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# Memory loss property for products of random matrices in the $(\max, +)$ algebra

Glenn MERLET  
glenn.merlet@univ-rennes1.fr

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## Abstract

Products of random matrices in the  $(\max, +)$  algebra are used as a model for a class of discrete event dynamical systems. J. Mairesse proved that such a system couples in finite times with a unique stationary regime if and only if it has a memory loss property.

When the system is driven by an i.i.d sequence, we prove that the memory loss property is generic in the following sense : if it is not fulfilled, the support of the common law of the sequence is included in a finite union of affine hyperplanes and in the discrete case the atoms of the measure are linearly related.

*Key words:* max-plus algebra; random matrices; discrete event systems

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## 1 Introduction

Products of matrices in the  $(\max, +)$  algebra are related to various fields such as discrete event dynamical systems, graph theory or asymptotic analysis. A review of various applications is given by Gaubert et al. [GP97].

The spectral and asymptotic theory of powers of a single matrix in the  $(\max, +)$  algebra is well known ( G. Cohen et al. [CDQV83]). Products of random matrices in the  $(\max, +)$  algebra have been investigated since J.E. Cohen [Coh88]. They have been applied as an example to modelise task graphs, cyclic Jackson networks ( Mairesse [Mai97]) or Job-Shop (G. Cohen et al.[CDQV85]). Some stability results have been proved under the so called memory loss property (Mairesse [Mai97], Gaubert and Hong [GH00]). When the system is driven by an i.i.d sequence, we prove that this property is generic in the following sense : if it is not fulfilled, the support of the common law of the sequence is included in a finite union of affine hyperplanes and in the discrete case the atoms of the measure are linearly related.

This article is divided into three sections. In the first one, we introduce the  $(\max, +)$  algebra and the matrices in that algebra and we recall the spectral and asymptotic theory. In the second one we define the system we study and the MLP, then we state and comment our main results. In the last section, we prove the results.

## 2 Matrices in $\mathbb{R}_{\max}$

For any  $k \in \mathbb{N}$ ,  $I_k$  will be the set of all integers between 1 and  $k$ .

**Definition 2.1.**

i) Let  $\mathbb{R}_{\max}$  be  $\mathbb{R} \cup \{-\infty\}$  and  $\otimes, \oplus$  be defined by :

$$\forall x, y \in \mathbb{R}_{\max}, \begin{cases} x \oplus y := \max(x, y) \\ x \otimes y := x + y \end{cases}$$

ii) For any  $k \in \mathbb{N}$ , it provides  $\mathbb{R}_{\max}^k$  with a linear-like structure :

$$\forall x, y \in \mathbb{R}_{\max}^k, \forall \alpha \in \mathbb{R}_{\max}, \forall 1 \leq i \leq k, \begin{cases} (x \oplus y)_i := x_i \oplus y_i \\ (\alpha \otimes y)_i := \alpha \otimes x_i \end{cases}$$

iii) For any  $k, l, m \in \mathbb{N}$ , the product of two matrices  $A \in \mathbb{R}_{\max}^{k \times l}$  and  $B \in \mathbb{R}_{\max}^{l \times m}$  is the matrix  $A \otimes B \in \mathbb{R}_{\max}^{k \times m}$  defined by :

$$\forall 1 \leq i \leq k, \forall 1 \leq j \leq m, (A \otimes B)_{ij} := \bigoplus_{p=1}^l A_{ip} \otimes B_{pj}.$$

Let us recall a few basic facts about  $\mathbb{R}_{\max}$ . All these facts can be checked by direct computations. After the first one, the computations are the same as in linear algebra.

i) These laws make  $\mathbb{R}_{\max}$  a semiring but not a ring because  $\oplus$  is not invertible. The neutral element of the addition (resp. the multiplication) is  $-\infty$  (resp. 0).

ii) A matrix  $A \in \mathbb{R}_{\max}^{k \times l}$  defines a  $(\max, +)$ -linear application  $\tilde{A}$  from  $\mathbb{R}_{\max}^l$  to  $\mathbb{R}_{\max}^k$  given by :

$$\forall x \in \mathbb{R}_{\max}^l, \tilde{A}(x) := A \otimes x,$$

and the product of the matrices corresponds to the composition of the functions. If  $A$  has no line of  $-\infty$ , it also defines an application from  $\mathbb{R}^k$  to  $\mathbb{R}^l$ .

iii) The image of  $\tilde{A}$  is stable under the operations of  $\mathbb{R}_{\max}^k$ . It is the  $(\max, +)$ -moduloid spanned by the columns of  $A$ . As in usual linear algebra :

$$\forall x \in \mathbb{R}_{\max}^k, A \otimes x = \bigoplus_{j \in I_k} x_j \otimes A_{.j}.$$

iv) A matrix  $A$  has rank 1 if all its columns are proportional. It happens if and only if there are  $a$  and  $b$  in  $\mathbb{R}_{\max}^k$  such that  $\forall (i, j) \in I_k^2, A_{ij} = a_i + b_j$ . We denote it by  $rk(A) = 1$ .

