



# Stability analysis of the ideal internal kink mode in a toroidally rotating tokamak plasma with ultra flat $q$ profile

Christer Wahlberg

## ► To cite this version:

Christer Wahlberg. Stability analysis of the ideal internal kink mode in a toroidally rotating tokamak plasma with ultra flat  $q$  profile. 2004. hal-00001872

**HAL Id: hal-00001872**

**<https://hal.science/hal-00001872>**

Preprint submitted on 19 Oct 2004

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Stability analysis of the ideal internal kink mode in a toroidally rotating tokamak plasma with ultra flat $q$ profile

C. Wahlberg

*Department of Astronomy and Space Physics, EURATOM/VR Fusion Association,  
P.O. Box 515, Uppsala University, SE-751 20 Uppsala, Sweden*

**Introduction** In some tokamak experiments the safety factor  $q$  is very close to unity in a wide area in the plasma center, and increases up to  $q_a \sim 3-5$  (or larger) in the edge region. This is the case for instance in tokamak discharges in the so-called “hybrid” scenario [1], and in the experiments with the spherical tokamak NSTX reported in Ref. [2]. Static equilibria with this type of  $q$ -profile are susceptible to an  $m = n = 1$  ideal, internal kink instability, with an eigenfunction of convective, or “quasi-interchange” character [3-5]. By the absence of normal sawteeth it seems, however, that the ideal quasi-interchange instability does not develop for instance in the hybrid scenario discharges in JET reported in Ref. [6]. In NSTX, on the other hand, onset of  $m = n = 1$  MHD activity is observed, but some sort of stabilization of this activity by the toroidal rotation of the plasma seems to take place [7]. The rotation in NSTX is so strong (rotation frequencies up to 30 % of the Alfvén frequency) that the modification of the plasma equilibrium due to the centrifugal force is detectable [2]. In the JET experiments [6] the rotation, due to neutral beam injection (NBI), is probably much weaker, but can nevertheless be expected to be of the order of a few percent of the Alfvén frequency [1].

**Brunt-Väisälä effect** A potential mechanism for rotational stabilization of the quasi-interchange mode in the experiments above, as well as in other, similar experiments, is the Brunt-Väisälä (BV) effect, previously found to be able to stabilize both the usual (Bussac), ideal internal kink mode [8] as well as the Mercier modes [9] in tokamaks. In the present work the BV effect on the quasi-interchange mode is investigated by extending the ideal MHD theory of this mode in toroidal plasmas, developed in Refs. [4-5], to equilibria with toroidal mass flow. The expression for the BV frequency defined in Refs. [8-9] is given by

$$\omega_{BV}^2 = \omega_s^2 \left\{ 1 + \frac{1}{2q^2} + \frac{M_s^2(M_s^2 + 4)}{\Gamma} - \left[ \left( 1 + \frac{1}{2q^2} + \frac{M_s^2(M_s^2 + 4)}{\Gamma} \right)^2 - \frac{2M_s^4}{q^2\Gamma} \left( 1 - \frac{1}{\Gamma} \right) \right]^{1/2} \right\}. \quad (1)$$

It was found in Refs. [8-9] that stabilization of the internal kink and Mercier instabilities occurs when the BV frequency above exceeds the growth rate of the instability without

plasma rotation. In Eq. (1), the rotation frequency is assumed to be of order  $\Omega \sim \omega_s \sim \varepsilon \omega_A$ , where  $\omega_s = (\Gamma p_0 / \rho_0)^{1/2} / R_0$  is the sound frequency,  $\varepsilon = r/R_0 \ll 1$  the inverse aspect ratio,  $\omega_A = B_0 / R_0 (\mu_0 \rho_0)^{1/2}$  the Alfvén frequency and  $B_0$  the toroidal magnetic field. Furthermore,  $M_s = (\rho_0 \Omega^2 R_0^2 / 2 p_0)^{1/2} \sim 1$  is the sonic Mach number,  $R_0$  the major radius of the plasma center and  $\Gamma (= 5/3)$  the adiabatic index.

