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Patrick Dehornoy

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Patrick Dehornoy. Geometric presentations for Thompson's groups and the multiscaled braid group. 2004. hal-00001636v1

HAL Id: hal-00001636

<https://hal.science/hal-00001636v1>

Preprint submitted on 27 May 2004 (v1), last revised 4 Feb 2005 (v3)

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GEOMETRIC PRESENTATIONS FOR THOMPSON'S GROUPS AND MULTISCALED BRAIDS

PATRICK DEHORNOY

ABSTRACT. This paper deals with Thompson's groups F and V , and two related groups, namely a subgroup V' of V and an extension FB_∞ of V' connected with the latter as Artin's braid group B_∞ is connected with the symmetric group \mathfrak{S}_∞ . The latter group also appears (under the name \widehat{BV}) in the independent work [2]. Our aim in this text is to investigate these groups from a geometric point of view relying on their connection with the associativity and commutativity laws. This approach leads in particular to new presentations. Using word reversing, a specific combinatorial method, we can derive a number of algebraic properties from the presentations. We prove that the group FB_∞ is torsion-free and even orderable, that it includes (and is generated by) both Thompson's group F and Artin's braid group B_∞ (whence our notation). Also it can be interpreted as a group of braids involving a fractal sequence of strands, and it is connected with a twisted version of commutativity and the self-distributivity law.

The current paper contents two parts. In Part 1, we investigate the so-called geometry groups corresponding to the associativity and the commutativity laws—these groups turn out to be Thompson's group F and V —and deduce from this approach new presentations. The presentation of V so obtained directly leads to introducing a new group here denoted FB_∞ , and the specific study of the latter group is the subject of Part 2.

Applying a given algebraic law to a formal expression—or, equivalently, to a decorated tree—can be seen as applying a (partial) operator. In this way, one associates with every algebraic law or, more generally, family of laws, a monoid of partial operators that captures specific geometrical features of these laws. This monoid will be called the geometry monoid of the considered laws. In good cases, in particular when the laws involve sufficiently simple expressions, the geometry monoid is closely related with some group, naturally called the *geometry group* of the considered algebraic laws.

In the case of the associativity law $x(yz) = (xy)z$, the geometry group turns out to be Richard Thompson's group F [34, 5]: this is just one way of formalizing the well-known connection between F and associativity. Other laws lead to quite different groups: for instance, the self-distributivity law $x(yz) = (xy)(xz)$ leads to a geometry group which is an extension of Artin's braid group B_∞ , and studying the latter group has led to unexpected results for B_∞ [12]. Although no general theory exists so far, the geometry groups or monoids seem to be promising objects in general—see, *e.g.*, [13] for the case of an exotic law. In this paper, we shall be concerned with the specific cases of associativity, commutativity, and a variant of the latter called semi-commutativity. We shall in particular address the question of finding presentations of the geometry groups, and develop a uniform approach.

The main results of Part 1 are as follows.

First, we show that the geometry group of associativity is Thompson's group F , and that this group admits a presentation consisting of more or less trivial geometric relations, plus the

1991 *Mathematics Subject Classification.* 20F05, 20F36, 20B07.

Key words and phrases. algebraic law, geometry group, associativity, commutativity, Thompson's groups, partial group action, decorated trees, Coxeter relations, braid groups, mapping class group, punctured disk, Cantor set, automorphisms of a free group, orderable group, group of fractions, word reversing.

so-called MacLane–Stasheff pentagon relations. This warm-up result—certainly not a surprise—illustrates a general method for recognizing a presentation of a group. A disadvantage of the presentation of F so obtained is that it is infinite, an advantage is that it enables one to express F as the group of fractions of an interesting monoid, and provides a double lattice structure on F connected with the lattice structure of the associahedra, as well as a quadratic isoperimetric inequality in terms of the considered (infinite) family of generators.

Then, we prove that the geometry group of associativity together with commutativity, *i.e.*, the algebraic law $xy = yx$, is Thompson’s group V . Once again, the connection belongs more or less to folklore, but the way to state it is perhaps new. At the least, it leads to a new presentation of V , in which, in addition to the geometric and pentagon relations, the only relations are the MacLane–Stasheff hexagon relations, plus the torsion relations expressing that applying commutativity is an involutive operation.

Next, we show that the geometry group of associativity together with a weak form of commutativity, namely the semi-commutativity law $x(yz) = y(xz)$, is some explicitly described subgroup V' of V . The interest of isolating V' inside V is that V' is not really smaller than V , but it is more manageable from a technical point of view, and, in particular, it naturally appears as an extension of the infinite symmetric group \mathfrak{S}_∞ .

In Part 2, we start from the above mentioned geometric presentation of the group V' , and investigate the group FB_∞ obtained by removing the torsion relations $\sigma_i^2 = 1$. So the connection between FB_∞ and V' is exactly similar to the connection between Artin’s braid group B_∞ and the symmetric group \mathfrak{S}_∞ . The group FB_∞ seems to be an interesting object, and, in particular, the deep connection existing between B_∞ and the self-distributivity law $x(yz) = (xy)(xz)$ extends to FB_∞ . The main results we prove in this part are as follows.

First, we establish that FB_∞ is a group of left fractions for some monoid FB_∞^+ that shares a number of properties with Garside’s monoid B_∞^+ , and we deduce in particular that FB_∞ includes a copy of Thompson’s group F and of Artin’s braid group B_∞ , and that it is generated by these subgroups. Also we show how to extend to FB_∞ the canonical linear ordering of B_∞ , as well as the acyclic left self-distributive operation that lives on B_∞ .

Then we describe several geometric realisations of the group FB_∞ . The first one involves multiscaled braid diagrams in which the strands are numbered by finite sequences of integers, instead of by integers as in the case of ordinary braids and B_∞ . By considering the natural isotopy notion and adding to the ordinary crossings new elementary diagrams corresponding to rescaling, one obtains a diagrammatic interpretation of FB_∞ .

Then, we come back to the geometry group approach of Part 1, and consider the case of associativity together with some twisted version of (semi)-commutativity, namely that obtained by weakening the ordinary semi-commutativity transformation $x(yz) \mapsto y(xz)$ into $x(yz) \mapsto x[y](xz)$, where the bracket denotes some left self-distributive binary operation. In this way, we go beyond the framework of algebraic laws, but a geometric group still exists, and we show that this geometry group is the group FB_∞ .

Still another realisation of FB_∞ comes when we consider the well-known realisation of Artin’s braid group B_n as the mapping class group of a disk with n punctures. We show that, similarly, FB_∞ can be realized as a subgroup of the mapping class group of a 2-sphere with a Cantor set of punctures, and, as a consequence, we obtain an explicit realization of FB_∞ as a subgroup of the group of automorphisms of free group of infinite rank.

The current approach is not the first attempt to marry Thompson’s group F with Artin’s group B_∞ : see [23, 20, 24], and, specially, [2]. The group here denoted FB_∞ turns out to be one of the groups considered by M. Brin in [2], namely the group denoted \widehat{BV} there. However, the approaches leading to that group, the technical treatments, as well as the results obtained and the geometric interpretations are mostly disjoint. In [2], the group FB_∞ (or \widehat{BV}) is introduced

in a context of categories, while the current paper involves a context of geometry groups. Of course, common geometrical features underly both approaches, but they lead to different developments, and, in particular, the geometry group approach seems specially suitable for providing explicit presentations. Similarly, the main algebraic tool used in [2] to study the algebraic structure of FB_∞ is the notion of Szep product of two monoids or groups, while we mostly appeal to word reversing, a combinatorial tool relevant for groups of fractions, and self-distributive operations, which should be seen here as extensions of group conjugacy. These tools, which had been developed for the study of self-distributivity and braids in [12], prove to be specially suitable here, and they lead to arguments that are conceptually simple and natural. To summarize, we can say that [2] mostly investigates the group FB_∞ as an extended Thompson's group, while our approach considers it as an extended braid group and investigates it accordingly.

Remark on notation. Here we work both with Thompson's group F and with Artin's braid group B_∞ . It turns out that different notational conventions are frequently used for these groups: in the case of F , it is customary to think of its elements as acting on the left (thus xy means "y then x"), while, in the case of B_∞ , the opposite convention is used (xy means "x then y"). We found it necessary to choose a unique convention, and, for the reasons listed in [3], we choosed the second one. Also, there is a tradition to number the strands of a braid, and, therefore the standard generators σ_i of B_∞ , starting from 1, while it is customary to number the standard generators x_i of F starting from 0. Once again, we choose the braid convention here, and to avoid confusion, we use a specific notation, namely a_i , for the generators of F . Thus the correspondence is that our generator a_i corresponds to the standard generator x_{i-1}^{-1} of F .

The author thanks Sean Cleary for drawing his attention to M. Brin's preprint [2] after a first version of this text was written, as well as Matthew Brin and Mark Lawson for helpful comments and suggestions.

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Part 1. The geometry group of associativity and other algebraic laws

1. A METHOD FOR FINDING PRESENTATIONS

Throughout the paper, \mathbb{N} denotes the set of all positive integers (0 excluded), \mathbb{R} denotes the set of all real numbers, and \mathbb{C} denotes the set of all complex numbers.

In the sequel, we address the problem of finding a presentation of a group or a monoid several times, and, at each time, we solve it using the same method. So it makes sense to describe this common method first. Although seemingly never described explicitly, the method was already used in [8], and it proved to be relevant in several similar situations [12, 13].

1.1. Partial group actions. The situation we investigate is essentially that of a group action. However, the framework we consider is both weaker and stronger than the standard one. The weakening is that the actions we consider are partial in the sense that every element of the group need not necessarily act on every element; the strengthening is that our actions satisfy a very strong freeness hypothesis, namely that, if two elements of the group act in the same way on one element, then they act in the same way on each element on which both of them act.

Several weak forms of the standard notion of an action may be thought of. The one convenient here is as follows. As pointed out by M. Lawson, it is essentially equivalent to the notion investigated in [25] (in the case of groups)—see also [27]—and in [29] (in the case of monoids).

Definition. Let G be a monoid. We define a *partial (right) action* of G on a set S to be a mapping ϕ of G into the partial injections of S into itself such that, writing $s \bullet g$ for the image of s under $\phi(g)$, the following conditions are satisfied:

- (PA₁) For every s in S , we have $s \bullet 1 = s$, *i.e.*, $\phi(1)$ is the identity mapping;
- (PA₂) For all g, h in G and s in S , if $s \bullet g$ is defined, then $(s \bullet g) \bullet h$ is defined if and only if $s \bullet gh$ is, and, in this case, they are equal;
- (PA₃) For each finite family g_1, \dots, g_n in G , there exists at least one element s in S such that $s \bullet g_1, \dots, s \bullet g_n$ are defined.

We say that a subset S_0 of S is *separating* for ϕ if, in (PA₃), we can require that s belongs to $S_0 \bullet G$, *i.e.*, the restriction of ϕ is a partial action of G on $S_0 \bullet G$, and, moreover, $g = g'$ holds whenever $s_0 \bullet g = s_0 \bullet g'$ holds for at least one element s_0 of S_0 .

Note that, in general, $s \bullet gh$ being defined does not guarantee that $s \bullet g$ is. In the context of a group, we have the following straightforward facts:

Lemma 1.1. *Assume that ϕ is a partial action of a group G on a set S . Then*

- (i) *The relations $s' = s \bullet g$ and $s = s' \bullet g^{-1}$ are equivalent;*
- (ii) *The relation $(\exists g \in G)(s' = s \bullet g)$ is an equivalence relation on S .*

Proof. As gg^{-1} is 1, Condition (PA₂) asserts that, if $s \bullet g$ is defined, then $(s \bullet g) \bullet g^{-1}$ is defined if and only if $s \bullet 1$ is, and the latter is always true by (PA₁). Then we find $(s \bullet g) \bullet g^{-1} = s \bullet 1 = s$. This implies that the relation of (ii) is symmetric; (PA₁) implies that it is reflexive, and (PA₂) that it is transitive. □

Thus a partial action of a group on a set S defines a partition of S into well-defined disjoint orbits. This need not be true in the case of a monoid action—but we still say that s' lies in the orbit of s if $s' = s \bullet g$ holds for some element g .

1.2. Presentations. As for monoid and group presentations, we use the following parallel notation throughout the paper.

Definition. (i) For \mathbf{x} a set, we denote by $W^+(\mathbf{x})$ the monoid of all words on \mathbf{x} , *i.e.*, of all finite sequences of elements of \mathbf{x} . We use ε for the empty word. If \mathbf{r} is a family of monoid relations $u = v$ with u, v in $W^+(\mathbf{x})$, we denote by $\equiv_{\mathbf{r}}^+$ the congruence on $W^+(\mathbf{x})$ generated by \mathbf{r} , and by $\langle \mathbf{x}; \mathbf{r} \rangle^+$ the monoid $W^+(\mathbf{x}) / \equiv_{\mathbf{r}}^+$.

(ii) For \mathbf{x} a set, we denote by $W(\mathbf{x})$ the monoid of all words constructed from $\mathbf{x} \cup \mathbf{x}^{-1}$, where \mathbf{x}^{-1} consists of one symbol x^{-1} for each x in \mathbf{x} . If \mathbf{r} is a family of group relations $u = v$ with u, v in $W(\mathbf{x})$, we denote by $\equiv_{\mathbf{r}}$ the congruence on $W(\mathbf{x})$ generated by \mathbf{r} together with all relations $xx^{-1} = x^{-1}x = \varepsilon$ with $x \in \mathbf{x}$, and by $\langle \mathbf{x}; \mathbf{r} \rangle$ the group $W(\mathbf{x}) / \equiv_{\mathbf{r}}$.

If \mathbf{x} is a subset of a group (*resp.* monoid) G and w is a word in $W(\mathbf{x})$ (*resp.* in $W^+(\mathbf{x})$), we usually denote by \overline{w} the evaluation of w in G , and then say that w is an expression of \overline{w} . In the case of a partial action on a set S , it will be convenient to extend the action to expressions of the elements of the monoid as follows:

Definition. Assume that ϕ is a partial action of a group (*resp.* monoid) G on a set S , and \mathbf{x} is a subset of G . For s in S and w in $W(\mathbf{x})$ (*resp.* $W^+(\mathbf{x})$), we define $s \bullet w$ to be $s \bullet \bar{w}$ if the latter exists and so does $s \bullet \bar{v}$ for each prefix v of w .

Our criterion for recognizing a presentation may be used when we have a (partial) action of a group or a monoid G on a set S , and there is a way to define inside G a copy of the set S in such a way that the counterpart to the action of G is a multiplication. We first state the result for a group.

Proposition 1.2. (i) Assume that G is a group partially acting on a set S , that \mathbf{x} is a subset of G , that a word u_s in $W(\mathbf{x})$ is chosen for each s in S , and

$$(1.1) \quad \mathbf{x} \text{ generates } G,$$

$$(1.2) \quad \text{for all } s, s' \text{ in } S \text{ and } x \text{ in } \mathbf{x} \text{ satisfying } s' = s \bullet x, \text{ we have } \overline{u_{s'}} = \overline{u_s} \cdot x.$$

Then a necessary and sufficient condition for a family of relations \mathbf{r} satisfied by the elements of \mathbf{x} in G to make a presentation of G is that

$$(1.3) \quad \text{for all } s, s' \text{ in } S \text{ and } x \text{ in } \mathbf{x} \text{ satisfying } s' = s \bullet x, \text{ we have } u_{s'} \equiv_{\mathbf{r}} u_s \cdot x.$$

(ii) Assume in addition that some subset S_0 of S is separating for the partial action of G . Then a sufficient condition for (1.1) and (1.2) to hold is that

$$(1.4) \quad \text{for all } s_0 \text{ in } S_0 \text{ and each } s \text{ in the orbit of } s_0, \text{ we have } s_0 \bullet \overline{u_s} = s.$$

Proof. (i) That (1.3) is a necessary condition is clear: if $(\mathbf{x}; \mathbf{r})$ is a presentation of G , the hypothesis that the words $u_{s'}$ and $u_s \cdot x$ represent the same element of G implies that these words are \mathbf{r} -equivalent.

Conversely, assume (1.3). We claim that

$$(1.5) \quad \text{for all } s, s' \text{ in } S \text{ and } w \text{ in } W(\mathbf{x}) \text{ satisfying } s' = s \bullet w, \text{ we have } u_{s'} \equiv_{\mathbf{r}} u_s \cdot w.$$

We prove the claim using induction on the length of w . If w has length 1, then either w is a single letter x of \mathbf{x} , and then $u_{s'} \equiv_{\mathbf{r}} u_s \cdot x$ is true by hypothesis, or w has the form x^{-1} for some x in \mathbf{x} . Then, by Lemma 1.1, $s' = s \bullet x^{-1}$ implies $s = s' \bullet x$, hence, by hypothesis, $u_s \equiv_{\mathbf{r}} u_{s'} \cdot x$, and, therefore, $u_{s'} \equiv_{\mathbf{r}} u_s \cdot x^{-1}$. Assume now that w has length 2 at least. Write $w = uv$ with u, v nonempty. By definition, the hypothesis that $s \bullet w$ is defined implies that both $s \bullet u$ and $(s \bullet u) \bullet v$ are defined. Applying the induction hypothesis and (PA_2) , we obtain

$$u_{s \bullet w} = u_{(s \bullet u) \bullet v} \equiv_{\mathbf{r}} u_{s \bullet u} \cdot v \equiv_{\mathbf{r}} u_s \cdot u \cdot v = u_s \cdot w.$$

Now assume that w and w' represent the same element of G , *i.e.*, $\bar{w} = \bar{w}'$ holds. Condition (PA_3) implies that there exists an element s of S such that both $s \bullet w$ and $s \bullet w'$ are defined. Then applying (1.5) we find

$$u_s \cdot w \equiv_{\mathbf{r}} u_{s \bullet w} = u_{s \bullet w'} \equiv_{\mathbf{r}} u_s \cdot w',$$

and we deduce $w \equiv_{\mathbf{r}} w'$ by cancelling u_s on the left. So $\bar{w} = \bar{w}'$ implies $w \equiv_{\mathbf{r}} w'$, which means that $(\mathbf{x}; \mathbf{r})$ is a presentation of G .

(ii) Let g be an arbitrary element of G . By (PA_3) , there exists an element s of S such that $s \bullet g$ exists. Moreover, as S_0 is an separating subset of S , we can assume that s belongs to the orbit of some element s_0 of S_0 . Then, by (1.5), we have $s = s_0 \bullet \overline{u_s}$, and, similarly, $s' = s_0 \bullet \overline{u_{s'}}$ for $s' = s \bullet g$, hence

$$(1.6) \quad s_0 \bullet \overline{u_{s'}} = s' = s \bullet g = (s_0 \bullet \overline{u_s}) \bullet g = s_0 \bullet (\overline{u_s} \cdot g),$$

and, using the hypothesis that S_0 is essential, we deduce $\overline{u_{s'}} = \overline{u_s} \cdot g$. For g in \mathbf{x} , we obtain (1.2). On the other hand, (1.6) implies $g = \overline{u_s}^{-1} \overline{u_{s'} \bullet g}$. By construction, all elements of the form $\overline{u_s}$ belong to the subgroup of G generated by \mathbf{x} , so g itself must belong to this subgroup. Hence \mathbf{x} generates G . \square

The criterion of Proposition 1.2 is in particular valid in the case of a (total) group action, in which case the proof is straightforward.

When we consider a monoid action, the statement has to be changed a little:

Proposition 1.3. (i) Assume that G is a left cancellative monoid partially acting on a set S , that \mathbf{x} is a subset of G , that a word u_s in $W^+(\mathbf{x})$ is chosen for each s in S , and

$$(1.7) \quad \mathbf{x} \text{ generates } G,$$

$$(1.8) \quad \text{for all } s, s' \text{ in } S \text{ and } g \text{ in } G \text{ satisfying } s' = s \cdot g, \text{ we have } \overline{u_{s'}} = \overline{u_s} \cdot g.$$

Then a necessary and sufficient condition for a family of relations \mathbf{r} satisfied in G by the elements of \mathbf{x} to make a presentation of G is that

$$(1.9) \quad \text{the monoid } \langle \mathbf{x}; \mathbf{r} \rangle^+ \text{ admits left cancellation,}$$

$$(1.10) \quad \text{for all } s, s' \text{ in } S \text{ and } x \text{ in } \mathbf{x} \text{ satisfying } s' = s \cdot x, \text{ we have } u_{s'} \equiv_{\mathbf{r}}^+ u_s \cdot x.$$

(ii) Assume in addition that some subset S_0 of S is separating for the partial action of G . Then a sufficient condition for (1.8) to hold is:

$$(1.11) \quad \text{for all } s_0 \text{ in } S_0 \text{ and } s \text{ in the orbit of } s_0, \text{ we have } s_0 \cdot u_s = s.$$

Proof. The argument is the same as in the case of a group. Apart from replacing $W(\mathbf{x})$ with $W^+(\mathbf{x})$ everywhere, the only changes are that we cannot deduce that \mathbf{x} generates G from (1.8) as $\overline{u_s}$ need not have an inverse in G , and that, in order to deduce $w \equiv_{\mathbf{r}}^+ w'$ from $u_s \cdot w \equiv_{\mathbf{r}}^+ u_s \cdot w'$, we have to explicitly assume that the monoid $\langle \mathbf{x}; \mathbf{r} \rangle^+$ admits left cancellation. \square

2. THOMPSON'S GROUP F AS THE GEOMETRY GROUP OF ASSOCIATIVITY

We describe now a realization of Thompson's group F as the group canonically associated with a monoid of partial associativity operators. This is one way of formalizing the well-known connection between F and the associativity law. This approach naturally leads to a presentation of F in terms of a family of generators indexed by binary addresses. Apart from more or less trivial geometric relations, the only relations of this presentation correspond to the MacLane–Stasheff pentagon relations.

2.1. Trees and associativity. In the sequel, we consider finite, rooted binary trees—simply called trees here. The number of leaves in a tree is called its *size*. We denote by \circ the tree consisting of a single vertex and by $t_1 \wedge t_2$ the tree with left subtree t_1 and right subtree t_2 . Every tree has a unique decomposition in terms of \circ and the operation \wedge .

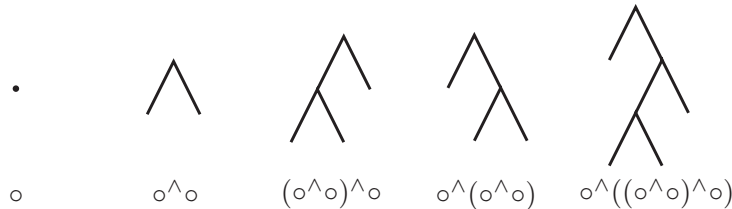


FIGURE 1. Typical trees with their decomposition in terms of \circ and \wedge

We shall also consider *decorated* trees, defined as trees in which the leaves wear labels. We write \circ_ℓ for the decorated tree consisting of a single vertex labelled ℓ , and, for S a nonempty set, we use T_S to denote the set of all S -decorated trees, *i.e.*, all decorated trees with labels in S . We shall occasionally denote by T_\emptyset the family of all undecorated trees. Note that T_S

equipped with the binary operation \wedge is an absolutely free algebra (or free magma) generated by S , or, more exactly, by the elements \circ_ℓ with ℓ in S .

The associativity law

$$(A) \quad x(yz) = (xy)z$$

gives rise to an equivalence relation on (decorated) trees: two trees t, t' are equivalent up to associativity if we can transform t into t' by iteratively replacing one subtree of the form $t_1 \wedge (t_2 \wedge t_3)$ with the corresponding tree $(t_1 \wedge t_2) \wedge t_3$, or *vice versa* (Figure 2).

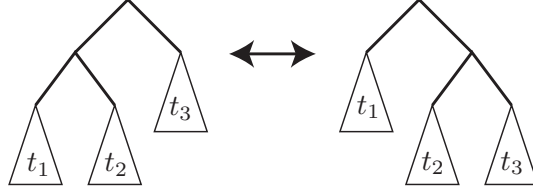


FIGURE 2. Applying the associativity law

In order to describe this action of associativity, we need an indexation for the subtrees of a tree. One solution is to start from the root of the tree and to describe the path down to the root of the considered subtree using for instance 0 for “forking to the left” and 1 for “forking to the right”.

Definition. A finite sequence of 0’s and 1’s is called an *address*; the empty address is denoted by ϕ . For t a (decorated) tree and α a short enough address, the α th subtree $t_{/\alpha}$ of t consists of the part of t that lies below α . Formally, it is defined by the inductive rules: (i) The tree $t_{/\phi}$ is defined for every t , and it equals t ; (ii) For $\alpha = 0\beta$ (resp. 1β), the tree $t_{/\alpha}$ exists only if t has the form $t' \wedge t''$ and $t'_{/\beta}$ (resp. $t''_{/\beta}$) is defined, and, then, we have $t_{/\alpha} = t'_{/\beta}$ (resp. $t''_{/\beta}$). The set of all α ’s for which $t_{/\alpha}$ exists is called the *skeleton* of t .

Example 2.1. Let $t = \circ \wedge ((\circ \wedge \circ) \wedge \circ)$ (the rightmost example in Figure 1). The subtree $t_{/\phi}$ is $\circ \wedge \circ$, while $t_{/01}$ and $t_{/111}$ are undefined. The skeleton of t consists of $\phi, 0, 1, 10, 100, 101, 11$.

In this framework, applying associativity to a tree t consists in choosing an address α in the skeleton of t and either replacing the α th subtree of t , supposed to have the form $t_1 \wedge (t_2 \wedge t_3)$, with the corresponding tree $(t_1 \wedge t_2) \wedge t_3$, or performing the inverse substitution. We can see this as applying a (partial) operator on trees.

Definition. (i) We denote by A the (partial) operator that maps every tree of the form $t_1 \wedge (t_2 \wedge t_3)$ to the corresponding tree $(t_1 \wedge t_2) \wedge t_3$.

(ii) For α an address and f a partial mapping on trees, we define the α -shift $\partial_\alpha f$ of f to be the partial mapping consisting in applying f to the α th subtree of its argument (when the latter exists). We write ∂ for ∂_1 .

(iii) For α an address, we put $A_\alpha = \partial_\alpha A$. We define $\mathcal{G}(A)$ to be the monoid generated by all A_α ’s and their inverses using reversed composition.

Example 2.2. (Figure 3) Let $t = \circ \wedge (((\circ \wedge \circ) \wedge \circ) \wedge (\circ \wedge \circ))$. Then t lies in the domain of A , as $t_{/\phi}$, i.e., t itself, can be written as $t_1 \wedge (t_2 \wedge t_3)$, with $t_1 = \circ$, $t_2 = (\circ \wedge \circ) \wedge \circ$, and $t_3 = \circ \wedge \circ$. Then the image of t under A is $(t_1 \wedge t_2) \wedge t_3$, i.e., $(\circ \wedge ((\circ \wedge \circ) \wedge \circ)) \wedge (\circ \wedge \circ)$. Similarly, t lies in the domain of A_1 , and in the images of A_1 and of A_{10} , hence in the domains of A_1^{-1} and A_{10}^{-1} . These are the only operators $A_\alpha^{\pm 1}$ applying to t .

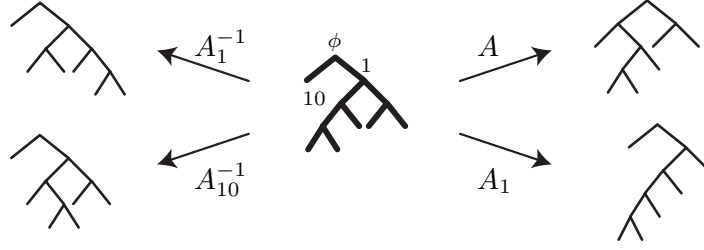


FIGURE 3. Two operators A_α and two operators A_α^{-1} apply to the tree $\circ^\wedge(((\circ^\wedge)^\wedge)^\wedge(\circ^\wedge))$

By construction, there exists a partial action of the monoid $\mathcal{G}(\mathcal{A})$ on trees; for f in $\mathcal{G}(\mathcal{A})$, we write $t \bullet f$ for the possible image of t under f . We use reversed composition in $\mathcal{G}(\mathcal{A})$ so as to make the monoid structure compatible with an action on the right.

By construction, two trees t, t' are equivalent up to associativity if and only if some element of $\mathcal{G}(\mathcal{A})$ maps t to t' . Thus the orbits with respect to the partial action of the monoid $\mathcal{G}(\mathcal{A})$ are the equivalence classes with respect to associativity: there is exactly one orbit for each size, and the number of trees in the orbit of a size n tree is the n th Catalan number.

2.2. Making $\mathcal{G}(\mathcal{A})$ into a group. Except the identity mapping, the elements of $\mathcal{G}(\mathcal{A})$ are partial mappings, and the monoid $\mathcal{G}(\mathcal{A})$ is not a group, but only an inverse monoid, *i.e.*, a monoid in which, for each element g , there exists an element g^{-1} satisfying $gg^{-1}g = g$ and $g^{-1}gg^{-1} = g^{-1}$. Indeed, the product of A and A^{-1} is the identity of its domain, but the latter does not contain the tree \circ for instance.

However, it is easy to quotient $\mathcal{G}(\mathcal{A})$ into a group by identifying all partial identity mappings. The construction will be used several times in the sequel, so we first state a general principle.

Definition. We say that two partial mappings g, g' are *near-equal*, denoted $g \approx g'$, if there exists at least one element s such that $s \bullet g$ and $s \bullet g'$ are defined, and $s \bullet g = s \bullet g'$ holds for every such s .

Lemma 2.3. *Assume that \mathcal{G} is a monoid consisting of partial injections of a set S into itself that is closed under inverse, and there exists a nonempty subset S_0 of S such that, for all g_1, \dots, g_n, g, g' in \mathcal{G} ,*

$$(2.1) \quad \text{Dom}(g_1) \cap \dots \cap \text{Dom}(g_n) \text{ contains at least one element of } S_0 \bullet \mathcal{G},$$

$$(2.2) \quad g \approx g' \text{ is true whenever } x \bullet g = x \bullet g' \text{ holds for some } x \text{ in } S_0 \bullet \mathcal{G}.$$

Then near-equality is a congruence on \mathcal{G} , the quotient-monoid is a group, the mappings of \mathcal{G} induce a partial action of this group on S , and the set S_0 is separating for this partial action.

Proof. Assume $g' \approx g'' \approx g'''$. By (2.1), there exists s in $S_0 \bullet \mathcal{G}$ such that $s \bullet g', s \bullet g''$, and $s \bullet g'''$ are defined. Then one necessarily has $s \bullet g' = s \bullet g'''$, hence $g' \approx g'''$ by (2.2), and \approx is an equivalence relation. Next, $g' \approx g''$ implies $gg' \approx gg''$ and $g'g \approx g''g$ for every g , because (2.1) guarantees that there exists s in $S_0 \bullet \mathcal{G}$ for which $s \bullet gg', s \bullet gg'', s \bullet g'g$, and $s \bullet g''g$ are defined, and, moreover, $s \bullet g$ belongs to $S_0 \bullet \mathcal{G}$. So \approx is a congruence on \mathcal{G} , and the quotient-monoid \mathcal{G}/\approx , henceforth denoted G , is well-defined. For each g in \mathcal{G} , we have $gg^{-1} \approx \text{id}$ because the domain of g is nonempty, so G is a group.

For g in \mathcal{G} , let us denote by \bar{g} the class of g in G . For s in S , and a in G , we define $s \bullet a$ to be s' if $s \bullet g = s'$ holds for some element g of \mathcal{G} satisfying $\bar{g} = a$, if such an element exists. Then $s \bullet a$ is well-defined by definition of \approx , and we claim that one obtains in this way a partial action of G on S . Indeed, Condition (PA_1) is trivial. As for (PA_2) , assume that $s \bullet a$ and $(s \bullet a) \bullet b$ are defined. This means that there exist g, h with $a = \bar{g}$ and $b = \bar{h}$ such that $s \bullet g$ and $(s \bullet g) \bullet h$ are defined. But, then, $s \bullet gh$ is defined, and, by construction, we have $\overline{gh} = \bar{g}\bar{h}$. Conversely,

assume that $s \bullet a$ and $s \bullet ab$ are defined, say $s \bullet a = s'$ and $s \bullet ab = z$. This means that there exist f, g in \mathcal{G} satisfying $s \bullet g = s'$, $s \bullet f = z$, with $\bar{g} = a$ and $\bar{f} = ab$. Let $h = g^{-1}f$. Then h belongs to \mathcal{G} , we have $\bar{h} = a^{-1}ab = b$, and $s' \bullet h = z$. This shows that $(s \bullet a) \bullet b$ is defined, and equal to z . So Condition (PA_2) is satisfied.

Then (2.1) implies (PA_3) directly, and we obtain a partial action of G on S . Finally, the subset S_0 is separating by (2.2). \square

In order to apply the previous construction to the monoid $\mathcal{G}(\mathcal{A})$ and its action on trees, we describe the domain and the image of a generic element of $\mathcal{G}(\mathcal{A})$ explicitly.

Definition. (i) Let S, S' be two sets. A mapping of S to $T_{S'}$ is called a S, S' -substitution. If t is a tree in T_S and φ is a S, S' -substitution, we denote by t^φ the tree (in $T_{S'}$) obtained by replacing each leaf \circ_s in t with the tree $\varphi(s)$.

(ii) A decorated tree is said to be *injective* if its labels are pairwise distinct.

(iii) For f a partial mapping of T_S into itself, we say that a pair of \mathbf{N} -decorated trees (t, t') is a *seed* for f if, as a set of pairs, f is the set of all (t^φ, t'^φ) with φ an \mathbf{N}, S -substitution.

The pair $(\circ_1 \wedge (\circ_2 \wedge \circ_3), (\circ_1 \wedge \circ_2) \wedge \circ_3)$ is a seed for A : this is just saying that A consists of all pairs of the form $(t_1 \wedge (t_2 \wedge t_3), (t_1 \wedge t_2) \wedge t_3)$. Then we have the following general result:

Lemma 2.4. *Each element of $\mathcal{G}(\mathcal{A})$ admits a seed consisting of injective trees.*

Proof. Let f be an element of $\mathcal{G}(\mathcal{A})$. We prove the property using induction on the (minimal) length of a decomposition of f in terms of the operators A_α and A_α^{-1} . If f is the identity mapping, the pair (\circ_1, \circ_1) is a seed for f . Otherwise, write $f = f_1 f_2$. By induction hypothesis, f_1 and f_2 admit seeds, say (t_1, t'_1) and (t_2, t'_2) . If t'_1 happens to coincide with t_2 , then (t_1, t'_2) is a seed for f . In the general case, because t'_1 and t_2 are injective, there exist substitutions φ_1 and φ_2 such that $t'_1 \varphi_1$ and $t_2 \varphi_2$ coincide, and, then, the pair $(t_1 \varphi_1, t'_2 \varphi_2)$ is a seed for f . \square

The seed can be made unique by adding the requirement that the labels make an initial segment of the positive integers—a result that we shall not use here.

Corollary 2.5. *The monoid $\mathcal{G}(\mathcal{A})$ satisfies Conditions (2.1) and (2.2) of Lemma 2.3, with respect to the subsets T and $T_{\mathbf{N}}$.*

Proof. Let f_1, \dots, f_n be arbitrary elements of $\mathcal{G}(\mathcal{A})$, and $(t_1, t'_1), \dots, (t_n, t'_n)$ be seeds for them. If t is a tree whose skeleton includes the skeletons of t_1, \dots, t_n , then there exist substitutions $\varphi_1, \dots, \varphi_n$ such that $t = t_i^{\varphi_i}$ holds for each i , which implies that $t \bullet f_i$ is defined for each i . So Condition (2.1) is satisfied in $\mathcal{G}(\mathcal{A})$.

Assume that f_1, f_2 belong to $\mathcal{G}(\mathcal{A})$, and $t \bullet f_1 = t \bullet f_2$ holds for some tree t in $T_{\mathbf{N}}$. Let $(t_1, t'_1), (t_2, t'_2)$ be seeds for f_1 and f_2 respectively. As above, there exist substitutions φ_1, φ_2 such that the trees $t_1^{\varphi_1}$ and $t_2^{\varphi_2}$ coincide, they are injective, and their common skeleton is the union of the skeletons of t_1 and t_2 . Then the hypothesis that $t \bullet f_1$ and $t \bullet f_2$ are defined implies that the skeleton of t includes those of t_1 and t_2 , hence their union. So there exists a substitution ψ satisfying $t = (t_1^{\varphi_1})^\psi = (t_2^{\varphi_2})^\psi$. Now the hypothesis that $t \bullet f_1$ and $t \bullet f_2$ are equal gives

$$(t_1^{\varphi_1})^\psi = t \bullet f_1 = t \bullet f_2 = (t_2^{\varphi_2})^\psi.$$

This implies that the skeletons of the terms $t_1^{\varphi_1}$ and $t_2^{\varphi_2}$ coincide. Moreover, the hypothesis that the trees $t_1^{\varphi_1}$ and $t_2^{\varphi_2}$ are equal implies that the sequence of labels in $t_1^{\varphi_1}$ and $t_2^{\varphi_2}$ are the same. Now, associativity does not change the order of the labels, so we deduce that the trees $t_1^{\varphi_1}$ and $t_2^{\varphi_2}$ must coincide. This means that f_1 and f_2 must agree on every tree whose skeleton includes that of $t_1^{\varphi_1}$, *i.e.*, on every tree in the intersection of the domains of f_1 and f_2 . In other words, $f_1 \approx f_2$ holds, and Condition (2.2) is satisfied. \square

By applying Lemma 2.3, we obtain:

Proposition 2.6. *Near-equality is a congruence on the monoid $\mathcal{G}(\mathcal{A})$, and the quotient-monoid is a group. For each set S (possibly $S = \emptyset$), the operators $A_\alpha^{\pm 1}$ induce a partial action of this group on T_S , and every subset of T_S containing trees of unbounded sizes is separating for this partial action.*

Definition. The quotient-monoid $\mathcal{G}(\mathcal{A})/\approx$ is called the *geometry group of associativity*. We denote it $G(\mathcal{A})$.

In the sequel, we still use A_α for the class of A_α in F . For t a tree, and a an element of F , we denote by $t \bullet a$ the result of letting a act on t , when it is defined; in this case, we simply say that a maps t to $t \bullet a$. It should be observed that there is a difference between the action of words in $W(\mathcal{A})$ and the action of the elements of F they represent: for instance, $(\circ \wedge \circ) \bullet AA^{-1}$ is not defined—since $(\circ \wedge \circ) \bullet A$ is not—while $(\circ \wedge \circ) \bullet \overline{AA^{-1}}$ is—since it is the action under 1.

It is straightforward to connect the group $G(\mathcal{A})$ with Thompson's group F :

Proposition 2.7. *The group $G(\mathcal{A})$ is isomorphic to Thompson's group F .*

Proof. Let us start with the definition of F as a group of diffeomorphisms of the unit interval, cf. [5]. Let f be an arbitrary element in $\mathcal{G}(\mathcal{A})$. We map f to a homeomorphism of $[0, 1]$ as follows: let (t, t') be a seed pair for f ; we associate with t a dyadic decomposition $0 = x_0 < x_1 < \dots < x_n = 1$ of $[0, 1]$, and, similarly, let $0 = x'_0 < x'_1 < \dots < x'_n = 1$ be the dyadic decomposition associated with t' ; then we map f to the unique piecewise linear homeomorphism that maps x_i to x'_i and interpolates the values. One obtains in this way a morphism π of $\mathcal{G}(\mathcal{A})$ to F . The homeomorphisms associated with (t, t') and (t^φ, t'^φ) coincide, and this implies that π factors through \approx . The injectivity of the resulting morphism follows from the fact that each element of F is determined by its values on a finite dyadic partition; its surjectivity follows from the fact that the images of A and A_1 generate F . \square

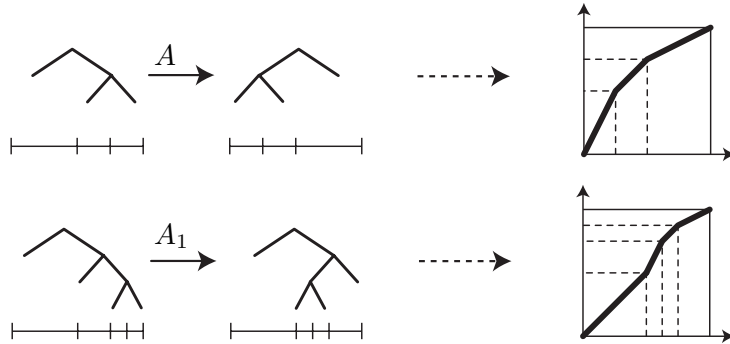


FIGURE 4. From $\mathcal{G}(\mathcal{A})$ to F : the action of A and A_1

From now on, we identify F with $G(\mathcal{A})$.

2.3. Guessing relations in $G(\mathcal{A})$. We show now how considering the group F as the geometry group of associativity naturally leads to a presentation of this group in terms of the generators A_α . We proceed in two steps: in a first step, we use the geometric definition of the operator A_α to guess a list of relations; then, in a second step, we prove that these relations actually make a presentation using the method sketched in Section 1.

So we are looking for possible relations between the operators A_α . We shall describe two types of relations successively: the *geometric relations*, and the *pentagon relations*. The geometric relations arise when we consider inheritance phenomena. Assume $t' = t \bullet A$, i.e., assume that the operator A maps t to t' . Then, by definition, the 1-subtree of t' is a copy of the 11-subtree

of t . It follows that performing any transformation in the latter subtree and then applying A has the same result as applying A first and performing the considered transformation in the 1-subtree of t' . Therefore, the equality

$$(2.3) \quad \partial f \cdot A = A \cdot \partial^2 f$$

holds for every (partial) mapping f defined on trees (Figure 5). In particular, when f is itself an operator of the form A_α , we obtain the relation

$$(2.4) \quad A_{11\alpha} \cdot A = A \cdot A_{1\alpha},$$

a typical example of what we shall call a *geometric relation*.

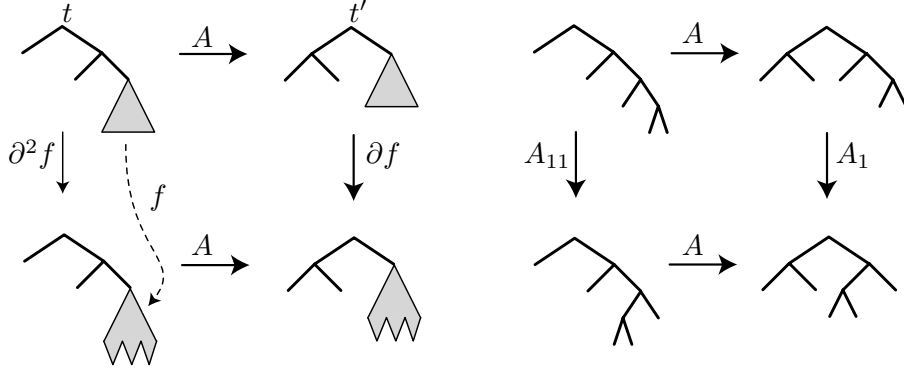


FIGURE 5. Geometric relations in $\mathcal{G}(\mathcal{A})$: the general scheme and one example

We can say that, under the action of A , the address 1 is the *heir* of the address 11, and, more generally, that 1α is the heir of 11α . Such inheritance phenomena are quite general. If we consider the operator A_α , we see that, for every β ,

- $\alpha 00\beta$ is the heir of $\alpha 0\beta$ under A_α ,
- $\alpha 01\beta$ is the heir of $\alpha 10\beta$ under A_α ,
- $\alpha 1\beta$ is the heir of $\alpha 11\beta$ under A_α .