**Remark 2.1.** The rank 1 notion is clear but there are several notions of rank (Cunningham-Green [CG79] and Wagneur [Wag91]).

We briefly review the spectral and asymptotic theory of matrices in  $(\max, +)$ . These results are proved by G. Cohen et al. [CDQV85]. The reader interested in proofs can find them in Baccelli et al. [BCOQ92].

**Definition 2.2.** A cycle on a graph is a closed path on the graph. Let  $A$  be a square matrix of size  $k$  with entries in  $\mathbb{R}_{\max}$ .

- i) The graph of  $A$  is the directed labelled graph whose vertices are the elements of  $I_k$  and whose edges are the  $(i, j)$  such that  $A_{ij} > -\infty$ . The label on  $(i, j)$  is  $A_{ij}$ . The graph will be denoted by  $\mathcal{G}(A)$  and the set of its minimal cycles by  $\mathcal{C}(A)$ .
- ii) The average weight of a cycle  $c = (i_1, \dots, i_n, i_{n+1}) \in \mathcal{C}(A)$  (where  $i_1 = i_{n+1}$ ) is  $aw(A, c) := \frac{1}{n} \sum_{j=1}^n A_{i_j i_{j+1}}$ .
- iii) The spectral radius of  $A$  is  $\rho(A) := \max_{c \in \mathcal{C}(A)} aw(A, c)$ .
- iv) The critical graph of  $A$  is obtained from  $\mathcal{G}(A)$  by keeping only vertices and edges belonging to cycles with average weight  $\rho(A)$ . It will be denoted by  $\mathcal{G}^c(A)$ .
- v) The cyclicity of a graph is the greatest common divisor of the length of its cycles if it is strongly connected (that is if any vertex can be reached from any other in both directions). Otherwise it is the least common multiple of the cyclicities of its strongly connected components. The cyclicity of  $A$  is that of  $\mathcal{G}^c(A)$ .
- vi) The type of  $A$  is **scsN-cycC**, where **N** is the number of connected components of  $\mathcal{G}^c(A)$  and **C** the cyclicity of  $A$ .

**Remark 2.2.** Interpretation of powers with  $\mathcal{G}(A)$ .

If  $(i_1, i_2, \dots, i_n)$  is a path on  $\mathcal{G}(A)$ , its weight is  $\sum_{1 \leq j \leq n-1} A_{i_j i_{j+1}}$ , so that  $(A^{\otimes n})_{ij}$  is the maximum of the weights of length  $n$  paths from  $i$  to  $j$ .

To state the spectral theorem, we will also need the following.  
For any  $A \in \mathbb{R}_{\max}^{k \times k}$  with  $\rho(A) \leq 0$ , we set :

$$A^+ := \bigoplus_{1 \leq n \leq k} A^{\otimes n}.$$

**Remark 2.3.** For any  $(i, j) \in I_k^2$ ,  $A_{ij}^+$  is the maximum of the weights of paths from  $i$  to  $j$ . Indeed, since  $\rho(A) \leq 0$  all cycles have nonpositive weights and removing cycles from a path makes its weight greater, so  $\bigoplus_{1 \leq n \leq k} A^{\otimes n} = \bigoplus_{n \geq 1} A^{\otimes n}$  and the remark follows from remark 2.2 .

**Proposition 2.1.** Eigenvectors

- i) If  $c$  is a cycle on  $\mathcal{G}^c(A)$ , its average weight is  $\rho(A)$ .
- ii) If  $\mathcal{G}(A)$  is strongly connected,  $\rho(A)$  is the only eigenvalue of  $A$ .
- iii) If  $\rho(A) = 0$ , for any  $i \in \mathcal{G}^c(A)$ ,  $A_{\cdot i}^+$  is an eigenvector of  $A$  with eigenvalue 0.

iv) If  $\rho(A) = 0$ , for any eigenvector  $y$  of  $A$  with eigenvalue 0, we have

$$y = \bigoplus_{i \in \mathcal{G}^c(A)} y_i \otimes A_{.i}^+.$$

v) If  $\rho(A) = 0$ , and if  $i$  and  $j$  are in the same connected component of  $\mathcal{G}^c(A)$ ,  $A_{.i}^+$  and  $A_{.j}^+$  are proportional.

vi) If  $\rho(A) = 0$ , no column vector  $A_{.i}^+$  with  $i \in \mathcal{G}^c(A)$  is a  $(\max, +)$ -linear combination of the  $A_{.j}^+$  with  $j$  in other connected components.

