1}{2\pi} \left[1 + \frac{1}{\Omega} \int^{\alpha} d\alpha (\vec{X} - \vec{X}_0) \cdot \partial_{\vec{x}} \right] G , \quad (52)$$

where the incompressibility property (41) has been taken into account. In the explicit form,

$$F' = \frac{1}{2\pi} \left\{ G + (\theta - \theta_0) \partial_{\epsilon_{\perp}} G + \right. \\ \left. + \frac{1}{\Omega} \int^{\alpha} d\alpha \left[(a_{\parallel} - a_{\parallel 0}) \partial_{v_{\parallel}} + \frac{1}{\Omega} (\vec{a} - \vec{a}_0) \times \vec{b} \cdot \vec{\nabla} \right] G \right\} . \quad (53)$$

6 Quasi-Particle Kinetic Equation

Let us consider a system of many charged particles with self-consistent interaction, such as the plasma, and a corresponding system of quasi-particles with the microscopic phase-space density $G = \sum_p G_p$. The relation between the particle (guiding-centre) phase-space density and the quasi-particle phase-space density makes it possible to relate self-consistent fields to the quasi-particle phase-space density (*reduced Poisson equation*).

In order to describe quasi-particle kinetics and fluctuations, we apply a statistical approach and formulate an infinite hierarchy of equations for quasi-particle correlation functions, similar to the Boholiubov-Born-Green-Kirkwood-Yvon hierarchy for the particle correlation functions. We introduce the quasi-particle distribution function g as the statistical mean value of the microscopic quasi-particle density in phase space: $g \equiv \langle G \rangle$. Fluctuations δg of the quasi-particle distribution function are introduced according to the definition $G \equiv g + \delta g$, where $\langle \delta g \rangle \equiv 0$.

The statistical averaging of the continuity equation (12) for the microscopic quasi-particle density space yields the first equation of the hierarchy (called *kinetic equation for quasi-particles*) which is still exact. Asymptotic approximations can be obtained from such an equation, or after statistical averaging of approximate microscopic continuity equations.

In the first-order adiabatic approximation, statistical averaging of (30) yields:

$$\partial_t g + \partial_{\vec{x}} \cdot (\langle \vec{X}_r \rangle g) = - \partial_{\vec{x}} \cdot \langle \delta \vec{X}_r \delta g \rangle. \quad (54)$$

Here, for any quantity Q , the statistical mean value of the renormalized quantity is

$$\begin{aligned} \langle Q_r \rangle = \langle Q_0 \rangle - \sum_{n \neq 0} [(\langle \nu_n \rangle + \frac{i}{n} \langle \vec{X}_n \rangle \cdot \partial_{\vec{x}}) \frac{1}{\Omega} \langle Q_n^* \rangle + \\ + \langle (\delta \nu_n + \frac{i}{n} \delta \vec{X}_n \cdot \partial_{\vec{x}}) \frac{1}{\Omega} \delta Q_n^* \rangle], \end{aligned} \quad (55)$$

where a fluctuation contribution determined by pair correlation functions is to be noticed. For the incompressible flow in the particle phase space,

$$\langle Q_r \rangle = \langle Q_0 \rangle - \partial_{\vec{x}} \cdot \sum_{n \neq 0} \frac{i}{n\Omega} [\langle \vec{X}_n \rangle \langle Q_n^* \rangle + \langle \delta \vec{X}_n \delta Q_n^* \rangle]. \quad (56)$$

The term on the right-hand side of (54) is also determined by a pair correlation function. This term can be interpreted as a collision integral for quasi-particles. It takes into account the quasi-particle collisions and the collective-fluctuation-induced relaxation of quasi-particle distribution function, similarly to the particle kinetic

integrals. The collision integral can be expressed in terms of pair correlation functions for the incoherent and collective fluctuations [8].

The relationship between the particle distribution function $f \equiv \langle F \rangle$ and the quasi-particle distribution function g is obtained after statistical averaging of the general microscopic relationship (17) between the guiding-centres and the quasi-particles. In this way we relate macroscopic (statistical) particle and quasi-particle properties.