Furthermore, if we say that two addresses α, β are incompatible, denoted $\alpha \perp \beta$, if neither is of prefix of the other, *i.e.*, if there exists γ such that $\gamma 0$ is a prefix of α and $\gamma 1$ is a prefix of β , or *vice versa*, then each address β with $\beta \perp \alpha$ is its own heir under the action of A_α .

The argument leading to (2.3) gives the relation $\partial_\gamma f \cdot A_\alpha = A_\alpha \cdot \partial_{\gamma'} f$ whenever γ' is the heir of γ under A_α . In this way, we deduce the following collection of geometric relations in $\mathcal{G}(\mathcal{A})$:

$$(2.5) \quad \left\{ \begin{array}{l} A_\beta \cdot A_\alpha = A_\alpha \cdot A_\beta \quad \text{for } \beta \perp \alpha, \\ A_{\alpha 0\beta} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 00\beta}, \\ A_{\alpha 10\beta} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 01\beta}, \\ A_{\alpha 11\beta} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 1\beta}. \end{array} \right.$$

The geometric relations are trivial in a sense, and one may wonder whether other, non-trivial relations hold in $\mathcal{G}(\mathcal{A})$. The answer is positive.

Lemma 2.8. *For each α , the following Pentagon Relation holds in $\mathcal{G}(\mathcal{A})$:*

$$(2.6) \quad A_\alpha \cdot A_\alpha = A_{\alpha 1} \cdot A_\alpha \cdot A_{\alpha 0}.$$

Relation (2.6) corresponds to the well-known MacLane–Stasheff pentagon property, and its proof is given (for $\alpha = \phi$) in Figure 6, which makes its name natural. Keeping the same name for the relations in $\mathcal{G}(\mathcal{A})$ and their counterparts in $G(\mathcal{A})$ —hence in F —we can summarize the results as follows.

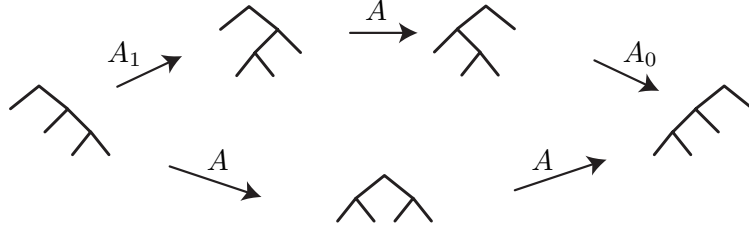


FIGURE 6. The pentagon relation

Definition. We denote by \mathbf{A} the family of all A_α 's, and by \mathbf{r}_A the family of all geometry relations involving A , namely the translated copies of

$$\begin{aligned}
 (\mathbf{g}_\perp) \quad & A_{0\alpha} \cdot A_{1\beta} = A_{1\beta} \cdot A_{0\alpha}, \\
 (\mathbf{g}_A) \quad & A_{11\alpha}A = AA_{1\alpha}, \quad A_{10\alpha}A = AA_{01\alpha}, \quad A_{0\alpha}A = AA_{00\alpha};
 \end{aligned}$$

plus the pentagon relations, *i.e.*, the translated copies of

$$(\mathbf{p}) \quad AA = A_1AA_0.$$

Proposition 2.9. *All relations of \mathbf{r}_A are satisfied by the elements A_α in the group $G(\mathcal{A})$, *i.e.*, in Thompson's group F .*

2.4. Constructing trees. Our next aim is to prove that the relations of Proposition 2.9 make a presentation of the geometry group $G(\mathcal{A})$, hence of F . To this end, we shall apply the method described in Section 1, using the partial action of $G(\mathcal{A})$ on trees. We saw that every family of trees containing trees of arbitrary large sizes is separating for this action, so, according to the criterion of Proposition 1.2(ii), two ingredients are needed, namely

- a family of trees containing one element in each orbit, and
- for every tree t , a distinguished word u_t in $W(\mathbf{A})$ connecting t with the distinguished element of its orbit.

Both steps are easy: two trees are equal up to associativity if and only if they have the same size, so each family of trees containing exactly one size n tree for each n is convenient. In the current case, we shall use the *right combs* of Figure 7.

Definition. For t_1, \dots, t_n trees, we put

$$\langle t_1, \dots, t_n \rangle = t_1 \wedge (t_2 \wedge \dots \wedge (t_{n-1} \wedge t_n) \dots);$$

we define the *right comb* $\langle n \rangle$ to be $\langle \circ, \dots, \circ \rangle$ with n times \circ .

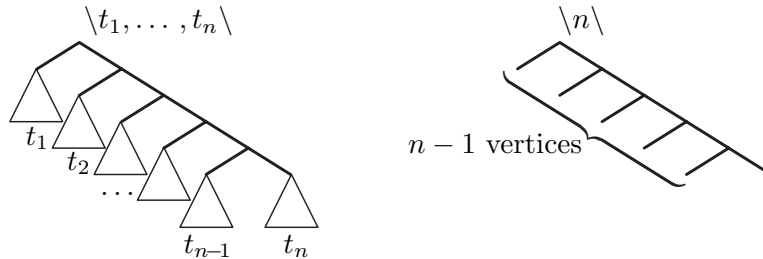


FIGURE 7. The notation $\langle t_1, \dots, t_n \rangle$ and the right comb $\langle n \rangle$

Note that, with the above notation, applying A means replacing $\langle t_1, t_2, \dots \rangle$ with $\langle t_1 \wedge t_2, \dots \rangle$. In the sequel, we shall use mixed expressions like $\langle p, t, q, \dots \rangle$ where p, q are numbers and t is a tree to mean $\langle \circ, \dots, \circ, t, \circ, \dots, \circ, \dots \rangle$ with p times \circ in the first block and q times \circ in the second.

If t is a size n tree, there exists a (unique) element of $G(\mathcal{A})$ mapping the right comb $\setminus n \setminus$ to t : in order to apply Proposition 1.2(ii), it suffices to select a distinguished word u_t representing that element, *i.e.*, to describe how t can be *constructed* from $\setminus n \setminus$ using associativity. Several solutions exist. We give now an inductive definition which leads to short computations, but requires that we introduce two words u_t, u_t^* for each tree t rather than one.

Definition. (i) For w a word involving letters indexed by addresses, we denote by $\partial_\alpha w$ the word obtained by appending α at the beginning of each index, and we use ∂w for $\partial_1 w$.

(ii) For each tree t , we define two words u_t, u_t^* using the inductive rules:

$$\begin{aligned} u_t &= u_t^* = \varepsilon && \text{for } t \text{ of size } 1, \\ u_t &= u_{t_1}^* \cdot \partial u_{t_2}, && u_t^* = u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A && \text{for } t = t_1 \wedge t_2. \end{aligned}$$

The following characterization of the words u_t and u_t^* is not needed in the sequel, but it helps making the definition concrete. As was mentioned above, each tree t admits a unique decomposition in terms of the basic tree \circ and the operation \wedge . Besides the algebraic notation $t_1 \wedge t_2$ for the product of t_1 and t_2 , we can also use the so-called *right Polish notation* in which the product of t_1 and t_2 is denoted $t_1 t_2 \bullet$ —we resort to a new symbol to avoid ambiguity. So, for instance, the (right) Polish decomposition of the tree $(\circ \wedge \circ) \wedge \circ$ is $\circ \circ \bullet \circ \bullet$. In the sequel, a length ℓ word w is considered as a sequence of symbols indexed by $\{1, \dots, \ell\}$, and we use $w(p)$ for the p th symbol in w .

Proposition 2.10. *For w a length ℓ word and $0 \leq p \leq \ell$, define the defect $\delta_w(p)$ of p in w by the rules: $\delta_w(0) = 0$, $\delta_w(p+1) = \delta_w(p) - 1$ for $w(p) = \bullet$, and $\delta_w(p+1) = \delta_w(p) + 1$ otherwise. Then, for each tree t in T , the word u_t^* is obtained from the Polish decomposition of t by deleting the symbols \circ , and replacing each defect i symbol \bullet with A_{1^i} . The word u_t is obtained similarly, except that the final symbols \bullet , *i.e.*, those followed by no \circ , do not contribute.*

Proof. It is standard that a word w is the Polish decomposition of a well-formed tree if and only if the defect of each symbol is nonnegative, and the defect of the last symbol is 0. For t a tree, define the enhanced decomposition of t to be the Polish decomposition with the defect of each symbol indicated in subscript. Then the enhanced decomposition of a product $t_1 \wedge t_2$ consists of the enhanced decomposition of t_1 , followed by the enhanced decomposition of t_2 with all defects shifted by 1, followed by the symbol \bullet with 0 defect. So the enhanced decomposition and the word u_t^* obey parallel inductive rules. Therefore, as the correspondence of the proposition clearly holds for the basic tree \circ , it inductively holds for every tree. A similar argument gives the connection between u_t^* and u_t . \square

Example 2.11. Let $t = ((\circ \wedge \circ) \wedge (\circ \wedge (\circ \wedge \circ))) \wedge ((\circ \wedge \circ) \wedge \circ)$. The enhanced decomposition of t is $\circ_0 \circ_1 \bullet_0 \circ_1 \circ_2 \circ_3 \bullet_2 \bullet_1 \bullet_0 \circ_1 \circ_2 \bullet_1 \circ_2 \bullet_1 \bullet_0$, and a direct translation yields $u_t^* = AA_{11}A_1AA_1A_1A$, and $u_t = AA_{11}A_1AA_1$ (for u_t the last two symbols \bullet in the Polish decomposition are dismissed).

Corollary 2.12. *For each tree t , we have*

$$(2.7) \quad u_t^* = u_t \cdot A_{1^{h-1}} \dots A_1 A,$$

where h is the length of the rightmost branch in t .

We are going to prove that the trees $\setminus n \setminus$ and the words u_t satisfy the requirements of Proposition 1.2(ii) and therefore can be used to obtain a presentation of the group $G(\mathcal{A})$.

Lemma 2.13. *For each size n tree t , we have*

$$(2.8) \quad \setminus n \setminus \xrightarrow{u_t} t \quad \text{and} \quad \setminus n, \dots \setminus \xrightarrow{u_t} \setminus t, \dots \setminus,$$

i.e., u_t constructs t from $\setminus n \setminus$, and u_t^* constructs $t \wedge t'$ from $\setminus n \setminus \wedge t'$ for every tree t' .

Proof. We use induction on n . For $n = 1$, the result is obvious. Otherwise, assume $t = t_1 \wedge t_2$, and let n_1 and n_2 be the sizes of t_1 and t_2 respectively. Then we have $u_t = u_{t_1}^* \cdot \partial u_{t_2}^*$. By induction hypothesis, $u_{t_1}^*$ maps $\setminus n \setminus$, *i.e.*, $\setminus n_1, n_2 \setminus$, to $t_1 \wedge \setminus n_2 \setminus$. Then, always by induction hypothesis, u_{t_2} maps $\setminus n_2 \setminus$ to t_2 , hence $\partial u_{t_2}^*$ maps $t_1 \wedge \setminus n_2 \setminus$ to $t_1 \wedge t_2$. So u_t maps $\setminus n \setminus$ to $t_1 \wedge t_2$, *i.e.*, to t (Figure 8 top):

$$\setminus n \setminus = \setminus n_1, n_2 \setminus \xrightarrow{u_{t_1}^*} \setminus t_1, n_2 \setminus \xrightarrow{\partial u_{t_2}^*} \setminus t_1, t_2 \setminus = t.$$

Similarly, we have $u_t^* = u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A$. The diagram is now:

$$\setminus n, \dots \setminus = \setminus n_1, n_2, \dots \setminus \xrightarrow{u_{t_1}^*} \setminus t_1, n_2, \dots \setminus \xrightarrow{\partial u_{t_2}^*} \setminus t_1, t_2, \dots \setminus \xrightarrow{A} \setminus t_1 \wedge t_2, \dots \setminus = t \wedge \dots$$

Indeed, by induction hypotheses, $u_{t_1}^*$ maps $\setminus n \wedge t'$, *i.e.*, $\setminus n_1, n_2, t' \setminus$, to $t_1 \wedge (\setminus n_2 \wedge t' \setminus)$, and $u_{t_2}^*$ maps $\setminus n_2 \wedge t' \setminus$ to $t_2 \wedge t'$, so $\partial u_{t_2}^*$ maps $t_1 \wedge (\setminus n_2 \wedge t' \setminus)$ to $t_1 \wedge (t_2 \wedge t')$. Finally, A maps $t_1 \wedge (t_2 \wedge t')$ to $(t_1 \wedge t_2) \wedge t'$, *i.e.*, to $t \wedge t'$ (Figure 8 bottom). \square

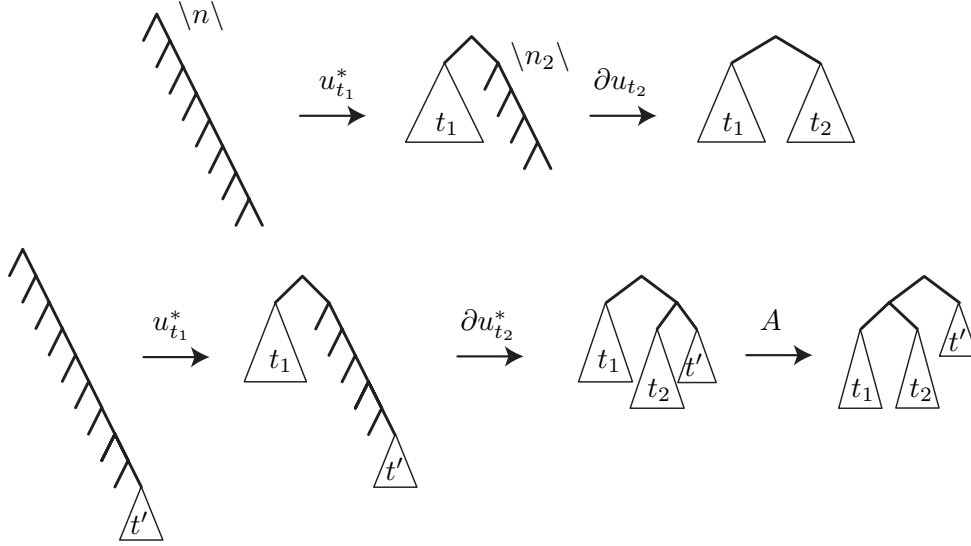


FIGURE 8. For t a size n tree, u_t describes how to construct t from $\setminus n \setminus$, and u_t^* describes how to construct $t \wedge t'$ from $\setminus n \wedge t' \setminus$; the figure illustrates the inductive argument for $t = t_1 \wedge t_2$

So Conditions (1.5), and therefore (1.2), of Proposition 1.2(ii) are satisfied. By construction, the family \mathbf{A} generates the group $G(\mathcal{A})$. We deduce that a family of relations involving the generators A_α makes a presentation of $G(\mathcal{A})$ if and only if it contains enough relations to make the words $u_{t'}$ and $u_t \cdot A_\alpha$ equivalent whenever t' is the image of t under A_α . We will show that this is the case for the relations r_A of Proposition 2.9. Due to our inductive construction, it will be convenient to prove two results simultaneously, namely one for u_t and one for u_t^* . Note that the argument proving $\overline{u_{t'}} = \overline{u_t} \cdot A_\alpha$ when A_α maps t to t' similarly proves $\overline{u_{t'}}^* = \overline{u_t}^* \cdot A_{0\alpha}$, as both sides of the latter equality map $\setminus k \setminus$ to $t' \wedge \setminus k - n \setminus$ for $k > n$. In the sequel, we use \equiv_g and \equiv_p to indicate that we specifically use a geometric or a pentagon relation.

Lemma 2.14. *Assume $t' = t \cdot A_\alpha$. Then we have*

$$(2.9) \quad u_{t'} \equiv_{r_A} u_t \cdot A_\alpha \quad \text{and} \quad u_{t'}^* \equiv_{r_A} u_t^* \cdot A_{0\alpha}.$$

Proof. We use induction on the length of α as a sequence of 0's and 1's. Assume first that α is the empty address. The hypothesis that $t' = t \cdot A$ holds, *i.e.*, that the operator A maps t to t' ,

means that there exist t_1, t_2, t_3 such that t is $t_1 \wedge (t_2 \wedge t_3)$ and t' is $(t_1 \wedge t_2) \wedge t_3$. Then we find

$$u_{t'} = u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A \cdot \partial u_{t_3}^* \equiv_{\mathbf{g}} u_{t_1}^* \cdot \partial u_{t_2}^* \cdot \partial^2 u_{t_3}^* \cdot A = u_t \cdot A,$$

$$u_{t'}^* = u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A \cdot \partial u_{t_3}^* \cdot A \equiv_{\mathbf{g}} u_{t_1}^* \cdot \partial u_{t_2}^* \cdot \partial^2 u_{t_3}^* \cdot AA \equiv_{\mathbf{p}} u_{t_1}^* \cdot \partial u_{t_2}^* \cdot \partial^2 u_{t_3}^* \cdot A_1 AA_0 = u_t^* \cdot A_0.$$

Assume now $\alpha = 0\beta$. The hypothesis that A_α maps t to t' means that there exist t_2, t_2', t_1' such that t is $t_1 \wedge t_2$, t' is $t_1' \wedge t_2$, and A_β maps t_1 to t_1' . Using the induction hypothesis, we find

$$u_{t'} = u_{t_1'}^* \cdot \partial u_{t_2}^* \equiv_{(IH)} u_{t_1'}^* \cdot A_{0\beta} \cdot \partial u_{t_2}^* \equiv_{\mathbf{g}} u_{t_1'}^* \cdot \partial u_{t_2}^* \cdot A_{0\beta} = u_t \cdot A_\alpha,$$

$$u_{t'}^* = u_{t_1'}^* \cdot \partial u_{t_2}^* \cdot A \equiv_{(IH)} u_{t_1'}^* \cdot A_{0\beta} \cdot \partial u_{t_2}^* \cdot A \equiv_{\mathbf{g}} u_{t_1'}^* \cdot \partial u_{t_2}^* \cdot A_{0\beta} A \equiv_{\mathbf{g}} u_{t_1'}^* \cdot \partial u_{t_2}^* \cdot AA_{00\beta} = u_t^* \cdot A_{0\alpha}.$$

Finally, assume $\alpha = 1\beta$. With similar notation, we have $t = t_1 \wedge t_2$ and $t' = t_1 \wedge t_2'$ with A_β mapping t_2 to t_2' . We find now

$$u_{t'} = u_{t_1}^* \cdot \partial u_{t_2'}^* \equiv_{(IH)} u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A_{1\beta} = u_t \cdot A_\alpha,$$

$$u_{t'}^* = u_{t_1}^* \cdot \partial u_{t_2'}^* \cdot A \equiv_{(IH)} u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A_{10\beta} A \equiv_{\mathbf{g}} u_{t_1}^* \cdot \partial u_{t_2}^* \cdot AA_{01\beta} = u_t^* \cdot A_{0\alpha},$$

which completes the proof. \square

We are now in position for concluding:

Proposition 2.15. *The relations \mathbf{r}_A , i.e., the geometric relations for A , plus the pentagon relations, make a presentation of the group $G(\mathcal{A})$, i.e., F , in terms of the generators A_α .*

Proof. All requirements of Proposition 1.2 are satisfied. Indeed, for each size n tree t , we have selected an element in the F -orbit of t , namely the right comb $\setminus n \setminus$, and a word u_t in $W(\mathbf{A})$ so that t is the image of $\setminus n \setminus$ under the action of u_t . So we deduce that \mathbf{A} generates F —which is obvious—and that a family of relations involving \mathbf{A} makes a presentation of $G(\mathcal{A})$ if and only if it contains enough relations to guarantee the equivalence of $u_t \cdot A_\alpha$ and $u_{t'}$ whenever $t' = t \cdot A_\alpha$ is satisfied: this is exactly what Lemma 2.14 asserts for the relation \mathbf{r}_A . \square

2.5. The standard presentation. As is well-known [5], there exists a presentation of F in terms of an infinite sequence of generators, usually denoted x_i or X_i , indexed by nonnegative integers. We can easily establish the connection between these generators and our current generators A_α , and, using Proposition 1.2 again, we shall re-obtain the standard presentation of F as a direct corollary.

Definition. (i) For $i \geq 1$, we put $a_i = A_{1^{i-1}}$, and we denote by \mathbf{a} the family of all a_i 's.

(ii) We denote by \mathbf{r}_a the subfamily of \mathbf{r}_A consisting of those relations in \mathbf{r}_A that involve the generators of \mathbf{a} exclusively, namely the relations $a_i a_j = a_{j+1} a_i$ for $j > i$.

Proposition 2.16. *The set \mathbf{a} generates $G(\mathcal{A})$, i.e., F , and the relations \mathbf{r}_a make a presentation of $G(\mathcal{A})$ in terms of the generators a_i .*

Proof. By construction, the words u_t belong to $W(\mathbf{a})$. So the first part of Proposition 1.2 shows that \mathbf{a} generates F . In order to prove that \mathbf{r}_a makes a presentation, it suffices to check that the relations of \mathbf{r}_a are sufficient to establish the equivalence of $u_{t'}$ and $u_t \cdot a_i$ when a_i maps t to t' . Looking at the proof of Lemma 2.14 immediately shows that this is the case. So, Proposition 1.2 applies again, and it shows that \mathbf{r}_a makes a presentation of F in terms of the a_i 's. \square

A consequence of the fact that the family \mathbf{a} generates F is that each element A_α has to be expressible in terms of the elements a_i . For future use, let us mention an explicit formula:

Lemma 2.17. *Assume that α is an address containing at least one 0, say $\alpha = 1^p 00^{e_0} 10^{e_1} 1 \dots 10^{e_q}$ with $p, q, e_0, \dots, e_q \geq 0$. Then we have*

$$(2.10) \quad A_\alpha = (a_{p+1}^{e_0+1} a_{p+2}^{e_1+1} \dots a_{p+q+1}^{e_q+1}) (a_{p+q+1} a_{p+q+2}^{-1}) (a_{p+1}^{e_0+1} a_{p+2}^{e_1+1} \dots a_{p+q+1}^{e_q+1})^{-1}.$$

For instance, for $\alpha = 01100$, we find $p = 0$, $q = 2$, $e_0 = e_1 = 0$, and $e_2 = 2$, hence $A_{01100} = a_1 a_2 a_3^4 a_4^{-1} a_3^{-3} a_2^{-1} a_1^{-1}$.

2.6. Connection with associahedra. Let K_n denote the Cayley graph of the orbit of the size n tree $\setminus n \setminus$ under the (partial) action of the group F with respect to the generators A_α . The vertices of K_n consist of all size n trees, and there is an edge from t to t' if and only if t' can be reached from t using a single transformation $A_\alpha^{\pm 1}$: such K_n is the n th associahedron [22].

By construction, the right combs make a separating family for the partial action of F when n varies, and there is exactly one element of the family in each orbit. Hence the Cayley graph of F with respect to the generators A_α is the union (or, rather, the direct limit) of the associahedra K_n .

It follows that studying the geodesics in the above Cayley graph is the same as investigating the rotation distance between binary trees—or, equivalently, the flip distance between triangulations [32].

One point showing (if needed) that the problem is difficult is the remark that positive words, *i.e.*, those containing only generators A_α and no A_α^{-1} , are not geodesic: one can show that, for every k , the word

$$(A_{1^k} A_{1^{k-1}} \dots A_1 A)(A_{011^{k-1}} \dots A_{011} A_{01}) \dots (A_{(01)^{k-1}1} A_{(01)^{k-1}})(A_{(01)^k}),$$

whose length is quadratic in k , is geodesic among positive words; now this word is equivalent to

$$A_{1^{k-1}} \dots A_1 A \cdot A A_0^{-1} A_{01} A_{010}^{-1} \dots A_{(01)^{k-1}0}^{-1} A_{(01)^k},$$

whose length is linear in k .

3. THE GROUP F AS A GROUP OF FRACTIONS

One advantage of using the geometric presentation of F in terms of the generators A_α is that it leads to an interesting submonoid F^+ of F , and, from there, to a lattice structure on F .

3.1. The positive geometry monoid. By construction, our associativity operators $A_\alpha^{\pm 1}$ come in pairs. Besides the monoid $\mathcal{G}(\mathcal{A})$ generated by all operators $A_\alpha^{\pm 1}$, it is also natural to consider the submonoid corresponding to selecting one of the two directions.

Definition. We denote by $\mathcal{G}^+(\mathcal{A})$ the monoid generated by all operators A_α , and call it the *positive geometry monoid* of associativity.

All results about the geometry monoid extend to the positive geometry monoid *mutatis mutandis*, and, in particular, we have:

Proposition 3.1. *Near-equality is a congruence on $\mathcal{G}^+(\mathcal{A})$, and the associated quotient-monoid is the submonoid $G^+(\mathcal{A})$ of $G(\mathcal{A})$ generated by \mathbf{A} .*

As the geometry group $G(\mathcal{A})$ has been identified with Thompson's group F , it is natural to give a related name to the monoid $G^+(\mathcal{A})$.

Definition. We denote by F^+ the submonoid of F generated by \mathbf{A} .

Our first task will be to find a presentation of the monoid F^+ . To this end, we shall apply the method of Section 1 again, still using the partial action of the operators A_α on trees and, this time, Proposition 1.3. The monoid F^+ admits left cancellation since it is a submonoid of a group. By looking at the definition, we see that the words u_t and u_t^* belong to $W^+(\mathbf{A})$. As the geometric and pentagon relations involve only words in $W^+(\mathbf{A})$, we are nearly in position of applying Proposition 1.3 and deducing that $\langle \mathbf{A}; \mathbf{r}_A \rangle^+$ and $\langle \mathbf{a}; \mathbf{r}_a \rangle^+$ are presentations of F^+ and F_L^+ respectively. Two points are missing:

- a proof that the monoid $\langle \mathbf{A}; \mathbf{r}_A \rangle^+$ admits left cancellation;
- a proof of Condition (1.10), which is *a priori* stronger than Condition (1.3) as established as (2.9) in Lemma 2.14.

The latter point is easy.

Lemma 3.2. *Assume that A_α maps t to t' . Then we have*

$$(3.1) \quad u_{t'} \equiv_{\mathbf{r}_A}^+ u_t \cdot A_\alpha \quad \text{and} \quad u_{t'}^* \equiv_{\mathbf{r}_A}^+ u_t^* \cdot A_{0\alpha}.$$

Proof. It suffices to look at the proof of Lemma 2.14 and to check that only positive words and relations are used, which is straightforward. \square

So the only remaining question is whether $\langle \mathbf{A}; \mathbf{r}_A \rangle^+$ admits left cancellation.

3.2. The word reversing technique. In order to study the monoids $\langle \mathbf{A}; \mathbf{r}_A \rangle^+$ we shall resort to general algebraic tools developed in [9, 14] and connected with Garside's seminal work [21]. This combinatorial approach applies to certain monoid presentations and it is relevant for establishing properties like the possible cancellativity of the monoid, or its embedding in a group of fractions.

The main notion is *word reversing*. In the sequel, we say that (\mathbf{x}, \mathbf{r}) is a *positive* group presentation if \mathbf{r} exclusively consists of relations of the form $u = v$ with u, v nonempty words in $W^+(\mathbf{x})$. Note that this condition is satisfied by the presentation $(\mathbf{A}, \mathbf{r}_A)$.

Definition. [9, 14] Assume that (\mathbf{x}, \mathbf{r}) is a positive group presentation, and that w, w' are words in $W(\mathbf{x})$. We say that w is *right \mathbf{r} -reversible* to w' in one step if w' is obtained from w either by deleting some length 2 subword $x^{-1}x$, or by replacing a length 2 subword $x^{-1}y$ with a word vu^{-1} such that $xv = yu$ is one of the relations of \mathbf{r} . We write $w \curvearrowright_{\mathbf{r}} w'$ if w is right \mathbf{r} -reversible to w' in finitely many steps, *i.e.*, if w' can be obtained from w using finitely many successive right \mathbf{r} -reversing steps.

The idea of right \mathbf{r} -reversing is to use the relations of \mathbf{r} to push the negative letters (those in \mathbf{x}^{-1}) to the right and the positive letters (those in \mathbf{x}) to the left by iteratively reversing the negative–positive patterns into positive–negative ones. Note that the first case of reversing, namely deleting some subword $x^{-1}x$, appears as a special case of the second case provided we assume that, for every letter x in \mathbf{x} , the trivial relation $x = x$ is implicit in \mathbf{r} .

Left \mathbf{r} -reversing is defined symmetrically: one step consists in deleting one subword xx^{-1} , or replacing one subword xy^{-1} with $v^{-1}u$ such that $vx = uy$ is a relation of \mathbf{r} .

Example 3.3. Let us consider the presentation $(\mathbf{A}; \mathbf{r}_A)$, and let w be the word $A^{-1}A_1A^{-1}$. Then w contains a unique negative–positive pair of letters, namely $A^{-1}A_1$. There exists a unique relation in \mathbf{r}_A with the form $A \dots = A_1 \dots$, namely $A^2 = A_1AA_0$. Therefore, the only way to \mathbf{r}_A -reverse w on the right is to replace $A^{-1}A_1$ with $AA_0^{-1}A^{-1}$, and we have $w \curvearrowright_{\mathbf{r}_A} AA_0^{-1}A^{-2}$. The latter word contains no negative–positive pair, so no further right \mathbf{r}_A -reversing is possible. Similarly, w is left \mathbf{r}_A -reversible to $A^{-2}A_{11}$. Note that, if we use \mathbf{r}_a instead of \mathbf{r}_A , the above left reversing of w remains possible, but no right reversing can be performed.

If $xu = yv$ is a relation of \mathbf{r} , then $x^{-1}y \equiv_{\mathbf{r}} vu^{-1}$ holds, and a straightforward induction shows that $w \curvearrowright_{\mathbf{r}} w'$ implies $w \equiv_{\mathbf{r}} w'$. A slightly more careful argument shows that, when only positive words are considered, \mathbf{r} -reversing gives rise to equivalence not only in the group, but even in the monoid:

Lemma 3.4. [14] *Assume $u, v, u', v' \in W^+(\mathbf{x})$. Then $u^{-1}v \curvearrowright_{\mathbf{r}} v'u'^{-1}$ implies $uv' \equiv_{\mathbf{r}}^+ vu'$.*

In particular, if u, v belong to $W^+(\mathbf{x})$, then $u^{-1}v \curvearrowright_{\mathbf{r}} \varepsilon$ (the empty word) implies $u \equiv_{\mathbf{r}}^+ v$. There is in general no reason why the converse implication should be true, but, when this happens to be the case, then word reversing proves to be a very useful tool, for instance to recognize cancellativity.

Definition. [14] We shall say that the presentation (\mathbf{x}, \mathbf{r}) is *complete with respect to right reversing* if right reversing always detects positive equivalence, *i.e.*, if $u \equiv_{\mathbf{r}}^+ v$ implies $u^{-1}v \curvearrowright_{\mathbf{r}} \varepsilon$ for all words u, v in $W^+(\mathbf{x})$. Symmetrically, we say that $(\mathbf{x}; \mathbf{r})$ is complete with respect to left reversing if $u \equiv_{\mathbf{r}}^+ v$ implies that uv^{-1} is left \mathbf{r} -reversible to ε .

A nice feature is that there exists a criterion which, under weak assumptions, enables one to recognize whether a given presentation is complete with respect to reversing—or to add new relations if it is not.

Definition. [14] We say that a positive presentation (\mathbf{x}, \mathbf{r}) is *homogeneous* if there exists an $\equiv_{\mathbf{r}}^+$ -invariant mapping $\lambda : W^+(\mathbf{x}) \rightarrow \mathbf{N}$ such that $\lambda(x) \geq 1$ holds for every x in \mathbf{x} , and $\lambda(uv) \geq \lambda(u) + \lambda(v)$ holds for all words u, v in $W^+(\mathbf{x})$.

If all relations in \mathbf{r} preserve the length of the words, then the length satisfies the requirements of homogeneity. The previous definition corresponds to relaxing some unessential properties of the length but keeping the core that makes inductive argument possible. Then we have the following criterion for recognizing complete presentations.

Proposition 3.5. [14] *Assume that (\mathbf{x}, \mathbf{r}) is a homogeneous positive presentation. Then (\mathbf{x}, \mathbf{r}) is complete with respect to right reversing if and only, for each triple (x, y, z) in \mathbf{x}^3 ,*

$$(3.2) \quad x^{-1}yy^{-1}z \curvearrowright_{\mathbf{r}} vu^{-1} \quad \text{with } u, v \text{ in } W^+(\mathbf{x}) \text{ implies } v^{-1}x^{-1}yu \curvearrowright_{\mathbf{r}} \varepsilon.$$

In other words: Whenever $x^{-1}yy^{-1}z$ reverses (on the right) to a word of the form vu^{-1} with u, v in $W^+(\mathbf{x})$, then the word $v^{-1}x^{-1}yu$ reverses (on the right) to the empty word.

Proposition 3.6. [14] *Assume that (\mathbf{x}, \mathbf{r}) is a positive presentation that is complete with respect to right reversing. Then the monoid $\langle \mathbf{x}; \mathbf{r} \rangle^+$ is left cancellative if and only if $u^{-1}v \curvearrowright_{\mathbf{r}} \varepsilon$ holds for every relation of the form $xu = xv$ in \mathbf{r} . Thus, in particular, a sufficient condition for $\langle \mathbf{x}; \mathbf{r} \rangle^+$ to be left cancellative is that \mathbf{r} contains no relation of the form $xu = xv$.*

To apply the previous criterion to the presentations $(\mathbf{A}, \mathbf{r}_A)$ and $(\mathbf{a}, \mathbf{r}_a)$, it suffices to check that these presentations are homogeneous and satisfy the criterion of Proposition 3.5.

Lemma 3.7. *The presentation $(\mathbf{A}, \mathbf{r}_A)$ is homogeneous.*

Proof. We cannot use the length, as it is not preserved under the pentagon relation: A_1AA_0 has length 3, while A^2 has length 2. Instead we can use the number of 0's in addresses. If α is an address, we shall denote by $\nu(\alpha)$ the number of 0's in α . For t a tree, let $\text{add}_k(t)$ be the address of the k th leaf from the left in t . Let us compare $\nu(\text{add}_k(t))$ and $\nu(\text{add}_k(t'))$ when t' is the image of t under some element of F^+ . For instance, assume that A maps t to t' . Then $\text{add}_k(t) = 11\alpha$ implies $\text{add}_k(t') = 1\alpha$, and $\text{add}_k(t) = 10\alpha$ implies $\text{add}_k(t') = 01\alpha$. So, in these cases, we have $\nu(\text{add}_k(t')) = \nu(\text{add}_k(t))$. Now $\text{add}_k(t) = 0\alpha$ implies $\text{add}_k(t') = 00\alpha$, hence $\nu(\text{add}_k(t')) = \nu(\text{add}_k(t)) + 1$. An obvious induction shows that, if t' is the image of t under a nontrivial element of F^+ , then $\nu(\text{add}_k(t')) \geq \nu(\text{add}_k(t))$ holds for each k , and there is at least one k such that the inequality is strict. So, if we define $\nu(t)$ to be $\sum_k \nu(\text{add}_k(t))$, we have $\nu(t') > \nu(t)$. For w a word in $W^+(\mathbf{A})$, define

$$\lambda(w) = \nu(t') - \nu(t),$$

where (t, t') is a seed for (the operator associated with) w . The previous argument shows that $\lambda(w)$ is a positive integer whenever w is nonempty, and, *e.g.*, we find $\lambda(A_\alpha) = 1$ for each α . Because the number of 0's in add_k is never decreasing under the action of a positive word, we necessarily have

$$\nu(t'^\sigma) - \nu(t^\sigma) \geq \nu(t') - \nu(t)$$

for every substitution σ when t' is the image of t under an operator in F^+ . So, by Lemma 2.4, we must have $\nu(t'_1) - \nu(t_1) \geq \lambda(w)$ whenever w maps t_1 to t'_1 .

Assume $w = w_1w_2$. Let (tti, t'_1) and (t_2, t'_2) be seeds for w_1 and w_2 , respectively. We saw in the proof of Lemma 2.4 that a seed for w is the pair (t_1^φ, t_2^ψ) , where φ and ψ are substitutions satisfying $t_1^\varphi = t_2^\psi$. Then we find

$$\begin{aligned} \lambda(w) &= \nu(t_2'^\psi) - \nu(t_1^\varphi) = \nu(t_2'^\psi) - \nu(t_2^\psi) + \nu(t_1'^\varphi) - \nu(t_1^\varphi) \\ &\geq \nu(t_2') - \nu(t_2) + \nu(t_2') - \nu(t_1) = \lambda(w_1) + \lambda(w_2). \end{aligned}$$

Finally, the compatibility of λ with the relations of \mathbf{r}_A is clear, as $\lambda(w)$ is defined using the action of w on trees only. \square

Lemma 3.8. *The presentation $(\mathbf{A}, \mathbf{r}_A)$ is complete with respect to right reversing.*

Proof. Let us consider $(\mathbf{A}, \mathbf{r}_A)$ first. By Proposition 3.5, it suffices to check that, if α, β, γ are three addresses and $A_\alpha^{-1}A_\beta A_\beta^{-1}A_\gamma$ reverses to a word of the form vu^{-1} with u, v in $W^+(\mathbf{x})$, then $v^{-1}A_\alpha^{-1}A_\gamma u$ reverses to the empty word. The systematic verification consists in considering all possible mutual positions of the addresses α, β, γ . An important point is that, for each pair of addresses (α, β) , there is exactly one relation of the form $A_\alpha \dots = A_\beta \dots$ in \mathbf{r}_A , which implies that word reversing is a deterministic process. So, in particular, there exists at most one pair of positive words u, v such that $A_\alpha^{-1}A_\beta A_\beta^{-1}A_\gamma$ reverses to vu^{-1} . To give an example, let us consider the case $\alpha = \phi, \beta = 1, \gamma = 11$, which is the most complicated one: the reader can check that $A^{-1}A_1 A_1^{-1}A_{11}$ reverses to $A^2 A_{00}^{-1} A_0^{-1} A^{-1} A_{10}^{-1} A_1^{-1}$, and, then, that $A^{-3} A_{11} A_1 A_{10} A A_0 A_{00}$ reverses to the empty word. For complete details, we refer to [10] where it is shown that most of the cases follow from a uniform geometric argument, and that the above particular case is essentially the only non-trivial one requiring a specific verification. \square

The presentation $(\mathbf{A}; \mathbf{r}_A)$ contains no relation of the form $A_\alpha u = A_\alpha v$, so, applying Proposition 3.6, we obtain

Lemma 3.9. *The monoid $\langle \mathbf{A}; \mathbf{r}_A \rangle^+$ admits left cancellation.*

All hypotheses of Proposition 1.3 are now satisfied, and we deduce:

Proposition 3.10. *The positive geometry monoid $G^+(\mathbf{A})$, i.e., F^+ , admits the presentation $\langle \mathbf{A}; \mathbf{r}_A \rangle^+$.*

Another consequence is the following strong faithfulness result for the action of F^+ on trees:

Proposition 3.11. *Assume that w, w' are words in $W^+(\mathbf{A})$ and there exists at least one tree t such that $t \bullet w$ and $t \bullet w'$ are defined and equal. Then $w \equiv_{\mathbf{r}_A}^+ w'$ holds, i.e., w and w' represent the same element of F^+ .*

Remark 3.12. The symmetry exchanging left and right induces an antiautomorphism of the group F and the monoid F^+ in which the image of A_α is $A_{\bar{\alpha}}$, where $\bar{\alpha}$ is obtained from α by exchanging 0 and 1. As the relations of \mathbf{r}_A are invariant under this transformation, we can deduce without new verification that the presentation $(\mathbf{A}; \mathbf{r}_A)$ is complete with respect to left reversing as well. So, for instance, the monoid $\langle \mathbf{A}; \mathbf{r}_A \rangle^+$ admits right cancellation as well as left cancellation.

3.3. Fractions. Once we know now that $(\mathbf{A}; \mathbf{r}_A)$ is a presentation for the monoid F^+ , we can easily establish further properties of this monoid and deduce new results about the group F .

Keeping the same notation as in Section 1, we denote by \bar{w} the image of a word w of $W(\mathbf{A})$ in F , i.e., its class under \equiv^+ . When w belongs to $W^+(\mathbf{A})$, i.e., contains only positive letters, \bar{w} belongs to F^+ , which we know embeds in F .

Lemma 3.13. *Two elements of F^+ always admit a common right multiple and a common left multiple.*

Proof. Let a, b be arbitrary elements of F^+ . By Lemma 2.4, there exists a tree t such that both $t \bullet a^{-1}$ and $t \bullet b^{-1}$ exists, i.e., there exist t_1, t_2 satisfying $t = t_1 \bullet a$ and $t = t_2 \bullet b$. Then (3.1) gives

$$\bar{u}_t = \bar{u}_{t_1} \cdot a = \bar{u}_{t_2} \cdot b,$$

hence \bar{u}_t is a common left multiple of a and b .

We deduce the existence of common right multiples using Remark 3.12. \square

It follows that the monoid F^+ satisfies the Ore conditions, on the left and on the right, and we deduce:

Proposition 3.14. *Thompson's group F is both a group of right and of left fractions for the monoid F^+ .*

The general results about word reversing enable us to say more about the monoid F^+ .

Proposition 3.15. [14] *Assume that (\mathbf{x}, \mathbf{r}) is a positive group presentation that is complete with respect to right reversing, and, for all x, y in \mathbf{x} , there is at most one relation of the form $x\dots = y\dots$ in \mathbf{r} . Then two elements a, b of $\langle \mathbf{x}; \mathbf{r} \rangle^+$ that admit a common right multiple admit a least common right multiple. Moreover, if u, v are words in $W^+(\mathbf{x})$ representing a and b respectively, the right reversing of $u^{-1}v$ leads in a finite number of steps to a word of the form $v'u'^{-1}$ with u', v' in $W^+(\mathbf{x})$, and, then, both uv' and vu' represent the right lcm of a and b .*

The criterion of Proposition 3.15 is clearly satisfied in the presentation $(\mathbf{A}; \mathbf{r}_A)$, and so is its left counterpart, so we obtain:

Proposition 3.16. *Any two elements of the monoid F^+ admit a right and a left lcm. The latter are computed using right and left reversing, respectively: if u, v are words in $W^+(\mathbf{A})$, the right lcm of \bar{u} and \bar{v} in F^+ is represented by the words uv' and vu' , where u' and v' are the (unique) positive words determined by $u^{-1}v \curvearrowright_{\mathbf{r}} v'u'^{-1}$.*

Another corollary is that left and right \mathbf{r}_A -reversing must always converge:

Lemma 3.17. *For each word w in $W(\mathbf{A})$, there exist unique words u, v, u', v' in $W^+(\mathbf{A})$ such that w is right \mathbf{r}_A -reversible to vu^{-1} and left \mathbf{r}_A -reversible to $u'^{-1}v'$.*

Definition. For w in $W(\mathbf{A})$, the words involved in Lemma 3.17 are called the *right denominator*, the *right numerator*, the *left denominator*, and the *left numerator* of w , respectively; we denote them $D_R(w), N_R(w), D_L(w), N_L(w)$.

Example 3.18. Let $w = A^{-1}A_1AA_1^{-1}A$. The reader can check that w is right reversible to $A_{11}AA_{11}^{-1}$, and left reversible to $A^{-1}A^{-1}A_0^{-1}AA_0AA$. So we have $N_R(w) = A_{11}A$, $D_R(w) = A_{11}$, $N_L(w) = AA_0A^2$, and $D_L(w) = A_0A^2$.