**Stability analysis** A system of equations describing the coupling of the  $(m, n) = (1, 1)$  and  $(m, n) = (2, 1)$  radial components  $\xi_1$  and  $\xi_2$  of the Lagrangian plasma displacement  $\xi \sim \exp(-i\omega t)$  in a rotating, toroidal plasma with large aspect ratio and circular cross section was derived from the Frieman-Rosenbluth equations in Ref. [8]. We apply this system to a situation where the magnetic shear is assumed small and  $q = 1 - \Delta q$  in the region  $0 \leq r \leq r_1$ , where  $\Delta q \sim \varepsilon$ , whereas in the edge region  $r_1 \leq r \leq a$ ,  $q - 1 \sim 1$  and the shear is of order unity. Furthermore, we assume that  $\Omega$  is independent of  $r$  (rigid rotation) and that the pressure profile is parabolic,  $p_0(r) = p_0(0)(1 - r^2/a^2)$ . In this case one obtains a constant value of  $\beta_p$  in the low-shear region, given by  $\beta_p = \beta_0 / (2\varepsilon_a^2)$ , where  $\beta_0 = 2\mu_0 p_0(0) / B_0^2$  and  $\varepsilon_a = a/R_0$ , and it turns out that the ideal MHD stability problem of the quasi-interchange mode in the rotating plasma is given by the following equation for  $\xi_1$  in the region  $0 \leq r \leq r_1$ , together with an integral condition on  $\xi_1$  coming from the boundary conditions at  $r = r_1$ :

$$\frac{d}{d\hat{r}} \left\{ \hat{r}^3 \left[ \left( \frac{\Delta q}{\varepsilon_a} \right)^2 - \hat{\omega}_D^2 + \hat{\Omega}^2 M_s^2 - \frac{(4\hat{\omega}_D^2 + \hat{\Omega}^2 / 2) \hat{\Omega}^2 + 2\hat{\omega}_D^2 \hat{\omega}_s^2}{\hat{\omega}_s^2 - \hat{\omega}_D^2} \right] \frac{d\xi_1}{d\hat{r}} \right\} - \frac{(2\hat{\omega}_D^2 - 2\hat{\omega}_D \hat{\Omega} + \hat{\Omega}^2)^2 \Gamma \beta_p \hat{r}^3}{(\hat{\omega}_s^2 - \hat{\omega}_D^2)^2} \xi_1 + 4D\beta_p \hat{r}^3 = 0, \quad (2a)$$

$$\int_0^{\hat{r}_1} \hat{r}^3 \xi_1 d\hat{r} = \frac{\hat{r}_1^3 (1 - C)}{16\beta_p}. \quad (2b)$$

Here,  $C = r_1 dx_2^+ / dr(r = r_1 + 0)$ ,  $D = (3 + C) / 4\hat{r}_1$ ,  $\xi_2 = \varepsilon_a x_2^+$  is the solution of the  $m = 2$  side-band equation for  $r_1 \leq r \leq a$ , and  $x_2^+$  is normalized as  $x_2^+(r_1) = 1$ . The  $m = 1$  amplitude  $\xi_1$  satisfies, in addition, the boundary conditions  $\xi_1(r_1) = 0$  and  $\xi_1'(0) = 0$ . Furthermore,  $\omega_D \equiv \omega + \Omega$  is the Doppler-shifted mode frequency as well as the eigenvalue of the problem (2a, b). Notice that  $\omega_D$  in general is complex,  $\omega_D = \omega_{Dr} + i\omega_{Di}$ , with positive  $\omega_{Di}$  indicating instability. The normalized radius and frequencies in Eq. (2a) are defined as  $\hat{r} = r/a$ ,  $\hat{\Omega} = \Omega / \varepsilon_a \omega_A$ ,  $\hat{\omega}_D = \omega_D / \varepsilon_a \omega_A$  and  $\hat{\omega}_s = \omega_s / \varepsilon_a \omega_A$ . A detailed derivation of the Eqs. (2a, b) will be published elsewhere [10].