In the adiabatic approximation, statistical averaging of (29) yields the following relationship between the distribution functions for guiding-centres, $f' \equiv \langle F' \rangle$, and quasi-particles, g :

$$\begin{aligned} f' = & \frac{1}{2\pi} g - \frac{1}{2\pi\Omega} \langle \nu - \nu_0 \rangle g + \frac{1}{2\pi\Omega} \int^\alpha d\alpha \partial_{\vec{x}} \cdot (\langle \vec{X} - \vec{X}_0 \rangle g) - \\ & - \frac{1}{2\pi\Omega} [\langle (\delta\nu - \delta\nu_0) \delta G \rangle - \int^\alpha d\alpha \partial_{\vec{x}} \cdot \langle (\delta\vec{X} - \delta\vec{X}_0) \delta G \rangle] . \end{aligned} \quad (57)$$

This relation includes the correlation functions for fluctuations. The fluctuations can be essential for the plasma, turbulent one especially. Therefore, the relation between the statistical distributions of particles and quasi-particles can be rather complicated because of fluctuations in the medium, which are especially important in turbulent states.

For the uniformly magnetized plasma, the relationship between the particle and guiding-centre distribution functions is especially simple in the case of guiding-centre transformation (4):

$$f(\vec{r}, \vec{v}, t) = f'(\vec{R}, v_{\parallel}, \epsilon_{\perp}, \alpha, t) . \quad (58)$$

In the case of potential interaction between particles via the self-consistent electric field $\vec{E} = -\vec{\nabla}\Phi$, and for the uniform magnetic field:

$$\tilde{a}_{\epsilon_{\perp}r} = 0 , \quad \tilde{a}_{\parallel r} = -\frac{q}{m} \vec{b} \cdot \partial_{\vec{R}} \tilde{\Phi}_s , \quad \vec{V}_r = v_{\parallel} \vec{b} + \frac{c}{B} \vec{b} \times \partial_{\vec{R}} \tilde{\Phi}_s .$$

The kinetic equation (54) takes the form

$$\begin{aligned} [\partial_t + v_{\parallel} \nabla_{\parallel} + \frac{c}{B} \langle \vec{E}_r \rangle \times \vec{b} \cdot \vec{\nabla} + \frac{q}{m} \langle E_{\parallel r} \rangle \partial_{v_{\parallel}}] g = \\ = - \frac{c}{B} \langle \delta \vec{E}_r \times \vec{b} \cdot \vec{\nabla} \delta G \rangle - \frac{q}{m} \langle \delta E_{\parallel r} \partial_{v_{\parallel}} \delta G \rangle . \end{aligned} \quad (59)$$

The relation between the statistical distributions of particles and quasi-particles simplifies as well:

$$f' = \frac{1}{2\pi} [1 + \frac{1}{\Omega} \int^\alpha d\alpha \langle \vec{X} - \vec{X}_0 \rangle \cdot \partial_{\vec{x}}] g + \frac{1}{2\pi\Omega} \int^\alpha d\alpha \langle (\delta\vec{X} - \delta\vec{X}_0) \cdot \partial_{\vec{x}} \delta G \rangle . \quad (60)$$

Equations for the space-time quasi-particle correlation functions can be derived from the nonlinear equation for fluctuations,

$$\partial_t \delta g + \partial_{\vec{x}} \cdot (\langle \vec{X}_r \rangle \delta g + \delta \vec{X}_r \delta g - \langle \delta \vec{X}_r \delta g \rangle) = - \partial_{\vec{x}} \cdot (\delta \vec{X}_r g) , \quad (61)$$

by means of the general renormalization approach [9, 12]. In the linear with respect to fluctuations approximation,

$$\partial_t \delta g + \partial_{\vec{x}} \cdot (\langle \vec{X}_r \rangle \delta g) = - \partial_{\vec{x}} \cdot (\delta \vec{X}_r g) . \quad (62)$$

Such an approximation results in a *quasi-linear* kinetic equation for quasi-particles. We can introduce also renormalized linear approximations for fluctuations resulting in a *renormalized* quasi-linear kinetic equation for quasi-particles.

If collisions are disregarded, as well as the fluctuation-induced relaxation of quasi-particle distribution function, the kinetic equation (54) takes the form

$$\partial_t g + \partial_{\vec{x}} \cdot (\langle \vec{X}_r \rangle g) = 0 , \quad (63)$$

where

$$\langle Q_r \rangle \approx \langle Q_0 \rangle - \sum_{n \neq 0} [(\langle \nu_n \rangle + \frac{i}{n} \langle \vec{X}_n \rangle \cdot \partial_{\vec{x}}) \frac{1}{\Omega} \langle Q_n^* \rangle] . \quad (64)$$

For the incompressible flow in the particle phase space,

$$\langle Q_r \rangle = \langle Q_0 \rangle - \partial_{\vec{x}} \cdot \sum_{n \neq 0} \frac{i}{n\Omega} \langle \vec{X}_n \rangle \langle Q_n^* \rangle . \quad (65)$$