By definition, the equivalences $w \equiv_{\mathbf{r}_A} N_R(w)D_R(w)^{-1} \equiv_{\mathbf{r}_A} D_L(w)^{-1}N_L(w)$ hold for every word in $W(\mathbf{A})$. It should be noted that the mappings N_R, \dots, D_L do *not* induce well-defined mappings of F to F^+ : if w and w' represent the same element of F , the words $N_R(w)$ and $N_R(w')$ need not represent the same element of F^+ . For instance, in the above example, we have $w \equiv_{\mathbf{r}_A} A_{11}AA_{11}^{-1}$, and the reader can check the value $D_L(A_{11}AA_{11}^{-1}) = A_{10}A_1 \not\equiv_{\mathbf{r}_A}^+ D_L(w) = A_0A^2$. However, this lack of compatibility disappears when both a right and a left reversing are performed.

Definition. For w in $W(\mathbf{A})$, we put

$$N_{\mathbb{R}\mathbb{L}}(w) = N_L(N_R(w)D_R(w)^{-1}), \quad D_{\mathbb{R}\mathbb{L}}(w) = D_L(N_R(w)D_R(w)^{-1}).$$

Proposition 3.19. (i) *The mappings $N_{\mathbb{R}\mathbb{L}}$ and $D_{\mathbb{R}\mathbb{L}}$ induce well-defined mappings (still denoted $N_{\mathbb{R}\mathbb{L}}$ and $D_{\mathbb{R}\mathbb{L}}$) of F into F^+ .*

(ii) *A word w represents 1 in F if and only if the words $N_{\mathbb{R}\mathbb{L}}(w)$ and $D_{\mathbb{R}\mathbb{L}}(w)$ are empty.*

(iii) *For each element c of F , the elements $N_{\mathbb{R}\mathbb{L}}(c)$ and $D_{\mathbb{R}\mathbb{L}}(c)$ have no common left divisor, and $D_{\mathbb{R}\mathbb{L}}(c)^{-1}N_{\mathbb{R}\mathbb{L}}(c)$ is the shortest decomposition of c as a left fraction w.r.t. F^+ : if we have $c = a^{-1}b$ with a, b in F^+ , there exists d in F^+ satisfying $a = dD_{\mathbb{R}\mathbb{L}}(c)$ and $b = dN_{\mathbb{R}\mathbb{L}}(c)$.*

3.4. The lattice structures on F . The monoid F^+ contains no invertible element excepted 1. Hence it is eligible to make the positive cone of a partial ordering on the group F . Actually, two orderings can be considered:

Definition. For a, b in F , we say that $a \leq_R b$ is true if $a^{-1}b$ belongs to F^+ . Symmetrically, we say that $a \leq_L b$ is true if ba^{-1} belongs to F^+

Note that $a \leq_L b$ is equivalent to $b^{-1} \leq_R a^{-1}$.

Lemma 3.20. *For u, v in $W(\mathbf{A})$, the relation $\bar{u} \leq \bar{v}$ is equivalent to $D_{\mathbb{R}}(u^{-1}v) = \varepsilon$.*

Proposition 3.21. *The relation \leq_R (resp. \leq_L) is a lattice ordering on F , which is compatible with multiplication on the left (resp. on the right).*

The Cayley graph of the monoid F^+ is the union of all associahedra, and the lattice structures on F^+ correspond to the lattice structures on the associahedra.

3.5. An alternative presentation. We proved that word reversing associated with the presentation $(\mathbf{A}; \mathbf{r}_A)$ always terminates in a finite number of steps: this follows from the fact that the presentation is complete with respect to reversing and from the existence of common multiples in the monoid F^+ . A direct argument is also possible. A trivial sufficient condition for reversing to always terminate is that all relations of the considered presentation have length 2 at most. At the expense of introducing more generators, we can easily fulfill this condition.

Definition. For α an address and $p \geq 0$, we put $A_\alpha^{(p)} = A_\alpha A_{0\alpha} \dots A_{\alpha 0^{p-1}}$, and denote by \mathbf{A}^* the family of all $A_\alpha^{(p)}$.

The action of $A_\alpha^{(p)}$ on trees is illustrated in Figure 9.

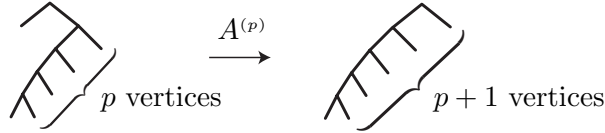


FIGURE 9. Action of $A^{(p)}$

Lemma 3.22. *The closure of \mathbf{A} under left \mathbf{r}_A -reversing, i.e., the smallest subset X of $W^+(\mathbf{A})$ such that, for all u, v in X , there exist u', v' in X satisfying $u^{-1}v \curvearrowright_{\mathbf{r}_A} v'u'^{-1}$, is $\{\varepsilon\} \cup \mathbf{A}^*$, and we have the following relations:*

$$\begin{aligned} (A_\alpha^{(q)})^{-1} A_\alpha^{(p)} &\curvearrowright_{\mathbf{r}_A} (A_{0^{q-p}}^{(q-p)})^{-1} && \text{for } p < q, \\ (A_\beta^{(q)})^{-1} A_\alpha^{(p)} &\curvearrowright_{\mathbf{r}_A} A_\alpha^{(p)} (A_\beta^{(q)})^{-1} && \text{for } \beta \perp \alpha, \\ (A_{\alpha 0 \beta}^{(q)})^{-1} A_\alpha^{(p)} &\curvearrowright_{\mathbf{r}_A} A_\alpha^{(p)} (A_{\alpha 0^{p+1} \beta}^{(q)})^{-1} \\ (A_{\alpha 10^p \beta}^{(q)})^{-1} A_\alpha^{(p)} &\curvearrowright_{\mathbf{r}_A} A_\alpha^{(p)} (A_{\alpha 0^p 1 \beta}^{(q)})^{-1} \\ (A_{\alpha 10^i 1 \beta}^{(q)})^{-1} A_\alpha^{(p)} &\curvearrowright_{\mathbf{r}_A} A_\alpha^{(p)} (A_{\alpha 0^i 1 \beta}^{(q)})^{-1} && \text{for } i < p, \\ (A_{\alpha 10^i}^{(q)})^{-1} A_\alpha^{(p)} &\curvearrowright_{\mathbf{r}_A} A_\alpha^{(p+q)} (A_{\alpha 0^i}^{(q)})^{-1} && \text{for } i < p. \end{aligned}$$

Definition. We define \mathbf{r}_{A^*} to consist of the following relations:

$$\begin{aligned} A_\alpha^{(p)} A_{0^{q-p}}^{(q-p)} &= A_\alpha^{(q)} && \text{with } p < q, \\ A_\alpha^{(p)} A_\beta^{(q)} &= A_\beta^{(q)} A_\alpha^{(p)} && \text{with } \beta \perp \alpha, \\ A_\alpha^{(p)} A_{\alpha 0^{p+1} \beta}^{(q)} &= A_{\alpha 0 \beta}^{(q)} A_\alpha^{(p)} \\ A_\alpha^{(p)} A_{\alpha 0^p 1 \beta}^{(q)} &= A_{\alpha 10^p \beta}^{(q)} A_\alpha^{(p)} \\ A_\alpha^{(p)} A_{\alpha 0^i 1 \beta}^{(q)} &= A_{\alpha 10^i 1 \beta}^{(q)} A_\alpha^{(p)} && i < p, \\ A_\alpha^{(p)} A_{\alpha 0^i}^{(q)} &= A_{\alpha 10^i}^{(q)} A_\alpha^{(p+q)} && i < p. \end{aligned}$$

Proposition 3.23. *The relations of \mathbf{r}_{A^*} make a presentation of the monoid F^+ , and of the group F , in terms of the generators $A_\alpha^{(p)}$.*

Proof. As we have $A_\alpha = A_\alpha^{(1)}$, the set \mathbf{A} is included in \mathbf{A}^* , and all relations of \mathbf{r}_A are relations of \mathbf{r}_A^* with $p = q = 1$. Thus the only point is to check that all relations in \mathbf{r}_A^* are consequences of the relations of \mathbf{r}_A : this follows from Lemmas 3.22 and 3.4. \square

The relations of \mathbf{r}_{A^*} are simple and natural: the first type is trivial, the next four types are geometric relations, and the last one, which is illustrated in Figure 10, is an extension of the pentagon relation: the latter corresponds to $p = q = 1$, and it is written $A^{(1)}A^{(1)} = A_1^{(1)}A^{(2)}$ in terms of the generators $A_\alpha^{(p)}$.

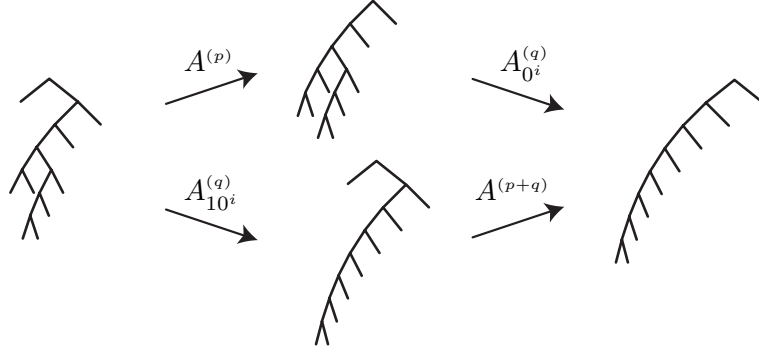


FIGURE 10. The extended pentagon relation; here $p = 4, i = 2, q = 3$.

Remark 3.24. If we are only interested in guessing the relations of \mathbf{r}_{A^*} and proving that they make a presentation of F^+ and F , the verification of Lemma 3.22 is not needed, and it is sufficient to check that the operators of $\mathcal{G}(\mathbf{A})$ associated with the left and the right term of each relation agree on at least one tree, which, by Lemma 3.2, is sufficient to deduce that the corresponding words represent the same element of F^+ . This follows from the geometric description of the action of $A^{(p)}$ directly.

3.6. Complexity of word reversing. By Lemma 3.17, the left and the right \mathbf{r}_A -reversing of an arbitrary word in $W(\mathbf{A})$ always converges, *i.e.*, it leads in finitely many steps to a word of the form uv^{-1} with u, v in $W^+(\mathbf{A})$.

Considering the generators $A_\alpha^{(p)}$ enables us to say more about the convergence of the right reversing process. Indeed, all relations in \mathbf{r}_{A^*} involve words of length 2 at most, hence the right \mathbf{r}_{A^*} -reversing of a word containing r positive letters and s negative letters requires at most rs steps, and all involved words contain at most r positive letters, and at most s negative letters. We deduce an upper bound for the length of the words obtained using right \mathbf{r}_A -reversing.

Proposition 3.25. *Assume that w is a word in $W(\mathbf{A})$ containing r positive letters and s negative letters. Then right \mathbf{r}_A -reversing w leads to a word of length at most $r + s + rs$; the latter upper bound is optimal.*

Proof. By Lemma 3.22—here Proposition 3.23 would not be sufficient—the \mathbf{r}_A -reversing steps in the right reversing of w can be gathered into \mathbf{r}_{A^*} -reversing steps, which are at most rs in number. Consider the sum of the exponents p of the involved generators $A_\alpha^{(p)}$. Each \mathbf{r}_{A^*} -reversing step increases this sum by 1 at most, so the total sum in the final $r + s$ generators $A_\alpha^{(p)}$ is at most $r + s + rs$. So, when the generators $A_\alpha^{(p)}$ are expressed as products of A_α 's, at most $r + s + rs$ of the latter occur.

The bound is optimal, as an easy induction gives

$$(A_{1^{r-1}} \dots A_1 A)^{-1} A_{1^r}^s \curvearrowright_{\mathbf{r}_A} A^s (A_{1^{r-1}} \dots A_1^{(s)} A^{(s)})^{-1},$$

thus an example of a word of length $r + s$ that reverses to a word of length $r + s + rs$. \square

Other upper bounds can be obtained by using the action of F^+ on trees. We already observed that, if two words w, w' of $W(\mathbf{A})$ represent the same element of F , the hypothesis that $t \bullet w$ is defined for some tree t does not guarantee that $t \bullet w'$ is also defined. However, this unpleasant phenomenon cannot happen when reversing is considered.

Lemma 3.26. *Assume that w, w' are words in $W(\mathbf{A})$ and w is left or right \mathbf{r}_A -reversible to w' . Then, for each tree t , the hypothesis that $t \bullet w$ is defined implies that $t \bullet w'$ is defined as well.*

Proof. The problem with arbitrary equivalences is that one can create a new pair $A_\alpha A_\alpha^{-1}$ or $A_\alpha^{-1} A_\alpha$, and it is not true that $t \bullet A_\alpha A_\alpha^{-1}$ is defined for every α . This however is impossible in the case of word reversing, as we can only delete pairs $A_\alpha A_\alpha^{-1}$ or $A_\alpha^{-1} A_\alpha$ and never create them. A complete formal proof requires that we check all possible reversing cases: this is easy, and we skip the details. \square

We deduce a new upper bound for the length of the words obtained by reversing from a given word in $W(\mathbf{A})$:

Proposition 3.27. *If w is a word in $W(\mathbf{A})$ and $t \bullet w$ is defined for some size n tree t , we have*

$$(3.3) \quad |N_L(w)| + |D_R(w)| \leq (n-1)(n-2)/2 \text{ and } |N_L(w)| + |D_R(w)| \leq (n-1)(n-2)/2.$$

Proof. Let $t' = t \bullet t$. By definition, the word w is right \mathbf{r}_A -reversible to $N_R(w)D_R(w)^{-1}$, and left \mathbf{r}_A -reversible to $D_L(w)^{-1}N_L(w)$. By Lemma 3.26, this implies that $t \bullet N_R(w)D_R(w)^{-1}$ and $t \bullet D_L(w)^{-1}N_L(w)$ are defined. Put $t_L = t \bullet D_L(w)^{-1}$ and $t_R = t \bullet N_R(w)$. By hypothesis, the terms t, t', t_L , and t_R all have size n . Hence there exists a word u_L in $W^+(\mathbf{A})$, namely u_{t_L} , mapping the right comb $\setminus n \setminus$ to t_L . By symmetry, there exists u_R in $W^+(\mathbf{A})$ mapping t_R to the left comb $/n/$ of size n . Then the words $u_L N_L(w) D_R(w) u_R$ and $u_L D_L(w) N_R(w) u_R$ belong to $W^+(\mathbf{A})$ by construction, and both map the right comb $\setminus n \setminus$ to its left counterpart $/n/$ (see Figure 11).

Now A^{n-2} also maps $\setminus n \setminus$ to $/n/$, so, by Proposition 3.11, we must have

$$(3.4) \quad A^{n-2} \equiv_{\mathbf{r}_A}^+ u_L N_L(w) D_R(w) u_R \equiv_{\mathbf{r}_A}^+ u_L D_L(w) N_R(w) u_R.$$

Then Lemma 3.7 gives an upper for the lengths of the words equivalent to a given word u , namely what is called $\lambda(u)$ there. In the case of A^{n-2} , we have $\lambda(A^{n-2}) = (n-1)(n-2)/2$, and (3.3) follows. \square

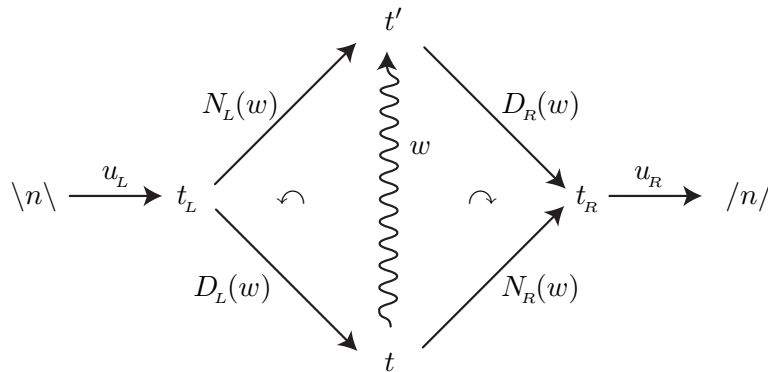


FIGURE 11. Bounding the lengths in terms of the size of a term t such that $t \bullet w$ is defined

Once again, the previous upper bound is (nearly) optimal: for $w = (A_{1^{r-1}} \dots A_1 A)^{-1} A_{1^r}^s$, the word $D_R(w)$ is $A_{1^{r-1}}^{(s)} \dots A_1^{(s)} A^{(s)}$, which has length rs in the letters A_α , so the sum of the lengths of $N_L(w)$ and $D_R(w)$ is $r+rs$, while the minimal size of a term t such that $t \bullet w$ is defined is $r+s+2$.

Remark 3.28. For each word w in $W^+(\mathbf{A})$, the parameter $\lambda(w)$ of Lemma 3.7 is an upper bound for the length of the words w' that are \mathbf{r}_A -equivalent to w . We do not know whether this bound is always sharp, but this is the case for the elements A^k at least. Indeed, we immediately obtain $\lambda(A^k) = k(k+1)/2$. Now we have

$$A^k \equiv_{\mathbf{r}_A}^+ (A_{1^{k-1}})(A_{1^{k-2}A_{1^{k-2}0}}) \dots (A_1A_{10} \dots A_{10^{k-2}})(AA_0 \dots A_{0^{k-1}}).$$

3.7. The monoid F_L^+ . We have seen in Section 2 that, besides the family \mathbf{A} , the group $G(\mathbf{A})$, *i.e.*, F , also admits the smaller generating family \mathbf{a} . This family gives rise to another submonoid of F .

Definition. We denote by F_L^+ the submonoid of F generated by \mathbf{a} .

By construction, we have $F_L^+ \subseteq F^+ \subseteq F$. The monoid F_L^+ is eligible for the same methods of investigation as the monoid F^+ , in particular for word reversing.

Lemma 3.29. *The presentation $(\mathbf{a}, \mathbf{r}_a)$ is complete with respect to left and to right reversing.*

Proof. Homogeneity is trivial in this case, as all relations in \mathbf{r}_a preserve the length of the words: in $a_i a_j = a_{j+1} a_i$, both sides are length 2 words. As for completeness, the verifications are similar, but more simple than in the case of $(\mathbf{A}, \mathbf{r}_A)$. Assume for instance $i \geq j+2$ and $j \geq k+2$. Then one can check that $a_i^{-1} a_j a_j^{-1} a_k$ is right \mathbf{r}_a -reversible to $a_j a_k a_{i-2}^{-1} a_{j-1}^{-1}$, and then that $a_k^{-1} a_j^{-1} a_i^{-1} a_k a_{j-1} a_{i-2}$ is right \mathbf{r}_a -reversible to the empty word. \square

By using the same analysing as for F^+ above (with even easier technical details), we obtain:

Proposition 3.30. (i) *The monoid $\langle \mathbf{a}; \mathbf{r}_a \rangle^+$ admits left and right cancellation.*

(ii) *The monoid F_L^+ admits the presentation $\langle \mathbf{a}; \mathbf{r}_a \rangle^+$.*

(iii) *Any two element in the monoid F_L^+ admit a least common left multiple. The latter can be computed using left \mathbf{r}_a -reversing, as explained in Proposition 3.15.*

(iv) *The group F is a group of left fractions for the monoid F_L^+ .*

Proof. For (i), we observe that \mathbf{r}_a contains no relation of the form $xu = xv$ or $ux = vx$ and apply Lemma 3.29.

For (ii), by construction, the words u_t belong to $W^+(\mathbf{a})$, and it is easy to check that the relations of \mathbf{r}_a are sufficient to establish the equivalence of the words $u_{t'}$ and $u_t \cdot a_i$ when $t' = t \cdot a_i$ holds. \square

However, the study does not continue much further, as the monoid F_L^+ lacks the symmetry satisfied by F^+ . In particular, the elements a_1 and a_2 have no common right multiple in F_L^+ , and right \mathbf{r}_a -reversing need not succeed in $W^+(\mathbf{a})$. That is why the balanced monoid F^+ might turn out to be more useful than the one-sided monoid F_L^+ .

4. THE GEOMETRIC PRESENTATION OF THOMPSON'S GROUP V

Our approach to Thompson's group F was based on its connection with the associativity. We now develop a similar approach for Thompson's group V . The latter corresponds to the case when the commutativity identity $xy = yx$ is added to the associativity identity. As in Section 2, the geometry of the commutativity operators leads to a natural presentation of the group: in addition to the geometric and pentagon relations, the only new relations are the MacLane–Stasheff hexagon relations, plus some torsion relations.

4.1. The geometry monoid of a family of algebraic laws. The approach developed in Section 2 for the special case of associativity extends to all other algebraic laws. The general form of an identity \mathcal{I} is $\tau_- = \tau_+$, where τ_-, τ_+ are formal combinations of variables, or, equivalently, decorated trees. Then, for each set of decorated trees T , we can consider the partial operator I on T such that a tree t belongs to the domain of I if it can be written as τ_-^φ for

some T -valued substitution φ , and, then, $t \bullet I$ is defined to be τ_+^φ . The operator I^{-1} is defined symmetrically, and, as above, we denote by $I_\alpha^{\pm 1}$ the translated copy $\partial_\alpha I^{\pm 1}$, *i.e.*, the result of letting $I^{\pm 1}$ act on the α th subtree of its argument.

Definition. For $\mathcal{I}, \mathcal{J}, \dots$ algebraic laws, we define the *geometry monoid* of $\mathcal{I}, \mathcal{J}, \dots$, denoted $\mathcal{G}(\mathcal{I}, \mathcal{J}, \dots)$, to be the monoid generated by all partial operators $I_\alpha^{\pm 1}, J_\alpha^{\pm 1}, \dots$ acting on decorated trees.

So, for instance, the monoid $\mathcal{G}(\mathcal{A})$ of Section 2 is the geometry monoid of the associativity identity.

Remark 4.1. Formally, the definition of the operators I_α and, therefore, of the geometry monoid, depends on choosing a specific family of (decorated) trees T . We shall forget about this point here, which amounts to assuming that we work once for all inside a sufficiently large family of decorated trees, for instance the set $T_{\mathbf{N}}$ of all \mathbf{N} -decorated trees. We shall come back to the question in Section 6 below.

The following fact is obvious:

Lemma 4.2. *Let $\mathcal{I}, \mathcal{J}, \dots$ be algebraic laws. Then two trees t, t' are $\{\mathcal{I}, \mathcal{J}, \dots\}$ -equal if and only if some element of $\mathcal{G}(\mathcal{I}, \mathcal{J}, \dots)$ maps t to t' .*

At this degree of generality, we cannot expect any really interesting result. Going further requires to restrict the considered algebraic laws. In particular, an unpleasant phenomenon is that, in general, the geometry monoid $\mathcal{G}(\mathcal{I}, \mathcal{J}, \dots)$ contains the empty mapping, *i.e.*, there exist compositions of operators $I_\alpha^{\pm 1}, J_\beta^{\pm 1}, \dots$, that apply to no tree, typically because of incompatibility between labels. This however is excluded when the considered laws are simple enough.

Definition. We say that an algebraic identity $\tau_- = \tau_+$ is *linear* if no variable is repeated twice or more in τ_- or τ_+ .

So, for instance, the associativity identity $x(yz) = (xy)z$ is linear, as x, y , and z occur only once on each side of the equality, while the self-distributivity identity $x(yz) = (xy)(xz)$ is not, as x is repeated twice in the right term.

Lemma 4.3. *Assume that $\mathcal{I}, \mathcal{J}, \dots$ are linear algebraic laws. Then each operator in $\mathcal{G}(\mathcal{I}, \mathcal{J}, \dots)$ admits a seed consisting of injective trees, *i.e.*, there exists a pair of injective trees (t, t') such that, as a pair of trees, f is the set of all substitutes of (t, t') .*

Proof. The point is that, if t_1, t_2 are injective trees, then there always exists substitutions φ_1, φ_2 such that $t_1^{\varphi_1}$ and $t_2^{\varphi_2}$ are equal, which need not be the case when some labels in t_1 or t_2 occur twice. Then the substitutions may be chosen so that the common skeleton of $t_1^{\varphi_1}$ and $t_2^{\varphi_2}$ is the union of the skeletons of t_1 and t_2 , and the proof is the same as that of Lemma 2.4 in the particular case of associativity. \square

In the previous case, Lemma 2.3 applies to the monoid $\mathcal{G}(\mathcal{I}, \mathcal{J}, \dots)$, and, exactly as in the case of $\mathcal{G}(\mathcal{A})$, it leads to a group.

Proposition 4.4. *Assume that $\{\mathcal{I}, \mathcal{J}, \dots\}$ is a family of linear algebraic laws. Then near-equality is a congruence on $\mathcal{G}(\mathcal{I}, \mathcal{J}, \dots)$, and the quotient-monoid is a group. The operators $I_\alpha^{\pm 1}, J_\alpha^{\pm 1}, \dots$ induce a partial action of this group on trees. Injective trees form a separating family for this partial action.*

Definition. Under the above hypothesis, the group $\mathcal{G}(\mathcal{I}, \mathcal{J}, \dots)/\approx$ is called the *geometry group* of the laws $\mathcal{I}, \mathcal{J}, \dots$, and it is denoted $G(\mathcal{I}, \mathcal{J}, \dots)$.

Remark 4.5. Here, we restrict to laws that involve a single binary operation, like associativity or commutativity. A similar approach is possible for laws involving more than one (binary) operation, like the distributivity identity $x(y + z) = xy + xz$, at the expense of considering trees in which the internal nodes are decorated with operation symbols. Also, the case of operations other than binary would be treated by considering rooted trees in which the degree of the internal nodes depend on their label.

4.2. Commutativity operators and Thompson's group V . The commutativity identity (C)

$$xy = yx$$

is eligible for the previous approach. In particular, commutativity is a linear identity, since the two variables x, y occur once in each member of the identity. Here the basic operator is the operator exchanging the left and the right subtrees of a tree:

Definition. We denote by C the (partial) operator that maps every tree of the form $t_1 \wedge t_2$ to the corresponding tree $t_2 \wedge t_1$. For each address α , we put $C_\alpha = \partial_\alpha C$. We define $\mathcal{G}(\mathcal{A}, \mathcal{C})$ to be the monoid generated by all operators A_α and C_α and their inverses.

Associativity and commutativity are linear laws, hence Lemma 4.3 and, therefore, Proposition 4.4 apply. So, near-equality is a congruence on the monoid $\mathcal{G}(\mathcal{A}, \mathcal{C})$, and we obtain a group, denoted $G(\mathcal{A}, \mathcal{C})$ by identifying near-equal operators. As in Section 2, we shall use A_α for the class of A_α in $G(\mathcal{A}, \mathcal{C})$, and, similarly, C_α for the class of C_α . We still denote by \mathbf{A} the family of all A_α 's, and, similarly, we use \mathbf{C} for the family of all C_α 's.

Proposition 4.6. *The geometry group $G(\mathcal{A}, \mathcal{C})$ of associativity and commutativity is isomorphic to Thompson's group V .*

Proof. We associate with each element of $G(\mathcal{A}, \mathcal{C})$ an element of V , i.e., a piecewise linear mapping of $[0, 1]$ into itself as in we did ofr $G(\mathcal{A})$ and F in Section 2: we associate to each tree a dyadic partition of $[0, 1]$, and we map f to the piecewise linear function that maps the partition associated to t' to the partition associated to t , where (tt', t') is a seed for f —we reverse the orientation to obtain a homomorphism with composition—and interpolates the values. The latter homomorphism is surjective since, as was shown in Section 2, its image includes F , and it contains the mappings denoted C and π_0 in [5], which correspond to AC_0A^{-1} and $AC_0A^{-1}C_1$ respectively. □

In the sequel, we identify V with $G(\mathcal{A}, \mathcal{C})$.

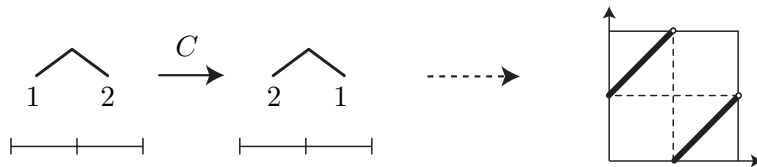


FIGURE 12. From $G(\mathcal{A}, \mathcal{C})$ to V : the action of C

4.3. Guessing relations in $\mathcal{G}(\mathcal{A}, \mathcal{C})$. As in the case of $\mathcal{G}(\mathcal{A})$, the notion of geometric inheritance provides a large family of relations in the monoid $\mathcal{G}(\mathcal{A}, \mathcal{C})$.

We observed that, if t' is the image of t under A_α , then the subtree $t'_{/\alpha 11}$ of t' reappears as $t'_{/\alpha 1}$, and it follows that, for every address β , applying $A_{\alpha 11\beta}$ before A_α is the same as applying A_α first, and then $A_{\alpha 1\beta}$. Now, for the same reason, applying $C_{\alpha 11\beta}$ before A_α is the same as

applying A_α first, and then $C_{\alpha 1\beta}$. So, if we use \square_α to represent either A_α or C_α , we obtain that, in $\mathcal{G}(\mathcal{A}, \mathcal{C})$, the following relations are satisfied:

$$(4.1) \quad \left\{ \begin{array}{l} \square_\beta A_\alpha = A_\alpha \square_\beta \quad \text{whenever } \beta \perp \alpha \text{ holds,} \\ \square_{\alpha 11\beta} A_\alpha = A_\alpha \square_{\alpha 1\beta}, \\ \square_{\alpha 10\beta} A_\alpha = A_\alpha \square_{\alpha 10\beta}, \\ \square_{\alpha 0\beta} A_\alpha = A_\alpha \square_{\alpha 0\beta}. \end{array} \right.$$

Similar phenomena appear when we replace the operator A_α with C_α : the action of C_α on a tree t consists in exchanging the subtrees $t_{/\alpha 0}$ and $t_{/\alpha 1}$, and we deduce the following relations, where \square_α still stands for A_α or C_α :

$$(4.2) \quad \left\{ \begin{array}{l} \square_\beta C_\alpha = C_\alpha \square_\beta \quad \text{whenever } \beta \perp \alpha \text{ holds,} \\ \square_{\alpha 0\beta} C_\alpha = C_\alpha \square_{\alpha 1\beta}, \\ \square_{\alpha 1\beta} C_\alpha = C_\alpha \square_{\alpha 0\beta}. \end{array} \right.$$

The relations mentioned in (4.1) and (4.2) will be called the *A-* and *C-geometric* relations, respectively.

Apart from the geometric relations, we know that the pentagon relations, *i.e.*,

$$(4.3) \quad A_0 A A_1 = A^2$$

and its shifted copies, are satisfied in $\mathcal{G}(\mathcal{A})$, hence in $\mathcal{G}(\mathcal{A}, \mathcal{C})$. Two additional types of relations arise naturally when commutativity is considered.

Lemma 4.7. *The following relations and their translated copies hold in $\mathcal{G}(\mathcal{A}, \mathcal{C})$:*

$$(4.4) \quad A C A = C_0 A C_1,$$

$$(4.5) \quad C^2 \approx \text{id}.$$

Proof. Relation (4.4) corresponds to two ways of going from $(t_1 \wedge t_2) \wedge t_3$ to $(t_2 \wedge t_3) \wedge t_1$, as shown in Figure 13.

The involutivity of C is obvious—but, as C is defined only on those trees that are not \circ , we obtain a \approx -relation, not an equality. \square

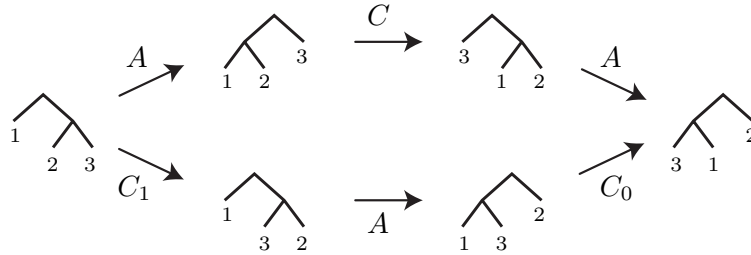


FIGURE 13. The Hexagon Relation

In the sequel, the relation (4.5) and its shifted copies will be called *torsion* relations, while (4.4) and its copies will be called the *hexagon* relations.

As the action of V on injective trees is free, the previous relations in $\mathcal{G}(\mathcal{A}, \mathcal{C})$ induce similar relations in $G(\mathcal{A}, \mathcal{C})$, *i.e.*, in V . We naturally use the same names for these induced relations—with A_α and C_α replacing A_α and C_α respectively, and $=$ instead of \approx . At this point, we may summarize the situation as follows.

Definition. We define r_{AC} to consist of all A - and C -geometric relations, namely the translated copies of

$$\begin{aligned} (\mathbf{g}_\perp) \quad & \square_{0\alpha} \cdot \diamond_{1\beta} = \diamond_{1\beta} \cdot \square_{0\alpha}, \\ (\mathbf{g}_A) \quad & \square_{11\alpha} \cdot A = A \cdot \square_{1\alpha}, \quad \square_{10\alpha} \cdot A = A \cdot \square_{01\alpha}, \quad \square_{0\alpha} \cdot A = A \cdot \square_{00\alpha}, \\ (\mathbf{g}_C) \quad & \square_{0\alpha} \cdot C = C \cdot \square_{1\alpha}, \quad \square_{1\alpha} \cdot C = C \cdot \square_{0\alpha}, \end{aligned}$$

with $\square, \diamond = A$ or C , plus the pentagon relations, *i.e.*, the translated copies of

$$(\mathbf{p}) \quad AA = A_1AA_0,$$

plus the hexagon relations, defined to be the translated copies of

$$(\mathbf{h}) \quad ACA = C_1AC_0 \quad \text{and} \quad A^{-1}CA^{-1} = C_0A^{-1}C_1.$$

Proposition 4.8. *All relations in r_{AC} plus the torsion relations $C_\alpha^2 = 1$ are satisfied by the elements of \mathbf{A} and \mathbf{C} in the group $G(\mathbf{A}, \mathbf{C})$, *i.e.*, in V .*

Remark 4.9. It may appear strange to distinguish two hexagon relations, which are equivalent when the torsion relations $C_\alpha^2 = 1$ are present. The reason of this option is that we shall investigate a torsion-free version of the group V in Section 6, and it is appropriate to keep track of the uses of the torsion relations separately.

4.4. Restricting the family of generators. As in the case of F , we shall consider two families of generators for the group V : besides the families \mathbf{A} and \mathbf{C} comprising all A_α 's and C_α 's, we shall also consider the proper subfamilies corresponding to right branch addresses.

Definition. For $i \geq 1$, we put $c_i = C_{1^{i-1}}$. We denote by \mathbf{c} the family of all c_i 's.

Thus c_i is an exact counterpart to a_i . We now list some relations satisfied by the elements of \mathbf{a} and \mathbf{c} in $G(\mathbf{A}, \mathbf{C})$. A disadvantage of restricting the families of generators is that expressing the geometric phenomena is less simple than with the whole families \mathbf{A} and \mathbf{C} .

Definition. We define r_{ac} to consist of the following relations:

$$(4.6) \quad a_i \square_j = \square_{j+1} a_i \quad \text{for } j \geq i+1 \text{ and } \square = a \text{ or } c,$$

$$(4.7) \quad c_i a_i^{-1} c_{i+1}^{-1} \square_j = \square_j c_i a_i^{-1} c_{i+1}^{-1} \quad \text{for } j \geq i+2 \text{ and } \square = a \text{ or } c,$$

$$(4.8) \quad a_{i+1} a_i c_i^e a_{i+1} = a_i^2 c_i^e \quad \text{for } e = \pm 1,$$

$$(4.9) \quad a_i c_i c_{i+1} a_i = c_{i+1} c_i,$$

$$(4.10) \quad c_{i+1} c_i a_i^{-1} c_{i+1} = c_i a_i^{-1} c_i a_i^{-1}.$$

Lemma 4.10. *All relations in r_{ac} follow from r_{AC} (and the definitions $a_i = A_{1^{i-1}}$, $c_i = C_{1^{i-1}}$).*

Proof. It is sufficient to establish the relations for $i = 1$ and then use ∂^{i-1} to deduce the general version. Relations (4.6) and (4.7) are of purely geometric nature: (4.6) is a A -geometric relation, and (4.7) follows from

$$CA^{-1}C_1^{-1}\square_{11\alpha} \equiv_{\mathbf{g}} CA^{-1}\square_{10\alpha}C_1^{-1} \equiv_{\mathbf{g}} C\square_{01\alpha}A^{-1}C_1^{-1} \equiv_{\mathbf{g}} \square_{11\alpha}CA^{-1}C_1^{-1},$$

which is valid both for $\square = A$ or C . Relations (4.8) use the pentagon relations: $A_1AC^eA_1 \equiv_{\mathbf{g}} A_1AA_0C^e \equiv_{\mathbf{p}} A^2C^e$. Finally, appealing to the hexagon relations, we find

$$ACC_1 \equiv ACA A^{-1}C_1 \equiv_{\mathbf{h}} C_1AC_0A^{-1}C_1 \equiv_{\mathbf{h}} C_1AA^{-1}CA^{-1} \equiv C_1CA^{-1},$$

$$C_1CA^{-1}C_1 \equiv_{\mathbf{g}} CC_0A^{-1}C_1 \equiv_{\mathbf{h}} CA^{-1}CA^{-1},$$

which gives (4.9) and (4.10). □

4.5. Constructing trees. Our aim is to prove that the relations r_{AC} and r_{ac} make presentations of the group V , and, as for F , we shall appeal to the criterion of Proposition 1.2. So, as in Section 2, the point is to introduce for each tree t a distinguished word u_t that describes the construction of t from some distinguished tree in its V -orbit.

In contrast to the case of associativity, considering commutativity requires that the labels are taken into account, because Condition (PA_4) in the definition of a free partial action is satisfied for injective trees only: for instance, the operators id and C do not coincide, but both map the tree $\circ \wedge \circ$ to itself. So, we use decorated versions of the right combs $\setminus n \setminus$.

Definition. For I_1, \dots, I_k finite subsets of \mathbf{N} , we define the *decorated right comb* $\setminus I_1, \dots, I_k \setminus$ by

$$\setminus I_1, \dots, I_k \setminus = [\circ_{\ell_1}, \circ_{\ell_2}, \dots, \circ_{\ell_n}],$$

where (ℓ_1, \dots, ℓ_n) is the increasing enumeration of I_1 , followed by the increasing enumeration of I_2 , etc. (Figure 14).

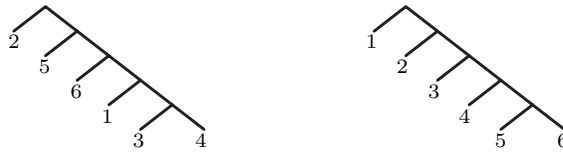


FIGURE 14. The decorated right combs $\setminus \{2, 5, 6\}, \{1, 3, 4\} \setminus$ and $\setminus \{2, 5, 6, 1, 3, 4\} \setminus$ (the latter also being $\setminus \{1, 2, 3, 4, 5, 6\} \setminus$)

In particular $\setminus I \setminus$ is the right comb in which the labels of the leaves are the elements of I enumerated in increasing order. What we need for our current inductive construction is an operator that maps $\setminus I \cup J \setminus$ to $\setminus I, J \setminus$ when I and J are disjoint. To this end, it will be useful to introduce new specific elements of $G(\mathcal{A}, \mathcal{C})$.

Definition. For each address α , we put $\Sigma_\alpha = C_\alpha A_\alpha^{-1} C_{\alpha 1}^{-1}$. We denote by Σ the family of all Σ_α 's, and by $r_{AC\Sigma}$ the family obtained by adding the definition of Σ_α to r_{AC} . For $i \geq 1$, we put $\sigma_i = \Sigma_{1^{i-1}}$, i.e., $\sigma_i = c_i a_i^{-1} c_{i+1}^{-1}$, and we denote by σ the family of all σ_i 's.

The action of the operator Σ associated with Σ on trees is illustrated in Figure 15. In terms of the notation $\setminus t_1, \dots, t_n \setminus$, the action of σ_i is to switch the i th and the $(i+1)$ th factors:

$$\sigma_i : \setminus t_1, \dots, t_i, t_{i+1}, \dots, t_n \setminus \mapsto \setminus t_1, \dots, t_{i+1}, t_i, \dots, t_n \setminus.$$

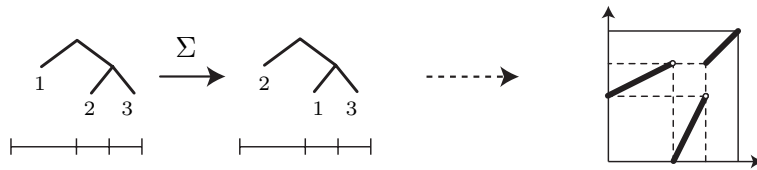


FIGURE 15. The action of Σ

Definition. For I, J finite disjoint subsets of \mathbf{N} , the word $c_{I,J}$ is inductively determined by $c_{\emptyset, \emptyset} = \varepsilon$ and the rule: for ℓ smaller than all elements of I and J ,

$$c_{\{\ell\} \cup I, J} = \partial c_{I, J},$$

$$c_{I, \{\ell\} \cup J} = \begin{cases} \sigma_1 \sigma_2 \dots \sigma_{p-1} c_p & \text{if } I \text{ has } p \text{ elements and } J \text{ is empty,} \\ \partial c_{I, J} \cdot \sigma_1 \sigma_2 \dots \sigma_p & \text{if } I \text{ has } p \text{ elements and } J \text{ is nonempty.} \end{cases}$$

The word $\sigma_{I, J}$ is defined similarly, except that $c_{I, \{\ell\}}$ is defined to be $\sigma_1 \sigma_2 \dots \sigma_p$.

Example 4.11. Let $I = \{2, 5, 6\}$ and $J = \{1, 3, 4\}$. By considering the elements of $I \cup J$ in decreasing order, we find successively $c_{\emptyset, \emptyset} = \varepsilon$, $c_{\{6\}, \emptyset} = \varepsilon$, $c_{\{5, 6\}, \emptyset} = \varepsilon$, $c_{\{5, 6\}, \{4\}} = \sigma_1 c_2$, $c_{\{5, 6\}, \{3, 4\}} = \partial(\sigma_1 c_2) \cdot \sigma_1 \sigma_2 = \sigma_2 c_3 \sigma_1 \sigma_2$, $c_{\{2, 5, 6\}, \{3, 4\}} = \partial(\sigma_2 c_3 \sigma_1 \sigma_2) = \sigma_3 c_4 \sigma_2 \sigma_3$, $c_{\{2, 5, 6\}, \{1, 3, 4\}} = \partial(\sigma_3 c_4 \sigma_2 \sigma_3) \cdot \sigma_1 \sigma_2 \sigma_3 = \sigma_4 c_5 \sigma_3 \sigma_4 \sigma_1 \sigma_2 \sigma_3$.

Similarly, we have $\sigma_{\{2, 5, 6\}, \{1, 3, 4\}} = \sigma_4 \sigma_5 \sigma_3 \sigma_4 \sigma_1 \sigma_2 \sigma_3$.

Lemma 4.12. For all sets I, J , and for every tree t , we have

$$\backslash I \cup J \backslash \xrightarrow{c_{I, J}} \backslash I, J \backslash \quad \text{and} \quad \backslash I \cup J, t \backslash \xrightarrow{\sigma_{I, J}} \backslash I, J, t \backslash.$$

Proof. We use induction on the cardinality of $I \cup J$. The result is clear if I and J are empty. Assume that ℓ is smaller than all elements in I and J . The induction hypothesis asserts that $c_{I, J}$ maps $\backslash I \cup J \backslash$ to $\backslash I, J \backslash$, hence $\partial c_{I, J}$ maps $\circ_\ell \wedge \backslash I \cup J \backslash$, which is $\backslash \{\ell\} \cup I \cup J \backslash$, to $\circ_\ell \wedge \backslash I, J \backslash$, i.e., to $\backslash \{\ell\} \cup I, J \backslash$, as expected for $c_{\{\ell\} \cup I, J}$.