**Proposition 2.2.** *Powers*

Assume  $\mathcal{G}(A)$  is strongly connected,  $\rho(A) = 0$  and the cyclicity of  $A$  is 1. Then for  $n$  big enough  $A^{\otimes n} = Q$ , where  $Q$  is defined by

$$\forall (i, j) \in I_k^2, Q_{ij} := \bigoplus_{l \in \mathcal{G}^c(A)} A_{il}^+ \otimes A_{lj}^+.$$

**Remark 2.4.**

1. When  $\rho(A) \neq 0$ , we use that  $\rho(A - \rho(A)O) = 0$ , where  $O$  is the  $k \times k$  matrix with all entries 1.
2. For any  $(i, j) \in I_k^2$ ,  $Q_{ij}$  is the maximum of the weight of paths from  $i$  to  $j$  that crosses  $\mathcal{G}^c(A)$ .

## 3 Results and applications

### 3.1 The model

Let  $(A(n))_{n \in \mathbb{N}}$  be a sequence of i.i.d. random variables with values in  $\mathbb{R}_{\max}^{k \times k}$ . We investigate the behaviour of the sequence  $(x(n, x_0))_{n \in \mathbb{N}}$  defined by  $x(0, x_0) = x_0$  and  $x(n+1, x_0) = A(n) \otimes x(n, x_0)$ .

As  $rk(A(n) \otimes \dots \otimes A(1)) = 1$  if and only if for any couple  $(i, j)$ ,  $x(n, x_0)_i - x(n, x_0)_j$  does not depend on the initial condition  $x_0$ , we state the following definition :

**Definition 3.1.** *The system is said to have the memory loss property (MLP) if there exist an  $n$  such that  $\mathbb{P}(rk(A(n) \otimes \dots \otimes A(1)) = 1) > 0$ .*

It is known since J.E. Cohen [Coh88], that such a system satisfy a law of large number, at least when the entries are finite and integrable. The limit of  $\frac{1}{n} \max_i x_i(n, x_0)$  is called the Lyapunov exponent.

The main reason to study the MLP, is the following result :

**Theorem 3.1.** *(Mairesse [Mai97, 6.15]) If the model has the MLP, then for any  $1 \leq i, j \leq k$  the sequences  $(x(n, x_0)_i - x(n, x_0)_j)_{n \in \mathbb{N}}$  and  $(x(n+1, x_0)_i - x(n, x_0)_j)_{n \in \mathbb{N}}$  converge in total variation, uniformly in  $x_0$ .*

Under the MLP, we also have the following result on the Lyapunov exponent :

**Theorem 3.2.** (Gaubert and Hong [GH00, 4.1]) Let  $(A_1 \cdots A_t)$  be a  $t$ -uple of matrices. For any probability vector  $p$ , we call  $L(p)$  the Lyapunov exponent of i.i.d random variables  $A(n)$  such that

$$\forall n \in \mathbb{N}, \forall i \in I_t, \mathbb{P}(A(n) = A_i) = p_i.$$

If the system has the MLP, then  $L$  is an analytical function of  $p$ . The analyticity domain only depends on the  $t$ -uple and includes all positive  $p$ .

### 3.2 The statements

The main aim of this article is to prove the two following results :

1. When  $A(n)$  follows a Bernoulli scheme with finite values  $A$  and  $B$ , the system has the MLP except for a negligible set of couples  $(A, B)$ .
2. When the support of  $A(1)$  is not included in a finite union of hyperplanes, the system has the MLP.

These results can be rephrased in a more abstract way in the two following theorems :

**Theorem 3.3.** For any  $k \geq 1$ , the complement set of

$$\{(A, B) \in (\mathbb{R}^{k \times k})^2 \mid \exists A_1, \dots, A_n \in \{A, B\}, rk(A_1 \otimes \cdots \otimes A_n) = 1\}$$

is included in a finite union of hyperplanes of  $(\mathbb{R}^{k \times k})^2$ .

**Theorem 3.4.** Let  $\mu$  be a probability on  $\mathbb{R}^{k \times k}$  whose support is not included in a finite union of affine hyperplanes of  $\mathbb{R}^{k \times k}$ . Then there exists  $n \in \mathbb{N}$  such that :

$$\mu^{\otimes n} (\{(A_i)_{i \in I_n} \in (\mathbb{R}^{k \times k})^n \mid rk(A_1 \otimes \cdots \otimes A_n) = 1\}) > 0.$$

**Corollary 3.1.** Let  $\mu$  be a measure on  $\mathbb{R}^{k \times k}$  for which affine hyperplanes are negligible, then we have :

$$\mu \otimes \mu ((\mathbb{R}^{k \times k})^2 \setminus \{(A, B) \in (\mathbb{R}^{k \times k})^2 \mid \exists A_1, \dots, A_n \in \{A, B\}, rk(A_1 \otimes \cdots \otimes A_n) = 1\}) = 0.$$

**Remark 3.1.** If  $k = 2$ , the hyperplanes of theorem 3.3 are explicit : if  $A_{11} \neq A_{22}$ ,  $B_{11} \neq B_{22}$  and  $A_{21} - A_{12} \neq B_{21} - B_{12}$ , then there is a word  $A_1 \cdots A_n$  in  $A$  and  $B$  such that  $rk(A_1 \otimes \cdots \otimes A_n) = 1$ . Moreover the product can be a power of  $A$ ,  $B$  or  $A \otimes B$ . This can be checked by a direct computation : in this case the critical graph of  $A$ ,  $B$  or  $A \otimes B$  has one vertex.