In the case  $\Omega = 0$ , Eq. (2a) can be integrated analytically because of the property  $\hat{\gamma}^2 \ll \hat{\omega}_s^2$  of the eigenvalue [4] (we use the notation  $\omega_{Di} \rightarrow \gamma$ , and  $\hat{\gamma} = \gamma / \varepsilon_a \omega_A$  in the non-rotating case). Assuming that  $\Delta q$  is a constant, the eigenvalue becomes

$$\left( \frac{\gamma}{\omega_A} \right)^2 = \frac{\beta_p^2 r_1^2}{18 R_0^2} \frac{3+C}{1-C} - \frac{(\Delta q)^2}{3}. \quad (3)$$

Expressed in terms of  $\beta_0 = 2\varepsilon_a^2 \beta_p$ , this result is consistent with the stability criterion (26) derived by Hastie and Hender in Ref. [4].

When the plasma rotates, we solve the eigenvalue problem (2a,b) numerically with a  $q$ -profile given by:

$$q(r) = \begin{cases} q_0 = 1 - \Delta q & 0 \leq r \leq r_1 \\ q_0 + \frac{(q_a - q_0)(r - r_1)^2}{(a - r_1)^2} & r_1 \leq r \leq a \end{cases} \quad (4)$$

An example of this  $q$ -profile, with  $\Delta q = 0.05$ ,  $r_1 = 0.5a$  and  $q_a = 4$ , is shown in Fig. 1. In Fig. 2 the growth rate  $\omega_{Di}$  of the quasi-interchange instability is shown as a function of  $\Delta q$ . With increasing rotation, the growth rate as well as the range of unstable  $\Delta q$  is seen to decrease, and at sufficiently strong rotation the mode becomes stable for all  $\Delta q$ . This behaviour is illustrated in more detail in Fig. 3, where both the real and imaginary parts of  $\omega_D$  are plotted as functions of the rotation frequency. It is seen that the mode becomes fully stabilized for  $\Omega > \Omega_{crit}$ , where  $\Omega_{crit} \approx 0.08\omega_A$ . In the unstable regime, there is also a small, negative real part of the mode frequency. At the rotation frequency where the mode is stabilized,  $\omega_{Dr}$  changes sign and approaches, as  $\Omega$  increases, the BV frequency shown by the dashed curve in Fig. 3. The BV-frequency is calculated from Eq. (1) with  $r = 0$  (and  $q = 1$ ), which gives the lowest BV-frequency in the low-shear region for the rotation frequency in the figure. Thus, the eigenfrequency of the quasi-interchange mode approaches the lowest BV-frequency in the low-shear region as the rotation frequency becomes large compared with  $\Omega_{crit}$ . The reason for this behaviour is explained in Ref. [10]. Furthermore, it is shown in Ref. [10] that  $\Omega_{crit}$  can be expressed in the form

$$\hat{\Omega}_{crit} = \hat{r}_1 \beta_p^{3/4} H, \quad (5)$$

where the quantity  $H$  depends on  $\hat{r}_1$  and on the  $q$ -profile in the edge region  $r_1 \leq r \leq a$ , and has to be calculated numerically in general. It turns out, however, that  $H \approx 1$  for a large class of  $q$ -profiles and radii  $r_1$  of the low-shear region, as shown in Fig. 4 for the values  $q_a = 3, 4$  and  $5$  in Eq. (4). Thus, Eq. (5) shows a generic scaling of the critical

rotation frequency with  $r_1$  and with  $\beta_p$  of the form  $\hat{\Omega}_{crit} \sim \hat{r}_1 \beta_p^{3/4}$ . The critical rotation frequency is in Fig. 5 shown as a function of  $\beta_p$  for  $\varepsilon_a = 0.3$  and  $q_a = 4.0$ , and for a few values of  $r_1/a$ . The solid curves are calculated from the exact values of  $H$ , whereas the dashed curves are obtained from an analytical approximation valid for small values of  $r_1/a$  given by  $\Omega_{crit} / \omega_A = 1.257 r_1 \beta_p^{3/4} / R_0$  [10]. It is seen that, unless  $\beta_p$  and  $r_1/a$  both are rather large, rotation frequencies of the order of a few percent of the Alfvén frequency are sufficient to stabilize the quasi-interchange mode.