Let us consider $c_{I, \{\ell\} \cup J}$. Let p be the cardinal of I . Assume first J nonempty. We have seen that $\partial c_{I, J}$ maps $\backslash \{\ell\} \cup I \cup J \backslash$ to $\backslash \{\ell\} \cup I, J \backslash$. Then the iterated transposition $\sigma_1 \sigma_2 \dots \sigma_p$ carries the leftmost leaf of $\backslash \{\ell\} \cup I, J \backslash$, i.e., \circ_ℓ , through p leaves to the right, i.e., we obtain $\backslash I, \{\ell\}, J \backslash$, which is also $\backslash I, \{\ell\} \cup J \backslash$. Finally, if I is empty, then $c_{I, J}$ is ε , as an induction shows, and $\sigma_1 \sigma_2 \dots \sigma_{p-1} c_p$ maps $\backslash \{\ell\}, J \backslash$ to $\backslash J, \{\ell\} \backslash$. So, in each case, $c_{I, \{\ell\} \cup J}$ maps $\backslash \{\ell\} \cup I \cup J \backslash$ to $\backslash I, \{\ell\} \cup J \backslash$.

The argument is similar for $\sigma_{I, J}$. □

Let us recall that our current aim is to select a distinguished word u_t in $W(\mathcal{A}, \mathcal{C})$ for each \mathbf{N} -decorated tree t so that (the operator associated to) u_t maps some distinguished element in the $G(\mathcal{A}, \mathcal{C})$ -orbit of t to t . The distinguished element we choose is the decorated right comb $\backslash I \backslash$, where I is the set of labels in t . Then we shall use the same inductive construction as in the case of associativity. The only change is that, before beginning the inductive construction, we first sort the labels in order to push to the initial positions the labels that correspond to the left subtree of the considered tree. This is exactly what (the operators associated with) $c_{I, J}$ and $\sigma_{I, J}$ do. So the following definition should be natural.

Definition. For each injective tree t , the words u_t, u_t^* are defined by the rules:

$$\begin{aligned} u_t &= u_t^* = \varepsilon && \text{for } t \text{ of size } 1, \\ u_t &= c_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2}, \quad u_t^* = \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A && \text{for } t = t_1 \wedge t_2 \text{ and } I_k \text{ the labels in } t_k. \end{aligned}$$

The following result is the exact counterpart to Lemma 2.13, and it describes how to construct an arbitrary decorated tree from the corresponding right comb.

Lemma 4.13. For each injective tree t with labels I , we have

$$(4.11) \quad \backslash I \backslash \xrightarrow{u_t} t \quad \text{and} \quad \backslash I, \dots \backslash \xrightarrow{u_t^*} \backslash t, \dots \backslash,$$

i.e., u_t constructs t from $\backslash I \backslash$, and u_t^* constructs $t \wedge t'$ from $\backslash I \wedge t' \backslash$ for every tree t .

Proof. The inductive verification is the same as for Lemma 2.13. The diagrams are now:

$$\begin{aligned} \backslash I \backslash &= \backslash I_1 \cup I_2 \backslash \xrightarrow{c_{I_1, I_2}} \backslash I_1, I_2 \backslash \xrightarrow{u_{t_1}^*} \backslash t_1, I_2 \backslash \xrightarrow{\partial u_{t_2}} \backslash t_1, t_2 \backslash = t, \\ \backslash I, \dots \backslash &= \backslash I_1 \cup I_2, \dots \backslash \xrightarrow{\sigma_{I_1, I_2}} \backslash I_1, I_2, \dots \backslash \xrightarrow{u_{t_1}^*} \backslash t_1, I_2, \dots \backslash \\ &\quad \xrightarrow{\partial u_{t_2}^*} \backslash t_1, t_2, \dots \backslash \xrightarrow{A} \backslash t_1 \wedge t_2, \dots \backslash = \backslash t, \dots \backslash \end{aligned}$$

for $t = t_1 \wedge t_2$ and I_1, I_2 the sets of labels in t_1 and t_2 respectively. □

4.6. Derived relations. In order to apply Proposition 1.2 to prove that the relations of Proposition 4.8 make a presentation of the group $G(\mathcal{A}, \mathcal{C})$, *i.e.*, of V , we have to check that these relations are sufficient to establish the equivalence of $u_{t'}$ and $u_t \cdot \square_\alpha$ under the hypothesis that \square_α maps t to t' , where \square_α denotes either A_α or C_α . The needed verifications are easy, but longer than in the case of $G(\mathcal{A})$, and we begin with some technical, but easy preparatory results consisting in verifying that certain relations involving the letters A_α , C_α , and Σ_α follow from $\mathbf{r}_{AC\Sigma}$ or derived lists of relations.

Lemma 4.14. *The following relations follow from $\mathbf{r}_{AC\Sigma}$:*

- (i) *The A - and C -geometric relations of \mathbf{r}_{AC} in which \square or \diamond is replaced with Σ ;*
- (ii) *The Σ -geometric relations, defined to be the translated copies of*

$$(\mathbf{g}_\Sigma) \quad \square_{11\alpha} \cdot \Sigma = \Sigma \cdot \square_{11\alpha}, \quad \square_{10\alpha} \cdot \Sigma = \Sigma \cdot \square_{0\alpha}, \quad \square_{0\alpha} \cdot \Sigma = \Sigma \cdot \square_{10\alpha},$$

in which \square stands for A, C or Σ ,

- (iii) *The translated copies of the relations*

$$(4.12) \quad \Sigma A = AC_0, \quad \Sigma A_1 A \equiv A_1 A \Sigma_0,$$

$$(4.13) \quad \Sigma_1 \Sigma A_1 = A \Sigma, \quad \Sigma \Sigma_1 A = A_1 \Sigma, \quad \Sigma \Sigma_1 \Sigma = \Sigma_1 \Sigma \Sigma_1.$$

Proof. The extension of the A - and C -geometric relations to Σ_α is obvious, as Σ_α is defined from C_α , A_α , and $C_{1\alpha}$. The Σ -geometric relations follow from the other geometric relations. For instance, we find

$$\Sigma \square_{11\alpha} = AC_0 A^{-1} \square_{11\alpha} \equiv_{\mathbf{g}_A} AC_0 \square_{1\alpha} A^{-1} \equiv_{\mathbf{g}_\perp} A \square_{1\alpha} C_0 A^{-1} \equiv_{\mathbf{g}_A} \square_{11\alpha} AC_0 A^{-1} = \square_{11\alpha} \Sigma.$$

The first relation in (4.12) follows from the definition and an hexagon relation:

$$\Sigma A_0 = CA^{-1} C_1^{-1} A \equiv_{\mathbf{h}} AC_0.$$

The second relation comes by cancelling A_0 on the right in

$$\Sigma A_1 A A_0 \equiv_{\mathbf{p}} \Sigma A A \equiv_{(4.12)} AC_0 A \equiv_{\mathbf{g}} A A C_0 \equiv_{\mathbf{p}} A_1 A A_0 C_0 \equiv_{(4.12)} A_1 A \Sigma_0 A_0.$$

Then we observe that the hexagon relation implies

$$(4.14) \quad C_1 \Sigma \equiv C_1 \Sigma A A^{-1} \equiv_{(4.12)} C_1 A C_0 A^{-1} \equiv_{\mathbf{h}} A C A A^{-1} \equiv AC.$$

Next, the first two relations in (4.13) are obtained by cancelling A on the right in

$$\begin{aligned} \Sigma_1 \Sigma A_1 A &\equiv_{(4.12)} \Sigma_1 A_1 A \Sigma_0 \equiv_{(4.12)} A_1 C_{10} A \Sigma_0 \\ &\equiv_{\mathbf{g}} A_1 A C_{01} \Sigma_0 \equiv_{(4.14)} A_1 A A_0 C_0 \equiv_{\mathbf{p}} A A C_0 \equiv_{(4.12)} A \Sigma A, \\ \Sigma \Sigma_1 A A &\equiv_{\mathbf{h}} \Sigma \Sigma_1 A_1 A A_0 \equiv_{(4.12)} \Sigma A_1 C_{10} A A_0 \equiv_{\mathbf{g}} \Sigma A_1 A C_{01} A_0 \\ &\equiv_{(4.12)} A_1 A \Sigma_0 C_{01} A_0 \equiv A_1 A C_0 \equiv_{(4.12)} A_1 \Sigma A. \end{aligned}$$

Finally, we have

$$\Sigma C_1 \Sigma \equiv_{(4.14)} \Sigma A C \equiv_{(4.12)} A C_0 C \equiv_{\mathbf{g}} A C C_1 \equiv_{(4.14)} C_1 \Sigma C_1,$$

so, using $\Sigma A_1 A \equiv_{(4.12)} A_1 A \Sigma_0$, and $\Sigma_1 A_1 A \equiv_{(4.12)} A_1 C_{10} A \equiv_{\mathbf{g}} A_1 A C_{01}$, we deduce

$$A_1 A \Sigma \Sigma_1 \Sigma \equiv \partial_0(\Sigma C_1 \Sigma) \cdot A_1 A \equiv \partial_0(C_1 \Sigma C_1) \cdot A_1 A \equiv A_1 A \Sigma_1 \Sigma \Sigma_1,$$

which implies the third relation in (4.13) by cancelling $A_1 A$ on the left. \square

On the other hand, we observe that, by construction, the words u_t and u_t^* involve the letters a_i , c_i , and σ_i only. So it will be convenient to work with the following restricted list of relations.

Definition. We define $\mathbf{r}_{ac\sigma}$ to consist of the following relations:

$$(4.15) \quad a_i \square_j = \square_{j+1} a_i, \quad \text{with } j \geq i+1 \text{ and } \square = a, c \text{ or } \sigma,$$

$$(4.16) \quad \sigma_i \square_j = \square_j \sigma_i, \quad \text{with } j \geq i+2 \text{ and } \square = a, c \text{ or } \sigma,$$

$$(4.17) \quad \sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i, \quad \text{and} \quad \sigma_{i+1} \sigma_i a_{i+1} = a_i \sigma_i,$$

$$(4.18) \quad \sigma_i \square_{i+1} \sigma_i = \square_{i+1} \sigma_i \square_{i+1}, \quad \text{with } \square = \sigma \text{ or } c.$$

Lemma 4.15. *All relations in $\mathbf{r}_{ac\sigma}$ are consequences of \mathbf{r}_{ac} , hence of \mathbf{r}_{AC} (plus the definitions of a_i, c_i and σ_i from \mathbf{A} and \mathbf{C}). Furthermore, the relation $\sigma_i^2 = 1$ follows from \mathbf{r}_{ac} completed with the relations $c_i^2 = 1$.*

Proof. When \square is a or c , (4.15) coincides with (4.6); for $\square_j = \sigma_j$, we apply (4.6) to c_j, a_j^{-1} , and c_{j+1}^{-1} successively. Similarly, (4.16) for $\square_j = a_j$ or c_j directly follows from (4.7) owing to the definition of σ_i ; the relation for $\square_j = \sigma_j$ then follows by replacing σ_j with its definition. As for (4.17), we find

$$\sigma_i \sigma_{i+1} a_i \equiv c_i a_i^{-1} a_{i+1}^{-1} c_{i+2}^{-1} a_i \equiv_{(4.6)} c_i a_i^{-1} a_{i+1}^{-1} a_i c_{i+1}^{-1} \equiv_{(4.8)} a_{i+1} c_i a_i^{-1} c_{i+1}^{-1} \equiv_{(4.19)} a_{i+1} \sigma_i.$$

Next, we observe that the relation (4.9) of \mathbf{r}_{ac} implies

$$(4.19) \quad \sigma_i = c_i a_i^{-1} c_{i+1}^{-1} \equiv_{\mathbf{r}_{ac}} c_{i+1}^{-1} a_i c_i,$$

and we deduce symmetrically

$$\sigma_{i+1} \sigma_i a_{i+1} \equiv c_{i+2}^{-1} a_{i+1} a_i c_i a_{i+1} \equiv_{(4.8)} c_{i+2}^{-1} a_i^2 c_i a_{i+1} \equiv_{(4.6)} a_i c_{i+1}^{-1} a_i c_i a_{i+1} = a_i \sigma_i.$$

For (4.18) with $\square_2 = c_2$, we have

$$\sigma_1 c_2 \sigma_1 \equiv c_1 a_1^{-1} \sigma_1 = c_1 a_1^{-1} c_1 a_1^{-1} c_2^{-1} \equiv_{(4.10)} c_2 c_1 a_1^{-1} \equiv c_2 \sigma_1 c_2.$$

As for (4.18) with $\square_2 = \sigma_2$, we have

$$\begin{aligned} \sigma_1 \sigma_2 \sigma_1 &\equiv \sigma_1 c_2 a_2^{-1} c_3^{-1} \sigma_1 \equiv_{(4.16)} \sigma_1 c_2 a_2^{-1} \sigma_1 c_3^{-1} \equiv_{(4.17)} \sigma_1 c_2 \sigma_1 a_1^{-1} \sigma_2^{-1} c_3^{-1}, \\ \sigma_2 \sigma_1 \sigma_2 &\equiv c_2 a_2^{-1} c_3^{-1} \sigma_1 \sigma_2 \equiv_{(4.16)} c_2 a_2^{-1} \sigma_1 c_3^{-1} \sigma_2 \equiv_{(4.17)} c_2 \sigma_1 a_1^{-1} \sigma_2^{-1} c_3^{-1} \sigma_2 \\ &\equiv c_2 \sigma_1 c_2 c_2^{-1} a_1^{-1} \sigma_2^{-1} c_3^{-1} \sigma_2 \equiv_{(4.15)} c_2 \sigma_1 c_2 a_1^{-1} c_3^{-1} \sigma_2^{-1} c_3^{-1} \sigma_2. \end{aligned}$$

Applying the relations $\sigma_1 c_2 \sigma_1 \equiv c_2 \sigma_1 c_2$ and $\sigma_2 c_3 \sigma_2 \equiv c_3 \sigma_2 c_3$ —hence $\sigma_2^{-1} c_3^{-1} \equiv c_3^{-1} \sigma_2^{-1} c_3^{-1} \sigma_2$ —which were established above, we deduce $\sigma_1 \sigma_2 \sigma_1 \equiv \sigma_2 \sigma_1 \sigma_2$.

Finally, we have seen that \mathbf{r}_{ac} implies $\sigma_1 = c_1 a_1^{-1} c_2^{-1} \equiv c_2^{-1} a_1 c_1$, hence $\sigma_1^2 \equiv c_1 c_2^{-2} c_1$: so $c_1^2 \equiv c_2^2 \equiv 1$ implies $\sigma_1^2 \equiv 1$. \square

4.7. Preparatory lemmas. We can now establish a technical result for the words $c_{I,J}$ and $\sigma_{I,J}$ that will make inductive arguments possible in the sequel.

Lemma 4.16. *For I, J, K disjoint with $p = \#I \geq 1$, we have*

$$(4.20) \quad \square_{I \cup J, K} \cdot \sigma_{I,J} \equiv_{\mathbf{r}_{ac\sigma}} \square_{I, J \cup K} \cdot \partial^p \square_{J,K} \quad \text{for } \square = \sigma \text{ and } \square = c.$$

Proof. We begin with the auxiliary formulas

$$(4.21) \quad \sigma_1 \sigma_2 \dots c_{k+1} \sigma_i \equiv_{\mathbf{r}_{ac\sigma}} c_{i+1} \sigma_1 \sigma_2 \dots c_{k+1}, \quad \text{for } 1 \leq i \leq k,$$

$$(4.22) \quad \sigma_1 \sigma_2 \dots \sigma_k c_{k+1} \sigma_k \equiv_{\mathbf{r}_{ac\sigma}} c_{k+1} \sigma_1 \sigma_2 \dots \sigma_k c_{k+1}, \quad \text{for } 1 \leq k.$$

A direct inductive verification is possible; we can also observe that Lemma 4.15 tells us that $(\sigma_1, \dots, \sigma_{k+1})$ and $(\sigma_1, \dots, \sigma_k, c_{k+1})$ satisfy the relations of Artin's presentation of the braid group B_{k+2} : therefore, every braid relation between the standard generators σ_i of B_{k+2} must hold between the σ_i 's, which is the case for the counterpart of (4.21) and (4.22).

Next, we claim that the following relations are true, where q denotes $\#J$:

$$(4.23) \quad \sigma_1 \sigma_2 \dots \sigma_{q+r} \sigma_{J,K} \equiv_{\mathbf{r}_{acc\sigma}} \partial \sigma_{J,K} \cdot \sigma_1 \sigma_2 \dots \sigma_{q+r},$$

$$(4.24) \quad \sigma_1 \sigma_2 \dots \sigma_{q+r-1} c_{q+r} \sigma_{J,K} \equiv_{\mathbf{r}_{acc\sigma}} \partial c_{J,K} \sigma_1 \sigma_2 \dots \sigma_{q+r-1} c_{q+r}.$$

Indeed, an easy induction shows that the word $c_{J,K}$ is a product of σ_i 's with $1 \leq i \leq q+r-2$, and, if it not empty, of c_{q+r-1} occurring once, and that $\sigma_{J,K}$ is obtained from $c_{I,J}$ by replacing the possible c_{q+r-1} with σ_{q+r-1} . Then (4.23) comes by applying (4.21) with $k = q+r-1$ to the successive letters σ_i in $c_{J,K}$, and so does (4.24) using (4.22) for the possible letter c_{q+r-1} of $\sigma_{J,K}$.

Let us turn to the proof of the first formula in (4.20). The result is trivial for $I = J = K = \emptyset$. For an induction, it is sufficient to prove that, if ℓ is smaller than all elements in $I \cup J \cup K$, the result is true for $(\{\ell\}, J, K)$, and it is true for $(\{\ell\} \cup I, J, K)$, $(I, \{\ell\} \cup J, K)$, and $(I, J, \{\ell\} \cup K)$ whenever it is for (I, J, K) and I is nonempty.

For the case of $(\{\ell\}, J, K)$, we find

$$\begin{aligned} c_{\{\ell\}, J \cup K} \cdot \sigma_{J,K} &= \sigma_1 \dots \sigma_{q+r-1} c_{q+r} \cdot \sigma_{J,K} \\ &\equiv_{(4.24)} \partial c_{J,K} \cdot \sigma_1 \dots \sigma_{q+r-1} c_{q+r} \\ &= \partial c_{J,K} \cdot \sigma_1 \dots \sigma_r \cdot \partial^r (\sigma_1 \dots \sigma_{q-1} c_q) = c_{\{\ell\} \cup J, K} \cdot \partial^r c_{\{\ell\}, J}. \end{aligned}$$

Assume $I \neq \emptyset$. For $(\{\ell\} \cup I, J, K)$, using the induction hypothesis, we find

$$\begin{aligned} c_{I \cup \{\ell\}, J \cup K} \sigma_{J,K} &= \partial c_{I, J \cup K} \cdot \sigma_1 \sigma_2 \dots \sigma_{q+r} \cdot \sigma_{J,K} \equiv_{(4.23)} \partial c_{I, J \cup K} \cdot \partial \sigma_{J,K} \cdot \sigma_1 \sigma_2 \dots \sigma_{q+r} \\ &\equiv_{(IH)} \partial \sigma_{I \cup J, K} \cdot \partial^{r+1} c_{I, J} \cdot \sigma_1 \sigma_2 \dots \sigma_{q+r} \\ &\equiv_{(4.16)} \partial \sigma_{I \cup J, K} \cdot \sigma_1 \sigma_2 \dots \sigma_r \cdot \partial^{r+1} c_{I, J} \cdot \sigma_{r+1} \dots \sigma_{q+r} \\ &= \partial \sigma_{I \cup J, K} \cdot \sigma_1 \sigma_2 \dots \sigma_r \cdot \partial^r (\partial c_{I, J} \cdot \sigma_1 \sigma_2 \dots \sigma_q) = c_{\{\ell\} \cup I \cup J, K} \cdot \partial^r c_{I \cup \{\ell\}, J}, \end{aligned}$$

The remaining cases are easy:

$$\begin{aligned} c_{I, \{\ell\} \cup J \cup K} \cdot \sigma_{\{\ell\} \cup J, K} &= \partial c_{I, J \cup K} \cdot \partial \sigma_{J, K} \cdot \sigma_1 \sigma_2 \dots \sigma_r \equiv_{(IH)} \partial \sigma_{I \cup J, K} \cdot \partial^{r+1} c_{I, J} \cdot \sigma_1 \sigma_2 \dots \sigma_r \\ &\equiv_{(4.16)} \partial \sigma_{I \cup J, K} \cdot \sigma_1 \sigma_2 \dots \sigma_r \cdot \partial^{r+1} c_{I, J} = c_{I \cup \{\ell\} \cup J, K} \cdot \partial^r c_{I, \{\ell\} \cup J}; \\ c_{I, J \cup \{\ell\} \cup K} \sigma_{J, \{\ell\} \cup K} &= \partial c_{I, J \cup K} \cdot \partial \sigma_{J, K} \equiv_{(IH)} \partial c_{I \cup J, K} \cdot \partial^r c_{I, J} = c_{I \cup J, \{\ell\} \cup K} \cdot \partial^{r+1} c_{I, J}, \end{aligned}$$

and the proof is complete. \square

Definition. For $p, q \geq 1$, we put $c_{p,q} = c_{\{q+1, \dots, q+p\}, \{1, \dots, q\}}$ and $\sigma_{p,q} = \sigma_{\{q+1, \dots, q+p\}, \{1, \dots, q\}}$.

So $\sigma_{p,q}$ is the iterated transposition that switches two blocks of p and q elements respectively, putting the p elements on the top. For instance, we have $\sigma_{p,1} = \sigma_1 \dots \sigma_{p-1}$, and $\sigma_{1,q} = \sigma_{q-1} \dots \sigma_1$.

Lemma 4.17. For all p, q, r , we have

$$(4.25) \quad c_{p+q,r} \equiv_{\mathbf{r}_{acc\sigma}} \sigma_{p,r} \cdot \partial^p c_{q,r} \quad \text{and} \quad \sigma_{p+q,r} \equiv_{\mathbf{r}_{acc\sigma}} \sigma_{p,r} \cdot \partial^p \sigma_{q,r},$$

$$(4.26) \quad c_{p,q+r} \equiv_{\mathbf{r}_{acc\sigma}} \partial^q c_{p,r} \cdot \sigma_{p,q} \quad \text{and} \quad \sigma_{p,q+r} \equiv_{\mathbf{r}_{acc\sigma}} \partial^q \sigma_{p,r} \cdot \sigma_{p,q},$$

$$(4.27) \quad a_{q+1} \cdot \sigma_{p+1,q} \equiv_{\mathbf{r}_{acc\sigma}} \sigma_{p+2,q} \cdot a_1.$$

Proof. Relation (4.25) and (4.26) follow from (4.20) by taking $I = \{r+1, \dots, r+p\}$, $J = \{r+p+1, \dots, r+p+q\}$, $K = \{1, \dots, r\}$, and $I = \{q+r+1, \dots, q+r+p\}$, $J = \{1, \dots, q\}$, $K = \{q+1, \dots, q+r\}$, respectively. In the first case, we have $c_{I, J \cup K} = \sigma_{p,r}$, and, in the second one, we have $c_{I \cup J, K} = \partial^q c_{p,r}$.

For (4.27), we use induction. For $q = 0$, the result is clear; for $q \geq 1$, we find

$$\begin{aligned} a_{q+1}\sigma_{p+1,q} &\stackrel{(4.25)}{=} a_{q+1} \cdot \partial\sigma_{p+1,q-1} \cdot \sigma_{p+1,1} = \partial(a_q\sigma_{p-1,q-1}) \cdot \sigma_{p+1,1} \\ &\stackrel{(IH)}{=} \partial\sigma_{p+2,q-1}a_1 \cdot \sigma_{p+1,1} = \partial\sigma_{p+2,q-1} \cdot a_2\sigma_1 \cdots \sigma_{p+1} \\ &\stackrel{(?)}{=} \partial\sigma_{p+2,q-1} \cdot \sigma_1\sigma_2a_1\sigma_2 \cdots \sigma_{p+1} \\ &\stackrel{\mathbf{r}_{ac\sigma}}{=} \partial\sigma_{p+2,q-1} \cdot \sigma_1\sigma_2\sigma_3 \cdots \sigma_{p+2}a_1 \stackrel{(4.25)}{=} \sigma_{p+2,q}a_1, \end{aligned}$$

which completes the computation. \square

Lemma 4.18. *Assume that t is a size n tree. Then, for $p, q \geq 0$, we have*

$$(4.28) \quad \square_{i+n} \cdot u_t^* \stackrel{\mathbf{r}_{ac\sigma}}{=} u_t^* \cdot \square_{i+1}, \quad \text{for } \square = a, c \text{ or } \sigma,$$

$$(4.29) \quad \partial^q u_t \cdot c_{1,q} \stackrel{\mathbf{r}_{ac\sigma}}{=} c_{n,q} \cdot u_t^* \quad \text{and} \quad \partial^q u_t^* \cdot \sigma_{p+1,q} \stackrel{\mathbf{r}_{ac\sigma}}{=} \sigma_{p+n,q} \cdot u_t^*,$$

$$(4.30) \quad u_t^* \cdot c_{p,q+1} \stackrel{\mathbf{r}_{ac\sigma}}{=} c_{p,q+n} \cdot \partial^q u_t, \quad \text{and} \quad u_t^* \cdot \sigma_{p,q+1} \stackrel{\mathbf{r}_{ac\sigma}}{=} \sigma_{p,q+n} \cdot \partial^q u_t^*.$$

Proof. We use induction on n . For $n = 1$, the words u_t and u_t^* are empty, and all formulas are equalities. Otherwise, assume $t = t_1 \wedge t_2$, with, as usual, n_k the size of t_k and I_k its set of labels. For (4.28), we find

$$\begin{aligned} \square_{i+n} \cdot u_t^* &= \square_{i+n} \cdot \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A \stackrel{\mathbf{g}}{=} \sigma_{I_1, I_2} \cdot \square_{i+n} \cdot u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A \\ &\stackrel{(IH)}{=} \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \square_{i+n_2} \cdot \partial u_{t_2}^* \cdot A \\ &\stackrel{(IH)}{=} \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2}^* \cdot \square_{i+2} \cdot A \stackrel{\mathbf{g}}{=} \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A \cdot \square_{i+1} = u_t^* \cdot \square_{i+1}. \end{aligned}$$

(The first equivalence holds because we consider σ_{I_1, I_2} , which consists of σ_i 's only.)

Consider now the second relation in (4.29). The relation follows from the commutativity of

$$\begin{array}{ccccccc} \langle q, I, p, \dots \rangle & \xrightarrow{\partial^q \sigma_{I_1, I_2}} & \langle q, I_1, I_2, p, \dots \rangle & \xrightarrow{\partial^q u_{t_1}^*} & \langle q, t_1, I_2, p, \dots \rangle & \xrightarrow{\partial^{q+1} u_{t_2}^*} & \langle q, t_1, I_2, p, \dots \rangle & \xrightarrow{a_{q+1}} & \langle q, t_1 \wedge t_2, p, \dots \rangle \\ \downarrow \sigma_{p+n, q} & & \downarrow \sigma_{p+n, q} & & \downarrow \sigma_{p+n_2+1, q} & & \downarrow \sigma_{p+2, q} & & \downarrow \sigma_{p+1, q} \\ \langle I, p, q, \dots \rangle & \xrightarrow{\sigma_{I_1, I_2}} & \langle I_1, I_2, p, q, \dots \rangle & \xrightarrow{u_{t_1}^*} & \langle t_1, I_2, p, q, \dots \rangle & \xrightarrow{\partial u_{t_2}^*} & \langle t_1, I_2, p, q, \dots \rangle & \xrightarrow{a_1} & \langle t_1 \wedge t_2, p, q, \dots \rangle \end{array}$$

The first (leftmost) square is commutative by (4.25). The second one is commutative by induction hypothesis. For the third, (4.25) tells us that $\sigma_{p+n_2+1, q}$ is $\mathbf{r}_{ac\sigma}$ -equivalent to $\sigma_{1, q} \cdot \partial\sigma_{p+n_2, q}$, and that $\sigma_{p+2, q}$ is $\mathbf{r}_{ac\sigma}$ -equivalent to $\sigma_{1, q} \cdot \partial\sigma_{p+1, q}$. As $\partial^{q+1} u_{t_2}^*$ $\mathbf{r}_{ac\sigma}$ -commutes with $\sigma_{1, q}$ by geometric relations, we are left with proving the $\mathbf{r}_{ac\sigma}$ -equivalence of $\partial^q u_{t_2}^* \cdot \sigma_{p+1, q}$ and $\sigma_{p+n_2, q} \cdot \partial u_{t_2}^*$, which is the induction hypothesis. Finally, the commutativity of the last square follows from (4.27).

The verification of the other three formulas is similar. \square

We are now in position for proving the counterpart to Lemma 2.14:

Lemma 4.19. *Assume $t' = t \bullet \square_\alpha$, where \square is A, C , or Σ . Then we have*

$$(4.31) \quad u_{t'} \stackrel{\mathbf{r}_{AC\Sigma}}{=} u_t \cdot \square_\alpha \quad \text{and} \quad u_{t'}^* \stackrel{\mathbf{r}_{AC\Sigma}}{=} u_t^* \cdot \square_{0\alpha}.$$

Proof. Clearly, it suffices to consider the cases of A_α and C_α , as Σ_α is defined from the latter. As for Lemma 2.14, we use induction on the length of α as a sequence of 0's and 1's. So assume first that α is the empty address. Let us consider the case of A . The hypothesis $t' = t \bullet A$ implies that there exist trees t_1, t_2, t_3 such that t is $(t_1 \wedge t_2) \wedge t_3$, and t' is $t_1 \wedge (t_2 \wedge t_3)$. We write I_1 (resp. I_2, I_3) for the labels in t_1 (resp. t_2, t_3), and n_1 (resp. n_2, n_3) for their size. We obtain

$$(4.32) \quad u_{t'} = c_{I_1 \cup I_2, I_3} \cdot \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A \cdot \partial u_{t_3}$$

$$(4.33) \quad u_t \cdot A = \sigma_{I_1, I_2 \cup I_3} \cdot u_{t_1}^* \cdot \partial c_{I_2, I_3} \cdot \partial u_{t_2}^* \cdot \partial^2 u_{t_3} \cdot A$$

$$(4.34) \quad u_{t'}^* = \sigma_{I_1 \cup I_2, I_3} \cdot \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A \cdot \partial u_{t_3}^* \cdot A$$

$$(4.35) \quad u_t^* \cdot A_0 = \sigma_{I_1, I_2 \cup I_3} \cdot u_{t_1}^* \cdot \partial \sigma_{I_2, I_3} \cdot \partial u_{t_2}^* \cdot \partial^2 u_{t_3}^* \cdot A_1 A A_0.$$

Using geometric relations, we may move the factor A to the right in (4.32), while, in (4.33), we may replace $u_{t_1}^* \cdot \partial c_{I_2, I_3}$ with $\partial^p c_{I_2, I_3} \cdot u_{t_1}^*$ using (4.28). Then, applying (4.25) gives the equivalence of $u_{t'}$ and $u_t \cdot A$. The argument is similar for (4.34) and (4.35), the only difference being an additional pentagon relation for replacing AA by $A_1 A A_0$ on the right.

For C , with similar notation, we have $t = t_1 \wedge t_2$ and $t' = t_2 \wedge t_1$, and we find now

$$(4.36) \quad u_{t'} = c_{I_2, I_1} \cdot u_{t_2}^* \cdot \partial u_{t_1},$$

$$(4.37) \quad u_t \cdot C = c_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2} \cdot C,$$

$$(4.38) \quad u_{t'}^* = \sigma_{I_2, I_1} \cdot u_{t_2}^* \cdot \partial u_{t_1}^* \cdot A,$$

$$(4.39) \quad u_t^* \cdot C_0 = \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2}^* \cdot A C_0.$$

By (4.28), we have $u_{t_2}^* \cdot \partial u_{t_1} \equiv_{\mathbf{r}_{ac\sigma}} \partial^{n_2} u_{t_1} \cdot u_{t_2}^*$, and the $\mathbf{r}_{ac\sigma}$ -equivalence of $u_{t'}$ and $u_t \cdot C$ follows from the commutativity of the diagram

$$\begin{array}{ccccccc} \backslash I \backslash & \xrightarrow{c_{I_1, I_2}} & \backslash I_1, I_2 \backslash & \xrightarrow{u_{t_1}^*} & \backslash t_1, I_2 \backslash & \xrightarrow{\partial u_{t_2}} & t_1 \wedge t_2 \\ \parallel & \mathbf{r}_{ac\sigma} + \text{torsion} & \downarrow c_{n_2, n_1} & (4.30) & \downarrow c_{n_2, 1} & (4.29) & \downarrow c_{1, 1} = C \\ \backslash I \backslash & \xrightarrow{c_{I_2, I_1}} & \backslash I_2, I_1 \backslash & \xrightarrow{\partial^{n_2} u_{t_1}} & \backslash I_2, t_1 \backslash & \xrightarrow{u_{t_2}^*} & t_2 \wedge t_1 \end{array}$$

The commutativity of the left square follow from the fact that both $c_{I_1, I_2} \cdot c_{n_2, n_1}$ and c_{I_2, I_1} induce the same permutation of the labels: it follows that these words must be equivalent with respect to any family of relations that makes a presentation of the symmetric group, and, therefore, they must be equivalent with respect to the Coxeter relations of $\mathbf{r}_{ac\sigma}$ completed with the torsion relations $c_i^2 = \sigma_i^2 = 1$.

The argument is similar for $u_{t'}^*$. First (4.28) gives $u_{t_2}^* \cdot \partial u_{t_1}^* \equiv_{\mathbf{r}_{ac\sigma}} \partial^{n_2} u_{t_1}^* \cdot u_{t_2}^*$, and the rest is the commutativity of

$$\begin{array}{ccccccc} \backslash I, .. \backslash & \xrightarrow{\sigma_{I_1, I_2}} & \backslash I_1, I_2, .. \backslash & \xrightarrow{u_{t_1}^*} & \backslash t_1, I_2, .. \backslash & \xrightarrow{\partial u_{t_2}^*} & \backslash t_1, t_2, .. \backslash \xrightarrow{A} & \backslash t_1 \wedge t_2, .. \backslash \\ \parallel & \mathbf{r}_{ac\sigma} + \text{torsion} & \downarrow \sigma_{n_2, n_1} & (4.30) & \downarrow \sigma_{n_2, 1} & (4.29) & \downarrow \sigma_{1, 1} = \Sigma & (4.12) & \downarrow C_0 \\ \backslash I, .. \backslash & \xrightarrow{\sigma_{I_2, I_1}} & \backslash I_2, I_1, .. \backslash & \xrightarrow{\partial^{n_2} u_{t_1}^*} & \backslash I_2, t_1, .. \backslash & \xrightarrow{u_{t_2}^*} & \backslash t_2, t_1, .. \backslash \xrightarrow{A} & \backslash t_2 \wedge t_1, .. \backslash \end{array}$$

The induction is now easy, and there is no need to consider the case of A_α and C_α separately. So we use \square_α to represent the two cases simultaneously. Assume $\alpha = 0\beta$. Then we have $t' = t_1' \wedge t_2$ with $t_1' = t_1 \bullet \square_\alpha$, and we find

$$\begin{aligned} u_{t'} &= c_{I_1, I_2} \cdot u_{t_1'}^* \cdot \partial u_{t_2} \equiv_{(IH)} c_{I_1, I_2} \cdot u_{t_1}^* \cdot \square_{0\beta} \cdot \partial u_{t_2} \equiv_{\mathbf{g}} c_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2} \cdot \square_{0\beta} = u_t \cdot \square_\alpha, \\ u_{t'}^* &= \sigma_{I_1, I_2} \cdot u_{t_1'}^* \cdot \partial u_{t_2}^* \cdot A \equiv_{(IH)} \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \square_{0\beta} \cdot \partial u_{t_2}^* \cdot A \\ &\equiv_{\mathbf{g}} \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2} \cdot \square_{0\beta} \cdot A \equiv_{\mathbf{g}} \sigma_{I_1, I_2} \cdot u_{t_1}^* \cdot \partial u_{t_2} \cdot A \cdot \square_{00\beta} = u_t^* \cdot \square_{0\alpha}. \end{aligned}$$

The argument is symmetric (and simpler: no commutation is needed) in the case $\alpha = 1\beta$. \square

Applying Proposition 1.2, we obtain

Proposition 4.20. *The relations $\mathbf{r}_{AC\Sigma}$ completes with the torsion relations $C_\alpha^2 = \Sigma_\alpha^2 = 1$, make a presentation of the group $G(\mathcal{A}, \mathcal{C})$, i.e., of Thompson's group V , in terms of the generators A_α, C_α and Σ_α .*

As the relations $\mathbf{r}_{AC\Sigma}$ follow from those of \mathbf{r}_{AC} and the definition of Σ_α , we immediately deduce:

Proposition 4.21. *The relations \mathbf{r}_{AC} , i.e., the geometric relations, completed with the pentagon and hexagon relations, and the torsion relations, make a presentation of V in terms of the generators A_α and C_α .*

As in the case of the group $G(\mathcal{A})$, *i.e.*, F , we can also restrict to the right branch generators a_i and σ_i . By looking at the proof of Lemma 4.22, we obtain:

Lemma 4.22. *Assume $t' = t \bullet \square_i$, where \square is a, c , or σ . Then we have*

$$(4.40) \quad u_{t'} \equiv_{\mathbf{r}_{ac\sigma}} u_t \bullet \square_i.$$

Always applying Proposition 1.2, we deduce

Proposition 4.23. *The relations $\mathbf{r}_{ac\sigma}$ completed with the torsion relations $c_i^2 = \sigma_i^2 = 1$ make a presentation of the group $G(\mathcal{A}, \mathcal{C})$, *i.e.*, of V , in terms of the generators a_i, c_i and σ_i .*

Finally, as all relations in $\mathbf{r}_{ac\sigma}$ follow from \mathbf{r}_{ac} , we also obtain

Proposition 4.24. *The relations \mathbf{r}_{ac} completed with the torsion relations $c_i^2 = 1$ make a presentation of the group $G(\mathcal{A}, \mathcal{C})$, *i.e.*, of V , in terms of the generators a_i and c_i .*

5. SEMI-COMMUTATIVITY AND THE GROUP V'

We have seen above how to naturally connect Thompson's group V with the associativity and commutativity laws. When we consider the computations of Section 4, we see that the main technical role is played by the elements Σ_α . This suggests to introduce the subgroup of $G(\mathcal{A}, \mathcal{C})$ generated by the elements A_α and Σ_α . We shall see now that the latter naturally arises as a geometry group of some algebraic laws, namely the geometry group of associativity together with the weakened form of commutativity.

5.1. The semi-commutativity identity.

Definition. We define *semi-commutativity* to be the identity

$$(\Sigma) \quad x(yz) = (yx)z.$$

As associativity and semi-commutativity are linear laws in the sense defined at the beginning of Section 4, they admit a geometry group.

Proposition 5.1. *The geometry group $G(\mathcal{A}, \Sigma)$ of associativity and semi-commutativity is the subgroup V' of $G(\mathcal{A}, \mathcal{C})$ generated by the elements A_α and Σ_α .*

Proof. Figure 15 shows that the operators associated with the semi-commutativity identity are the operators Σ_α of Section 4, so the geometry monoid $\mathcal{G}(\mathcal{A}, \Sigma)$ is the sumonoid pf $\mathcal{G}(\mathcal{A}, \mathcal{C})$ generated by the operators $A_\alpha^{\pm 1}$ and $\Sigma_\alpha^{\pm 1}$. Quotienting under near-equality gives the corresponding result for the geometry groups. \square

So, in particular, if we extract from the relations established for $G(\mathcal{A}, \mathcal{C})$ those that involve the generators A_α and Σ_α only, the latter have to be satisfied in the group $G(\mathcal{A}, \Sigma)$.

Definition. We define $\mathbf{r}_{A\Sigma}$ to be the family of all A and Σ -geometry relations, namely the translated copies of

$$\begin{aligned} (\mathbf{g}_\perp) \quad & \square_{0\alpha} \cdot \diamond_{1\beta} = \diamond_{1\beta} \cdot \square_{0\alpha}, \\ (\mathbf{g}_A) \quad & \square_{11\alpha} \cdot A = A \cdot \square_{1\alpha}, \quad \square_{10\alpha} \cdot A = A \cdot \square_{01\alpha}, \quad \square_{0\alpha} \cdot A = A \cdot \square_{00\alpha}, \\ (\mathbf{g}_\Sigma) \quad & \square_{11\alpha} \cdot \Sigma = \Sigma \cdot \square_{11\alpha}, \quad \square_{10\alpha} \cdot \Sigma = \Sigma \cdot \square_{0\alpha}, \quad \square_{0\alpha} \cdot \Sigma = \Sigma \cdot \square_{10\alpha}, \end{aligned}$$

with \square, \diamond standing for A or Σ , together with the translated copies of

$$(5.1) \quad \Sigma A_1 A = A_1 A \Sigma_0, \quad \Sigma_1 \Sigma A_1 = A \Sigma, \quad \Sigma \Sigma_1 A = A_1 \Sigma, \quad \Sigma \Sigma_1 \Sigma = \Sigma_1 \Sigma \Sigma_1.$$

Proposition 5.2. *All relations of $\mathbf{r}_{A\Sigma}$, as well as the torsion relations $\Sigma_\alpha^2 = 1$, are satisfied in the group $G(\mathcal{A}, \Sigma)$.*

We shall also consider the subfamily of $\mathbf{r}_{A\Sigma}$ containing only those relations that involve the elements of \mathbf{a} and $\boldsymbol{\sigma}$.

Definition. We define $\mathbf{r}_{a\sigma}$ to consist of the following relations:

$$\begin{aligned} a_i \square_{j+1} &= \square_j a_i, & \text{with } j \geq i+1 \text{ and } \square = a \text{ or } \sigma, \\ \sigma_i \square_j &= \square_j \sigma_i, & \text{with } j \geq i+2 \text{ and } \square = a \text{ or } \sigma, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & \sigma_{i+1} \sigma_i a_{i+1} = a_i \sigma_i, \quad \sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i. \end{aligned}$$

Lemma 5.3. *All relations of $\mathbf{r}_{a\sigma}$ are satisfied in V' .*

Actually, it is easy to check

Lemma 5.4. *All relations of $\mathbf{r}_{a\sigma}$ follow from $\mathbf{r}_{A\Sigma}$ plus the definitions $\sigma_i = \Sigma_{1^{i-1}}$.*

5.2. **Presentations of V' .** Our aim is to prove:

Proposition 5.5. *The family $\mathbf{r}_{A\Sigma}$ completed with the torsion relations $\Sigma_\alpha^2 = 1$ make a presentation of the group $G(\mathcal{A}, \Sigma)$, i.e., V' , in terms of the generators A_α and Σ_α .*

Proof. The method should be clear: we have to select a family of trees containing one element in each V' -orbit, then define distinguished words in $W(\mathcal{A}, \Sigma)$ describing how to construct a tree starting from the distinguished element of its orbit, and, finally, check that there are enough relations in $\mathbf{r}_{A\Sigma}$ to witness for the relations (1.3) of Proposition 1.2.

