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Memory loss property for products of random matrices in the $(\max, +)$ algebra

Glenn MERLET

May 24, 2004

Abstract

Products of random matrices in the $(\max, +)$ algebra are used as a model for a class of discrete event dynamical systems. J. Mairesse proved that such a system couples in finite times with a unique stationary regime if and only if it has a memory loss property.

We prove that the memory loss property is generic in the following sense : if it is not fulfilled, the support of the measure is included in a finite union of affine hyperplanes and in the discrete case the atoms of the measure are linearly related.

Keywords: max-plus algebra; random matrices; DEDS

AMS 2000 Subject Classification: Primary 93C65

Secondary 93B25;68R10;15A52

1 Introduction

Products of matrices in the $(\max, +)$ algebra are related to various fields such as discrete event dynamical systems, graph theory or asymptotic analysis. A review of various applications is given in [GP97].

The spectral and asymptotic theory of powers of a single matrix in the $(\max, +)$ algebra is well known.(cf [CDQV83]) Products of random matrices in the $(\max, +)$ algebra have been investigated since [Coh88]. Some results have been proved under the so called "memory loss property" ([Mai97] and [GH00]). In this article we prove that this condition is generic in the following sense : if it is not fulfilled, the support of the measure is included in a finite union of affine hyperplanes and in the discrete case the atoms of the measure are linearly related.

This article is divided into three sections. In the first one, we introduce the $(\max, +)$ algebra and the matrices in that algebra and we recall the spectral and asymptotic theory. In the second one we state our main results and apply them to the cases of [Mai97] and [GH00]. In the last section, we prove the results.

2 Matrices in \mathbb{R}_{\max}

For any $k \in \mathbb{N}$, I_k will be the set of all integers between 1 and k .

Definition 2.1.

i) Let \mathbb{R}_{\max} be $\mathbb{R} \cup \{-\infty\}$ and \otimes, \oplus be defined by :

$$\forall x, y \in \mathbb{R}_{\max}, \begin{cases} x \oplus y := \max(x, y) \\ x \otimes y := x + y \end{cases}$$

ii) For any $k \in \mathbb{N}$, it provides \mathbb{R}_{\max}^k with a linear-like structure :

$$\forall x, y \in \mathbb{R}_{\max}^k, \forall \alpha \in \mathbb{R}_{\max}, \forall i \leq k, \begin{cases} (x \oplus y)_i := x_i \oplus y_i \\ (\alpha \otimes y)_i := \alpha \otimes y_i \end{cases}$$

iii) For any $k, l, m \in \mathbb{N}$, the product of two matrices $A \in \mathbb{R}_{\max}^{k \times l}$ and $B \in \mathbb{R}_{\max}^{l \times m}$ is the matrix $A \otimes B \in \mathbb{R}_{\max}^{k \times m}$ defined by :

$$\forall i \leq k, \forall j \leq m, (A \otimes B)_{ij} := \bigoplus_{p=1}^l A_{ip} \otimes B_{pj}.$$

Proposition 2.1.

i) *These laws make \mathbb{R}_{\max} a semiring but no ring because \oplus is not invertible. The neutral element of the addition (resp. the multiplication) is $-\infty$ (resp. 0).*

ii) *A matrix $A \in \mathbb{R}_{\max}^{k \times l}$ defines a $(\max, +)$ -linear application \tilde{A} from \mathbb{R}_{\max}^l to \mathbb{R}_{\max}^k given by :*

$$\forall x \in \mathbb{R}_{\max}^l, \tilde{A}(x) := A \otimes x,$$

and the product of the matrices corresponds to the composition of the functions. If A has no line of $-\infty$, it also defines an application from \mathbb{R}^k to \mathbb{R}^l .

iii) The image of \tilde{A} is stable under the operations of \mathbb{R}_{\max}^k . It is the $(\max, +)$ -moduloid spanned by the columns of A . As in usual linear algebra :

$$\forall x \in \mathbb{R}_{\max}^k, A \otimes x = \bigoplus_{j \in I_k} x_j \otimes A_{.j}.$$

iv) A matrix A has rank 1 if all its columns are proportional. It happens if and only if there are a and b in \mathbb{R}^k such that $\forall (i, j) \in I_k^2, A_{ij} = a_i + b_j$. We denote it by $rk(A) = 1$.