By omitting the second term in Eq. (2a), the eigenvalue problem can be formulated as the integral condition ( $\Delta q = 0$  is assumed here) [10]

$$\int_0^{\hat{r}_1} \frac{(\hat{\omega}_s^2 - \hat{\omega}_D^2) \hat{r}^5 d\hat{r}}{(\hat{\omega}_{BV}^2 - \hat{\omega}_D^2)(\hat{\omega}_{HF}^2 - \hat{\omega}_D^2)} = \frac{\hat{r}_1^4}{\beta_p^2} \frac{1 - C}{3 + C}. \quad (6)$$

Here,  $\omega_{BV}^2$  is given by Eq. (1) and  $\omega_{HF}^2$  is the (high-frequency) root obtained if a plus sign is used in front of the square root in Eq. (1). Furthermore,  $\hat{\omega}_{BV}$  and  $\hat{\omega}_{HF}$  are normalized in the same way as the other frequencies in Eq. (2a), and  $\hat{\omega}_{BV}^2 - \hat{\omega}_D^2$  is a positive quantity that becomes small as the rotation frequency becomes large compared with  $\Omega_{crit}$ , as seen in Fig. 3. Furthermore, the BV-frequency is always smaller than, and for moderate Mach numbers ( $< 1$ ) much smaller than, the sound frequency. This is shown in Fig. 6, which illustrates the radial dependence of  $\omega_{BV}^2$  and the ratio  $\omega_{BV}^2(r)/\omega_s^2(0)$  for several Mach numbers  $M_s^2(\hat{r}) = M_s^2(0)/(1 - \hat{r}^2)$ . Obviously,  $\hat{\omega}_{BV}^2 \ll \hat{\omega}_s^2$  holds up to relatively high Mach numbers, and we therefore also have  $\hat{\omega}_s^2 - \hat{\omega}_D^2 \approx \hat{\omega}_s^2$  in Eq. (6). As concerns the frequency  $\hat{\omega}_{HF}^2$ , this quantity is given by  $3\hat{\omega}_s^2$  when the Mach number is small [8]. Then, for Mach numbers that are not too large, Eq. (6) is approximated by

$$\int_0^{\hat{r}_1} \frac{\hat{r}^5 d\hat{r}}{3(\hat{\omega}_{BV}^2 - \hat{\omega}_D^2)} = \frac{\hat{r}_1^4}{\beta_p^2} \frac{1 - C}{3 + C}. \quad (7)$$

Thus, with increasing rotation frequency, and thereby increasing BV-frequency, the other parameters remaining fixed,  $\omega_D^2$  has to increase also in order to fulfill the integral condition (7), and at sufficiently strong rotation the mode therefore becomes stable. If we assume, in addition, that  $r_1/a$  is small, we can approximate  $\omega_{BV}^2$  in the integral by its value for  $r = 0$ . This leads, by using Eq. (3), to the eigenvalue  $\omega_D^2 = \omega_{BV}^2(0) - \gamma^2$ . Thus, we obtain the same stability condition for the quasi-interchange mode,  $\omega_{BV} > \gamma$ , as was found previously for the stabilization of the Bussac and Mercier modes in Refs. [8-9].

**Conclusions** The effect of toroidal rotation on the ideal MHD stability of the “quasi-interchange” mode in tokamaks with  $q \approx 1$  in a wide area in the plasma core has been analyzed. This stability problem can be formulated as Eq. (2a) for the  $m = 1$  amplitude