The construction is a slight modification of the one used in Section 4. The difference between commutativity and semi-commutativity is that the latter cannot change the rightmost label of a tree. To keep the same conventions as in Section 4, let $T_{\mathbf{N}}^\infty$ denote the set of all decorated trees whose leaves except the rightmost one wear labels in \mathbf{N} , and whose rightmost leaf is labelled ∞ . Then every tree in $T_{\mathbf{N}}^\infty$ is equivalent up to associativity and semi-commutativity to some right comb $\setminus I, \infty \setminus$. For such a tree t , the word u_t maps $\setminus I \setminus$ to t , and, by construction, u_t consists of letters a_i and σ_j exclusively, since the rightmost leaf is never changed. Indeed, the only letter c_i possibly occurring in u_t comes from the factors $c_{I,J}$ in the inductive construction, and this happens only when I contains the largest element of $I \cup J$. We can therefore use the words u_t and u_t^* without change. Then the only point is to check that $t' = t \cdot \square_\alpha$ implies

$$(5.2) \quad u_{t'} \equiv_{\mathbf{r}_{a\sigma}} u_t \cdot \square_\alpha \quad \text{and} \quad u_{t'}^* \equiv_{\mathbf{r}_{a\sigma}} u_t^* \cdot \square_{0\alpha}$$

both in the case $\square = A$ and $\square = \Sigma$. For the case of A_α , it suffices to look at the proof of Lemma 4.22. The case of Σ has not been considered in Section 4, and we consider it now. So we assume $t = t_1 \wedge (t_2 \wedge t_3)$ and $t' = t_2 \wedge (t_1 \wedge t_3)$. We obtain

$$(5.3) \quad u_{t'} = \sigma_{I_2, I_1 \cup I_3} \cdot u_{t_2}^* \cdot \partial \sigma_{I_1, I_3} \cdot \partial u_{t_1}^* \cdot \partial^2 u_{t_3},$$

$$(5.4) \quad u_t \cdot \Sigma = \sigma_{I_1, I_2 \cup I_3} \cdot u_{t_1}^* \cdot \partial \sigma_{I_2, I_3} \cdot \partial u_{t_2}^* \cdot \partial^2 u_{t_3} \cdot \Sigma.$$

By (4.28), we have $u_{t_2}^* \cdot \partial \sigma_{I_1, I_3} \equiv_{\mathbf{r}_{a\sigma}} \partial^{n_2} \sigma_{I_1, I_3} \cdot u_{t_2}^*$, $u_{t_1}^* \cdot \partial \sigma_{I_2, I_3} \equiv_{\mathbf{r}_{a\sigma}} \partial^{n_1} \sigma_{I_2, I_3} \cdot u_{t_1}^*$, and $u_{t_2}^* \cdot \partial u_{t_1}^* \equiv_{\mathbf{r}_{a\sigma}} \partial^{n_2} u_{t_1}^* \cdot u_{t_2}^*$. Then the $\mathbf{r}_{A\Sigma}$ -equivalence of $u_{t'}$ and $u_t \cdot \Sigma$ follows from the commutativity of the diagram

$$\begin{array}{ccccccc} \setminus I \setminus & \xrightarrow{\sigma_{I_1, I_2 \cup I_3} \cdot \partial^{n_1} \sigma_{I_2, I_3}} & \setminus I_1, I_2, I_3 \setminus & \xrightarrow{u_{t_1}^*} & \setminus t_1, I_2, I_3 \setminus & \xrightarrow{\partial u_{t_2}^*} & \setminus t_1, t_2, I_3 \setminus & \xrightarrow{\partial^2 u_{t_3}^*} & \setminus t_1, t_2, t_3 \setminus \\ \parallel & \text{\scriptsize } \mathbf{r}_{a\sigma} + \text{torsion} & \downarrow \sigma_{n_2, n_1} & (4.30) & \downarrow \sigma_{n_2, 1} & (4.29) & \downarrow \sigma_{1, 1} & (5.1) & \downarrow \sigma_{1, 1} = \Sigma \\ \setminus I \setminus & \xrightarrow{\sigma_{I_2, I_1 \cup I_3} \cdot \partial^{n_2} \sigma_{I_1, I_3}} & \setminus I_2, I_1, I_3 \setminus & \xrightarrow{\partial^{n_2} u_{t_1}^*} & \setminus I_2, t_1, I_3 \setminus & \xrightarrow{u_{t_2}^*} & \setminus t_2, t_1, I_3 \setminus & \xrightarrow{\partial^2 u_{t_3}^*} & \setminus t_2, t_1, t_3 \setminus \end{array}$$

The relations of $\mathbf{r}_{A\Sigma}$ are sufficient to obtain the commutativity of the last three squares. As for the first square, the associated permutations are equal, so the relations of $\mathbf{r}_{a\sigma}$ completed with the torsion relations $\sigma_i^2 = 1$ must give the result.

The argument is similar for the words u_t^* , with an associated diagram coinciding with the above one up to an additional square on the right whose commutativity is provided by the relation $\Sigma A_1 A = A_1 A \Sigma_0$. The induction along addresses is similar to the one we used for the groups $G(\mathcal{A})$ and $G(\mathcal{A}, \mathcal{C})$, i.e., F and V . \square

As in Section 2 and 4, looking at the previous proof enables one to extract the following result:

Proposition 5.6. *The group V' is generated by \mathbf{a} and σ , and the relations $\mathbf{r}_{a\sigma}$ completed with $\sigma_i^2 = 1$ make a presentation of V' in terms of these generators.*

Part 2. The group of multiscaled braids

The presentations of Section 4 show that the groups V and V' are connected with the infinite symmetric group \mathfrak{S}_∞ , and make it natural to consider the groups with similar presentations but with the torsion relations $\sigma_i^2 = 1$ removed, as was in particular done in independent work by M. Brin [2]. The relationship between these new groups and the groups V and V' is then parallel to that between Artin's braid group B_∞ and \mathfrak{S}_∞ . Here we shall concentrate on the case of the group associated with V' , which is technically simpler as only two families of generators are to be considered, instead of three in the case of V . A mixture of Thompson's group F and Artin's braid group B_∞ —whence our notation FB_∞ —the group so obtained can be seen both as a braided version of Thompson's group V —whence the notation \widehat{BV} of [2], with BV for the similar braided version of V —and a sort of fractal version of Artin's braid group B_∞ , as its elements can adequately be represented by braid diagrams in which the ordinary sequence of integer indexed strands is replaced with a multiscaled sequence of strands indexed by finite sequences of integers. The study of the group FB_∞ turns out to be rich and interesting, and it is the subject of the second part of this text.

6. ALGEBRAIC PROPERTIES OF FB_∞

In this section, we start from the presentation of FB_∞ analogous to that of V' and establish various algebraic properties, in particular that FB_∞ is the group of fractions for the monoid with the same presentation, and that FB_∞ comes equipped with a linear ordering and a self-distributive operation extending similar objects previously defined on Artin's braid group B_∞ .

6.1. The group FB_∞ and the monoid FB_∞^+ . We saw in Section 5 that the group V' admits two natural presentations, namely one in terms of the families of generators \mathbf{A} and Σ , and one in terms of the restricted families \mathbf{a} and σ . Here we start from the latter, which makes the verifications more simple, and remove the torsion relations $\sigma_i^2 = 1$. Then we see that the remaining relations between the generators σ_i exactly are those of the standard presentation of Artin's braid group B_∞ in terms of the generators σ_i , namely $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $j \geq i + 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, cf. [1, 4].

Definition. We define FB_∞ to be the group $\langle \mathbf{a}, \sigma; \mathbf{r}_{a\sigma} \rangle$, i.e., the group generated by two infinite sequences $a_1, a_2, \dots, \sigma_1, \sigma_2, \dots$ with the relations i.e.,

$$(6.1) \quad \left\{ \begin{array}{l} a_i \square_j = \square_{j+1} a_i, \quad \text{with } j \geq i + 1, \text{ and } \square = a \text{ or } \sigma, \\ \sigma_i \square_j = \square_j \sigma_i, \quad \text{with } j \geq i + 2, \text{ and } \square = a \text{ or } \sigma, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_{i+1} \sigma_i a_{i+1} = a_i \sigma_i, \quad \sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i. \end{array} \right.$$

The relations (6.1) include the relations of the standard presentation of Thompson's group F in terms of the a_i generators, those of the standard presentations of Artin's braid group B_∞ in terms of the generators σ_i , and mixed relations connecting the generators a_i and σ_i . The relations in the first line can be called shifted commutation relations, those in the second line are ordinary commutation relations, while in the third line can be called braid relations, as we shall see in Section 7.

The group FB_∞ is a sort of twisted product of Thompson's group F and Artin's braid group B_∞ , and it is not surprising that it can be investigated by the same methods as F

and B_∞ , in particular in terms of monoid and fractions. The tools presented in Section 3 prove to be specially relevant here.

Definition. We define FB_∞^+ to be the monoid $\langle \mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}} \rangle^+$.

So, as a monoid, FB_∞^+ is defined by the same presentation as the group FB_∞ . All relations in $\mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}}$ are of the type $u = v$ where u and v are nonempty words, so $\mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}}$ is eligible for the word reversing method of Section 3. The main technical point is:

Proposition 6.1. *The presentation $(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}})$ is complete with respect to both right and left reversing.*

We shall establish Proposition 6.1 using the method of Section 3, *i.e.*, we first show that the presentation is homogeneous, and, then, we check that the criterion of Proposition 3.5 holds for all triples of generators.

Lemma 6.2. *The presentation $(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}})$ is homogeneous.*

Proof. As the relations $\sigma_i \sigma_{i+1} a_i = a_{i+1} \sigma_i$ and $\sigma_{i+1} \sigma_i a_{i+1} = a_i \sigma_i$ in $\mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}}$ do not preserve the length of the words, the length itself cannot be used. Instead we shall define a convenient function λ so that the contribution of σ_i depend on the letters that precede them. First we define an action of $W^+(\mathbf{a}, \boldsymbol{\sigma})$ on \mathbb{N} -indexed sequences of integers as follows:

$$\begin{aligned} (\dots, n_{i-1}, n_i, n_{i+1}, n_{i+2}, \dots) \cdot a_i &= (\dots, n_{i-1}, n_i + n_{i+1}, n_{i+2}, \dots), \\ (\dots, n_{i-1}, n_i, n_{i+1}, n_{i+2}, \dots) \cdot \sigma_i &= (\dots, n_{i-1}, n_{i+1}, n_i, n_{i+2}, \dots). \end{aligned}$$

It is easily verified that this action is compatible with the relations of $\mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}}$. For instance, we have $(n_1, n_2, n_3, \dots) \cdot a_1 \sigma_1 = (n_3, n_1 + n_2, \dots) = (n_1, n_2, n_3, \dots) \cdot \sigma_2 \sigma_1 a_2$. Then, for u a word in $W^+(\mathbf{a}, \boldsymbol{\sigma})$, we define $\text{wt}(a_i, u)$ to be 1 and $\text{wt}(\sigma_i, u)$ to be the product of the i th and $(i+1)$ -th entries in the sequence $(1, 1, \dots) \cdot u$. Finally, for w in $W^+(\mathbf{a}, \boldsymbol{\sigma})$, we define $\lambda(w) = \sum_p \text{wt}(w(p), w_p)$, where $w(p)$ denotes the p th letter in w and w_p denotes the length $p-1$ prefix of w . Then λ has the required properties. For instance, assuming $(1, 1, 1, \dots) \cdot u = (n_1, n_2, n_3, \dots)$, we have

$$\begin{aligned} \lambda(u \sigma_1 \sigma_2 \sigma_1) &= \lambda(u) + n_1 n_2 + n_2 n_3 + n_1 n_3 = \lambda(u \sigma_2 \sigma_1 \sigma_2), \\ \lambda(u \sigma_1 \sigma_2 a_1) &= \lambda(u) + n_1 n_2 + n_1 n_3 + 1 = \lambda(u a_2 \sigma_1), \\ \lambda(u \sigma_2 \sigma_1 a_2) &= \lambda(u) + n_1 n_3 + n_2 n_3 + 1 = \lambda(u a_1 \sigma_1). \end{aligned}$$

(The intuition underlying this definition will become obvious in the next section: we attribute to each σ_i a weight that is the total number of elementary crossings involved in the associated multiscaled braid diagram.) \square

Lemma 6.3. *For each triple of letters (x, y, z) , the right and the left cube conditions at (x, y, z) are satisfied in the presentation $(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}})$.*

Proof. As there are infinitely many letters, infinitely many cases should be considered *a priori*. However, it is clear that the only point that matters is the mutual distance of the indices, and, therefore, only finitely many types occur. The verification is purely mechanic, and we postpone it to an appendix. \square

By Proposition 3.5, Proposition 6.1 is established. This in turn enables us to read some properties of the monoid FB_∞^+ and of the group FB_∞ almost directly.

Proposition 6.4. (i) *The monoid FB_∞^+ admits left and right cancellation.*

(ii) *Any two elements of FB_∞^+ that admit a right (resp. left) common multiple admit a right (resp. a left) least common multiple.*

Proof. (i) For each letter x , the family $r_{a\sigma}$ contains no relation of the form $x \dots = x \dots$ or $\dots x = \dots x$, so Proposition 3.6 applies.

(ii) For all letters x, y , the family $r_{a\sigma}$ contains at most one relation of the form $x \dots = y \dots$, and at most one relation of the form $\dots x = \dots y$ so Proposition 3.15 applies. \square

The next step is to prove that any two elements of FB_∞^+ admit a common left multiple. By Proposition 3.15 again, this amounts to proving that left reversing always terminates, which is not *a priori* obvious as the length of the words that appear in the reversing process may increase. By Garside's theory, it is known that any two elements in the braid monoid B_∞^+ admit a common left multiple, and, therefore, the left reversing of any word of the form uv^{-1} with u, v in $W^+(\sigma)$ terminates in a finite number of steps. The same is true for u, v in $W^+(\mathbf{a})$, since, in this case, the length cannot increase. So the only remaining case is that of mixed words involving both σ and \mathbf{a} . We shall treat this case directly, by explicitly describing the result of reversing.

Adding the relations $\sigma_i^2 = 1$ to the relations of the standard presentation of the group B_∞ yields the Coxeter presentation of the symmetric group \mathfrak{S}_∞ in terms of transpositions. One obtains in this way a surjective homomorphism of B_∞ onto \mathfrak{S}_∞ , so every word in $W^+(\sigma)$ defines a bijection of \mathbb{N} . We denote by $w[k]$ the image of k under the bijection associated in this way with the word w .

Definition. For w in $W^+(\sigma)$ and k in \mathbb{N} , we inductively define $d_k(w)$ by $d_k(\varepsilon) = \varepsilon$ and

$$(6.2) \quad d_k(w\sigma_i) = d_{\sigma_i[k]}(w) \cdot d_k(\sigma_i), \text{ with } d_k(\sigma_i) = \begin{cases} \sigma_{k+1} & \text{for } k < i, \\ \sigma_i\sigma_{i+1} & \text{for } k = i, \\ \sigma_{i+1}\sigma_i & \text{for } k = i + 1, \\ \sigma_k & \text{for } k > i + 1. \end{cases}$$

It is easy to check that, with the standard diagrammatic interpretation of braid words (cf. Section 7), $w[k]$ is the initial position of the strand finishing at position k in the diagram coded by w , and $d_k(w)$ encodes the braid diagram obtained from the diagram coded by w by doubling the strand that finishes at position k (Figure 16).

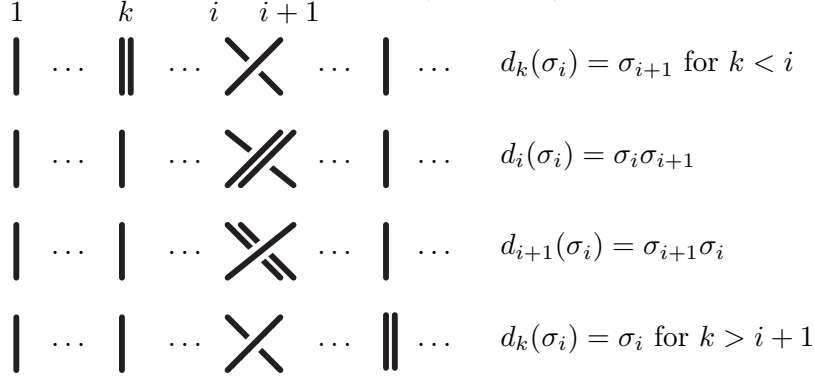


FIGURE 16. Doubling the k th strand (from the bottom) in a braid.

Lemma 6.5. For w in $W^+(\sigma)$ and $k \geq 1$, the word $w \cdot a_k^{-1}$ is left $r_{a\sigma}$ -reversible to $a_{w[k]}^{-1} \cdot d_k(w)$.

Proof. It suffices to check the formula when w has length 1, and to show that the induction rule preserves it. Looking at the various cases shows that the result is true when w consists of a single letter σ_i . For instance, $\sigma_1 a_1^{-1}$ is left reversible to $a_2^{-1} \sigma_2 \sigma_1$ on the one hand, and we have $\sigma_1[1] = 2$ and $d_1(\sigma_1) = \sigma_1 \sigma_2$ on the other hand. Then the inductive rule for left reversing is that $u_1 u_2 v^{-1}$ is left reversible to $v''^{-1} u'_1 u'_2$ if and only if, for some v' , the word $u_2 v$ is left reversible to $v'^{-1} u'_2$ and $u_1 v'^{-1}$ is left reversible to $v''^{-1} u'_1$. \square

Proposition 6.6. (i) *Left $\mathbf{r}_{a\sigma}$ -reversing always terminates in finitely many steps.*

(ii) *Any two elements in the monoid FB_∞^+ admit a left lcm.*

(iii) *The monoid FB_∞^+ embeds in the group FB_∞ , and FB_∞ is a group of left fractions of FB_∞^+ , i.e., every element of FB_∞ can be expressed as $x^{-1}y$ with x, y in FB_∞^+ .*

Proof. We know that left reversing inside $W^+(\mathbf{a})$ and $W^+(\boldsymbol{\sigma})$ terminate in a finite number of steps, so Lemma 6.5 is what is needed for (i). Then (ii) follows from (the left counterpart to) Proposition 3.15. It follows that the monoid FB_∞^+ satisfies Ore's conditions on the left, and, therefore, it embeds in the group FB_∞ and the latter is a group of left fractions for FB_∞^+ . \square

As FB_∞ is the group of fractions of a monoid where least common multiples exist and which has no torsion, we immediately deduce, using the result of [15] (or [?]):

Corollary 6.7. *The group FB_∞ is torsion-free.*

This result will be strengthened in the sequel, as we shall prove in Section 8 that FB_∞ is actually an orderable group.

Proposition 6.8. (i) *Two words u, v in $W^+(\mathbf{a}, \boldsymbol{\sigma})$ represent the same element in the monoid FB_∞^+ if and only if the left $\mathbf{r}_{a\sigma}$ -reversing of the word uv^{-1} ends up with the empty word, if and only if the right $\mathbf{r}_{a\sigma}$ -reversing of the word $u^{-1}v$ ends up with the empty word.*

(ii) *A word w in $W(\mathbf{a}, \boldsymbol{\sigma})$ represents 1 in FB_∞ if and only if its double left $\mathbf{r}_{a\sigma}$ -reversing ends up with an empty word, where double left reversing consists in left reversing w into $u^{-1}v$ with u, v in $W^+(\mathbf{a}, \boldsymbol{\sigma})$, and then left reversing vu^{-1} .*

Proof. Point (i) follows from the completeness of the presentation $(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{a\sigma})$ with respect to left and right reversing. Point (ii) is a consequence: starting with w in $W(\mathbf{a}, \boldsymbol{\sigma})$, we know by Proposition 6.6(i) that there exist two words u, v in $W^+(\mathbf{a}, \boldsymbol{\sigma})$ such that w is left $\mathbf{r}_{a\sigma}$ -reversible to $u^{-1}v$. Then w represents 1 in FB_∞ if and only if u and v represent the same element of FB_∞ , hence the same element of FB_∞^+ , as FB_∞^+ embeds in FB_∞ . Now, by (i), the latter is true if and only if the left reversing of vu^{-1} ends up with the empty word. \square

Corollary 6.9. (i) *The Dehn function of FB_∞ relative to the (infinite) presentation $(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{a\sigma})$ is quadratic, i.e., every length ℓ word in $W^+(\mathbf{a}, \boldsymbol{\sigma})$ representing 1 in FB_∞ can be connected to the empty word via $O(\ell^2)$ relations of $\mathbf{r}_{a\sigma}$.*

(ii) *The subgroup of FB_∞ generated by the elements a_i is a copy of Thompson's group F ; the subgroup of FB_∞ generated by the elements σ_i is a copy of Artin's group B_∞ .*

(iii) *The mapping ∂ is an injective endomorphism of FB_∞ into itself.*

Proof. For (ii), it is clear that the subgroup of FB_∞ generated by the family \mathbf{a} is a homomorphic image, i.e., a quotient, of F . The problem is to show that the kernel is trivial, i.e., that a word w of $W(\mathbf{a})$ that represents 1 in FB_∞ also represents 1 in F . Now this follows from Proposition 6.8: if w belongs to $W(\mathbf{a})$, its (double) left reversing is the same as a word of $W(\mathbf{a}, \boldsymbol{\sigma})$ and as a word of $W(\mathbf{a})$, because the only relations of $\mathbf{r}_{a\sigma}$ that can be used in this reversing are those of \mathbf{r}_a .

The argument is the same for the inclusion of B_∞ in FB_∞ , and for the injectivity of ∂ : in the latter case, the point is that all words involved in the reversing of a word in the image of ∂ , i.e., containing only letters a_i and σ_i with $i \geq 2$, themselves belong to the image of ∂ , as shows an inspection of the relations of $\mathbf{r}_{a\sigma}$. \square

6.2. An alternative presentation. We have seen that each of the groups F, V , or V' admits two natural presentations, namely one in terms of generators indexed by addresses, and one in terms of restricted subfamilies indexed by positive integers. In the case of the group FB_∞ , we started with the latter approach. We shall see now that the other approach is also possible.

Proposition 6.10. *The group FB_∞ admits the presentation $\langle \mathbf{A}, \boldsymbol{\Sigma}; \mathbf{r}_{A\Sigma} \rangle$.*

Proof. We first define elements A_α and Σ_α in FB_∞ as follows. For A_α , we use the formula of Lemma 2.17, *i.e.*, for $p \geq 0$, we put $A_{1^p} = a_{p+1}$, and, for $\alpha = 1^p 0^{e_0} 10^{e_1} 1 \dots 10^{e_q}$ with $p, q, e_0, \dots, e_q \geq 0$, we put

$$(6.3) \quad A_\alpha = (a_{p+1}^{e_0+1} a_{p+2}^{e_1+1} \dots a_{p+q+1}^{e_q+1}) (a_{p+q+1} a_{p+q+2}^{-1}) (a_{p+1}^{e_0+1} a_{p+2}^{e_1+1} \dots a_{p+q+1}^{e_q+1})^{-1}.$$

For the Σ_α , we define similarly $\Sigma_{1^p} = \sigma_{p+1}$ for $p \geq 1$, and, for $\alpha = 1^p 0^{e_0} 10^{e_1} 1 \dots 10^{e_q}$ with $p, q, e_0, \dots, e_q \geq 0$, we put

$$(6.4) \quad \Sigma_\alpha = (a_{p+1}^{e_0+1} a_{p+2}^{e_1+1} \dots a_{p+q+1}^{e_q+1}) (\sigma_{p+q+1}) (a_{p+1}^{e_0+1} a_{p+2}^{e_1+1} \dots a_{p+q+1}^{e_q+1})^{-1}.$$

It is easy to check that (6.4) follows from $\mathbf{r}_{A\Sigma}$, so this is the only possible choice.

It remains to prove that the elements A_α and Σ_α so defined satisfy all relations of $\mathbf{r}_{A\Sigma}$, which is *a priori* not obvious, as we established that the relations of $\mathbf{r}_{a\sigma}$ follow from $\mathbf{r}_{A\Sigma}$, but not the converse implication. As for the elements A_α , we know that the elements a_i generate a copy of the group F inside FB_∞ , so the elements A_α must satisfy all relations of \mathbf{r}_A . As for the other relations, the fact that $\mathbf{r}_{a\sigma}$ plus the torsion relations σ_i^2 make a presentation of V' implies that all relations of $\mathbf{r}_{A\Sigma}$ follow from $\mathbf{r}_{a\sigma}$ plus the torsion relations. The problem is to eliminate the torsion relations. This can be done as follows. In order to prove that some relation $u = v$ follows from $\mathbf{r}_{a\sigma}$, where u and v involve the letters A_α and Σ_α , it is sufficient to exhibit two injective trees t, t' such that we have $t' = t \bullet u = t \bullet v$, and to establish the equivalences $u_{t'} \equiv_{\mathbf{r}_{a\sigma}} u_t \cdot u$ and $u_{t'} \equiv_{\mathbf{r}_{a\sigma}} u_t \cdot v$. This was done in Section 5, with the proviso that the torsion relations σ_i^2 were available. Now, when we look at the argument of Section 5, we see that the torsion relations are used at one place only, namely to justify that two words of the form $\sigma_{I,J}$ giving rise to the same permutation are equivalent. Now, if we start with a term t in which the labels are 1 to n when enumerated from left to right, the word u_t^* contains no letter σ_i at all, and the only question is to control the relations needed for the words $u_{t'}^*$. The point is that if the words u and v correspond to braids in which any two strands cross at most once, *i.e.*, to what is called simple braids, then there exists a one-to-one correspondence between the permutations and the involved braids, and, in order to establish the needed permutation equivalences, only the braid relations, *i.e.*, $\mathbf{r}_{a\sigma}$, are needed. This is precisely the case for all relations of $\mathbf{r}_{A\Sigma}$, as a direct inspection shows. \square

6.3. A self-distributive operation on FB_∞ . One of the most intriguing properties of the group B_∞ is the existence of an exotic binary operation that obeys the self-distributivity law $x(yz) = (xy)(xz)$. This operation (here denoted as a bracket) is defined by the formula

$$(6.5) \quad x[y] = x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1},$$

thus a sort of twisted conjugacy. There are several ways of making this definition natural. The importance of this operation comes from the fact that the subsystem generated by any element of B_∞ , so in particular by 1, is a free LD-system, which is directly connected with the existence of a canonical ordering of B_∞ [12, 16].

In this section, we shall see that the self-distributivity properties of B_∞ extend to FB_∞ , in an even stronger form.

Definition. An *LD-system* is defined to be a set equipped with a binary operation $x, y \mapsto x[y]$ satisfying the left self-distributivity law

$$(6.6) \quad x[y[z]] = x[y][x[z]].$$

An *enhanced LD-system* is defined to be an LD-system equipped with a second binary operation \otimes satisfying the mixed laws

$$(6.7) \quad x[y[z]] = (x \otimes y)[z] \quad \text{and} \quad x[y \otimes z] = x[y] \otimes x[z].$$

It is easy to check that any group equipped with conjugacy and product is an enhanced LD-system. The self-distributive operation defined on B_∞ by (6.5) is rather different from a group conjugacy in that $x[x] = x$ is never true. However, there is no way to enhance the LD-system made by B_∞ equipped with its bracket.

The interesting fact is that not only can the self-distributive structure of B_∞ be extended to FB_∞ , but also it can be enhanced.

Definition. For x, y in FB_∞ , we set

$$(6.8) \quad x[y] = x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1},$$

$$(6.9) \quad x \otimes y = x \cdot \partial y \cdot a_1,$$

where we recall ∂ denotes the endomorphism of FB_∞ that maps σ_i to σ_{i+1} and a_i to a_{i+1} for each i .

Proposition 6.11. *When equipped with $[]$ and \otimes , the set FB_∞ is an enhanced LD-system. Furthermore, it is left-cancellative, i.e., $x[y] = x[z]$ implies $y = z$.*

Proof. A simple verification:

$$\begin{aligned} x[y][x[z]] &= (x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1})(x \cdot \partial z \cdot \sigma_1 \cdot \partial x^{-1}) \\ &= x \cdot \partial y \cdot \sigma_1 \cdot \partial x^{-1} \cdot \partial x \cdot \partial^2 z \cdot \sigma_2 \cdot \partial^2 x^{-1} \cdot \sigma_1 \cdot \partial^2 x \cdot \sigma_2^{-1} \cdot \partial^2 y^{-1} \cdot \partial x^{-1} \\ &=^{(*)} x \cdot \partial y \cdot \partial^2 z \cdot \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \cdot \partial^2 y^{-1} \cdot \partial x^{-1} \\ &= x \cdot \partial y \cdot \partial^2 z \cdot \sigma_2 \sigma_1 \cdot \partial^2 y^{-1} \cdot \partial x^{-1} = x[y \cdot \partial z \cdot \sigma_1 \cdot \partial y^{-1}] = x[y[z]]. \end{aligned}$$

The reason for $(*)$ is that $\partial^2 x$ commutes with σ_1 for every x ,

As for left cancellativity, $x[y] = x[z]$ implies $\partial y \cdot \sigma_1 = \partial z \cdot \sigma_1$, hence $\partial y = \partial z$, and, therefore, $y = z$ since, by Corollary 6.9(iii), the mapping ∂ is injective.

Similarly, we find:

$$\begin{aligned} x[y[z]] &= x \cdot \partial y \cdot \partial^2 z \cdot \sigma_2 \sigma_1 \cdot \partial^2 y^{-1} \cdot \partial x^{-1} = x \cdot \partial y \cdot a_1 a_1^{-1} \cdot \partial^2 z \cdot \sigma_2 \sigma_1 \cdot a_2 a_2^{-1} \cdot \partial^2 y^{-1} \cdot \partial x^{-1} \\ &= (x \otimes y) \cdot a_1^{-1} \cdot \partial^2 z \cdot \sigma_2 \sigma_1 a_2 \cdot \partial(x \otimes y)^{-1} = (x \otimes y) \cdot \partial^2 z \cdot a_1^{-1} \sigma_2 \sigma_1 a_2 \cdot \partial(x \otimes y)^{-1} \\ &= (x \otimes y) \cdot \partial z \cdot \sigma_1 \cdot \partial(x \otimes y)^{-1} = (x \otimes y)[z], \\ x[y \otimes z] &= x \cdot \partial y \cdot \partial^2 z \cdot a_2 \sigma_1 \cdot \partial x^{-1} = x[y] \cdot \partial x \cdot \sigma_1^{-1} \cdot \partial^2 z \cdot a_2 \sigma_1 \cdot \partial x^{-1} \\ &= x[y] \cdot \partial x \cdot \partial^2 z \cdot \sigma_1^{-1} a_2 \sigma_1 \cdot \partial x^{-1} = x[y] \cdot \partial(x[z]) \cdot \partial^2 x \cdot \sigma_2^{-1} \sigma_1^{-1} a_2 \sigma_1 \cdot \partial x^{-1} \\ &= x[y] \cdot \partial(x[z]) \cdot \partial^2 x \cdot a_1 \cdot \partial x^{-1} = x[y] \cdot \partial(x[z]) \cdot a_1 = x[y] \otimes x[z], \end{aligned}$$

which completes the proof. \square

The self-distributive structure on FB_∞ so constructed will be instrumental in the next sections.

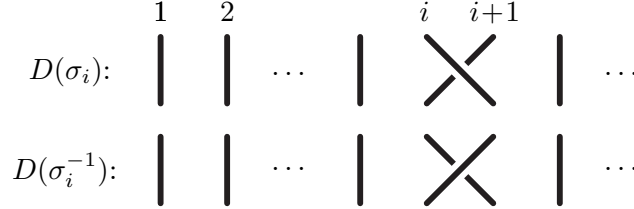
6.4. Further results. It is well-known that Thompson's group F is generated by the elements we call here a_1 and a_2 , all other a_i 's can be defined from them by conjugacy; moreover, F admits a finite presentation in terms of a_1 and a_2 , as it turns out to be sufficient to express in terms of a_1 and a_2 the shifted commutation relations that connect them to a_3 and a_4 [5]. As explained in [2], the same argument works for the group FB_∞ , which admits a finite presentation in terms of the generators a_1 , a_2 , σ_1 , and σ_2 .

7. MULTISCALED BRAIDS

It is standard to interpret the elements of the braid group B_∞ in terms of (three-dimensional) geometric braids and of (planar) braid diagrams. Here we shall define similar interpretations for the elements of FB_∞ . The idea is still to use braid diagrams, but, instead of starting

from positions that are indexed by integers, we start from positions that are indexed by finite sequence of integers and we interpret the additional generators a_i as rescaling operators.

7.1. Diagrams for B_∞ . By definition, every element of the braid group B_∞ is represented by words that are finite sequences of letters $\sigma_i^{\pm 1}$. It is customary to associate with the letters σ_i and σ_i^{-1} the following elementary diagrams



and, then, to associate with every braid word w the diagram $D(w)$ obtained by stacking one above the other the elementary diagrams $D(\sigma_i^{\pm 1})$ associated with the successive letters of w .

As shown by E. Artin in the 1920's, two braid words represent the same element of B_∞ , *i.e.*, they are equivalent under the congruence defined by the braid relations, if and only if the diagrams they encode are the 2D-projections of isotopic 3D-figures, *i.e.*, of figures that be continuously deformed one into the other without moving the ends of the strands.

It is important to observe that, although we start with an infinite sequence of strands, only finitely many strands are really braided in a braid diagram of the type above: for i large enough, all strands beyond position i are straight lines. Thus, if we define $D_n(\sigma_i)$ to be the diagram similar to $D(\sigma_i)$ starting from positions 1 to n only, and if $D_n(w)$ is defined accordingly, then, for every braid word w , the complete diagram $D(w)$ can be replaced without loss of generality by $D_n(w)$ with n large enough. More precisely, for $n' \geq n$, let $c_{n,n'}$ denote the completion mapping $c_{n,n'}$ adding $n' - n$ unbraided new strands on the right of an n strand diagram. Then, for each word w , there exists an integer n such that $D_n(w)$ is defined and $D(w)$ is the union—or, in an obvious sense, the direct limit—of the finite diagrams $c_{n,n'}(D_n(w))$ for $n' \geq n$ (Figure 17).

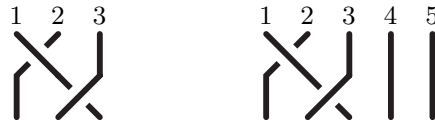


FIGURE 17. The braid diagram $D_3(\sigma_1\sigma_2^{-1})$ and its completion $D_5(\sigma_1\sigma_2^{-1})$ under $c_{3,5}$: one adds two unbraided strands numbered 4 and 5 at the right; then the infinite diagram $D(\sigma_1\sigma_2^{-1})$ is the union (or direct limit) of all finite diagrams $D_n(\sigma_1\sigma_2^{-1})$.

7.2. Diagrams for FB_∞ . In order to associate braid diagrams to the elements of FB_∞ , we consider strands that start from positions that are no longer indexed by elements of \mathbb{N} , *i.e.*, by positive integers, but by finite sequences of integers.

Definition. We denote by \mathbb{N}^* the set of all finite sequences of positive integers, and by $\mathbb{N}^{(*)}$ the quotient of \mathbb{N}^* obtained by identifying s with $(s, 1)$ —*i.e.*, s followed by 1—for every finite sequence s . The elements of $\mathbb{N}^{(*)}$ will be called *positions*.

Thus the positions (1) , $(1, 1)$, and $(1, 1, 1)$ coincide, and so do $(2, 3)$ and $(2, 3, 1)$. In the sequel, we consider braid diagrams in which the strands begin and end at positions belonging to $\mathbb{N}^{(*)}$. We equip \mathbb{N}^* with the lexicographical ordering, *i.e.*, $(i_1, \dots, i_p) < (j_1, \dots, j_q)$ holds if either (i_1, \dots, i_p) is a beginning of (j_1, \dots, j_q) or, for some r , we have $i_t = j_t$ for $t < r$ and $i_r < j_r$. This ordering induces a well-defined ordering on $\mathbb{N}^{(*)}$.

There are two adequate ways of visualizing the set $\mathbb{N}^{(*)}$. The first one is to embed it into the real line. To this end, we choose a base $b > 2$ and we map the sequence (i_1, \dots, i_p) to the

rational whose base b expansion is $0.1^{i_1-1}01^{i_2-1} \dots 01^{i_r-1}$, *i.e.*, the sum of all b^{-p} for which there is a 1 at position p . For instance, for $b = 3$, the sequence (1) is mapped to 0, (1, 2) is mapped to $0, 01$, *i.e.*, to $1/9$, while (2) is mapped to 0.1 , *i.e.*, $1/3$, and (3, 1, 2) is mapped to 0.11001 , *i.e.*, to $109/243$. Requiring $b \geq 2$ guarantees the compatibility of the standard ordering of the reals with the lexicographical ordering on $\mathbb{N}^{(*)}$, and requiring in addition $b > 2$ guarantees that the image of $\mathbb{N}^{(*)}$ is discrete, which fits better to our current approach (Figure 18, left).

Another way of thinking of $\mathbb{N}^{(*)}$ is to consider distances in the semiring $\mathbb{N}[\epsilon]$ where ϵ obeys the rule $\epsilon + 1 = 1$, *i.e.*, is infinitely small compared with 1 on the right. Then we map (i_1, \dots, i_p) to $i_1 + (i_2 - 1)\epsilon + \dots + (i_r - 1)\epsilon^{r-1}$. Here (1) is mapped to 1, (1, 2) is mapped to $1 + \epsilon$, while (2) is mapped to 2 and (3, 1, 2) is mapped to $3 + \epsilon^2$ (Figure 18, right).

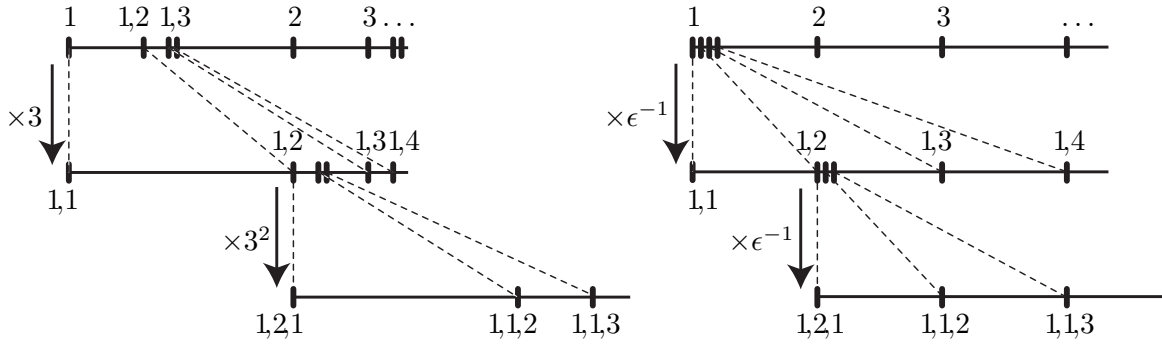


FIGURE 18. Realization of the set of positions $\mathbb{N}^{(*)}$ using a base, here $b = 3$, and using infinitesimals.

Once the set of positions has been fixed, we can define the diagrams. As in the case of B_∞ , we shall first describe infinite diagrams: although the latter only are direct limits of finite diagrams and the latter are the main objects, it is simpler to begin with the infinite version.

In the case of B_∞ , defining $D(\sigma_i)$ means describing what happens to the strand starting from each position. Here we do the same. Keeping the intuition of infinitesimal distances, we say that the positions of the form (i, s) lie *near* i . Then, we define $D(\sigma_i)$ to be the diagram where all strands starting near i cross over all strands starting near $i + 1$, and all relative distances are preserved. So this corresponds to the picture:



Formally, this means that, for all finite sequences s, s' , the strand starting at (i, s) goes to $(i + 1, s)$, and the strand starting at $(i + 1, s')$ goes to (i, s') , the former going in front of the latter; for $j \neq i, i + 1$, the strand starting at (j, s) goes to (j, s) .

The action of a_i is defined to be a pinching—or a rescaling—near i : the strands at the left of i remain vertical, the strands starting near i are pinched by a factor ϵ , and the next strands are translated so as to keep their mutual distances and avoid gaps:



Formally, this means that, for $j < i$, the strand starting at (j, s) goes to (j, s) , the strand starting at (i, s) goes to $(i, 1, s)$, the strand starting at $(i + 1, j, s)$ goes to $(i, j + 1, s)$, and, for $j \geq i + 2$, the strand starting at (j, s) goes to $(j - 1, s)$.

7.3. Finite approximations. Although intuitively natural, the previous infinite diagrams are misleading, as they involve infinitely many crossings, making in particular the definition of isotopy problematic. Actually, as in the case of B_∞ , the important objects are the finite approximations of the infinite diagrams. The only difference with B_∞ is that, instead of having to consider a 1-dimensional sequence of finite approximations $D_n(w)$ for an infinite diagram, we have now to consider a 2-dimensional sequence of finite approximations indexed by (finite) binary trees.

Up to now, we used a binary system of addresses for the nodes in binary trees. It will be convenient here to switch to an alternative system where addresses are finite sequences of integers. The principle is that the right direction is given priority over the left direction: an address in the new system indicates how many intervals are crossed on the right branch before the next left forking.

Definition. For s a finite sequence of positive integers, say $s = (i_1, \dots, i_p)$, we define \bar{s} to be the binary address $1^{i_1-1}01^{i_2-1} \dots 01^{i_p-1}$.

So, for instance, we have $\overline{(1,2)} = 010$ and $\overline{(3,1,1)} = 1100$. Clearly, the correspondence so defined is a bijection (see Figure 19), and, for α a binary address, we denote by $\underline{\alpha}$ the preimage of α in this correspondence.

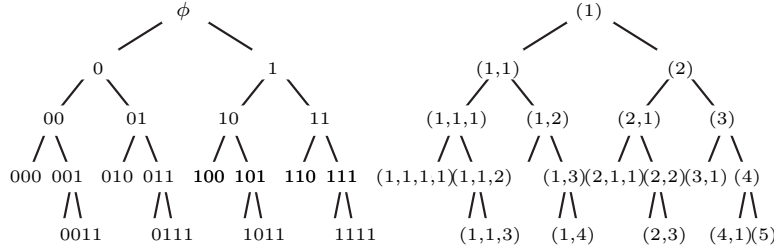


FIGURE 19. Correspondence between the two systems of addresses for the nodes in a binary tree

Definition. For t a finite binary tree, we denote by P_t the set of all sequences $\underline{\alpha}$ for α in the outer skeleton of t , i.e., for α address of a leaf in t .

For instance, for $t = (\circ \wedge \circ) \wedge \circ$, the outer skeleton of t consists of the three addresses 00, 01, and 1, hence P_t consists of $(1, 1, 1)$, which is identified with (1) in \mathbb{N}^* , of $(1, 2)$, and of (2) .

In the case of B_∞ , for w in $W(\sigma)$, we have defined the diagram $D_n(w)$ provided n is bigger than all indices i such that $\sigma_i^{\pm 1}$ occurs in w . For w in $W(\sigma, \mathbf{a})$, we shall define a similar finite diagram $D_t(w)$ provided t is large enough. The difference is that, in the case of B_∞ , the action of σ_i does not change the set of positions we are considering, while, in the case of FB_∞ , the action of σ_i and a_i may change that set. That is why the definition of the diagram $D_t(w)$ has to be slightly more complicated.

Definition. Assume that t is a size n tree such that $t \bullet \sigma_i$ is defined in the sense of Section 4. Then $D_t(\sigma_i)$ is defined to be the finite diagram with n strands starting from the positions P_t , and such that each strand starting at (i, s) goes to $(i + 1, s)$, each strand starting at $(i + 1, s)$ goes to (i, s) , all the latter under all the former, and each strand starting at (j, s) goes to (j, s) for $j \neq i, i + 1$.