If  $k \neq 2$ , we explicitly construct products of rank 1 as powers of  $A^{\otimes n} \otimes B \otimes A^{\otimes p}$  for some  $n$  and  $p$  but there are too many hyperplanes to write them down with formulas.

We notice that in this case there is also a power of  $A^{\otimes n+p} \otimes B$  with rank 1, but we can not bound  $n$ .

For sake of simplicity, the theorems have been stated and will be proved for matrices with finite entries, but in some applications, such as heap of pieces we want to allow  $-\infty$ .

The set of matrices with given entries equal to  $-\infty$  is a vector space with dimension less than  $k^2$ . In the extension of theorem 3.3  $A$  and  $B$  take values in

that spaces. To extend the proof of theorem 3.3, we have to assume that  $A$  has a power with every entry finite. It is easy to see that this condition only depends on the position of the  $-\infty$ , that is on the vector space. Under this condition, the same proof can be conducted in that spaces and leads to the same theorem with each occurrence of  $\mathbb{R}^{k \times k}$  replaced by that spaces.

For theorem 3.4 the condition will be that the support of the restriction of  $A(1)$  to one of that spaces is not contained in a finite union of affine hyperplanes.

**Remark 3.2.** *Under the conditions of the theorem, the sequence  $(A(-1) \otimes \cdots \otimes A(-n))_{n \in \mathbb{N}}$  contracts the  $(\max, +)$  projective space in a finite number of steps. That behaviour looks like that of classic products of random matrices : Y. Guivarc'h and A. Raugi [GR85] proved that products of i.i.d. invertible matrices contract the projective space under generic hypothesis.*

*In the classic case, Le Page [LP82] used the contraction property to prove a CLT and other limit theorems. Following the same way, we are currently writing a proof of these theorems for what we called  $x(n, x_0)$ . This proof requires the MLP.*

To conclude this section, we give an example of system without the MLP :

**Example 3.1.** *If  $A(n)$  takes values  $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ , then the system has not the MLP.*

*Proof.* Matrix  $A$  maps  $u = \begin{bmatrix} 0 & -1 \end{bmatrix}'$  and  $v = \begin{bmatrix} -1 & 0 \end{bmatrix}'$  on themselves while  $B$  exchange them. Therefore the image of  $A(n) \otimes \cdots \otimes A(1)$  always contains  $u$  and  $v$  that are not collinear.  $\square$

## 4 Proofs

### 4.1 Proof of theorems 3.3 and 3.4.

Both theorems follow from the next lemma which will be proved in section 4.3.

**Lemma 4.1.** *For any couple  $(A, B) \in (\mathbb{R}^{k \times k})^2$  outside a finite union of hyperplanes, there exist two integers  $m$  and  $n$  such that the critical graph of the matrix  $A^{\otimes m} \otimes B \otimes A^{\otimes n}$  has only one vertex. As a consequence that matrix is **scs1-cyc1**.*

Theorem 3.3 directly follows from lemma 4.1 and the following :

**Lemma 4.2.** *Every matrix **scs1-cyc1**  $A \in \mathbb{R}^{k \times k}$  has a power with rank 1 .*

*Proof.* Let  $O$  be the  $k \times k$  matrix with all entries 1. From proposition 2.2, when  $n$  is big enough, the column vectors of  $A^{\otimes n} - n\rho(A)O = (A - \rho(A)O)^{\otimes n}$  are eigenvectors of  $A - \rho(A)O$ . Moreover this last matrix has spectral radius zero so it follows from proposition 2.1 that all these vectors are proportional.  $\square$

**Notation** We will denote a  $n$ -uple of matrices by  $(\mathcal{A})_{i \in I_n}$  instead of  $(A_i)_{i \in I_n}$  to use indices for entries of matrices.

Theorem 3.4, will be deduced from lemma 4.1 and the following :

**Lemma 4.3.** *If  $\mathcal{G}^c(A)$  has only one vertex, then there is a neighbourhood  $V$  of  $A$  and an integer  $n$  such that :*

$$\forall (\overset{\#}{A})_{i \in I_n} \in V^n, rk(\overset{\#}{A} \otimes \cdots \otimes \overset{\#}{A}) = 1.$$