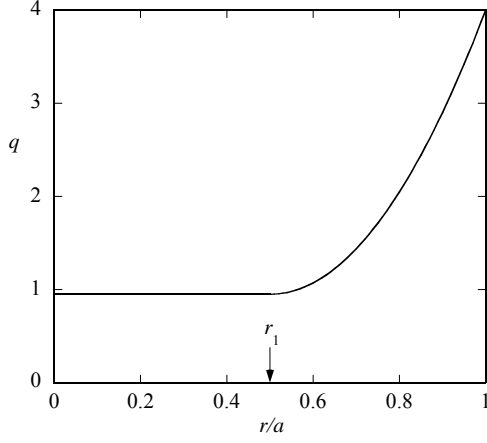
$\xi_1$  in the region where  $q \approx 1$  ( $0 \leq r \leq r_1$ ), together with the integral condition in Eq. (2b) and the boundary conditions  $\xi_1(r_1) = 0$  and  $\xi_1'(0) = 0$  for  $\xi_1$ . For a static plasma, the eigenvalue problem (2a,b) reproduces the pressure-driven, quasi-interchange,  $m = n = 1$  instability previously analyzed in Refs. [4-5]. The numerical solution of Eqs. (2a,b) presented here shows that this mode is stabilized by rigid, toroidal rotation. The stabilization is caused by the modified plasma equilibrium, and associated Brunt-Väisälä (BV) frequency, created by the centrifugal force. In the regime of plasma rotation where the stabilization occurs, the eigenvalue is given approximately by  $\omega_D^2 = \omega_{BV}^2 - \gamma^2$ , where  $\gamma$  is the growth rate of the mode in the absence of rotation and  $\omega_{BV}$  is the lowest BV frequency in the low-shear region of the plasma. Thus, the stabilizing BV mechanism competes with the drive from the quasi-interchange instability, and stability is achieved when  $\omega_{BV}$  exceeds  $\gamma$ , which is the same stability condition as was found earlier for rotational stabilization of the usual ideal, internal kink mode and for the stabilization of Mercier modes that are unstable in the absence of rotation in Refs. [8-9].

Comparison with experiments with low-shear  $q$ -profiles in JET [6] and in NSTX [2, 7] indicates that the critical rotation for stabilization of the ideal  $m = n = 1$  mode found here appears to be of the same order of magnitude as the actual plasma rotation, driven by neutral beam injection, in these experiments (with some uncertainty, though, for the actual rotation frequency in the JET experiments). The Brunt-Väisälä mechanism could therefore be of interest for the interpretation of the  $m = n = 1$  activity in these, and in similar tokamak discharges.

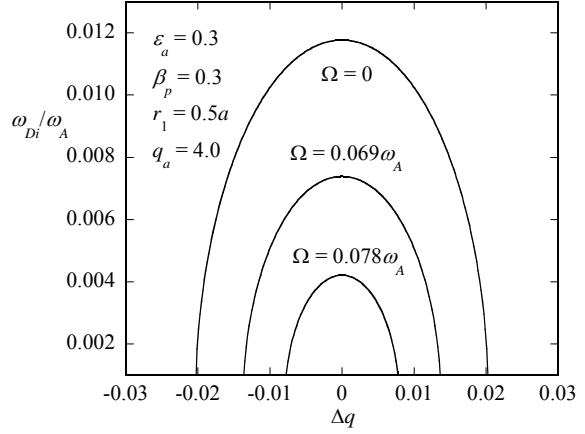
**Acknowledgement** This work has been supported by the European Communities under an association contract between EURATOM and the Swedish Research Council VR.

## References

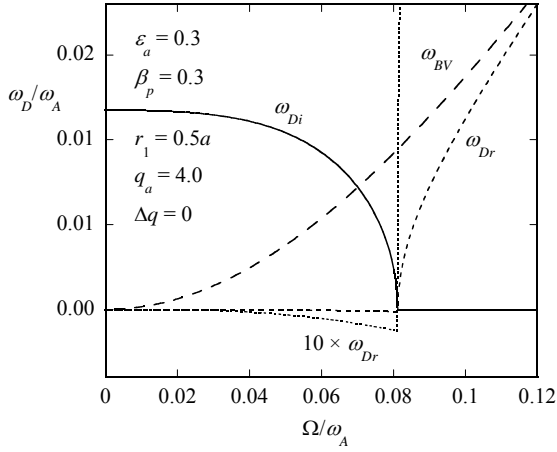
- [1] Litaudon X *et al* 2004 *Plasma Phys. Control. Fusion* **46** A19
- [2] Menard J E *et al* 2003 *Nucl. Fusion* **43** 330
- [3] Wesson J A 1986 *Plasma Phys. Control. Fusion* **28** 243
- [4] Hastie R J and Hender T C 1988 *Nucl. Fusion* **28** 585
- [5] Waelbroeck F L and Hazeltine R D 1988 *Phys. Fluids* **31** 1217
- [6] Buratti P *et al* 2004 *Proc. 31st EPS Conf. on Plasma Phys. (London, 2004)* vol 28B, P-1.165
- [7] Menard J E *et al* 2003 *Proc. 30th EPS Conf. on Contr. Fusion and Plasma Phys. (St Petersburg, 2003)* vol 27A, P-3.101
- [8] Wahlberg C and Bondeson A 2000 *Phys. Plasmas* **7** 923
- [9] Wahlberg C and Bondeson A 2001 *Phys. Plasmas* **8** 3595
- [10] Wahlberg C paper in preparation