The diagram $D_t(a_i)$ is defined similarly when $t \bullet a_i$ exists; the rules are now that each strand starting at (i, s) goes to $(i, 1, s)$, each strand starting at $(i + 1, j, s)$ goes to $(i, j + 1, s)$, each strand starting at (j, s) goes to (j, s) for $j < i$, and, finally, each strand starting at (j, s) goes to $(j - 1, s)$ for $s > i + 1$.

The diagrams $D_t(\sigma_i)$ and $D_t(a_i^{-1})$ are defined to be the mirror images of $D_{t \bullet \sigma_i}(\sigma_i)$ and $D_{t \bullet a_i}(a_i)$, respectively. Finally, for w in $W(\mathbf{a}, \boldsymbol{\sigma})$, and t a binary tree such that $t \bullet w$ is defined, the diagram $D_t(w)$ is inductively defined by the rule that, if w is xw' where x is one of $\sigma_i^{\pm 1}$, $a_i^{\pm 1}$, then $D_t(w)$ is obtained by stacking $D_t(x)$ over $D_{t \bullet x}(w')$.

Example 7.1. Let $w = a_1^{-1}\sigma_1 a_1$. The smallest tree t for which $t \bullet w$ is defined is $(\circ^\wedge \circ)^\wedge \circ$, and the diagram $D_t(w)$ is displayed in Figure 20, left. Now $t' \bullet w$ exists for every tree t' whose skeleton includes that of t , for instance $t' = ((\circ^\wedge \circ)^\wedge \circ)^\wedge (\circ^\wedge \circ)$. The corresponding braid diagram is displayed in Figure 20, right.

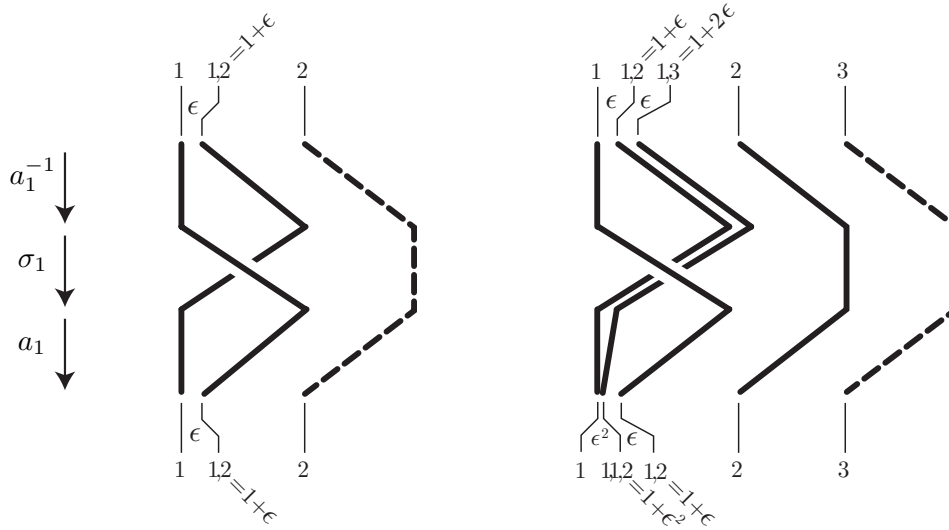


FIGURE 20. The braid diagram $D_t(a_1^{-1}\sigma_1 a_1)$ and its completion $D_{t'}(a_1^{-1}\sigma_1 a_1)$, for $t = (\circ^\wedge \circ)^\wedge \circ$ and $t' = ((\circ^\wedge \circ)^\wedge \circ)^\wedge (\circ^\wedge \circ)$. Each additional strand in $D_{t'}$ is obtained by following the strand that starts immediately on its left and adapting the distance to that strand by looking at the strand that lies immediately on its right, if there is one, or preserving it otherwise (the rightmost strand are dashed because, in this construction, they are always superfluous)

Although the construction may seem more involved, it is exactly similar to that of ordinary braid diagrams, and, in particular, the infinite braid diagram $D(w)$ is nothing but the union of the finite diagrams $D_t(w)$ for t large enough.

7.4. The group of multiscaled braids. As in the case of ordinary braid diagrams, we have a natural notion of isotopy for multiscaled braid diagrams. In order to avoid any unessential problem, it is convenient to use finite diagrams and to think of $\mathbb{N}^{(*)}$ as embedded in the reals, *i.e.*, to appeal to a finite base rather than to infinitesimal distances, which are kept as a heuristic device. As our diagrams are piecewise linear, isotopy is generated by Δ -moves (see for instance [?] or [4]), or, equivalently, by the Reidemeister moves of type II and III displayed in Figure 21.



FIGURE 21. Reidemeister moves of type II (left) and III (right); the only requirement is that the endpoints remain fixed

Our aim is to show that, as in the case of ordinary braids diagrams, the isotopy classes of multiscaled braid diagrams form a group, and that this group is (isomorphic) to FB_∞ . A minor problem is that, contrary to the case of ordinary braids, we cannot fix a finite integer and construct a group using n strand diagrams: for t a fixed binary tree, the diagrams starting from t , *i.e.*, from the positions of P_t , do not lead to a group, as the positions change. So, we must define the complete group of all multiscaled braids directly, and, to this end, we must be able to multiply (finite) diagrams with different numbers of strands. This is easy. In the case of ordinary braid diagrams, we can define the product of two diagrams $D_{n_1}(w_1)$, $D_{n_2}(w_2)$ with possibly different numbers of strands by taking n to be the sup of n_1 and n_2 and completing both diagrams into n strand diagrams by adding trivial new strands. Here we shall do the same thing.

Definition. For t a binary tree, we define a t strand braid diagram to be a (finite) multiscaled braid diagram in which the initial positions of the strands are P_t . If t' is another tree whose skeleton includes that of t , we denote by $c_{t,t'}$ the completion that transforms any t strand diagram into a t' strand diagram by adding trivial new strands.

With this notation, what we define to be the product of two finite diagrams $D_{t_1}(w_1)$ and $D_{t_2}(w_2)$ is the concatenation of the diagrams $c_{t_1,t}(D_{t_1}(w_1))$ and $c_{t_2,t \bullet w_1}(D_{t_2}(w_2))$, where t is the smallest tree such that t includes t_1 and $t \bullet w_1$ includes t_2 —we say that a tree includes another one when the skeleton of the former includes the skeleton of the latter. The product so defined induces the usual concatenation on the associated infinite diagrams.

Lemma 7.2. *Completion preserves isotopy, i.e., if D, D' are isotopic t strand braid diagrams, then, for each t' including t , the t' strand diagrams $c_{t,t'}(D)$ and $c_{t,t'}(D')$ are isotopic.*

Proof. Isotopy does not take the mutual distances of the strands into account. The completion procedure $c_{t,t'}$ consists in iterately doubling a strand, and it is easy to check that doubling any strand in one of the two basic Reidemeister moves gives an isotopy. \square

Therefore there is no ambiguity in saying that two infinite multiscaled braid diagrams are *isotopic* if they are the completions of isotopic finite diagrams—and, in this way, we avoid all possible difficulties arising from considering infinitely many crossings. Then multiscaled braid diagrams up to isotopy make a group, and we set:

Definition. The group of multiscaled braid diagrams up to isotopy is called the *multiscaled braid group*.

Our aim for the rest of this section is to prove

Proposition 7.3. *The multiscaled braid group is (isomorphic to) FB_∞ .*

In other words, we aim at showing that the relations $r_{A\Sigma}$ make a presentation of the multiscaled braid group in terms of the generators σ_i and a_i . As in the case of ordinary braids, one direction is trivial.

Lemma 7.4. *All relations of $r_{A\Sigma}$ induce isotopies.*

Proof. The graphical verification is given in Figure 22. \square

7.5. Diagram colouring. Owing to Lemma 7.4, it remains to establish that the morphism of FB_∞ to the multiscaled braid group is injective. Our method will consist in showing that, for any word w in $W(\mathbf{a}, \boldsymbol{\sigma})$, the class of w in FB_∞ can be recovered from the isotopy class of any diagram $D_t(w)$. To do that, we use diagram colourings.

The principle, which can be traced back at least to Alexander, is to fix a nonempty set S (the colours), to attribute colours from S to the input strands of a braid diagram D , and to push the colours along the strands. If the colours never change, the sequence of output colours is just

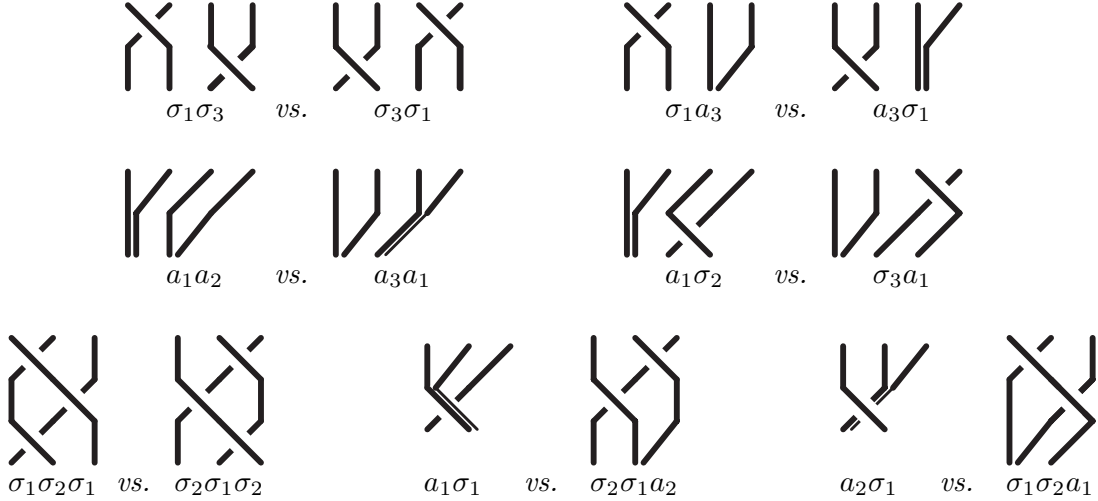
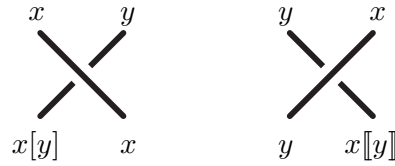


FIGURE 22. The relations of $r_{A\Sigma}$ and the corresponding multiscaled braid diagrams isotopy.

a permutation of the sequence of input colours, and we do not gain much information about the diagram. Now, assume that the set of colours S is equipped with two binary operations, here denoted $x, y \mapsto x[y]$ and $x, y \mapsto x[[y]]$ to suggest that products are images of y under x . Then we require that, when a x -coloured strand makes a positive (*resp.* negative) crossing over a y -coloured strand, then the colour of the latter becomes $x[y]$ (*resp.* $x[[y]]$), as below:



In this way, for each sequence of input colours \vec{x} and each braid diagram D , one obtains a well-defined sequence of output colours denoted $\vec{x} \bullet D$. If D is the n strand diagram encoded by a word w in $W(\sigma)$, we shall write $\vec{x} \bullet w$ for $\vec{x} \bullet D_n(w)$, and, similarly, we write $\vec{x} \bullet w$ for $\vec{x} \bullet D_t(w)$ when w is a word in $W(\mathbf{a}, \sigma)$ and t is a tree such that $t \bullet w$ is defined.

The following observation was made in several different frameworks [2, ?, ?, 12], and it is follows from an easy graphical verification:

Lemma 7.5. [16] *Assume that $(S, [], [[]])$ is a rack, i.e., $(S, [])$ is an LD-system and $[[]]$ is a left inverse for $[]$, i.e., for all x, y , the element $x[[y]]$ is the unique element z satisfying $x[z] = y$. Then S -colourings are invariant under Reidemeister moves II and III, i.e., for every diagram D and every sequence of input colours \vec{x} , the output colours $\vec{x} \bullet D$ only depend on the isotopy class of D .*

In the sequel, we consider a slightly more general situation, namely the case when $(S, [])$ is still an LD-system but we only assume that $(S, [])$ admits left cancellation, i.e., $x[y] = x[z]$ implies $y = z$. Then we are not sure that all pairs of colours are eligible for negative crossings: we can still define $x[[y]]$ to be the unique element z satisfying $x[z] = y$ when it exists, but, now, $x[[y]]$ need not be everywhere defined. The following lemma gathers the results we need here:

Lemma 7.6. *Assume that $(S, [])$ is a left cancellative LD-system.*

(i) *For every finite family of multiscaled braid diagrams D_1, \dots, D_p , there exists at least one sequence \vec{x} of colours from S such that $\vec{x} \bullet D_k$ is defined for each k .*

(ii) *If D, D' are isotopic diagrams, and \vec{x} is a sequence of colours from S such that both $\vec{x} \bullet D$ and $\vec{x} \bullet D'$ are defined, then the latter are equal.*

Proof. The result is known in the case of ordinary braid diagrams [16]. Now finite multiscaled braid diagrams are ordinary braid diagrams: the only difference lies in the encoding in terms of σ_i 's and a_i 's, but, geometrically, there is no difference. This is formalized in Lemma 8.18, which states that every multiscaled braid diagram can be deformed into an ordinary braid diagram, preceded and followed by rescaling operations that do not interfere with colourings. \square

It follows that the LD-system consisting of FB_∞ equipped with the bracket of Section 6.3 is eligible for colouring multiscaled braid diagrams. An example is displayed in Figure ?? below.

In the case of an ordinary n strand braid diagram D , the input positions and the output positions are $1, 2, \dots, n$ for some n , and an S -colouring consists of a sequence of elements of S indexed by $1, \dots, n$, *i.e.*, equivalently, of pairs of the form (i, x) with i an integer and x an element of S . In the case of multiscaled diagrams, an S -colouring consists of a sequence of elements of S indexed by positions in $\mathbb{N}^{(*)}$, *i.e.*, of a family of pairs (s, x) where s is a finite sequence of integers and x is an element of S . Such pairs will simply be called *S -coloured positions*. For instance, in Figure 23, the initial coloured positions are $((1), x)$, $((1, 2), y)$, and $((2), z)$, while the final one are $((1), x[y])$, $((1, 2), x)$, and $((2), z)$. If \vec{x} is a sequence of S -coloured positions and w is a word in $W(\mathbf{a}, \boldsymbol{\sigma})$, we still denote by $\vec{x} \bullet w$ the sequence of S -coloured positions obtained by propagating the colours through the diagram of w starting from the positions of \vec{x} , when it exists. The following result is a direct consequence of the rule for changes of colours in crossings:

Lemma 7.7. *Assume that \vec{x} is a sequence of S -coloured positions and $\vec{x} \bullet \sigma_i$ is defined. Let x_1, \dots, x_q be the colours attributed to positions of the form (i, s) enumerated in increasing order. Then $\vec{x} \bullet \sigma_i$ is obtained from \vec{x} by replacing $((i, s), x)$ with $((i+1, s), x)$ and replacing $((i+1, s), x)$ with $((i, s), x_1[\dots[x_q[x]]\dots])$.*

We shall now switch from the language of sequences of S -coloured positions to that of S -decorated trees. We introduced above a one-to-one mapping P^{-1} from positions, *i.e.*, finite sequences of integers, to binary addresses. Using this correspondence, we can map every coloured position (s, x) to a pair (\bar{s}, x) consisting of a binary address and a colour. As, in $\mathbb{N}^{(*)}$, we identify the positions s and $(s, 1)$, the number of final 0's in \bar{s} remains open. The ambiguity disappears when the addresses are adjusted so as to make the outer skeleton of a tree.

Definition. (Figure 23) Assume that \vec{x} is a sequence of S -coloured positions. Then we define $\text{tr}(\vec{x})$ to be the S -decorated tree t such that \vec{x} consists of all pairs $(P(\alpha), x)$ for α an outer address in t and x the corresponding label. We define $\text{tr}^*(\vec{x})$ to be the finite sequence (t_1, \dots, t_n) such that $\text{tr}(\vec{x})$ is $t_1 \wedge \dots \wedge t_n \wedge \circ_x$, *i.e.*, t_i is the $i^{i-1}0$ th subtree of $\text{tr}(\vec{x})$ —we recall that $t_1 \wedge t_2 \wedge t_3$ stands for $t_1 \wedge (t_2 \wedge t_3)$.

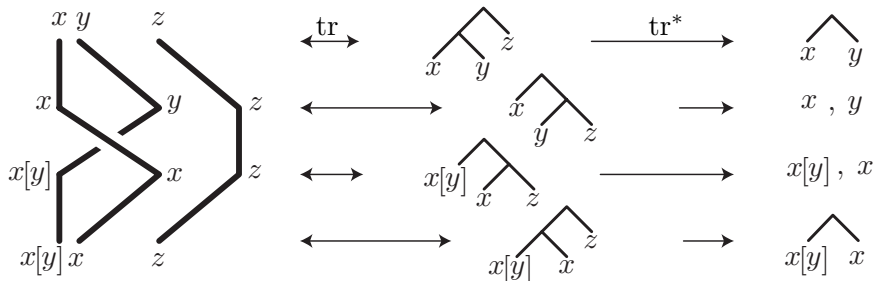


FIGURE 23. Correspondence between sequences of S -coloured positions and S -decorated trees: for instance, the initial sequence of coloured positions consists of $((1), x)$, $((1, 2), y)$, and $((2), z)$, corresponding to the coloured addresses $(00, x)$, $(01, y)$, and $(1, z)$, hence to the tree $(\circ_x \wedge \circ_y) \wedge \circ_z$

The generators a_i and σ_i act on sequences of S -coloured positions, and it is easy to describe the corresponding action on S -decorated trees.

Lemma 7.8. *Assume that \vec{x} is a sequence of S -coloured positions and we have $\text{tr}^*(\vec{x}) = (t_1, \dots, t_n)$. Then, whenever $\vec{x} \bullet a_i$ (resp. $\vec{x} \bullet \sigma_i$) is defined, we have*

$$(7.1) \quad \text{tr}^*(\vec{x} \bullet a_i) = (t_1, \dots, t_{i-1}, t_i \wedge t_{i+1}, t_{i+2}, \dots, t_n),$$

$$(7.2) \quad \text{tr}^*(\vec{x} \bullet \sigma_i) = (t_1, \dots, t_{i-1}, t', t_i, t_{i+2}, \dots, t_n),$$

where t' denotes the tree obtained from t_{i+1} by replacing every label x with the corresponding label $x_1[\dots[x_p[x]]\dots]$, where x_1, \dots, x_p are the labels of the leaves in t_i enumerated from left to right.

Proof. For (7.1), we compare the action of a_i on positions and on trees, and the result directly follows as the colours and labels are not changed. For (7.2), the principle is the same: the action on positions and on trees are similar. As for colours and labels, the changes are described in Lemma 7.7, and the current formula follows. \square

Figure 23 gives an easy example for the previous situation.

The key tool is to introduce a convenient evaluation function that associated with every sequence of FB_∞ -coloured positions a specific element of FB_∞ . This function is constructing using FB_∞ -decorated trees and the operation \otimes .

Definition. (Figure 24) Assume that \vec{x} is a sequence of FB_∞ -coloured positions. Let (t_1, \dots, t_n) be $\text{tr}^*(\vec{x})$, and let x_i be the \otimes -evaluation of t_i , *i.e.*, the image of t_i under the mapping e defined by

$$(7.3) \quad e(t) = x \quad \text{for } t = \circ_x, \quad e(t) = e(t_1) \otimes e(t_2) \quad \text{for } t = t_1 \wedge t_2.$$

Then $E^*(\vec{x})$ is defined to be (x_1, \dots, x_n) , and $E(\vec{x})$ is defined to be $x_1 \cdot \partial x_2 \cdot \dots \cdot \partial^{n-1} x_n$.

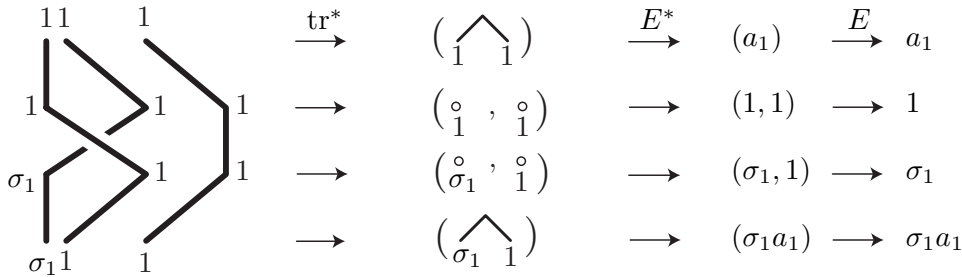


FIGURE 24. Evaluation of sequences FB_∞ -coloured positions *via* a sequence of FB_∞ -decorated trees and their \otimes -evaluation

It is now easy to describe the action of the generators a_i and σ_i on E^* and E .

Lemma 7.9. *Assume that \vec{x} is a sequence of FB_∞ -coloured positions. Then we have*

$$(7.4) \quad E(\vec{x} \bullet a_i) = E(\vec{x}) \cdot a_i, \quad E(\vec{x} \bullet \sigma_i) = E(\vec{x}) \cdot \sigma_i,$$

whenever the sequence $\vec{x} \bullet a_i$ or $\vec{x} \bullet \sigma_i$ is defined.

Proof. Let us assume $\text{tr}(\vec{x}) = t_1 \wedge \dots \wedge t_n \wedge \circ_x$ and $E^*(\vec{x}) = (x_1, \dots, x_n)$. First, we find

$$(7.5) \quad E^*(\vec{x} \bullet a_i) = (x_1, \dots, x_{i-1}, x_i \otimes x_{i+1}, x_{i+2}, \dots, x_n),$$

$$(7.6) \quad E^*(\vec{x} \bullet \sigma_i) = (x_1, \dots, x_{i-1}, x_i[x_{i+1}], x_i, x_{i+2}, \dots, x_n).$$

Indeed, (7.5) directly follows from (7.1). As for (7.6), it follows from (7.2) by using the relations (6.7) of enhanced LD-systems: an induction on the size of t_i using the left formula in (6.7)

first shows that each label x in t_{i+1} gives a label $x_i[x]$ in t' , and, then, an induction on the size of t_{i+1} shows that the \otimes -evaluation of t' is $x_i[x_{i+1}]$ as x_{i+1} is the \otimes -evaluation of t_{i+1} .

We can now establish (7.4). For readability, we assume $i = 1$, but the argument is the same for other values of i . For a_i , we find

$$\begin{aligned} E(\vec{x} \bullet a_1) &= (x_1 \otimes x_2) \cdot \partial x_3 \cdot \dots \cdot \partial^{n-2} x_n \\ &= x_1 \cdot \partial x_2 \cdot a_1 \cdot \partial x_3 \cdot \dots \cdot \partial^{n-2} x_n \\ &= x_1 \cdot \partial x_2 \cdot \partial^2 x_3 \cdot \dots \cdot \partial^{n-2} x_n \cdot a_1 = E(\vec{x}) \cdot a_1, \end{aligned}$$

as $a_1 \cdot \partial^k x = \partial^{k+1} x \cdot a_1$ holds for $k \geq 1$. For σ_i , we find similarly

$$\begin{aligned} E(\vec{x} \bullet \sigma_1) &= x_1[x_2] \cdot \partial x_1 \cdot \partial^2 x_3 \cdot \dots \cdot \partial^{n-1} x_n \\ &= x_1 \cdot \partial x_2 \cdot \sigma_1 \cdot \partial x_1^{-1} \cdot \partial x_1 \cdot \partial^2 x_3 \cdot \dots \cdot \partial^{n-1} x_n \\ &= x_1 \cdot \partial x_2 \cdot \sigma_1 \cdot \partial^2 x_3 \cdot \dots \cdot \partial^{n-1} x_n \\ &= x_1 \cdot \partial x_2 \cdot \partial^2 x_3 \cdot \dots \cdot \partial^{n-2} x_n \cdot \sigma_1 = E(\vec{x}) \cdot \sigma_1, \end{aligned}$$

as $\sigma_1 \cdot \partial^k x = \partial^k x \cdot \sigma_1$ holds for $k \geq 2$. □

We are now able to conclude:

Proof of Proposition 7.3. We wish to prove that, if w and w' are words in $W(\mathbf{a}, \boldsymbol{\sigma})$ and there is a tree t such that the diagrams $D_t(w)$ and $D_t(w')$ are isotopic, then the words w and w' are equivalent *modulo* the relations $\mathbf{r}_{A\Sigma}$, *i.e.*, they represent the same element of the group FB_∞ . Now, by Lemma 7.6, we know that there exists at least one way to attribute colours from FB_∞ to the initial positions of the diagrams in such a way that the colours can be propagated through w and w' , and the hypothesis that $D_t(w)$ and $D_t(w')$ are isotopic implies that the final sequences of colours are the same. In other words, we have a sequence of FB_∞ -coloured positions \vec{x} such that both $\vec{x} \bullet w$ and $\vec{x} \bullet w'$ exist and they are equal to some sequence \vec{y} . Now an obvious induction from (7.4) gives the equality

$$E(\vec{x}) \cdot \bar{w} = E(\vec{y}) = E(\vec{x}) \cdot \bar{w}'$$

in FB_∞ —we recall that \bar{w} denotes the image of w in FB_∞ —and we deduce that both \bar{w} and \bar{w}' are equal to $E(\vec{x})^{-1} E(\vec{y})$. □

Thus multiscaled braids give an accurate diagrammatic realization of the group FB_∞ .

8. THE GROUP FB_∞ AS THE GEOMETRY GROUP OF TWISTED SEMI-COMMUTATIVITY

In the first part of this work, we introduced the general notion of a geometry group for a family of algebraic laws, and we interpreted Thompson's group F and V , as well as the subgroup V' of V , as the geometry groups of associativity, associativity plus commutativity, and associativity plus semi-commutativity, respectively. The group FB_∞ has been introduced as an extension of the group V' , and it is natural to wonder whether it can also be realized as the geometry group of some algebraic laws. In this section, we shall give a positive answer to that question, at the expense of generalizing the notion of a geometry group to transformations that are slightly more general than those corresponding to algebraic laws. Indeed, FB_∞ turns out to be, in some convenient sense, the geometry group of associativity together with a twisted form of semi-commutativity.

8.1. Twisted (semi)-commutation. As applying (semi)-commutativity is an involutive operation, while FB_∞ is a torsion-free group, we cannot expect that FB_∞ be interpreted so as to include (semi)-commutativity operators. So we are led to consider variants of (semi)-commutativity. A natural way for making commutativity operators non-involutive is to assume that subtrees are changed when they are switched. The simplest case is when only one of the subtrees is changed, and it is then natural to consider the case when the new subtree depends on the two subtrees that have been exchanged, *i.e.*, to assume that there exists a binary operation on the considered family of trees.

Definition. (Figure 25) Assume that T is a set of trees equipped with a binary bracket operation $-[-]$. Then we define the T -twisted commutation operator to be the partial operator C^T on T given by

$$(8.1) \quad C^T : t_1 \wedge t_2 \longmapsto t_1[t_2] \wedge t_1.$$

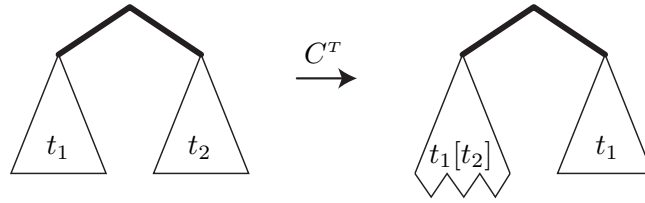


FIGURE 25. The twisted commutation operator C^T

Observe the similarity between this approach and the definition of braid diagram colourings in Section 7.

So the idea remains to switch the left and the right subtrees but, in the transformation, the right subtree is (possibly) changed when it crosses the left subtree. The bracket notation is chosen to emphasize that $t_1[t_2]$ is the image of t_2 under the action of t_1 . Note that the standard commutation operator C simply corresponds to using the trivial operation $t_1[t_2] = t_2$.

As in the case of the operators C_α , we define C_α^T to be the translated operator $\partial_\alpha C^T$, *i.e.*, C^T acting on the α th subtree. As for inverses, the operators C_α^T need not be injective in general, but we have the following obvious criterion:

Lemma 8.1. *Assume that T is a set of trees equipped with a bracket operation. Then the associated operators C_α^T are injective if and only if the bracket on T is left cancellative, *i.e.*,*

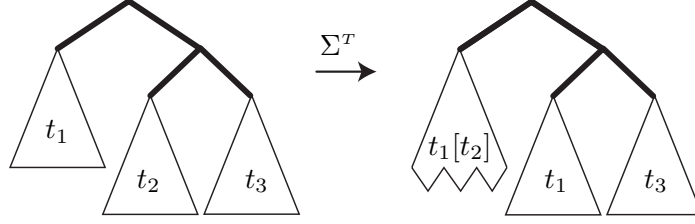
$$(8.2) \quad t[t_1] = t[t_2] \quad \text{implies} \quad t_1 = t_2.$$

Under such an hypothesis, the inverse operator of C_α^T is still a partial operator on T . In Section 6, we chose to investigate the torsion-free version FB_∞ of the group V' rather than that of the group V , we are led to considering a twisted version of semi-commutation. In order to preserve the relation between C_α and Σ_α established in Section 4, we define the twisted version Σ^T of Σ by $\Sigma^T = C^T A^{-1} (C_1^T)^{-1}$, which corresponds to setting

Definition. (Figure 26) Assume that T is a set of decorated trees equipped with a bracket operation. Then we define the T -twisted semi-commutation operator Σ^T by

$$(8.3) \quad \Sigma^T : t_1 \wedge (t_2 \wedge t_3) \longmapsto t_1[t_2] \wedge (t_1 \wedge t_3).$$

We naturally define Σ_α^T to be the α -translated copy of Σ^T . Under the hypothesis that the bracket on T is left cancellative, the operator Σ_α^T is injective, and its inverse $(\Sigma_\alpha^T)^{-1}$ is a partial operator. The (semi)-commutation operators correspond to no algebraic identity, but we still have a family of partial injections of a set of decorated trees into itself, and, on the shape of what we did in the Part 1, it is natural to consider the monoids they generate:


 FIGURE 26. The twisted semi-commutation operator Σ^T

Definition. Assume that T is a family of trees equipped with a left cancellative bracket operation. Then we define $\mathcal{G}(\mathcal{A}, \mathcal{C}^T)$ (resp. $\mathcal{G}(\mathcal{A}, \Sigma^T)$) to be the monoid generated by the operators $A_\alpha^{\pm 1}$ and $C_\alpha^{T \pm 1}$ (resp. the operators $A_\alpha^{\pm 1}$ and $\Sigma_\alpha^{T \pm 1}$) acting on T .

Our aim is now to investigate the monoids $\mathcal{G}(\mathcal{A}, \mathcal{C}^T)$ and, mainly, $\mathcal{G}(\mathcal{A}, \Sigma^T)$ for appropriate choices of the bracket operation. Note that, if T is equipped with the trivial bracket $t_1[t_2] = t_2$, we just have

$$\mathcal{G}(\mathcal{A}, \mathcal{C}^T) = \mathcal{G}(\mathcal{A}, \mathcal{C}) \quad \text{and} \quad \mathcal{G}(\mathcal{A}, \Sigma^T) = \mathcal{G}(\mathcal{A}, \Sigma),$$

i.e., we come back to the framework of Sections 4 and 5.

8.2. LD-systems. In general, the twisted operators C_α^T and Σ_α^T need not satisfy the same relations as their standard versions. However it is easy to list the requirements for the bracket operation on T ensuring that the relations of $\mathbf{r}_{AC\Sigma}$ are valid in the monoid $\mathcal{G}(\mathcal{A}, \Sigma^T)$.

Lemma 8.2. (i) *The relations $A_1 \Sigma^T = \Sigma^T \Sigma_1^T A_1$, $A \Sigma^T = \Sigma_1^T \Sigma^T A_1$, and $\Sigma^T \Sigma_1^T \Sigma^T = \Sigma_1^T \Sigma^T \Sigma_1^T$ hold in the monoid $\mathcal{G}(\mathcal{A}, \Sigma^T)$ if and only if, for all trees t_1, t_2, t_3 in T , we have*

$$(8.4) \quad t_1[t_2 \wedge t_3] = t_1[t_2] \wedge t_1[t_3],$$

$$(8.5) \quad (t_1 \wedge t_2)[t_3] = t_1[t_2[t_3]],$$

$$(8.6) \quad t_1[t_2[t_3]] = t_1[t_2][t_1[t_3]].$$

(ii) *Assume $T = T_S$, i.e., T is the set of all S -decorated trees for some set S . Then the conditions of (i) are satisfied if and only if there exists a left cancellative left self-distributive bracket operation on S such that, for all trees t_1, t_2 in T_S , the tree $t_1[t_2]$ is obtained by replacing each label y in t_2 with the corresponding label $x_1[x_2[\dots x_n[y]\dots]]$, where (x_1, \dots, x_n) is the left-to-right enumeration of the labels in t_1 .*

(iii) *In this case, all relations of $\mathbf{r}_{AC\Sigma}$ are satisfied by the operators A_α , C_α^T , and Σ_α^T , and the torsion relations $C_\alpha^{T^2} \approx \Sigma_\alpha^{T^2} \approx \text{id}$ are satisfied if and only if, for all trees t_1, t_2 , we have*

$$(8.7) \quad t_1[t_1[t_2]] = t_2.$$

Proof. For (i), the verifications are given in Figures 27, 28, and 29, respectively. Then (ii) follows from an induction on the size of the trees t_1 and t_2 . Finally, in order to establish (iii), it suffices to check the \mathcal{C} -geometric relations, and the hexagon relations, which is done in Figures 30 and 31. \square

Remark 8.3. As we are mostly interested in the group FB_∞ , we adjusted the constraints about the bracket operation on trees so as to guarantee that the relations of $\mathbf{r}_{A\Sigma}$ are satisfied, and we saw that all relations of $\mathbf{r}_{AC\Sigma}$ are then valid. If we start with the operators C_α^T and require that the relations of \mathbf{r}_{AC} be satisfied, we come up with exactly the same constraints, as can be read in Figures 30 and 31.

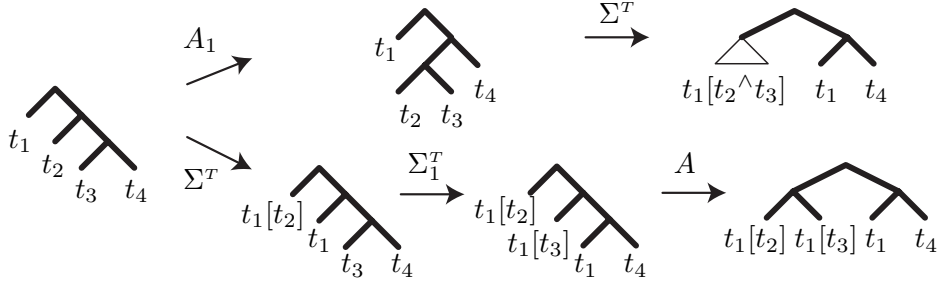


FIGURE 27. The relation $A_1 \Sigma^T = \Sigma^T \Sigma_1^T A$ requires $t_1[t_2 \wedge t_3] = t_1[t_2] \wedge t_1[t_3]$

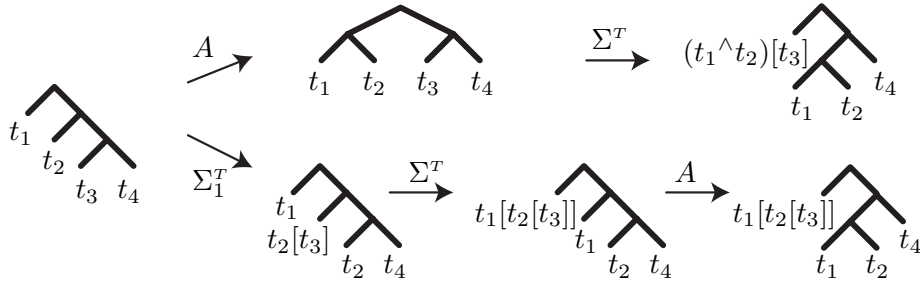


FIGURE 28. The relation $A \Sigma^T = \Sigma_1^T \Sigma^T A_1$ requires $(t_1 \wedge t_2)[t_3] = t_1[t_2[t_3]]$

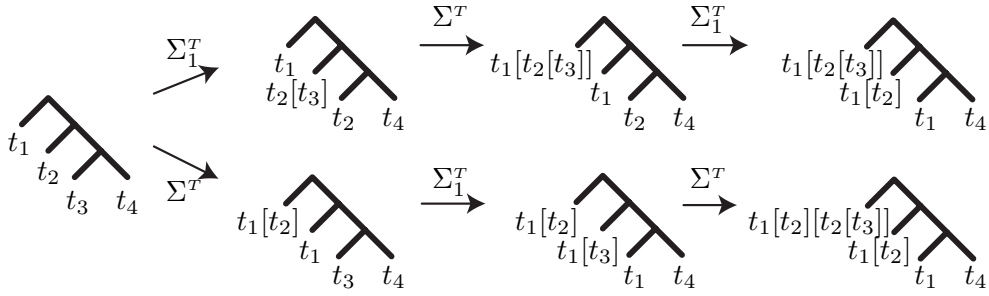


FIGURE 29. The relation $\Sigma_1^T \Sigma^T \Sigma_1^T = \Sigma^T \Sigma_1^T \Sigma^T$ requires $t_1[t_2[t_3]] = t_1[t_2][t_1[t_3]]$

We shall therefore be interested in the sequel with sets equipped with a left self-distributive operation, *i.e.*, a binary operation that satisfies the algebraic identity

$$(LD) \quad x[y[z]] = x[y][x[z]]$$

—or $x(yz) = (xy)(xz)$ when the operation symbol is omitted.

Definition. An algebraic system consisting of a set equipped with a left self-distributive operation is called an *LD-system*. An LD-system is said to be left cancellative if its left translations are injective, *i.e.*, if (8.2) holds; it is called an *LD-quasigroup* (in [12]) or a *rack* (in [19]) if its left translations are bijective. An LD-system is said to be *involutory* if (8.7) holds. Note that an involutory LD-system is necessarily an LD-quasigroup.

If S is an LD-quasigroup, it can be equipped with a second self-distributive operation, namely the operation such that $x[[y]]$ is the unique z satisfying $x[z] = y$ (assuming that $-[-]$ denotes the first operation). In this case, one has $x[[x[y]]] = x[x[[y]]] = y$ for all x, y . With this notation, saying that S is involutory amounts to saying that $-[-]$ coincides with $-[-]$.

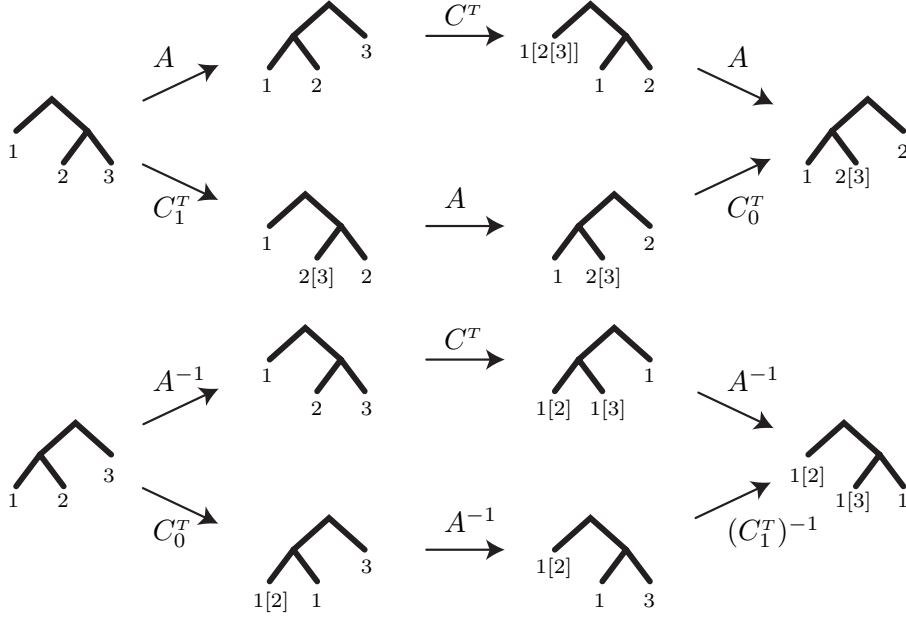


FIGURE 30. The twisted hexagon relations

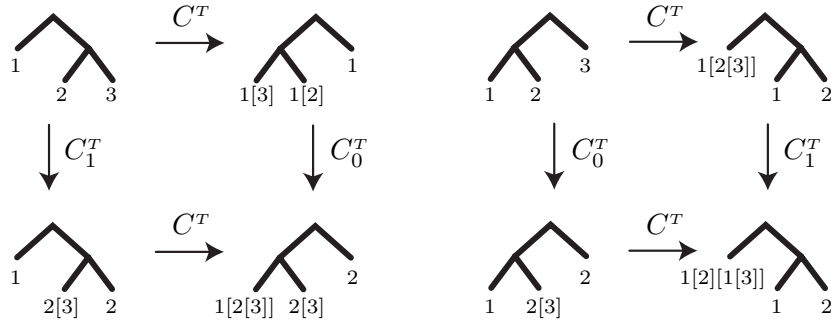


FIGURE 31. The twisted C -geometric relations

Example 8.4. We have seen that any set S equipped with $x[y] = y$ is a (trivial) LD-system; it is clearly involutory.

If G is a group, then G equipped with $x[y] = xyx^{-1}$ and $x[[y]] = x^{-1}yx$ is an LD-quasigroup. Still assuming that G is a group, let S be a subset of G . On $G \times S$ define $(x, a)[(y, b)] = (xax^{-1}y, b)$ and $(x, a)[[(y, b)]] = (xa^{-1}x^{-1}y, b)$. Then $(G \times S, [-])$ is an LD-quasigroup. When G is a free group based on S , the resulting LD-quasigroup is the free LD-quasigroup based on S , *i.e.*, any other LD-quasigroup based on S is a homomorphic image of this LD-quasigroup.

Another left cancellative LD-system of a completely different flavour has appeared in Section 6, namely the set FB_∞ equipped with the bracket of (6.6).

From now on, we shall always restrict to the context of Lemma 8.2(ii), *i.e.*, consider the twisted (semi)-commutation operators on the set T_S that stem from some left cancellative LD-system S . Accordingly, we shall simplify our notation, and write $\mathcal{G}(\mathcal{A}, \Sigma^S)$ for $\mathcal{G}(\mathcal{A}, \Sigma^{T_S})$, and, similarly, $\mathcal{G}(\mathcal{A}, \mathcal{C}^S)$ for $\mathcal{G}(\mathcal{A}, \mathcal{C}^{T_S})$ from now on.

8.3. Making groups. As in the case of the geometry monoids of algebraic laws, we can derive some groups from the monoids $\mathcal{G}(\mathcal{A}, \Sigma^S)$ by identifying near-equal operators. The question is again to prove that every operator in the above monoids admits a seed, *i.e.*, that, for an appropriate notion of substitution, the operator consists of all substitutes of some distinguished

pair of trees. However, some care is needed because the framework is not exactly that of Section 4, and the definition of a substitution has to be revisited.

In Section 4, we defined a substitution to be a mapping φ from \mathbf{N} to a set of (decorated) trees T , so that, for each \mathbf{N} -decorated tree t , we obtain a tree in T denoted t^φ by replacing each leaf labelled k in t with $\varphi(k)$. In our current framework, we shall start with decorated trees in which the labels are no longer integers, but more complicated data, namely abstract terms constructed using integers and binary operations, here the bracket operations $-[-]$ and $-[\![\]\!]$.