*Proof of theorem 3.4.*

Every hyperplane  $\alpha$  given by lemma 4.1 is the kernel of a linear form  $f^\alpha$  on  $(\mathbb{R}^{k \times k})^2$ . This linear form can be written  $f_1^\alpha + f_2^\alpha$  where  $f_1^\alpha$  depends only on the first matrix and  $f_2^\alpha$  on the second one. The support of  $\mu$  is not included in the union of  $\ker f_1^\alpha$ . Therefore there exists a matrix  $A$  in the support of  $\mu$  such that  $\forall \alpha, f_1^\alpha(A) \neq 0$ . For any  $\alpha$ , the set  $\{B \in \mathbb{R}^{k \times k} | f_2^\alpha(B) = -f_1^\alpha(A)\}$  is an affine hyperplane of  $\mathbb{R}^{k \times k}$  or the emptyset. Therefore there exists a  $B$  in the support of  $\mu$  such that  $B$  is not in  $\cup_\alpha \{B \in \mathbb{R}^{k \times k} | f_2^\alpha(B) = -f_1^\alpha(A)\}$ . Finally  $(A, B) \notin \bigcup_\alpha \ker f^\alpha$ .

By lemma 4.1 there exist  $m$  and  $n$  such that  $\mathcal{G}^c(A^{\otimes m} \otimes B \otimes A^{\otimes n})$  has only one vertex. By lemma 4.3 there exists a neighbourhood  $V$  of  $A^{\otimes m} \otimes B \otimes A^{\otimes n}$  and an integer  $N$  such that every matrix in  $V^{\otimes N}$  has rank 1. Let  $V_1 \times V_2$  be a neighbourhood of  $(A, B)$  such that  $V_1^{\otimes m} \otimes V_2 \otimes V_1^{\otimes n} \subset V$ . The matrices in  $(V_1^{\otimes m} \otimes V_2 \otimes V_1^{\otimes n})^{\otimes N}$  have rank 1. As  $A$  and  $B$  are in the support of  $\mu$ ,  $\mu(V_1) > 0$  and  $\mu(V_2) > 0$ , so

$$\mu^{\otimes (n+m+1)N} (\{(A_i)_{i \in I_{(n+m+1)N}} | rk(A_1 \otimes \cdots \otimes A_{(n+m+1)N}) = 1\}) \geq \mu(V_1)^{(n+m)N} \mu(V_2)^N > 0.$$

□

The end of this section will be devoted to the proof of lemma 4.3 by putting together the proofs of the results we used in lemma 4.2.

To understand the powers of  $A$ , we used that their entries are the weights of paths on  $\mathcal{G}(A)$ , as explained in remark 2.2. We want to do the same for products of several matrices, which means the edges weights can be different at every step.

From now on,  $\mathcal{G}$  will be the complete directed graph with vertices the elements of  $I_k$ . For any finite sequence of matrices  $(\overset{\#}{A})_{i \in I_n}$  we give the following notations :

- The weight of a path  $pth = (i_j)_{j \in I_{n+1}}$  on  $\mathcal{G}$  (with respect to  $(\overset{\#}{A})_{i \in I_n}$ ) is  $w(pth) := \sum_{j \in I_n} \overset{\#}{A}_{i_j i_{j+1}}$
- A path is maximising if its weight is maximal among the weights of paths with the same origin, the same end, and the same length.

With these definitions  $(i_j)_{j \in I_{n+1}}$  is maximising if and only if its weight is  $(\overset{\#}{A} \otimes \cdots \otimes \overset{\#}{A})_{i_1 i_{n+1}}$ .

*Proof of lemma 4.3.* As  $\mathcal{G}^c(A)$  has only one vertex, there exists  $l \in I_k$  such that :  $\forall c \in \mathcal{C}(A) \setminus \{(l, l)\}, A_{ll} > aw(A, c)$ . So there exists an  $\varepsilon > 0$  such that

$$\forall c \in \mathcal{C}(A) \setminus \{(l, l)\}, A_{ll} - aw(A, c) > 3\varepsilon.$$

Let  $V$  be the open ball with center  $A$  and radius  $\varepsilon$  for infinity norm and  $M$  be the maximum of the infinity norm on  $V$ .



Let us notice that every matrix  $B \in V$  has the same critical graph as  $A$ . Let  $\tilde{B}$  be the matrix with spectral radius 0 defined by  $\tilde{B}_{ij} = B_{ij} - B_{ll}$ . Then  $\|\tilde{B} - \tilde{A}\|_\infty < 2\varepsilon$ .

From now on,  $(\tilde{A})_{i \in I_n}$  will be in  $V^n$  and the weights of paths will always be with respect to  $(\tilde{A})_{i \in I_n}$ .

Let  $pth = (i_j)_{j \in I_{n+1}}$  be a path of length  $n$  that does not cross  $l$ . It can split into a path of length less than  $k$  and minimal cycles. As a minimal cycle that is not  $(l, l)$  and whose length is  $t$  has weight  $w(c) \leq aw(\tilde{A}, c)t + 2t\varepsilon < -t\varepsilon$ , we have :

$$w(pth) < -(n - k)\varepsilon + 2kM.$$

But for any  $i, j \in I_k$ ,

$$\left( \tilde{A} \otimes \cdots \otimes \tilde{A} \right)_{ij} \geq w((i, l, \dots, l, j)) > -2M,$$

so there exists an  $N$  such that every maximising path of length  $n \geq N$  crosses  $l$ .