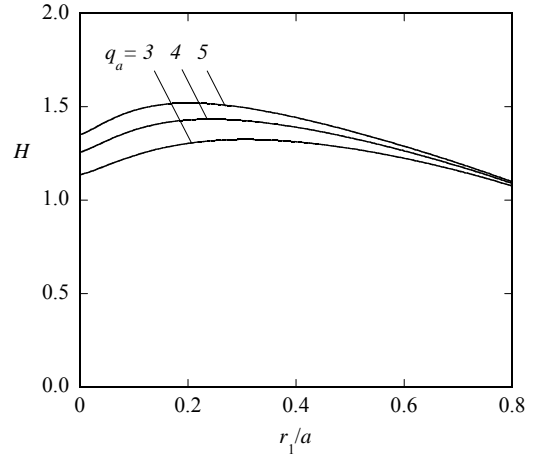
**Fig. 1.** Example of the model  $q$ -profile, defined in Eq. (4), used in the numerical calculations. The parameters of the  $q$ -profile shown are  $\Delta q = 0.05$ ,  $r_1/a = 0.5$  and  $q_a = 4$ .



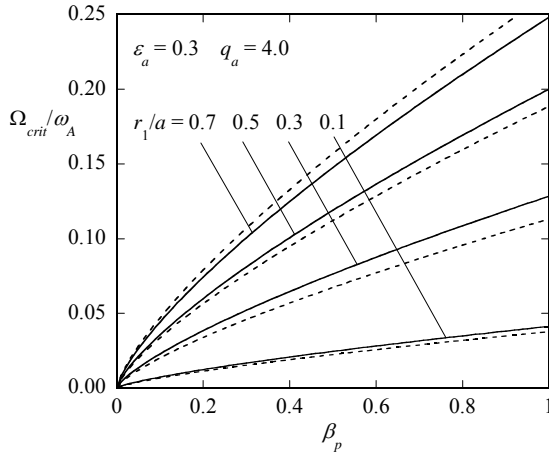
**Fig. 2.** The growth rate  $\omega_{Di}$  of the quasi-interchange instability as a function of  $\Delta q$  for a plasma with the parameters shown in the figure



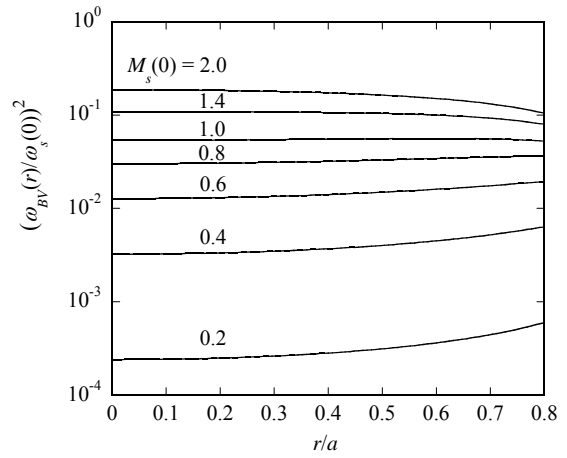
**Fig. 3.** The real ( $\omega_{Dr}$ ) and imaginary ( $\omega_{Di}$ ) parts of the eigenvalue  $\omega_D$  as functions of the rotation frequency  $\Omega$ .



**Fig. 4.** The parameter  $H$  in Eq. (5) as a function of the radius  $r_1$  of the shear-free region.



**Fig. 5.** The critical rotation frequency as a function of  $\beta_p$  for a few values of the radius  $r_1$  of the shear-free region.



**Fig. 6.** The Brunt-Väisälä frequency  $\omega_{BV}$  in Eq. (1) as a function of  $r$  for a few values of the sonic Mach number at  $r = 0$ .