Definition. We define $\widehat{\mathbf{N}}$ to be the closure of \mathbf{N} under the brackets operations $-[-]$ and $-[\![\]\!]$; the elements of $\widehat{\mathbf{N}}$ are called *terms*.

Thus typical terms are $1, 3, 1[3], 1[2[3]]$, etc. Terms make an absolutely free algebra based on \mathbf{N} , which means that, if S is any set equipped with two binary operations, and f is any mapping of \mathbf{N} to S , there exists a unique way to *evaluate* each term τ into an element $\widehat{f}(\tau)$ of S by mapping k to $f(k)$ and then using the operations of S to inductively evaluate more complicated terms.

Definition. Assume that S is a set equipped with two binary operations, and f is a mapping of \mathbf{N} to S . For each $\widehat{\mathbf{N}}$ -decorated tree t , we define the *f-evaluation* of t to be the S -decorated tree obtained by replacing each label in t with its f -evaluation.

Thus, for instance, if t is the tree $1[2]^\wedge 2$ —from now on, we simply write ℓ instead of \circ_ℓ for the tree consisting of a single vertex labelled ℓ —if S is a group G equipped with the two conjugacy operations as in Example 8.4, and if we have $f(1) = x$ and $f(2) = y$, the f -evaluation of t is the G -decorated tree $(xyx^{-1})^\wedge y$. The following result is obvious:

Lemma 8.5. *The pair $(1^\wedge 2, 1[2]^\wedge 1)$ is a local seed for the operators C^T in the sense that, for every left cancellative LD-system S , the restriction of the action of C^T to trees of T_S with the same skeleton as $1^\wedge 2$ is the set of all pairs of S -decorated trees obtained by evaluating this pair in S . Similarly, $(1^\wedge(2^\wedge 3), 1[2]^\wedge(1^\wedge 3))$ is a local seed for Σ^T .*

There are two problems for extending the result to arbitrary operators in $\mathcal{G}(\mathcal{A}, \mathcal{C}^S)$ or $\mathcal{G}(\mathcal{A}, \Sigma^S)$. The first problem is that, as in the case of standard commutativity, using tree substitutions is necessary for enlarging the skeletons. Consider for instance the product $C^T A$: by Lemma 8.5, the S -decorated trees in the image of C^T are the evaluations of the $\widehat{\mathbf{N}}$ -decorated tree $1[2]^\wedge 1$, whose skeleton is $\{\phi, 0, 1\}$. Now, in order to apply A , we need a larger skeleton comprising the two additional addresses 10 and 11. In the case of standard commutativity, the solution is to apply the substitution $1 \mapsto 1^\wedge 2, 2 \mapsto 3$ and to conclude that the pair $(1^\wedge 2)^\wedge 3, (2^\wedge 1)^\wedge 3$ is a seed for CA . In the current case, the solution is similar, but applying the substitution not only changes the skeleton, but also the labels: when we replace 1 with $1^\wedge 2$ and 2 with 3, then the label $1[2]$ occurring in the right tree of the seed of C^T has to be replaced with $(1^\wedge 2)[3]$, which, by (8.4), is $1[2[3]]$. The conclusion is that the pair $((1^\wedge 2)^\wedge 3, (1[2[3]])^\wedge 1)^\wedge 2$ is a local seed for $C^T A$.

Similarly, let us consider $C^T A^{-1}$. In the untwisted case, we replace the vertex 2 with $2^\wedge 3$, and conclude that $(1^\wedge(2^\wedge 3), 2^\wedge(3^\wedge 1))$ is a seed. In the twisted case, the solution is the same, but the label of the considered vertex is now $1[2]$: we still apply the substitution $2 \mapsto 2^\wedge 3$, but, according to (8.5), we then distribute 1, *i.e.*, we replace $1[2]$ with $1[2]^\wedge 1[3]$. Finally, the pair $(1^\wedge(2^\wedge 3), 1[2]^\wedge(1[3]^\wedge 1))$ is a local seed for $C^T A^{-1}$.

Definition. Assume that φ is a mapping of \mathbf{N} to $T_{\widehat{\mathbf{N}}}$. First we inductively extend φ to $\widehat{\mathbf{N}}$ so that $\varphi(\tau[\tau'])$ (*resp.* $\varphi(\tau[\![\tau']\!])$) is obtained by replacing each label y in $\varphi(\tau')$ with $\tau_1[\dots[\tau_k[y]]\dots]$ (*resp.* $\tau_1[\![\dots[\tau_k[\![y]\!]]\dots]\!]$), where τ_1, \dots, τ_k is the left-right enumeration of the labels in $\varphi(\tau)$. Then we define the *substitute* t^φ or an arbitrary term t of $T_{\widehat{\mathbf{N}}}$ using induction as in Section 4, *i.e.*, we define $t^\varphi = \varphi(t)$ if t has size 1, and $t^\varphi = t_1^\varphi \wedge t_2^\varphi$ for $t = t_1 \wedge t_2$.

With this extended notion of substitution, it is easy to improve Lemma 8.5 as follows:

Lemma 8.6. *The pair $(1^{\wedge}2, 1[2]^{\wedge}1)$ is a seed for the operators C^T in the sense that, for every left cancellative LD-system S , the action of C^T on T_S is the set of all pairs of S -decorated trees obtained by evaluating a substitute of the above pair in S . Similarly, $(1^{\wedge}(2^{\wedge}3), 1[2]^{\wedge}(1^{\wedge}3))$ is a seed for Σ^T .*

When we consider the action of twisted (semi)-commutation on trees decorated using an LD-quasigroup S , it is routine to extend the previous result to any operator in $\mathcal{G}(\mathcal{A}, C^T; T_S, -[-])$ or $\mathcal{G}(\mathcal{A}, \Sigma^T; T_S, -[-])$. The point is that we can always assume that the left tree involved in a seed is an injective \mathbf{N} -decorated tree.

Things are more delicate when we consider a left cancellative LD-system that is not an LD-quasigroup. The problem in this case is to restrict to terms that involve the bracket operation exclusively, and this in turn makes it impossible to assume that the left tree in a seed is \mathbf{N} -decorated: for instance, a seed for the operator $(C^T)^{-1}$ is the pair $(1^{\wedge}2, 2^{\wedge}2[[1]])$, but, without $-[-]$, the only possible solution is $(1[2]^{\wedge}1, 1^{\wedge}2)$ or one of its substitute. The problem in this case is that, in order to carry out the proof of the counterpart of Lemma 2.4, we must be able to unify two $\widehat{\mathbf{N}}$ -decorated trees. When one of the trees to unify is an injective \mathbf{N} -decorated tree, the existence of a solution is trivial. In the case of general $\widehat{\mathbf{N}}$ -decorated trees, the question is not trivial, and it relies on specific results about the self-distributivity identity. The results of Chapter VIII of [12] show that a seed exists provided there is at least one way of unifying the involved trees: so, when we consider two operators in $\mathcal{G}(\mathcal{A}, C^S)$ or $\mathcal{G}(\mathcal{A}, \Sigma^S)$ that admit a seed, either their product is never defined, or it admits a seed. So we are left with the question of proving that an operator in $\mathcal{G}(\mathcal{A}, C^S)$ or $\mathcal{G}(\mathcal{A}, \Sigma^S)$ never has an empty domain. Here we shall consider the second case only and prove the following result, which is sufficient:

Lemma 8.7. *Assume that S is a left cancellative LD-system, and that w_1, \dots, w_n are words in $W(\mathbf{A}, \Sigma)$. Then there exists an S -decorated tree t such that $t \bullet w_i$ is defined for each i , the action being that of $\mathcal{G}(\mathcal{A}, \Sigma^S)$.*

The proof of Lemma 8.7 decomposes into two easier results.

Lemma 8.8. *Assume that w, w' are words in $W(\mathbf{a}, \sigma)$ and w is left $\mathbf{r}_{\mathbf{a}\sigma}$ -reversible to w' . Then, for each tree t , if $t \bullet w'$ is defined, so is $t \bullet w$.*

Proof. It suffices to consider the various possible cases. Consider for instance the case of $a_1\sigma_1^{-1}$, which is left reversible to $\sigma_2^{-1}\sigma_1^{-1}a_2$. So assume that $t \bullet \sigma_2^{-1}\sigma_1^{-1}a_2$ is defined. Write t as $t_1^{\wedge}(t_2^{\wedge}(t_3^{\wedge}t_4))$. The hypothesis that $t \bullet \sigma_2^{-1}$ is defined implies that t_2 can be expressed as $t_3[t_2']$ for some t_2' , and then we have $t \bullet \sigma_2^{-1} = t_1^{\wedge}(t_3^{\wedge}(t_2'^{\wedge}t_4))$. The hypothesis that the latter tree lies in the domain of σ_1^{-1} then implies that t_1 in turn can be expressed as $t_3[t_1']$ for some t_1' . But, in this case, we have $t \bullet a_1 = (t_3[t_1']^{\wedge}t_3[t_2'])^{\wedge}(t_3^{\wedge}t_4)$, a term explicitly in the domain of σ_1^{-1} . The other cases are similar. \square

Lemma 8.9. *Assume that u_1, \dots, u_n are words in $W^+(\mathbf{a}, \sigma)$. Then there exists a tree t such that $t \bullet u_i^{-1}$ is defined for each i .*

Proof. By Proposition 6.6(ii), any two elements in the monoid FB_{∞}^+ admit a left (least) common multiple, so there must exist words v_1, \dots, v_n in $W^+(\mathbf{a}, \sigma)$ such that v_1u_1, \dots, v_nu_n all represent the same element of FB_{∞}^+ . Let t' be any tree in the common domain of the positive equivalent words v_iu_i , and let t be the common image of t' under the (operators associated with) these words. By construction, we have $t \bullet u_i^{-1} = t' \bullet v_i$ for each i . \square

Proof of Lemma 8.7. Let w_1, \dots, w_n be arbitrary words in $W(\mathbf{a}, \sigma)$. By Proposition 6.6(i), each word w is left $\mathbf{r}_{\mathbf{a}\sigma}$ -reversible to some word $u_i^{-1}v_i$ with u_i, v_i in $W^+(\mathbf{a}, \sigma)$. By Lemma 8.8, it suffices to show that there exists a tree t such that $t \bullet u_i^{-1}v_i$ is defined for each i . The only problem is with the negative factors u_i^{-1} , and, there, the result is given by Lemma 8.9. \square

As was said above, the previous result was the missing part in the proof of:

Lemma 8.10. *Assume that S is a left cancellative LD-system. Then every operator in the monoid $\mathcal{G}(\mathcal{A}, \Sigma^S)$ admits a seed.*

It follows that, if S is a left cancellative LD-system, the monoid $\mathcal{G}(\mathcal{A}, \Sigma^S)$ satisfies the hypothesis of Lemma 2.17, and we thus obtain:

Proposition 8.11. *Assume that S is a left cancellative LD-system. Then near-equality is a congruence on the monoid $\mathcal{G}(\mathcal{A}, \Sigma^S)$. The quotient-monoid is a group. The operators A_α and Σ_α^T induce a partial action of this group on T_S .*

Definition. For S a left cancellative LD-system, we define $G(\mathcal{A}, \Sigma^S)$ to be the quotient-group of $\mathcal{G}(\mathcal{A}, \Sigma^S)$ under near-equality.

Remark 8.12. We skipped the details of the unification arguments needed to completely establish Lemma 8.10, and the reader may feel uncomfortable with the result. However, the results of this section are needed to justify the existence of the group $G(\mathcal{A}, \Sigma^S)$ in the general case only; in the specific case that we shall investigate in the next section, a direct argument shows that the action of $\mathcal{G}(\mathcal{A}, \Sigma^S)$ is free, and there is no problem to quotient the monoid into a group.

8.4. Using the self-distributive operation on FB_∞ . The results of Sections 5 show that, when $-[-]$ is the trivial operation $x[y] = y$, the group $G(\mathcal{A}, \Sigma^S)$ is the group $G(\mathcal{A}, \Sigma)$, *i.e.*, the group V' . We can now reset the initial question of this section more precisely as:

Question 8.13. *Does there exist a left cancellative LD-system S such that the associated geometry group $G(\mathcal{A}, \Sigma^S)$ is the group FB_∞ ?*

A positive answer would exactly correspond to what can be called a geometric realization of FB_∞ , *i.e.*, a realization of FB_∞ as the geometry group of associativity and twisted semi-commutativity. We shall now answer Question 8.13 in the positive. To do so, we have to exhibit a convenient LD-system, and we shall do it here by using the bracket on FB_∞ . Another solution using the conjugacy operation on a free group will be given in Section 9 below. Our current aim is to prove:

Proposition 8.14. *The group $G(\mathcal{A}, \Sigma^{FB_\infty})$ is isomorphic to FB_∞ , *i.e.*, FB_∞ is the geometry group of associativity and FB_∞ -twisted semi-commutativity.*

Proving Proposition 8.14 amounts to proving that the relations $\mathbf{r}_{A\Sigma}$ make a presentation of the group $G(\mathcal{A}, \Sigma^{FB_\infty})$ in terms of the generators A_α and Σ_α^T , or, equivalently, that the relations $\mathbf{r}_{a\sigma}$ make a presentation in terms of the generators a_i and σ_i . We shall do this using Proposition 1.2(i). The argument is actually very close to the one we used in Section 7 to show that FB_∞ is the group of multiscaled braids, and most of the computations have already been done. We use a similar notation here.

Definition. For t an FB_∞ -decorated tree, we put

$$E(t) = x_1 \cdot \partial x_2 \cdot \dots \cdot \partial^{n-1} x_n,$$

where $t = t_1 \wedge \dots \wedge t_n \wedge \circ_x$ is the decomposition of t along its right branch and x_i is the \otimes -evaluation of t_i .

Lemma 8.15. *Assume that t is an FB_∞ -decorated tree. Then we have*

$$(8.8) \quad E(t \bullet a_i) = E(t) \cdot a_i, \quad E(t \bullet \sigma_i) = E(t) \cdot \sigma_i,$$

whenever the tree $t \bullet a_i$ or $t \bullet \sigma_i$ is defined.

Proof. For $t = t_1 \wedge \dots \wedge t_n \wedge \circ_x$, let us denote the sequence (t_1, \dots, t_n) by $D(t)$, and let $E^*(t)$ be the sequence (x_1, \dots, x_n) where x_i is the \otimes -evaluation of t_i . Then, whenever they exist, we have the equalities

$$\begin{aligned} D(t \bullet a_i) &= (t_1, \dots, t_{i-1}, t_i \wedge t_{i+1}, t_{i+2}, \dots, t_n), \\ D(t \bullet \sigma_i) &= (t_1, \dots, t_{i-1}, t_i[t_{i+1}], t_i, t_{i+2}, \dots, t_n), \end{aligned}$$

hence, assuming $E^*(t) = (x_1, \dots, x_n)$,

$$\begin{aligned} E^*(t \bullet a_i) &= (x_1, \dots, x_{i-1}, x_i \otimes x_{i+1}, x_{i+2}, \dots, x_n), \\ E^*(t \bullet \sigma_i) &= (x_1, \dots, x_{i-1}, x_i[x_{i+1}], x_i, x_{i+2}, \dots, x_n). \end{aligned}$$

The latter exactly are the formulas (7.5) and (7.6) of the proof of Lemma 7.9, and we can then repeat the rest of the computation used there. \square

We are now able to conclude.

Proof of Proposition 8.14. Using an induction, we deduce from (8.8) the equality

$$(8.9) \quad E(t \bullet w) = E(t) \cdot \bar{w}$$

for each FB_∞ -decorated tree t and every word w in $W(\mathbf{a}, \boldsymbol{\sigma})$ such that $t \bullet w$ is defined, where we recall \bar{w} denotes the class of w in FB_∞ . Assume that two words w, w' represent the same element of $G(\mathcal{A}, \Sigma^{FB_\infty})$, *i.e.*, the associated operators are near-equal. Then there exists at least one tree t such that both $t \bullet w$ and $t \bullet w'$ are defined, and, then, by hypothesis, these trees are equal. From (8.9) we deduce

$$\bar{w} = E(t)^{-1} \cdot E(t \bullet w) = E(t)^{-1} \cdot E(t \bullet w') = \bar{w}',$$

i.e., w and w' represent the same element of FB_∞ . \square

8.5. The general group of twisted semi-commutativity. To conclude with a simple statement, we can define Σ_α^* to be the union of all operators Σ_α^T (considered as sets of pairs) for all possible sets T_S associated with a left cancellative LD-system, and define $\mathcal{G}(\mathcal{A}, \Sigma^\#)$ to be the monoid generated by all operators $A_\alpha, \Sigma_\alpha^*$ and their inverses. Then, by construction, each specific monoid $\mathcal{G}(\mathcal{A}, \Sigma^S)$ is a quotient of $\mathcal{G}(\mathcal{A}, \Sigma^\#)$. The same argument as above shows that near-equality is a congruence on $\mathcal{G}(\mathcal{A}, \Sigma^\#)$, and we can introduce the corresponding group $G(\mathcal{A}, \Sigma^\#)$. Thus $G(\mathcal{A}, \Sigma^\#)$ can be considered naturally as the geometry group of associativity and twisted semi-commutativity. Then we can state:

Proposition 8.16. *The geometry group $G(\mathcal{A}, \Sigma^\#)$ of associativity and twisted semi-commutativity is isomorphic to FB_∞ .*

Proof. By construction, the group $G(\mathcal{A}, \Sigma^{FB_\infty})$ is a quotient of the general group $G(\mathcal{A}, \Sigma^\#)$. By Lemma 8.2(iii), the group $G(\mathcal{A}, \Sigma^\#)$ is a quotient of FB_∞ . Now, Proposition 8.14 shows that the canonical mapping of FB_∞ to $G(\mathcal{A}, \Sigma^{FB_\infty})$ is an isomorphism, so the two surjective homomorphisms of which the latter is the product must be isomorphisms as well. \square

This leads to a natural question. We have introduced, for each particular left cancellative LD-system S , a geometry group of associativity together with S -twisted semi-commutativity. We saw in Section 5 that, if S is trivial, this geometry group is V' , while, if S is FB_∞ equipped with its bracket, the geometry group is FB_∞ itself. Now, another natural example of left cancellative LD-system is provided by the conjugacy operation of a group. The natural question is: what is the associated geometry group, in particular what is the one corresponding to conjugacy in a free group? The question will be answered in Section 9 below.

8.6. An ordering on the group FB_∞ . We conclude the current with an application, namely the orderability of the group FB_∞ . The Twisted semi-commutativity is involved in the technical argument.

One of the consequences of the existence of a self-distributive structure on B_∞ is the existence of an explicit ordering compatible with multiplication on the left [16]. On the other hand, it is known that Thompson's group F is even bi-orderable, *i.e.*, it admits a linear ordering with is compatible with multiplication on both sides. So it is not surprising that the group FB_∞ turns out to be orderable, although a specific argument is needed to prove that the action of F introduces no cycle in the ordering of B_∞ . We shall prove:

Proposition 8.17. *The group FB_∞ is left-orderable, i.e., there exists a linear ordering $<$ on FB_∞ that is compatible with multiplication on the left: $x < y$ implies $zx < zy$ for all x, y, z in FB_∞ .*

As in the case of B_∞ , a linear ordering on FB_∞ could be derived from the self-distributive operation described in the previous section. Actually, we shall resort to a more simple approach consisting in using the braid ordering, thanks to a decomposition result of every element of FB_∞ in terms of one braid and two elements of Thompson's group F .

Lemma 8.18. *Every element of the group FB_∞ admits an (non necessarily unique) expression of the form $u^{-1}vw$ with u, v in $W^+(\mathbf{a})$ and v in $W(\sigma)$.*

Proof. As FB_∞ is a group of left fractions of FB_∞^+ , it suffices to prove that every element in the monoid FB_∞^+ as an expression consisting of a word in $W^+(\sigma)$ followed by a word in $W^+(\mathbf{a})$. Using an induction on the number of a_i 's, we deduce this from the fact that, for each pair a_i, σ_j , there exists a relation in $r_{a\sigma}$ transforming $a_i\sigma_j$ into a word of the form $u \dots a_{i'}$, where u consists of one or two letters σ_k . \square

We fix some linear ordering on trees: a tree t is said to be smaller than t' , denoted $t \prec t'$, if either t has size 1 and t' has size 2 at least, or t and t' have size at least 2 and $t_{/0} \prec t'_{/0}$ holds, or or t and t' have size at least 2 and $t_{/0} = t'_{/0}$ and $t_{/1} \prec t'_{/1}$ hold.

Definition. We say that an element of FB_∞ is *positive* if it admits an expression $u^{-1}vw$ with u, w in $W^+(\mathbf{a})$, v in $W(\sigma)$, and

- (i) either some generator σ_i occurs in v , but neither σ_i^{-1} nor any $\sigma_j^{\pm 1}$ with $j < i$ occurs, or
- (ii) the word v is empty and, for n large enough, $\setminus n \setminus \bullet u$ is \prec -smaller than $\setminus n \setminus \bullet v$.

As explained in [16], defining $x < y$ to mean that $x^{-1}y$ has an expression satisfying (i) yields a linear ordering $<$ on B_∞ . On the other hand, it is clear that (ii) defines a linear ordering on F .

Lemma 8.19. *A positive element of FB_∞ cannot be trivial, i.e., 1 is not positive.*

Proof. By construction, the bracket operation on the group FB_∞ induces a well-defined operation on the subset B_∞ of FB_∞ . So we can consider the (partial) action of FB_∞ on B_∞ -decorated trees involving associativity and B_∞ -twisted semi-commutativity. In order to prove that some element x of FB_∞ is non-trivial, it is sufficient to exhibit a B_∞ -decorated tree t such that $t \bullet x$ exists and is not t .

Now, for t a B_∞ -decorated special tree, we denote by L_t the infinite sequence in $B_\infty^{\mathbb{N}}$ consisting of the labels in t enumerated from left to right, and completed with an infinite sequence of 1's. For instance, if t is the tree $(\circ_{\sigma_1} \wedge \circ_1) \wedge \circ_{\sigma_2}$, the sequence L_t is $(\sigma_1, 1, \sigma_2, 1, 1, \dots)$.

Assume that x is an element of FB_∞ that be expressed as $u^{-1}vw$ with u, w in $W^+(\mathbf{a})$ and v in $W(\sigma)$, and assume that we are in case (i) of the definition, *i.e.*, the braid word v is what is called σ_i -positive. By Lemma 8.8, there exists at least one B_∞ -decorated tree t such that

$t \bullet u^{-1}vw$ is defined. Use $<_{Lex}$ to denote the lexicographic extension of the braid ordering $<$ to $B_\infty^{\mathbb{N}}$. Then we have

$$L_t = L_{t \bullet u^{-1}} <_{Lex} L_{t \bullet u^{-1}v} = L_{t \bullet u^{-1}vw} = L_{t \bullet x}.$$

This is enough to conclude that x is not 1. □

We deduce

Proposition 8.20. *For x, y in FB_∞ , say that $x < y$ holds if $x^{-1}y$ is positive. Then the relation $<$ is a linear ordering on FB_∞ that is compatible with multiplication on the left, and extends the braid ordering.*

Proof. Lemma 8.19 guarantees that the relation $<$ has no cycle. The fact that $<$ is linear follows from the property that every nontrivial element of FB_∞ has to be positive or to have a positive inverse. This follows from Lemma 8.18, and from the corresponding facts in B_∞ and F . □

So, in particular, the group FB_∞ is orderable, and Proposition 8.17 is established. Observe that, contrary to the case of B_∞ , there is no global characterization of the ordering on FB_∞ in terms of σ -positivity: for instance, $a_1\sigma_2a_1^{-1}\sigma_3^{-1}$ is an expression of 1 in which the σ_i generator with lower index, namely σ_2 , has one positive occurrence and no negative occurrence.

9. HOMEOMORPHISMS OF A PUNCTURED SPHERE

There is a well-known realization of Artin's braid group B_n as the mapping class group of a disk with n punctures [1], and the induced action on the fundamental group leads to Artin's representation of B_n in the automorphisms of a rank n free group. In this section, we prove similar results for the group FB_∞ . Starting from the fact that FB_∞ can be mapped inside the mapping class group of a sphere with a Cantor set of punctures—as already observed by M. Brin and J. Meier—we prove here that FB_∞ embeds in the groups of automorphisms of a free group of countable rank using the ordering constructed in Section 6.

9.1. The mapping class group of a sphere with a Cantor set of punctures. As FB_∞ includes B_∞ , one can expect to use a disk with infinitely many punctures, and therefore a free group of infinite rank. Actually, the tree-like structure of FB_∞ should make it natural to use a set of punctures that is dense in the diameter of the disk. A suitable choice is to take this set of punctures to be a Cantor set. Also, it is convenient to collapse the boundary of the disk, *i.e.*, to start with a 2-sphere. In the sequel, we fix a real number ρ in $(0, 1)$ —for instance $\rho = 1/3$ —and we denote by \mathbb{K} the Cantor subset of $[0, 1]$ obtained by iteratively removing the median intervals of sizes ρ^k .

Definition. (Figure 32) We denote by $S_{\mathbb{K}}$ the topological space obtained from the disk of diameter $[-\rho, 1 + \rho]$ in \mathbb{R}^2 by removing all points of \mathbb{K} , and collapsing the outer circle to a point.

We denote by $MCG(S_{\mathbb{K}})$ the mapping class group of $S_{\mathbb{K}}$, *i.e.*, the group of all homeomorphisms of $S_{\mathbb{K}}$ up to isotopy. As in the case of a finite set of punctures, a continuous motion of the points of $(0, 1) \setminus \mathbb{K}$ inside D^2 that finishes with $(0, 1) \setminus \mathbb{K}$ gives rise to a well-defined element of $MCG(S_{\mathbb{K}})$. We can then mimic the standard construction and define elements of $MCG(S_{\mathbb{K}})$ corresponding to the usual Dehn's half-twists on the one side and to Thompson's piecewise linear homeomorphisms on the other side. Because of the Cantor set \mathbb{K} , the full group $MCG(S_{\mathbb{K}})$ is huge, and we shall always work with small subgroups of that huge group.

By construction, the complement of \mathbb{K} in $(0, 1)$ consists of a countable collection of open intervals indexed by dyadic numbers. For further explicit constructions, we fix the following notation.

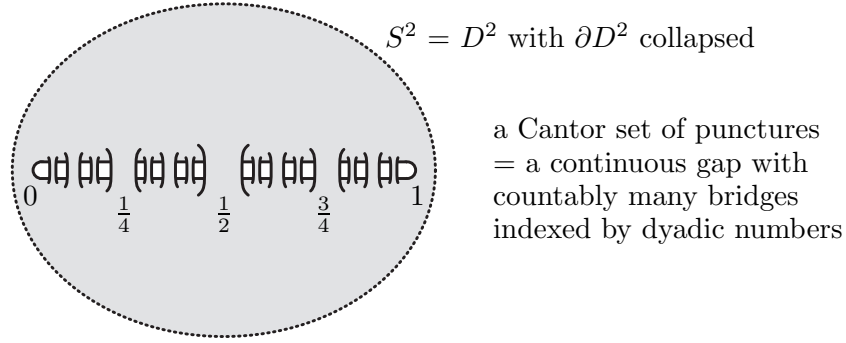


FIGURE 32. The space $S_{\mathbb{K}}$: a sphere in which the equator is replaced with the complement of a Cantor set, or, equivalently, two disks connected by a countable family of bridges (here the ratio α used to construct \mathbb{K} is about $1/6$)

Definition. (Figure 33) For α a binary address (*i.e.*, a finite sequence of 0's and 1's) of length ℓ , we define Δ_α to be the image in $S_{\mathbb{K}}$ of the disk centered at the point whose binary expansion is $0.\alpha 1$ and whose diameter is $(1/2)^\ell + \rho^{\ell+1}$ (where ρ is the distance used in the construction of the Cantor set \mathbb{K}). For s a finite sequence of positive integers, we define D_s to be $\Delta_{\bar{s}0}$.

For instance, D_1 , *i.e.*, Δ_0 , is (the image of) the disk centered at $1/4$ with diameter $1/2 + \rho^2$, while $D_{1,1}$, which is Δ_{00} is (the image of) the disk centered at $1/8$ and has diameter $1/4 + \rho^3$. The diameters are adjusted so that $\Delta_{\alpha 0}$ (*resp.* $\Delta_{\alpha 1}$) includes the left (*resp.* right) half of $\Delta_\alpha \cap \mathbb{K}$. Note that only the disks Δ_α with α finishing with 0 are disks D_s : for instance, Δ_1 is not a D_s disk.

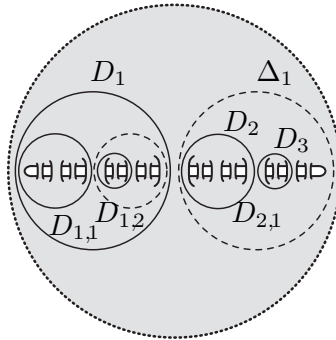


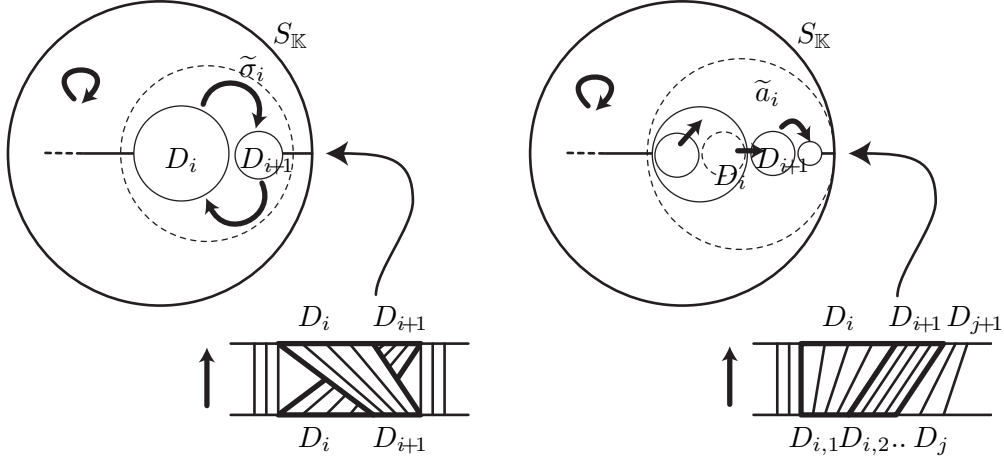
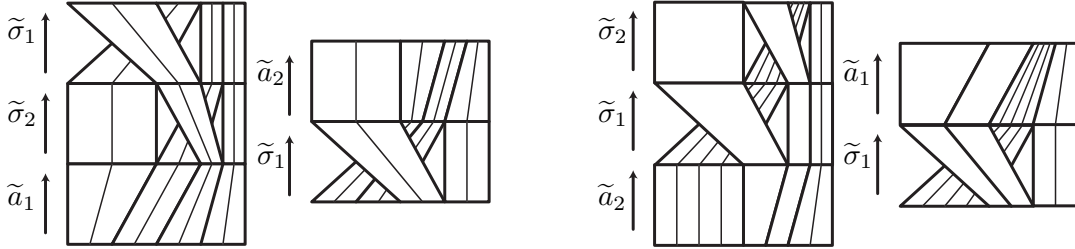
FIGURE 33. The disks Δ_α and D_i in $S_{\mathbb{K}}$. By construction, all disks $D_{s,i}$ are included in D_s .

Definition. (Figure 34) For $i \geq 1$, we define $\tilde{\sigma}_i$ to be the class in $MCG(S_{\mathbb{K}})$ of a clockwise half-turn (with rescaling) that exchanges D_i and D_{i+1} and is the identity on all other D_j 's. We define \tilde{a}_i to be the class in $MCG(S_{\mathbb{K}})$ of a motion that fixes D_j for $j < i$, dilates $D_{i,1}$ to D_i , translates $D_{i,j+1}$ to $D_{i+1,j}$ for every j , and contracts D_j to D_{j+1} for $j > i$.

Lemma 9.1. Mapping σ_i to $\tilde{\sigma}_i$ and a_i to \tilde{a}_i defines a homomorphism of FB_∞ into $MCG(S_{\mathbb{K}})$.

Proof. It suffices to check that all relations in $r_{\alpha\sigma}$ induce isotopies. The result is well-known for Coxeter relations; the case of the Thompson relations is clear. Finally, the diagrams of Figure 35 treat the mixed relations. \square

In the sequel, the subgroup of $MCG(S_{\mathbb{K}})$ generated by all $\tilde{\sigma}_i$ and \tilde{a}_i , *i.e.*, the image of FB_∞ under the homomorphism of Lemma 9.1, will be denoted by \widehat{FB}_∞ .


 FIGURE 34. Action of $\tilde{\sigma}_i$ and \tilde{a}_i , with the detail on the equator

 FIGURE 35. Relations in $MCG(S_{\mathbb{K}})$ (the order of the factors corresponds to composition of functions)

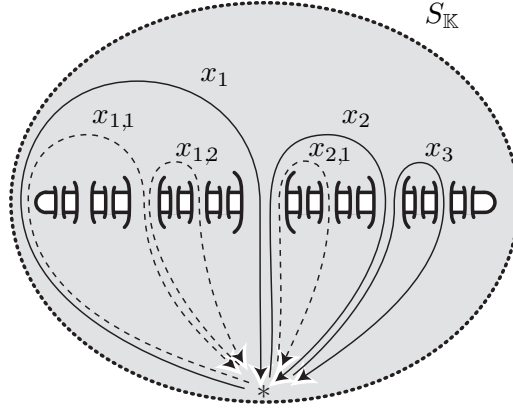
9.2. Action on the fundamental group. As the set $S_{\mathbb{K}}$ has a perfect set of punctures, its fundamental group is a group of uncountable type. Here we shall be interested only in the subgroup of $\pi_1(S_{\mathbb{K}})$ corresponding to loops that cross the equator finitely many times only. This subgroup will be called the *tame* fundamental group of $S_{\mathbb{K}}$, denoted $\pi_1^{\text{tame}}(S_{\mathbb{K}})$. The latter group is easily described. The basepoint of all loops will be the South pole of $S_{\mathbb{K}}$.

Definition. (Figure 36) For s a finite sequence of positive integers, we define x_s to be the class in $\pi_1^{\text{tame}}(S_{\mathbb{K}})$ of the loop that starts from the basepoint, reaches the South pole of D_s , turns once clockwise around D_s , and returns to the basepoint.

Lemma 9.2. *The group $\pi_1^{\text{tame}}(S_{\mathbb{K}})$ is a free group based on the family of all x_s for s a finite sequence of positive integers.*

Proof. In order to prove that the family $\{x_s; s \in \mathbb{N}^*\}$ generates $\pi_1^{\text{tame}}(S_{\mathbb{K}})$, it is sufficient to prove that, for each nonempty binary address α , the loop γ_α crossing the equator at 0 and returning by the bridge centered at the rational with binary expansion $0.\alpha$ can be expressed as a product of x_s 's. Indeed, the boundary of the initial disk D^2 was collapsed, so the loop crossing at 0 and returning at 1 is trivial, and, if we can obtain γ_α , then, by considering $\gamma_\beta^{-1}\gamma_\alpha$, we can obtain any loop crossing the equator twice, and, from there, any loop crossing the equator finitely many times.

So let us inductively define a word w_α in the x_s 's for α a nonempty address. We start with $w_0 = 1$ and $w_1 = x_1$. Then, for $\alpha = 0\beta$, we construct w_α from w_β by appending 1 at the beginning of each index s occurring in w_β . For $\alpha = 1\beta$, we construct w_α from w_β by shifting the first entry by $+1$ in every index s occurring in w_β and adding x_1 on the left. For instance,

FIGURE 36. Generators of $\pi_1^{\text{tame}}(S_{\mathbb{K}})$

for $\alpha = 1101$, we successively find $w_1 = x_1$, next $w_{01} = x_{1,1}$, then $w_{101} = x_1x_{2,1}$, and, finally, $w_\alpha = x_1x_2x_{3,1}$. It is easy to check inductively that w_α so constructed represents the loop γ_α .

It remains to show that the elements x_s form a free family. Assume that we have a relation in $\pi_1^{\text{tame}}(S_{\mathbb{K}})$, say $W(x_{s_1}, \dots, x_{s_n}) = 1$ with W a freely reduced word. Assume first that the disks D_{s_1}, \dots, D_{s_n} are pairwise disjoint. Then collapsing each disk D_{s_i} to a point induces a surjective homomorphism of the subgroup of $\pi_1^{\text{tame}}(S_{\mathbb{K}})$ generated by x_{s_1}, \dots, x_{s_n} onto the fundamental group of a disk with n punctures. The latter is known to be a free group of rank n , so the only possibility is that W be trivial.

Finally, consider the case when some disk D_{s_i} may include another disk D_{s_j} , *i.e.*, when some index s_i may be a prefix of another index s_j . Then for each i , we define $y_i = x_{s_i,1}x_{s_i,2} \dots x_{s_i,p_i}$, where p_i is the minimal p such that (s_i, p) is a prefix of no other index s_j . Note that the process creates no new inclusions. Let f be the result of collapsing all generators $x_{s_i,p}$ with $p > p_i$. Then, by construction, we have $f(x_{s_i}) = y_i$, and, therefore, the hypothesis $W(x_{s_1}, \dots, x_{s_n}) = 1$ implies $W(y_1, \dots, y_n) = 1$. Now, for each i , the generator x_{s_i,p_i} occurs in y_i only, and the disks D_{s_i,p_i} are disjoint. Then the same argument as above shows that W must be trivial. \square

The homeomorphisms of $S_{\mathbb{K}}$ induce automorphisms of its fundamental group $\pi_1(S_{\mathbb{K}})$, and, possibly, of its subgroup $\pi_1^{\text{tame}}(S_{\mathbb{K}})$. In particular, this is the case for the homeomorphisms in \widehat{FB}_∞ , *i.e.*, those generated by the elements \tilde{a}_i and $\tilde{\sigma}_i$. In this way, we obtain a homomorphism of \widehat{FB}_∞ into the automorphisms of a free group of countable rank.

Definition. We denote by ϕ the homomorphism of FB_∞ into $\text{Aut}(\pi_1^{\text{tame}}(S_{\mathbb{K}}))$ induced by the action of FB_∞ on $S_{\mathbb{K}}$ described in Lemma 9.1. For w a word in $W(\mathbf{a}, \boldsymbol{\sigma})$, we denote by \hat{w} the image of the element of FB_∞ represented by w under ϕ .

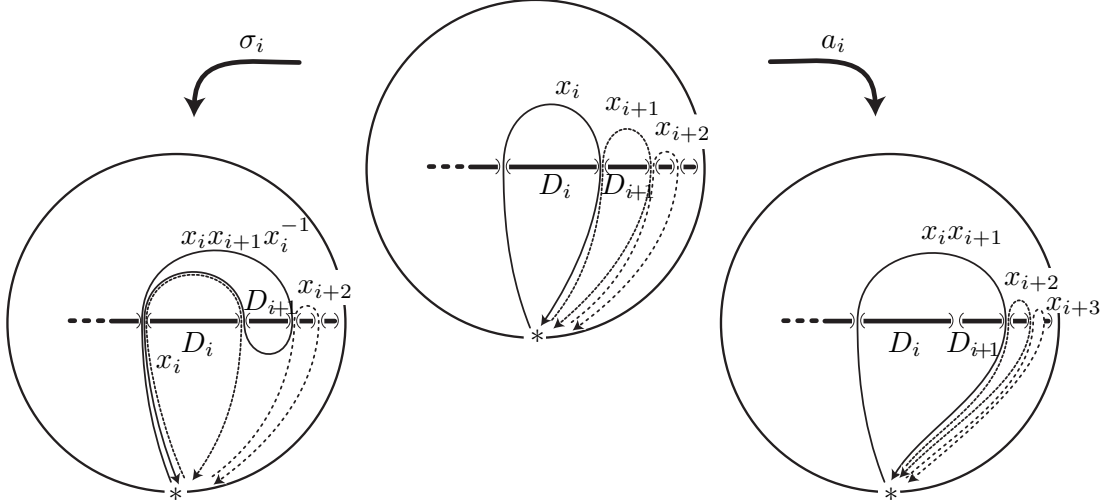
Explicit formulas are easy to obtain:

Lemma 9.3. *The automorphisms of $\pi_1^{\text{tame}}(S_{\mathbb{K}})$ associated with σ_i and a_i are:*

$$(9.1) \quad \hat{\sigma}_i : \quad x_{j,s} \mapsto x_{j,s} \quad \text{for } j \neq i, i+1, \quad x_{i,s} \mapsto x_i x_{i+1,s} x_i^{-1}, \quad x_{i+1,s} \mapsto x_{i,s},$$

$$(9.2) \quad \hat{a}_i : \quad \begin{cases} x_{j,s} \mapsto x_{j,s} \text{ for } j < i, & x_{j,s} \mapsto x_{j+1,s} \text{ for } j > i, \\ x_i \mapsto x_i x_{i+1}, & x_{i,1,s} \mapsto x_{i,s}, & x_{i,j+1,s} \mapsto x_{i+1,j,s} \text{ for } j \geq 2. \end{cases}$$

Proof. The formulas can be read in Figure 37 directly. \square


 FIGURE 37. Action of σ_i and a_i on the generators of $\pi_1^{\text{tame}}(S_{\mathbb{K}})$

It will be also convenient to have for the inverses of the generators; these are:

$$(9.3) \quad \widehat{\sigma}_i^{-1} : x_{j,s} \mapsto x_{j,s} \text{ for } j \neq i, i+1, \quad x_{i,s} \mapsto x_{i+1,s}, \quad x_{i+1,s} \mapsto x_{i+1}^{-1} x_{i,s} x_{i+1},$$

$$(9.4) \quad \widehat{a}_i^{-1} : \begin{cases} x_{j,s} \mapsto x_{j,s} \text{ for } j < i, & x_{j+1,s} \mapsto x_{j,s} \text{ for } j > i, \\ x_{i,s} \mapsto x_{i,1,s}, & x_{i+1} \mapsto x_{i,1}^{-1} x_i, & x_{i+1,j,s} \mapsto x_{i,j+1,s} \text{ for } j \geq 2. \end{cases}$$

The representation of FB_{∞} in $\text{Aut}(\pi_1^{\text{tame}}(S_{\mathbb{K}}))$ defined above extends the standard Artin representation of the braid group B_{∞} . Let us denote by $F_{\mathbb{N}^*}$ the free group generated by the elements x_s —hence a copy of $\pi_1^{\text{tame}}(S_{\mathbb{K}})$ —and by $F_{\mathbb{N}}$ the subgroup of $F_{\mathbb{N}^*}$ generated by the elements x_i with i a positive integer. Then the restriction of $\widehat{\sigma}_i$ defines an automorphism of $F_{\mathbb{N}}$, namely the standard Artin representation of braids: $\widehat{\sigma}_i$ maps x_i to $x_i x_{i+1} x_i^{-1}$ and x_{i+1} to x_i and leaves all other x_j 's unchanged. On the other hand, the restriction of \widehat{a}_i defines an endomorphism of $F_{\mathbb{N}}$ that is not surjective: neither x_i nor x_{i+1} belong to the image of \widehat{a}_i . Restoring surjectivity requires that new generators $x_{i,j}$ be added so that $x_{i,1}$ is $\widehat{a}_i^{-1}(x_i)$ and $x_{x,j+1}$ is $\widehat{a}_i^{-1}(x_{i+1,j})$, and, by and by, one arrives to the full group $F_{\mathbb{N}^*}$.

Formulas (9.1) and (9.2) enable one to explicitly determine the automorphism associated with an element of FB_{∞} . In the sequel, it will be also necessary to use the following alternative recipe, which connects the computation of the automorphism with twisted semi-commutativity as defined in Section 8.