Let  $pth = (i_j)_{j \in I_{n+1}}$  be a maximising path of length  $n \geq 2N + 1$ . As  $(i_j)_{j \in I_{N+1}}$  is also maximising, there is a  $j_0 \leq N$  such that  $i_{j_0} = l$ . As  $(i_j)_{n-N \leq j \leq n+1}$  is maximising for  $(\tilde{A})_{n-N \leq j \leq n+1}$ , there exists  $n - N \leq j_1 \leq n + 1$  such that  $i_{j_1} = l$ . The path  $(i_j)_{j_0 \leq j \leq j_1}$  is a cycle, so it can split into minimal cycles. As minimal cycles have a negative weight, except for  $(l, l)$ , the only subcycle of  $(i_j)_{j_0 \leq j \leq j_1}$  is  $(l, l)$ . Consequently for any  $j$  between  $j_0$  and  $j_1$ , and practically for  $N + 1 \leq j \leq n - N$ ,  $i_j = l$ , so that

$$w(pth) = w((i_j)_{1 \leq j \leq N+1}) + w((i_j)_{n-N \leq j \leq n+1}).$$

This means that :

$$\forall n \geq 2N+1, \forall i, j \in I_k, \left( \tilde{A} \otimes \cdots \otimes \tilde{A} \right)_{i,j} = \left( \tilde{A} \otimes \cdots \otimes \tilde{A} \right)_{il} + \left( \tilde{A}^{\otimes n-N} \otimes \cdots \otimes \tilde{A} \right)_{lj},$$

so that  $rk \left( \tilde{A} \otimes \cdots \otimes \tilde{A} \right) = 1$  and also  $rk \left( \tilde{A} \otimes \cdots \otimes \tilde{A} \right) = 1$ .  $\square$

## 4.2 Reduced matrix

To prove lemma 4.1, we introduce the notion of reduced matrix.

**Definition 4.1.** A matrix  $A \in \mathbb{R}_{\max}^{k \times k}$  is said reduced if :

$$\begin{cases} \forall (i, j) \in I_k^2, A_{ij} \leq 0 \\ \forall i \in I_k, \exists j \in I_k A_{ij} = 0. \end{cases} \quad (1)$$

It is said strictly reduced if :

$$\begin{cases} \forall (i, j) \in I_k^2, A_{ij} \leq 0 \\ \forall i \in I_k, \exists ! j \in I_k A_{ij} = 0. \end{cases} \quad (2)$$

**Proposition 4.1.**

- i) The set of reduced matrices is a semigroup. The set of strictly reduced matrices too.
- ii) Reduced matrices have spectral radius zero and a critical graph with 0 on each edge.

*Proof.* The proofs are straightforward.  $\square$

To simplify the statements, we will use the following notion of genericity.

**Definition 4.2.** We say that a property  $P$  of a matrix is generic (or a generic matrix has property  $P$ ) if every matrix outside a finite union of hyperplanes of  $\mathbb{R}^{k \times k}$  has property  $P$ .

We say that a property  $P$  of a couple of matrices is generic if every couple of matrices outside a finite union of hyperplanes of  $(\mathbb{R}^{k \times k})^2$  has property  $P$ .

**Proposition 4.2.** There is a finite set  $\mathcal{A}$  and for any  $\alpha \in \mathcal{A}$  there are  $\rho^\alpha$  and  $V_i^\alpha$  linear forms in the entries of  $A$ , such that for any  $A \in \mathbb{R}^{k \times k}$ , there exists an  $\alpha \in \mathcal{A}$  such that  $\bar{A}$  defined by  $\bar{A}_{ij} := A_{ij} - \rho^\alpha(A) - V_i^\alpha(A) + V_j^\alpha(A)$  is reduced. Moreover, for a generic  $A$ ,  $\bar{A}$  is strictly reduced.

To prove this proposition, we need the following definition.

**Definition 4.3.** For any  $\xi \in I_k^k$ , let  $A^\xi$  be  $\{C \in \mathbb{R}^{k \times k} \mid \forall i \in I_k, C_{i\xi(i)}\} = 0$  and let  $\varphi_\xi$  be the linear application from  $A^\xi \times \mathbb{R} \times \mathbb{R}^{k-1}$  to  $\mathbb{R}^{k \times k}$  defined by :

$$\varphi_\xi(C, \rho, V)_{ij} = C_{ij} + V_i - V_j + \rho$$

where  $V_k = 0$  by definition.

*Proof of proposition 4.2.* If  $V$  is an eigenvector associated with eigenvalue  $\rho$ , the matrix with entries  $A_{ij} - \rho - V_i + V_j$  is reduced : it is equivalent to the relation  $A \otimes V = \rho \otimes V$ .