The equation (63) is the kinetic equation for quasi-particles in the *Vlasov (self-consistent field) approximation*. The latter must be supplemented with the relationship between the distribution functions for particles (guiding-centres) and quasi-particles. When the fluctuations are disregarded, such a relationship has the following form:

$$f' = \frac{1}{2\pi} g - \frac{1}{2\pi\Omega} \langle \nu - \nu_0 \rangle g + \frac{1}{2\pi\Omega} \int^{\alpha} d\alpha \partial_{\vec{x}} \cdot (\langle \vec{X} - \vec{X}_0 \rangle g) . \quad (66)$$

For the incompressible flow in the particle phase space,

$$f' = \frac{1}{2\pi} [1 + \frac{1}{\Omega} \int^{\alpha} d\alpha \langle \vec{X} - \vec{X}_0 \rangle \cdot \partial_{\vec{x}}] g . \quad (67)$$

In the case of potential interaction between particles, and for the uniform magnetic field, the kinetic equation in the self-consistent field approximation takes the form

$$[\partial_t + v_{\parallel} \nabla_{\parallel} + \frac{c}{B} \langle \vec{E}_r \rangle \times \vec{b} \cdot \vec{\nabla} + \frac{q}{m} \langle E_{\parallel r} \rangle \partial_{v_{\parallel}}] g = 0 . \quad (68)$$

When the fluctuations are disregarded, the relationship between the statistical distributions of particles and quasi-particles is the following:

$$f' = \frac{1}{2\pi} [1 + \frac{q}{m} (\Phi - \Phi_0) \partial_{\epsilon_{\perp}} + \frac{1}{\Omega} \int^{\alpha} d\alpha (\frac{c}{B} \langle \vec{E} - \vec{E}_0 \rangle \times \vec{b} \cdot \vec{\nabla} + \frac{q}{m} \langle E_{\parallel} - E_{\parallel 0} \rangle \partial_{v_{\parallel}})] g . \quad (69)$$

7 Discussion

The Vlasov-type kinetic equation for quasi-particles is formulated from the first principles, taking into account both mean self-consistent fields and plasma fluctuations. Such an equation can be used for the reduced description of the magnetized plasma when the slow time evolution (as compared to rapid particle gyrations in the magnetic field) is of interest. The quasi-particles are introduced within the context of microscopic description, and particles are related consistently to guiding-centres and quasi-particles.

The basic quantities of the theory are the microscopic densities in the particle, guiding-centre and quasi-particle phase spaces. Statistical averaging yields the relation between the distribution functions for particles, guiding-centres and quasi-particles. Such relations take into account plasma fluctuations, which can be especially important in turbulent plasma states.

The fundamental idea of the reduced theory is to avoid detailed information on rapid particle gyrations in the magnetic field. Such an information is contained in the dependence of the particle density on the gyrophase, or in the gyrophase harmonics of the guiding-centre density. The infinite set of harmonics provides complete information on the gyrophase dependence within the exact microscopic description. The reduced description of slow plasma motions results from the closure relating these harmonics to the quasi-particle density. As a result of closure, only incomplete information on the gyrophase dependence becomes available in the form of a probability distribution.

One can suggest the following basic idea for better understanding of reduced microscopic and kinetic descriptions. Let us consider the simplest situation of a particle in a stationary and uniform magnetic field. In a suitable frame the particle trajectory is a circle. If one measures the particle position with a tool which can not distinguish the time steps of gyration period order, one finds the particle here and there in a random manner. Gradually, in the course of measurements it will become clear that particle positions are distributed randomly on a circle. One can introduce a particle probability distribution which is non-zero and constant on that circle. The dipole moment of such a "probability circle" in the frame with the origin at the centre will appear to be zero.

The probabilistic approach appears to be natural. When the magnetic field is weakly non-uniform, and there are other forces, the original "probability circle" is distorted, and the problem is to describe this. For instance, one could suggest the "probability circle" becoming more like a ring with some dipole moment, since the particle deviates from the strictly circular orbit and spends more time within some parts of the ring. The magnetic moment is also perturbed. It is possible to contemplate some thermal energy of particle "random" motion.

The kinetic equation for quasi-particles can be applied to develop nonlinear low-

frequency electrodynamics of the magnetized plasma by means of approach [8, 9]. The successive approximation method enables one to find the formal solution to the quasi-particle kinetic equation as a series in ascending powers of the electric field strength. The particle density in phase space follows from the particle-guiding-centre and guiding-centre-quasi-particle relations. Then one can calculate consistently the low-frequency approximations for plasma electrodynamic coefficients (linear and nonlinear electric susceptibilities) and their fluctuations.

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