Definition. For α a binary address, we define y_{α} in $F_{\mathbb{N}^*}$ inductively by the following rules: $y_{\emptyset} = 1$, $y_{\alpha 0} = x_{\alpha}$, and $y_{\alpha 1} = y_{\alpha 0}^{-1} y_{\alpha}$. We say that an $F_{\mathbb{N}^*}$ -labeled tree t is *natural* if, for each address α of a leaf in t , the label of α is y_{α} .

Thus, for instance, we have $y_{00} = x_0 = x_{1,1}$, $y_{01} = y_{00}^{-1} y_0 = x_{1,1}^{-1} x_0 = x_{1,1}^{-1} x_1$, $y_{10} = x_1 = x_2$, and $y_{11} = y_{10}^{-1} y_1 = x_2^{-1} y_0^{-1} y_{\emptyset} = x_2^{-1} x_1^{-1}$. The tree $(\circ_{x_{1,1}} \wedge_{\circ_{x_{1,1}^{-1} x_1}})^{\wedge_{\circ_{x_1^{-1}}}}$ is a natural $F_{\mathbb{N}^*}$ -labeled tree of size 3.

Proposition 9.4. For w in $W(\mathbf{a}, \boldsymbol{\sigma})$, the automorphism \widehat{w} can be computed as follows:

- (i) Choose a natural $F_{\mathbb{N}^*}$ -labeled tree t such that $t \bullet w$ is defined;
- (ii) Then we have $t \bullet w = t' \widehat{w}$, where t' is the natural tree with the same skeleton as $t \bullet w$, i.e., $\widehat{w}(y_{\alpha})$ is the label at α in $t \bullet w$ whenever α is the address of a leaf in $t \bullet w$.
- (iii) The latter relation extends to each α in the skeleton of $t \bullet w$ if the labels are propagated to interior nodes using the rules of Lemma 9.2.

Proof. It is straightforward to directly check the result when w consists of a single letter $\sigma_i^{\pm 1}$ or $a_i^{\pm 1}$. So the point is to show that the result is true for $w = w_1 w_2$ assuming that it is true for w_1 and w_2 . Assume that t is a natural tree such that $t \bullet w$ exists. Denote by t' and t'' the natural trees whose skeletons are those of $t \bullet w_1$ and of $t \bullet w$, respectively. By induction hypothesis, we have $t \bullet w_1 = t'^{\widehat{w}_1}$, hence $t \bullet w = t'^{\widehat{w}_1} \bullet w_2$. By induction hypothesis again, we have $t' \bullet w_2 = t''^{\widehat{w}_2}$, which means that, for each α in the outer skeleton of t'' , $\widehat{w}_2(y_\alpha)$ is the label at α in $t' \bullet w_2$. By construction, this label is an expression $E(y_{\beta_1}, \dots, y_{\beta_q})$ involving some variables $y_{\beta_1}, \dots, y_{\beta_q}$ with products and inverses. When we substitute t' with $t'^{\widehat{w}_1}$ and apply the same associativity and semi-commutativity operators, the result is the expression $E(\widehat{w}_1(y_{\beta_1}), \dots, \widehat{w}_1(y_{\beta_q}))$, which is also $\widehat{w}_1(E(y_{\beta_1}, \dots, y_{\beta_q}))$ as \widehat{w}_1 is a group automorphism. We deduce that $t \bullet w$ is $t''^{\widehat{w}_1 \circ \widehat{w}_2}$, *i.e.*, $t''^{\widehat{w}}$, as was expected. \square

A detailed example is described in Figure 38.

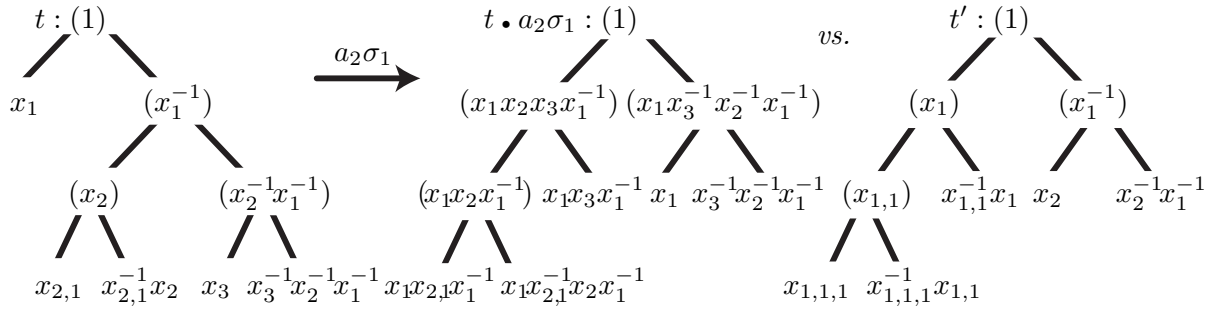


FIGURE 38. Computing the automorphism of $F_{\mathbb{N}^*}$ associated with $a_2 \sigma_1$: starting with t wearing the “natural” labels (expressed in x -coordinates), we apply associativity and conjugacy-twisted semi-commutativity to obtain $t \bullet a_2 \sigma_1$, and we compare it with the natural tree t' with the same skeleton. For each address α , the image of y_α (expressed in x -coordinates) is the label written at α in $t \bullet a_2 \sigma_1$; originally, only the leaves have labels, but we can propagate the labels to inner nodes using the rules of Lemma 9.2, *i.e.*, labelling each node with the product of the labels of its sons. So we read that x_1 is mapped to $x_1 x_2 x_3 x_1^{-1}$, that x_2 is mapped to x_1 , and that $x_{1,1,1}$ is mapped to $x_1 x_{2,1} x_1^{-1}$. For addresses outside the skeleton of t' , it suffices to extend t into a bigger tree so that the skeleton of the corresponding image contains that address. For instance, to determine the image of $x_{1,1,2}$ (*i.e.*, y_{0010}), we enlarge t by splitting the leaf at 101 into two new leaves with labels $x_{2,2}$ and $x_{2,2}^{-1} x_2$, and we read that $x_{1,1,2}$ is mapped to $x_1 x_{2,2} x_1^{-1}$.

9.3. The injectivity result. It is well-known that Artin’s representation is an embedding of B_∞ into $\text{Aut}(F_{\mathbb{N}})$. Our aim now is to extend the result and to prove the following result, which gives a realization of FB_∞ as a group of automorphisms of a (countable rank) free group:

Proposition 9.5. *The representation ϕ of FB_∞ in $\text{Aut}(F_{\mathbb{N}^*})$ is an embedding.*

The method for proving Proposition 9.5 relies on the possibility of considering words w of a specific form, in connection with the linear ordering of FB_∞ described in Section ???. In the case of braids, the method was first used by D. Larue in [26] (see also [16]), and it gives a useful method for proving the possible injectivity of a representation [30, 7, 31].

For u a word in the letters $x_s^{\pm 1}$, we use $\text{red}(u)$ for the freely reduced word obtained from u by removing all pairs $x_s x_s^{-1}$ and $x_s^{-1} x_s$. Thus $F_{\mathbb{N}^*}$ identifies with the set of all freely reduced words equipped with the product $u_1 * u_2 = \text{red}(u_1 u_2)$.

We begin with two auxiliary results. The first one is essentially identical to the property established as [16] Proposition 5.1.6 in the case of braids. The only change is the possible occurrence of variables x_s with s of length more than 1.

Lemma 9.6. *For each i , the image of a word ending with x_i^{-1} under $\widehat{\sigma}_i$ or under $\widehat{\sigma}_j^{\pm 1}$ with $j > i$ is a word ending with x_i^{-1} .*

Proof. Assume that u is a word ending with x_i^{-1} , say $u = u'x_i^{-1}$. Then we have

$$(9.5) \quad \widehat{\sigma}_i(u) = \text{red}(\widehat{\sigma}_i(u')x_ix_{i+1}^{-1}x_i^{-1}).$$

In order to prove that the right hand word in (9.5) ends with x_i^{-1} , it is sufficient to check that the final x_i^{-1} cannot be cancelled during the reduction process by some x_i coming from $\widehat{\sigma}_i(u')$. By (9.1), a letter x_i in $\widehat{\sigma}_i(u')$ must come from some letter in u' , namely x_i , x_i^{-1} , or x_{i+1} . We consider the three cases separately and display the involved letter in u' . Assume first $u' = u''x_iu'''$. Then (9.5) becomes

$$\widehat{\sigma}_i(u) = \text{red}(\widehat{\sigma}_i(u'')x_ix_{i+1}x_i^{-1}\widehat{\sigma}_1(u''')x_ix_{i+1}^{-1}x_i^{-1}).$$

The hypothesis that the first x_i cancels the final x_i^{-1} implies that the word $\widehat{\sigma}_i(u''')$ must be empty, hence so must be u''' . But this contradicts the hypothesis that $u''x_iu'''x_i^{-1}$ is reduced.

Assume now $u' = u''x_i^{-1}u'''$. Then (9.5) becomes

$$\widehat{\sigma}_i(u) = \text{red}(\widehat{\sigma}_i(u'')x_ix_{i+1}^{-1}x_i^{-1}\widehat{\sigma}_1(u''')x_ix_{i+1}^{-1}x_i^{-1}).$$

The hypothesis that the first x_i cancels the final x_i^{-1} implies now that $x_{i+1}^{-1}x_i^{-1}\widehat{\sigma}_i(u''')x_ix_{i+1}^{-1}$ freely reduces to the empty word, which implies that $\widehat{\sigma}_i(u''')$ must be $x_ix_{i+1}^2x_i^{-1}$, and, therefore, that u''' must be x_i^2 . But this contradicts the hypothesis that $u''x_i^{-1}u'''$ is reduced.

Finally assume $u' = u''x_{i+1}u'''$. Then (9.5) becomes

$$\widehat{\sigma}_i(u) = \text{red}(\widehat{\sigma}_i(u'')x_i\widehat{\sigma}_1(u''')x_ix_{i+1}^{-1}x_i^{-1}).$$

The hypothesis that the first x_i cancels the final x_i^{-1} implies now that $\widehat{\sigma}_i(u''')x_ix_{i+1}^{-1}$ reduces to the empty word, hence $\widehat{\sigma}_i(u''')$ must be $x_{i+1}x_i^{-1}$, and, therefore, that u''' must be $x_{i+1}^{-1}x_i$. But again this contradicts the hypothesis that $u''x_{i+1}u'''$ is reduced.

We consider now similarly the action of $\widehat{\sigma}_j^e$ with $j > i$ and $e = \pm 1$. We find

$$(9.6) \quad \widehat{\sigma}_j(u) = \text{red}(\widehat{\sigma}_j^e(u')x_i^{-1}),$$

and we still wish to show that the final x_i^{-1} cannot vanish in the reduction process. Now it could do it only with some x_i in $\widehat{\sigma}_j^e(u')$, itself coming from some x_i in u' . For a contradiction, we display the latter as $u' = u''x_iu'''$. Then (9.6) becomes

$$\widehat{\sigma}_j(u) = \text{red}(\widehat{\sigma}_j^e(u'')x_i\widehat{\sigma}_j^e(u''')x_i^{-1}),$$

and, as above, we see that $\widehat{\sigma}_j^e(u''')$ must be empty, hence that u''' must be so, contradicting the hypothesis that $u''x_iu'''x_i^{-1}$ is reduced. \square

The second preliminary result is specific to our current situation.

Definition. A word in the letters $x_s^{\pm 1}$ is said to be *special* if it is freely reduced and it admits a suffix of the form $x_s^{-1}s_{s,j_1,s_1} \dots x_{s,j_r,s_r}$, where r is nonnegative, s, s_1, \dots, s_r are finite sequences of positive integers, and j_1, \dots, j_r are positive integers.

Thus x_1^{-1} and $x_1x_2^{-1}x_{2,1}$ are special words.

Lemma 9.7. *For each i , the image of a special word under \widehat{a}_i^{-1} is a special word.*

Proof. Let $u = u'x_{j,s}^{-1}s_{j,s,j_1,s_1} \dots x_{j,s,j_r,s_r}$ be a special word. We consider the image of u under \widehat{a}_i^{-1} , according to the various mutual positions of i and j . Assume first $j < i$. Then we have $\widehat{a}_i^{-1}(x_{j,s}) = x_{j,s}$, and, similarly, $\widehat{a}_i^{-1}(x_{j,s,j_k,s_k}) = x_{j,s,j_k,s_k}$ for each k , hence

$$(9.7) \quad \widehat{a}_i^{-1}(u) = \text{red}(\widehat{a}_i^{-1}(u')x_{j,s}^{-1}x_{j,s,j_1,s_1} \dots x_{j,s,j_r,s_r}).$$

In order to conclude that the right hand word in (9.7) is special, it suffices to prove that the displayed letter $x_{j,s}^{-1}$ cannot vanish during the reduction process. Now assume that it does. The letter $x_{j,s}^{-1}$ is cancelled by some letter $x_{j,s}$, which must come from $\widehat{a}_i^{-1}(u')$. By (9.4), such a letter must come from a letter $x_{j,s}$ of u' . Let us display the considered letter and write $u' = u''x_{j,s}u'''$. Then $\widehat{a}_i^{-1}(u')$ is $\text{red}(\widehat{a}_i^{-1}(u'')x_{j,s}\widehat{a}_i^{-1}(u'''))$. Now the hypothesis that the final $x_{j,s}^{-1}$ in $\widehat{a}_i^{-1}(u')x_{j,s}^{-1}$ is cancelled by the $x_{j,s}$ above means that $\widehat{a}_i^{-1}(u''')$ is the empty word, hence, as \widehat{a}_i is an automorphism, that u''' itself is the empty word. But this means that u' finishes with $x_{j,s}$, contradicting the hypothesis that $u'x_{j,s}^{-1}$ is a reduced word.

The argument is similar in the case $j > i + 1$, and, more generally, it works in all cases except those of x_i and x_{i+1} . Indeed, in these cases, \widehat{a}_i^{-1} maps $x_{j,s}$ to a (possibly different) letter $x_{j',s'}$ so that a letter $x_{j',s'}$ in a word $\widehat{a}_i^{-1}(v)$ must come from a letter $x_{j,s}$ in v . In each such case, the previous argument shows that the letter $x_{j,s}^{-1}$ witnessing for specialness becomes a letter $x_{j',s'}^{-1}$ that cannot be cancelled. On the other hand, (9.4) shows that, in all considered cases, the final letters x_{j,s,j_k,s_k} become letters x_{j',s',j_k,s_k} , and the word $\widehat{a}_i^{-1}(u)$ is special.

So there remain the cases of x_i and x_{i+1} . To simplify reading, we assume $i = 1$. Let us first consider x_1 , *i.e.*, $u = u'x_1^{-1}x_{1,j_1,s_1} \dots x_{1,j_r,s_r}$, which gives

$$(9.8) \quad \widehat{a}_1^{-1}(u) = \text{red}(\widehat{a}_1^{-1}(u')x_{1,1}^{-1}x_{1,1,j_1,s_1} \dots x_{1,1,j_r,s_r}).$$

The question is to study whether the letter $x_{1,1}^{-1}$ can vanish in the reduction. Now (9.4) shows that a letter $x_{1,1}$ in $\widehat{a}_1^{-1}(u')$ must come either from a letter x_1 or from a letter x_2^{-1} in u' . By the same argument as above, the first case is excluded. As for the second one, let us display the involved letter x_2^{-1} and write $u' = u''x_2^{-1}u'''$. Then (9.8) becomes

$$\widehat{a}_1^{-1}(u) = \text{red}(\widehat{a}_1^{-1}(u'')x_1^{-1}x_{1,1}\widehat{a}_1^{-1}(u''')x_{1,1}^{-1}x_{1,1,j_1,s_1} \dots x_{1,1,j_r,s_r},$$

and the hypothesis is that $\widehat{a}_1^{-1}(u''')$ is the empty word. So, as above, we deduce that u''' is empty, and obtain $u' = u''x_2^{-1}$, which is not forbidden, and then

$$(9.9) \quad \widehat{a}_1^{-1}(u) = \text{red}(\widehat{a}_1^{-1}(u'')x_1^{-1}x_{1,1,s_1} \dots x_{1,1,s_r}).$$

In order to show that the right hand side of (9.9) is a special word, it is sufficient to prove that the letter x_1^{-1} cannot disappear. Now the only way x_1^{-1} could vanish is with some x_1 in $\widehat{a}_1^{-1}(u'')$, necessarily coming from some x_2 in u'' . Write $u'' = u'''x_2u''''$. As above, we obtain $\widehat{a}_1^{-1}(u''''') = \varepsilon$, hence $u'''' = \varepsilon$, implying that u'' finishes with x_2 , and contradicting that $u''x_2^{-1}$ be a reduced word. So the study for x_1 is complete: the critical letter x_1^{-1} becomes a letter $x_{1,1}^{-1}$ that may vanish in the reduction process, but, in this case, there remains instead a letter x_1^{-1} that still witnesses for specialness.

Finally, let us consider the case of x_2 . The problem here is that \widehat{a}_1^{-1} maps x_2 to $x_{1,1}^{-1}x_1$, which is not a single letter. So assume $u = u'x_2^{-1}x_{2,j_1,s_1} \dots x_{2,j_r,s_r}$. We obtain

$$(9.10) \quad \widehat{a}_1^{-1}(u) = \text{red}(\widehat{a}_1^{-1}(u')x_1^{-1}x_{1,1}x_{1,j_1+1,s_1} \dots x_{1,j_r+1,s_r}).$$

In order to conclude that $\widehat{a}_1^{-1}(u)$ is a special word, it suffices to prove that the letter x_1^{-1} cannot vanish. Now a letter x_1 in $\widehat{a}_1^{-1}(u')$ must come from a letter x_2 in u' , and we argue as above. This completes the proof. \square

With this preliminary result at hand, we can now prove the injectivity of the homomorphism ϕ of FB_∞ into $\text{Aut}(F_{\mathbb{N}^*})$.

Proof of Proposition 9.5. Our aim is to show that, if w is a word in $W(\mathbf{a}, \boldsymbol{\sigma})$ that represents a non-trivial element of FB_∞ , then the automorphism \widehat{w} is not the identity mapping of $F_{\mathbb{N}^*}$, *i.e.*, there exists at least one letter x_s such that $\widehat{w}(x_s)$ is not x_s . Proposition ?? tells us that every element of FB_∞ can be represented by a word $w_1^{-1}w_2w_3$, with w_1, w_3 in $W^+(\mathbf{a})$ and w_2

in $W(\boldsymbol{\sigma})$, so we can assume that w has the form above. We separate two cases, according to whether the braid word w_2 is trivial or not.

Case 1: $w_2 = \varepsilon$, *i.e.*, $w \in W(\mathbf{a})$. Let (t, t') be the pair of trees associated with w as in Proposition ??, *i.e.*, the seed of the corresponding associativity operator. Let $\alpha_1, \dots, \alpha_n$ (*resp.* $\alpha'_1, \dots, \alpha'_n$) be the left-to-right enumeration of the addresses of leaves in t (*resp.* t'). Then Proposition 9.4 shows that, for each k , the automorphism \widehat{w} maps $y_{\alpha'_k}$ to y_{α_k} , and $y_{\alpha'_k 0}$ to $y_{\alpha_k 0}$. The latter variables are x_s -variables. The hypothesis that w is non-trivial implies that α'_k is different from α_k for some k , so we conclude that there must exist a variable x_s such that \widehat{w} maps x_s to $x_{s'}$ with $s' \neq s$, and, therefore, \widehat{w} is not the identity mapping.

Case 2: $w_2 \neq \varepsilon$. According to the characterization of the braid ordering recalled in Proposition ?? and at the expense of possibly replacing w with w^{-1} , we can assume that there exists an index i such that at least one letter σ_i occurs in w' but no letter σ_i^{-1} or $\sigma_j^{\pm 1}$ with $j < i$ occurs in w_2 .

Let (t, t') be a pair of trees satisfying $t' = t \bullet w_3$ in the sense of the associativity action of Section 2, and such that t is a right comb of size at least $i + 1$: such a tree t certainly exists as w_1 consists of positive letters only. Let α be the i th leaf address in t' starting from the left: by Proposition 9.4, we have $\widehat{w}_3(y_\alpha) = x_i$, and, therefore, $\widehat{w}_3(y_{\alpha 0}) = x_{i,1}$, *i.e.*, $\widehat{w}_3(x_s) = x_{i,1}$, where x_s is the name of $y_{\alpha 0}$ in the x -basis (using $y_{\alpha 0}$ instead of y_α is necessary only when α does not end with 0).

We now consider $\widehat{w}_2(\widehat{w}_3(x_s))$, *i.e.*, $\widehat{w}_2(x_{i,1})$. Let us display the occurrences of σ_i in w_2 and write $w_2 = w_{2,0}\sigma_i w_{2,1}\sigma_i \dots \sigma_i w_{2,r}$, where $w_{2,k}$ contains no letter $\sigma_j^{\pm 1}$ with $j \leq i$. Then $\widehat{w}_{2,r}$ fixes $x_{i,1}$, while σ_i maps it to $x_i x_{i+1, i} x_i^{-1}$, a reduced word ending with x_i^{-1} . Applying Lemma 9.6 repeatedly, we then deduce that the final x_i^{-1} cannot disappear, and, so, $\widehat{w}_2(\widehat{w}_3(x_s))$ is a reduced word ending with x_i^{-1} .

It remains to consider the action of \widehat{w}_1^{-1} . Now every reduced word ending with x_i^{-1} is a special word, hence, by Lemma 9.7, its image under \widehat{w}_1^{-1} is still a special word. We deduce that $\widehat{w}(x_s)$ is a special word. As x_s itself is not a special word, we conclude that \widehat{w} cannot be the identity mapping. \square

9.4. Back to twisted commutativity. In Section 8, we introduced the notion of twisted (semi)-commutativity and showed that, for each left cancellative LD-system S , associativity together with S -twisted semi-commutativity gives rise to a geometry group $G(\mathcal{A}, \Sigma^S)$ which is a quotient of the group FB_∞ . A natural example of left cancellative LD-system is provided by conjugacy in a group, in particular in a free group, and we raised the question of determining the corresponding group $G(\mathcal{A}, \Sigma^S)$. The results of the current section enable us to answer the question:

Proposition 9.8. *Assume that G is a free group of rank at least 2. Then the geometry group of associativity together with G -twisted semi-commutativity is FB_∞ .*

Proof. As every free group of rank at least 2 contains a free group of countable rank, we can assume that G is $F_{\mathbb{N}^*}$ without loss of generality. We know that $G(\mathcal{A}, \Sigma^{F_{\mathbb{N}^*}})$ is a quotient of FB_∞ , and the question is to prove that, if w is a word in $W(\mathbf{a}, \boldsymbol{\sigma})$ representing a non-trivial element of FB_∞ , then the corresponding $F_{\mathbb{N}^*}$ -twisted semi-commutativity operator on $F_{\mathbb{N}^*}$ -labeled trees, say \widetilde{w} , is not the identity. Now Proposition 9.5 shows that the automorphism \widehat{w} is not the identity mapping, so, in order to prove that \widetilde{w} is not trivial, it is sufficient to show that \widehat{w} can be constructed from \widetilde{w} . But this is precisely what Proposition 9.4 asserts. \square

Finally, we proved in Section 7 that the multiscaled braid group is isomorphic to FB_∞ using FB_∞ -colourings. This result can also be deduced from the results of the current section. Indeed, it is sufficient to prove that, if w is a word in $W(\mathbf{a}, \boldsymbol{\sigma})$, then the automorphism \widehat{w} is determined by the diagram associated with w , and, in particular, that a non-trivial diagram

leads to a non-trivial automorphism. Indeed, as we know that \widehat{w} is non-trivial whenever w does not represent 1 in FB_∞ , we deduce that, in this case, the diagram associated with w is not trivial, *i.e.*, the morphism from FB_∞ to the multiscaled braid group is injective. So it just remains to explain how to compute \widehat{w} from the diagram associated with w . This can be done using $F_{\mathbb{N}^*}$ -coloured multiscaled diagrams following the same scheme as in the proof of Proposition 9.4. Assuming that D is a t strand diagram for w , we attribute to each position $\underline{\alpha}$ the colour y_α , and we propagate the colours using conjugacy of $F_{\mathbb{N}^*}$ for the crossings; then \widehat{w} is determined by the property that the image of y_α is the final label of the strand finishing at position α in the diagram. The details are easy.

9.5. Further questions. There are so many interesting results about Thompson's group F and Artin's group B_∞ , and the group FB_∞ seems to be so well suited to extend most of them that the list of potential problems about FB_∞ is virtually infinite, and we shall mention only very few here.

Firstly, the group similar to FB_∞ but involving twisted commutativity instead of semi-commutativity was not addressed here, thus a torsion-free version of Thompson's group V . The presentations of V described in Section 4 clearly indicate the candidate group—which is the group BV of [2]. The details should be easy, although probably tedious.

Next, we did not discuss much the geometry of the Cayley graph of FB_∞ here. Both in terms of the infinite sequences of generators a_i, σ_i or in terms of the finite subfamily $a_1, a_2, \sigma_1, \sigma_2$, or even of the extended families A_α, Σ_α , this study is certainly interesting.

Then, the study of the homological properties both of the braid groups and of Thompson's groups led to deep results, and extending the study to FB_∞ is an obvious task. The methods of [6] or [17] should be relevant there.

Finally, tensor braided categories should provide an alternative framework for the computations developed in this paper. Establishing that the pentagon relations make a presentation for the geometry group of associativity is another way of proving a coherence theorem for monoidal categories [28]. Is there a similar counterpart for the results established in this paper, for instance the embeddability of FB_∞ into a group of automorphisms of free groups, or the FB_∞ -evaluability of FB_∞ -decorated trees by means of the self-distributive structure on this set?

10. APPENDIX: THE CUBE CONDITION FOR THE PRESENTATION $(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}})$

The algebraic results of Section 6 mainly rely on the fact that the presentation $(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}})$ satisfies the so-called left and right *cube conditions*. There are combinatorial properties whose verification involves systematically considering all possible triples of letters. The alphabet we use here, namely the union of \mathbf{a} and $\boldsymbol{\sigma}$, is infinite, but it is easy to see that only finitely many different patterns may appear, so that the needed verifications are finite in number. Here we give these verifications.

The left cube condition. We recall from Section 3 that checking the left cube condition for a triple of letters (x, y, z) means proving that, whenever the word $xy^{-1}yz^{-1}$ can be left reversed to some word $v^{-1}u$ with u, v containing no negative letter, then the word $vxz^{-1}u^{-1}$ can be left reversed to the empty word.

In the specific case of $(\mathbf{a}, \boldsymbol{\sigma}; \mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}})$, we observed that left reversing is a deterministic process, *i.e.*, there exists at most one way to reverse a given word w to a word of the form $v^{-1}u$ with u, v positive.

Definition. For u, v in $W^+(\mathbf{a}, \boldsymbol{\sigma})$, we denote by u/v the unique positive word u' such that uw^{-1} is left $\mathbf{r}_{\mathbf{a}\boldsymbol{\sigma}}$ -reversible to $v'^{-1}u'$ for some positive word v' , if such words exist.

If w is left reversible to w' , then w^{-1} is left reversible to w'^{-1} , and it follows that, if uv^{-1} is left reversible to some word $v'^{-1}u'$, then the latter word is $(v/u)^{-1}(u/v)$. So, for instance, we have $\sigma_1/\sigma_2 = \sigma_2\sigma_1$ and $\sigma_2/\sigma_1 = \sigma_1\sigma_2$, and Lemma 6.5 can be rephrased as the equalities

$$(10.1) \quad \sigma_i/a_j = d_j(\sigma_i), \quad a_j/\sigma_i = a_{\sigma_i[j]}.$$

In the case of the a -generators, the formulas always take the form $a_i/a_j = a_{i'}$, and the involved index i' will be simply denoted i/j , so that we always have $a_i/a_j = a_{i/j}$. For instance, one has $1/2 = 1$ and $2/1 = 3$. We shall appeal to two easy auxiliary formulas.

Lemma 10.1. *For all i, j, k , we have*

$$(10.2) \quad d_k(\sigma_i)/d_k(\sigma_j) \equiv d_{\sigma_j[k]}(\sigma_i/\sigma_j),$$

$$(10.3) \quad \sigma_k[i]/\sigma_k[j] = d_j(\sigma_k)[i/j].$$

Proof. As for (10.2), the only critical case is when i and j are neighbours, and k is either i or j . So it is sufficient to consider the cases $i = 1, j = 2$, and $k = 1$ or 2 and to simultaneously consider $d_k(\sigma_i)/d_k(\sigma_j)$ and $d_k(\sigma_j)/d_k(\sigma_i)$. We start with $\sigma_1/\sigma_2 = \sigma_2\sigma_1$ and $\sigma_2/\sigma_1 = \sigma_1\sigma_2$. For $k = 1$, we obtain

$$d_1(\sigma_1)/d_1(\sigma_2) = \sigma_1\sigma_2/\sigma_3 = \sigma_2\sigma_1\sigma_3\sigma_2, \quad d_2(\sigma_2\sigma_1) = \sigma_2\sigma_3\sigma_1\sigma_2,$$

which are equivalent braid words, and, similarly, $d_1(\sigma_2)/d_1(\sigma_1) = \sigma_1\sigma_2\sigma_3 = d_2(\sigma_1\sigma_2)$. For $k = 2$, we find $d_2(\sigma_1)/d_2(\sigma_2) = \sigma_2\sigma_1 = d_3(\sigma_2\sigma_1)$ and $d_2(\sigma_2)/d_2(\sigma_1) = \sigma_1\sigma_2\sigma_3 = d_1(\sigma_1\sigma_2)$.

The argument is similar for (10.3), □

We are ready to consider all possible triples of letters. We sort them according to the numbers of σ 's and a 's. First, in the case of three σ 's or of three a 's, it is already known that the left (and right) cube condition is satisfied. So, we have only to consider the cases involving

two letters of one family and one of the other family. We shall use the notation $v' \begin{array}{c} \curvearrowright \\ \hline u \end{array} v$ to express that uv^{-1} is left reversible to $v'^{-1}u'$. Then checking the left cube condition at (x, y, z)

means that, when we reverse $xy^{-1}yz^{-1}$ to $v_1^{-1}v_2^{-1}u_2u_1$ by filling the diagram $\begin{array}{ccccc} & & u_2 & & u_1 \\ & & \curvearrowright & & \curvearrowright \\ v_2 & & & & z \\ & & \curvearrowright & & \\ v_1 & & & & y \\ & & \curvearrowright & & \\ & & x & & \end{array}$,

then the word $v_1v_2xz^{-1}u_1^{-1}u_2^{-1}$ is left reversible to the empty word, *i.e.*, filling the following

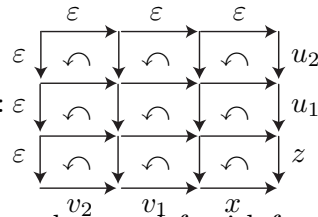


diagram ends with ε edges on the left and the top: ε So, in each case, it is

sufficient to give the corresponding two diagrams, and we are left with four cases, corresponding to the patterns (σ, σ, a) , (σ, a, σ) , (a, a, σ) , and (a, σ, a) respectively. Figures 39 to 42 give the details.

The right cube condition. The verifications for the right cube condition are similar, except that we use right reversing, denoted \curvearrowleft , instead of left reversing, *i.e.*, we push the negative letters to the right. Again, right reversing is a deterministic process in the case of the presentation $(W(\mathbf{a}, \boldsymbol{\sigma}), r_{a\sigma})$, so it leads to at most one final word of the form uv^{-1} with u, v positive. For u, v in $W^+(\mathbf{a}, \boldsymbol{\sigma})$, we shall denote by $u \setminus v$ and $v \setminus u$ the unique positive words such that $u^{-1}v$ is right reversible to $(u \setminus v)(v \setminus u)^{-1}$, if such words exist. Observe that, in contrast to the case of

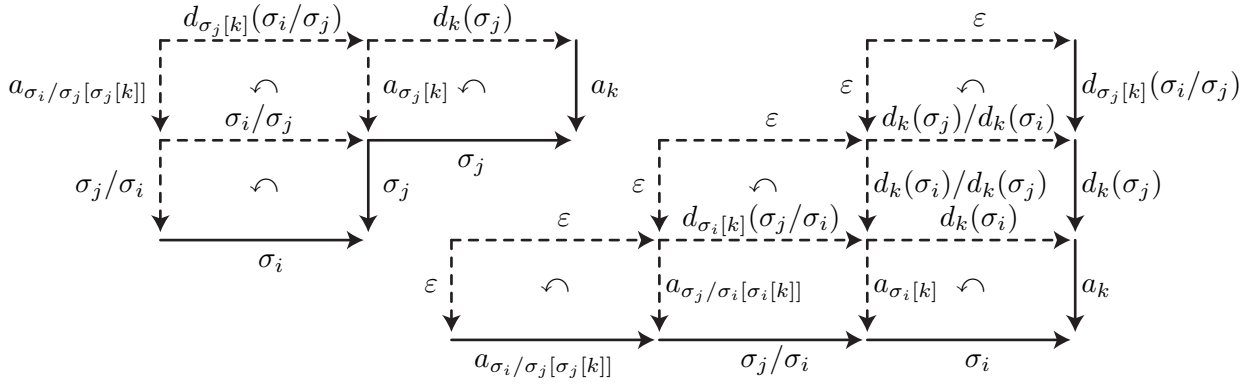


FIGURE 39. The (σ, σ, a) case: we start from $\sigma_i \sigma_j^{-1} \sigma_j a_k^{-1}$, let reverse it to $(\sigma_j/\sigma_i)^{-1} (a_{\sigma_i/\sigma_j[\sigma_j[k]]})^{-1} (d_{\sigma_j[k]}(\sigma_i/\sigma_j)) ((d_k(\sigma_j)))$, then restart from $(a_{\sigma_i/\sigma_j[\sigma_j[k]]}) (\sigma_j/\sigma_i) (\sigma_i) (\sigma_k)^{-1} (d_k(\sigma_j))^{-1} (d_{\sigma_j[k]}(\sigma_i/\sigma_j))^{-1}$ and check that the latter word left reverses to the empty word using (10.2) and the fact that the permutations associated with $(\sigma_i/\sigma_j)\sigma_j$ and $(\sigma_j/\sigma_i)\sigma_i$ coincide, as both come from the left lcm of the involved braid (once ε is obtained everywhere, there is no need to complete the last three squares).

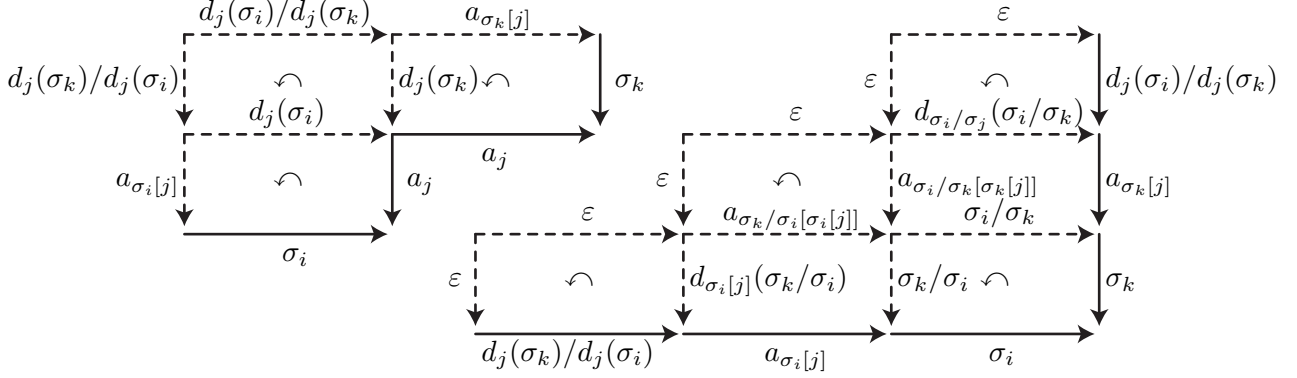


FIGURE 40. The (σ, a, σ) case: a similar method, using the same auxiliary results.

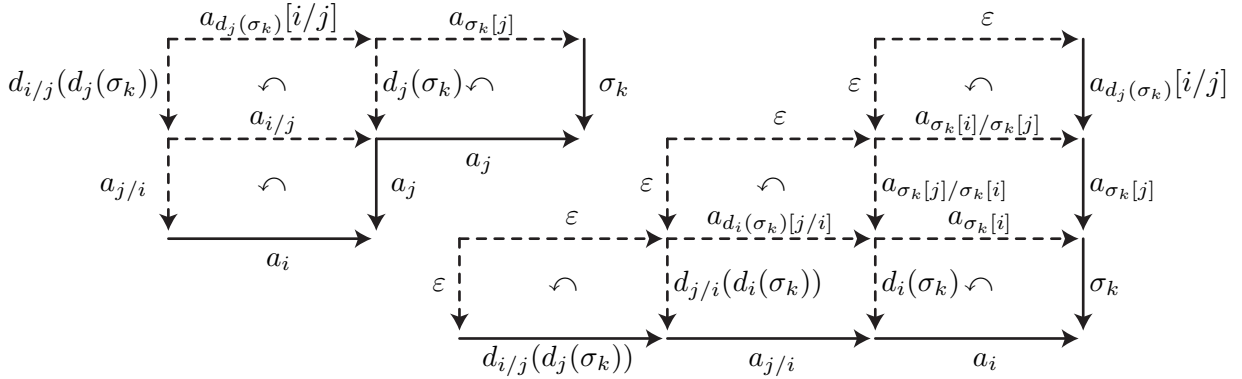


FIGURE 41. The (a, a, σ) case: again the same method, but using (10.3).

left reversing, right reversing need not converge in every case: indeed, $r_{a\sigma}$ contains no relation of the form $a_i \dots = a_{i+1} \dots$ or $\sigma_i \dots = a_i \dots$, which means that the patterns $a_i^{-1} a_{i+1}$ and $\sigma_i^{-1} a_i$ cannot be reversed on the right.

It is possible to establish general formulas similar to (10.1) to (10.3). In particular, it is easy to inductively establish the following:

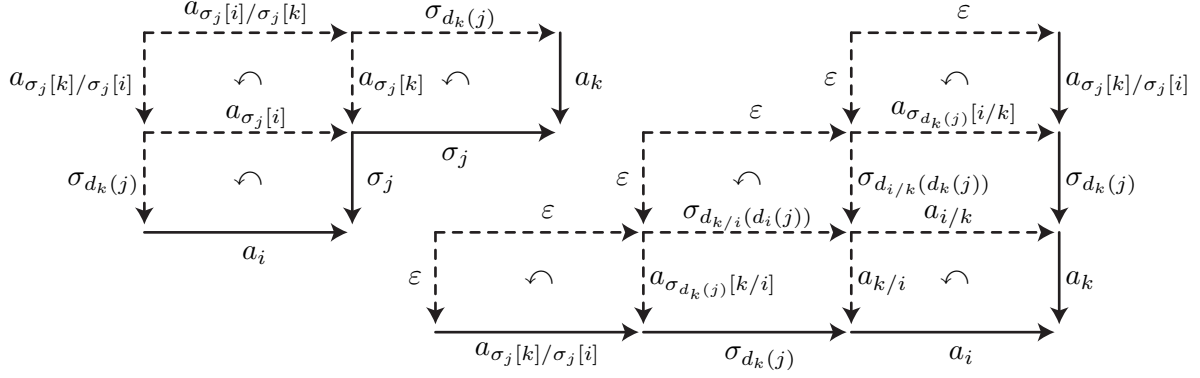


FIGURE 42. The (a, σ, a) case: similar to the (a, a, σ) case, using (10.3) again.

Lemma 10.2. *For u, v in $W^+(\sigma)$, the word $u \setminus (va_j)$, when it exists, is obtained from $u \setminus (v\sigma_j)$ by replacing the last letter σ_k with the corresponding letter a_k . The word $a_j \setminus u$, when it exists, is obtained from u by erasing the j -th strand (in the braid diagram coded by u).*

However, using such formulas is uneasy as one has to carefully check that the considered words do exist. Although seemingly longer, it is actually easier to systematically consider all possible cases, which are not so many owing to the symmetries—and to double-check using a computer program. As in the case of left reversing, we can forget about the cases involving three σ 's or three a 's; also, the cases when two letters coincide automatically hold.

For instance, we shall consider here the case (a_i, σ_j, a_k) . By symmetry, we may assume $i \leq k$, and even $i < k$ as the case when two letters coincide automatically holds, and, then, $i < k$. Assume first $k = i + 1$. Then considering the possible j 's, namely $j = i + 1$, and $j \geq i + 2$, shows that no right reversing terminates as one always reaches some pattern $a_{i'+1}^{-1} a_i$. Next assume $k = i + 2$. The cases $j \geq i + 3$ are straightforward due to commutation relations, $j = i + 2$ leads to nothing, so the only non-trivial case is $j = i + 1 = k - 1$. Then one checks that $a_i^{-1} \sigma_j \sigma_j^{-1} a_k$ right reverses to $\sigma_i \sigma_{i+1} a_i a_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}$, from where the result is easy. Assume now $k = i + 3$. Then the cases $j = i + 1$ and $j = i + 2$ are similar to the case $j = i + 1$ with $k = i + 2$ (actually simpler), while $j \leq i + 4$ is trivial owing to commutation relations. Finally, $k \geq i + 4$ is trivial, and a_j then commutes with each factor. This completes the case of the triples (a_i, σ_j, a_k) .

The three other kinds of triples are treated similarly—with no possible use of symmetry for j in the case of the non-symmetric triples $(\sigma_i, \sigma_j, a_k)$ and (a_i, a_j, σ_k) . Almost all cases are trivial. As one can expect, the only more complicated cases are those where i, j, k are neighbours. Lemma 10.2 explains that no word of length more than 6 appears in the diagrams, as all words appearing here are substitutes of words involved in the cube condition for braids, and the latter essentially are divisors of the fundamental braid Δ_4 , which has length 6 with respect to the generators σ_j . We give in Figure 43 an typical example.

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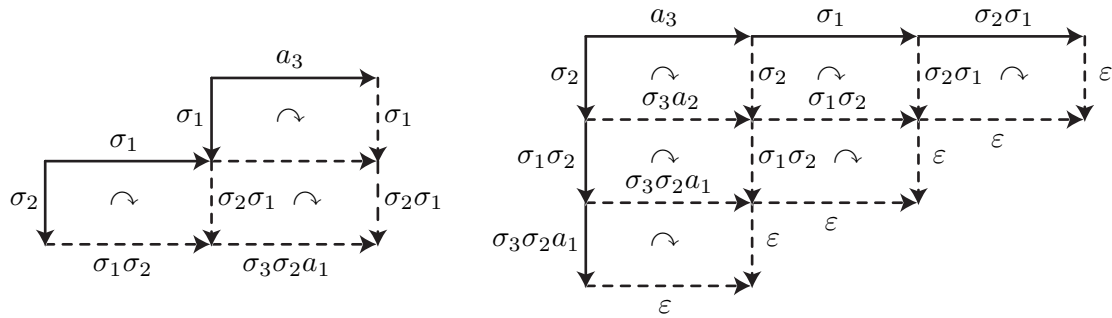


FIGURE 43. An example of verification of the right cube condition, here the triple $(\sigma_2, \sigma_1, a_3)$: one first reverses $\sigma_2^{-1}\sigma_1\sigma_1^{-1}a_3$ to a positive-negative word, here $\sigma_1\sigma_2\sigma_3\sigma_2a_1\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$, and, then, one checks that right reversing $a_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}a_3\sigma_1\sigma_2\sigma_1$ leads to the empty word.

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