For any cycle  $c = (i_1, i_2, \dots, i_n)$  of  $I_k$ , the application that maps  $A$  on  $aw(A, c) = \frac{1}{n} \sum_{j=1}^n A_{i_j i_{j+1}}$  is a linear form of  $A$ . So let us choose as  $\rho^\alpha$  the  $aw(\cdot, c)$ , for  $c \in \mathcal{C}(A)$ . Since the length of  $c$  is always less than  $k$ , there are finitely many  $c$ .

By proposition 2.1, we can take  $V_i = A_{ij}^+$  for some  $j$ . As  $A_{ij}^+ = A_{ij}^{\otimes n} = \sum_{j=1}^n A_{i_j i_{j+1}}$  for some  $n \leq k$  and some finite sequence of  $i_l$ , we have found the  $V_i^\alpha$  among the applications of the form  $A \mapsto \sum_{j=1}^n A_{i_j i_{j+1}}$ . The set of this applications is finite.

It remains to show that for a generic  $A$ ,  $\bar{A}$  is strictly reduced. Let  $V_i := V_i^\alpha(A) - V_k^\alpha(A)$ . Then

$$\forall A \in \mathbb{R}^{k \times k}, \exists \xi, A = \varphi_\xi(\bar{A}, \rho(A), V).$$

If  $\bar{A}$  is not strictly reduced, there exist  $i$  and  $j \neq \xi(i)$ , such that  $\bar{A}_{ij} = 0$ , so  $A \in \varphi_\xi(\{(C, \rho, V) \mid C_{ij} = 0\})$ . Since  $A^\xi \times \mathbb{R} \times \mathbb{R}^{k-1}$  has dimension  $k^2$ ,  $\varphi_\xi(\{(C, \rho, V) \mid C_{ij} = 0\})$  has dimension at most  $k^2 - 1$ . Therefore the set of matrices  $A$  with  $\bar{A}$  not strictly reduced is a subset of the union on  $\xi$ ,  $i$  and  $j \neq \xi(i)$  of these strict subspaces of  $\mathbb{R}^{k \times k}$ . A generic  $A$  is outside this union so generically  $\bar{A}$  is strictly reduced.  $\square$

In the following, we will use without repeating that the critical graph of  $\bar{A}$  has the same vertices and edges as the one of  $A$  with 0 on each edge.

### 4.3 Proof of lemma 4.1

Let  $A$  and  $B$  be two square matrices of size  $k$ . By proposition 4.2 there are  $\bar{A}$  and  $\alpha$  such that :

$$\bar{A}_{ij} := A_{ij} - \rho^\alpha(A) - V_i^\alpha(A) + V_j^\alpha(A).$$

Let  $\bar{B}$  be such that :

$$\bar{B}_{ij} := B_{ij} - V_i^\alpha(A) + V_j^\alpha(A).$$

Moreover we assume that  $\bar{A}$  is strictly reduced and denote by  $\xi$  the application that maps  $i$  to the unique  $j$  such that  $\bar{A}_{ij} = 0$ . We also assume that there is only one cycle on  $\mathcal{G}^c(A)$ . The first assumption is generic from proposition 4.2. The second one is generic because if  $c_1$  and  $c_2$  are two minimal cycles on  $\mathcal{G}^c(A)$ , the non-zero linear form  $aw(\cdot, c_1) - aw(\cdot, c_2)$  vanishes in  $A$ .

By proposition 2.2 applied to  $\bar{A}^{\otimes c(A)}$  there is  $P = \bar{A}^{\otimes N}$  such that every great power of  $\bar{A}$  is equal to  $\bar{A}^{\otimes p} \otimes P$  for some  $p \leq k$ . As in the proof of lemma 4.3, we investigate the paths on  $\mathcal{G}$ . The decomposition of the paths in minimal cycles shows that, up to taking a bigger  $N$ , every maximising path of length  $N$  on  $\mathcal{G}(\bar{A})$  crosses  $\mathcal{G}^c(A)$ .

Let us search for the maximal entries of  $\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)}$ . Such an entry  $(\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)})_{ij}$  is the weight of a path  $(i_r)_{r \in I_{N+k+p+1}}$  from  $i$  to  $j$  (figure 1). We denote  $i_{N+1}$  by  $l$  and  $i_{N+2}$  by  $m$ .

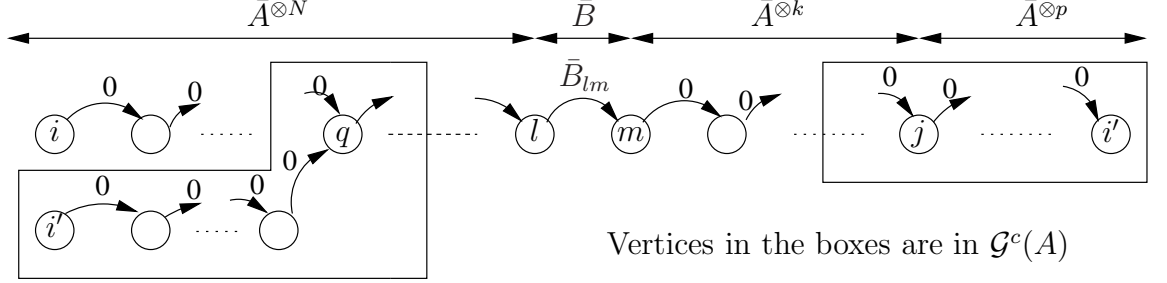
If we replace every  $i_r$  with  $r \geq N+3$  by  $\xi^{r-N-2}(m)$ , we replace edges with negative weights by edges with weight 0, so we get a path with greater weight. Since this weight can not be strictly greater, and since  $\bar{A}$  is strictly reduced, this means that  $i_r = \xi^{r-N-2}(m)$ . Therefore a maximal entry is equal to  $P_{il} + \bar{B}_{lm}$  and does not depend on  $p$ . Moreover,  $(i_r)_{N+2 \leq r \leq N+k+p+2}$  crosses twice the same vertex, so it contains a cycle. Since all edges have zero weight, the cycle is on  $\mathcal{G}^c(A)$ . As for any  $r \geq N+3$ ,  $i_{r+1} = \xi(i_r)$ , the path stays on this cycle so  $j \in \mathcal{G}^c(A)$ .

On the other side,  $(i_r)_{r \in I_{N+1}}$  crosses  $\mathcal{G}^c(A)$ . Let  $q$  be the first vertex in  $\mathcal{G}^c(A)$ . We can find a new path such that every edge before  $q$  is in  $\mathcal{G}^c(A)$  : we construct it backward from  $q$ . The first vertex of this path is called  $i'$  and belongs to  $\mathcal{G}^c(A)$ . By construction  $(\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)})_{i'j}$  is greater than  $(\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)})_{ij}$  so it is maximal. Since  $\mathcal{G}^c(A)$  is strongly connected, there is a  $p$  such that  $j = i'$ . From now on, we will assume it is the case.

The matrix  $M := \bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes k+p}$  has a maximum entry on its diagonal. This entry is the spectral radius of  $M$  and the weight of every edge of  $\mathcal{G}^c(M)$ . Let us show that generically all edges with this weight have the same end.

First  $A^+$  is a  $(\max, +)$  polynomial in  $A$ , so there is a finite set  $\mathcal{B}$  and for any  $\beta \in \mathcal{B}$  a linear form  $f^\beta$  such that  $A_{ij}^+ = f^\beta(A)$  for some  $\beta \in \mathcal{B}$ . Second  $P_{ij} = (\bar{A}^{\otimes c(A)})_{ih}^+ + (\bar{A}^{\otimes c(A)})_{hj}^+$  is also the image of  $\bar{A}$  by a linear form, and the image of  $A$  by another linear form. At each step, the linear form is chosen in a finite set, so there exists a finite set  $\Gamma$  and linear forms  $P_{ij}^\gamma$  and  $V_i^\gamma$  such that for every  $A$  there is a  $\gamma \in \Gamma$  such that  $P = P^\gamma(A)$  and  $V = V^\gamma(A)$ .

Figure 1: Maximal paths of  $\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)}$ .



We have proved that the maximal  $M_{ij}$  are equal to  $P_{il} + \bar{B}_{lm}$  for some  $l$  and  $m$ . If there are two indices such that  $(l, m)$  are different, then the linear form

$$P_{i_1 l_1}^\gamma + B_{l_1 m_1}^\circ + V_{m_1}^\gamma - V_{l_1}^\gamma - P_{i_2 l_2}^\gamma - B_{l_2 m_2}^\circ - V_{m_2}^\gamma + V_{l_2}^\gamma$$

where  $B_{lm}^\circ$  is the  $(l, m)$ -coordinate application, vanishes in  $(A, B)$ . But this linear form is not zero, because the  $B$  component is the difference of the two coordinates applications  $B_{l_1 m_1}^\circ$  and  $B_{l_2 m_2}^\circ$ . If no such linear form vanishes in  $(A, B)$ , then all maximal entries of  $M$  are equal to  $P_{il} + \bar{B}_{lm}$  with the same  $(l, m)$ . Since there are finitely many  $\gamma$  and indices, this condition is fulfilled outside a finite union of hyperplanes.

So generically if  $M_{ij}$  is a maximal entry of  $M$  we have  $j = \xi^{k+p}(m) = i'$ . That means that all edges of  $\mathcal{G}^c(M)$  end in  $i'$ . The only possible cycle with such edges is  $(i', i')$ .

To conclude we notice that  $\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)}$  and  $A^{\otimes N} \otimes B \otimes A^{\otimes(k+p)}$  have the same critical graph.

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