Proof. All the assumptions can be checked by direct computations. After the first one, the computations are the same as in linear algebra. \square

Remark 2.1. The rank 1 notion is clear but there are several notions of rank.(cf [CG79] and [Wag91])

We briefly review the spectral and asymptotic theory of matrices in $(\max, +)$. These results are from [CDQV85]. The reader interested in proofs can find them in [BCOQ92].

Definition 2.2. A cycle on a graph is a closed path on the graph and a loop is a cycle of length 1. Let A be a square matrix of size k with coefficients in \mathbb{R}_{\max} .

- i) The graph of A is the directed labelled graph whose vertices are the elements of I_k and whose edges are the (i, j) such that $A_{ij} > -\infty$. The label on (i, j) is A_{ij} . The graph will be denoted by $\mathcal{G}(A)$ and the set of its minimal cycles by $\mathcal{C}(A)$.
- ii) The average weight of a cycle $c = (i_1, \dots, i_n, i_{n+1}) \in \mathcal{C}(A)$ (where $i_1 = i_{n+1}$) is $aw(A, c) := \frac{1}{n} \sum_{j=1}^n A_{i_j i_{j+1}}$.
- iii) The spectral radius of A is $\rho(A) := \max_{c \in \mathcal{C}} aw(A, c)$.
- iv) The critical graph of A is obtained from $\mathcal{G}(A)$ by keeping only vertices and edges that belong to cycles with average weight $\rho(A)$. It will be denoted by $\mathcal{G}^c(A)$.
- v) The cyclicity of a graph is the greatest common divisor of the length of its cycles if it is connected. Otherwise it is the least common multiple of the cyclicities of its connected components. The cyclicity of A is that of $\mathcal{G}^c(A)$.

vi) The type of A is **scsN-cycC**, where \mathbf{N} is the number of connected components of $\mathcal{G}^c(A)$ and \mathbf{C} the cyclicity of A .

Remark 2.2. Interpretation of powers with $\mathcal{G}(A)$.

If (i_1, i_2, \dots, i_n) is a path on $\mathcal{G}(A)$, its weight is $\sum_{1 \leq j \leq n-1} A_{i_j i_{j+1}}$, so that $A_{ij}^{\otimes n}$ is the maximum of the weights of length n paths from i to j .

To state the spectral theorem, we will also need the following :

Definition 2.3.

For any $A \in \mathbb{R}_{\max}^{k \times k}$ with $\rho(A) \leq 0$,

$$A^+ := \bigoplus_{1 \leq n \leq k} A^{\otimes n}.$$

Remark 2.3. For any $(i, j) \in I_k^2$, A_{ij}^+ is the maximum of the weights of paths from i to j . Indeed, since $\rho(A) \leq 0$ all cycles have nonpositive weights and removing cycles from a path makes its weight greater, so $\bigoplus_{1 \leq n \leq k} A^{\otimes n} = \bigoplus_{n \geq 1} A^{\otimes n}$ and the remark follows from remark 2.2 .

Proposition 2.2. Eigenvectors

- i) If c is a cycle on $\mathcal{G}^c(A)$, its average weight is $\rho(A)$.
- ii) If $\mathcal{G}(A)$ is connected, $\rho(A)$ is the only eigenvalue of A .
- iii) If $\rho(A) = 0$, for any $i \in \mathcal{G}^c(A)$, $A_{\cdot i}^+$ is an eigenvector of A with eigenvalue 0.
- iv) If $\rho(A) = 0$, for any eigenvector y of A with eigenvalue 0, we have

$$y = \bigoplus_{i \in \mathcal{G}^c(A)} y_i \otimes A_{\cdot i}^+.$$

- v) If $\rho(A) = 0$, and if i and j are in the same connected component of $\mathcal{G}^c(A)$, $A_{\cdot i}^+$ and $A_{\cdot j}^+$ are proportional.
- vi) If $\rho(A) = 0$, no column vector $A_{\cdot i}^+$ with $i \in \mathcal{G}^c(A)$ is a $(\max, +)$ -linear combination of the $A_{\cdot j}^+$ with j in other connected components.

Proposition 2.3. Powers

Assume $\mathcal{G}(A)$ is connected, $\rho(A) = 0$ and the cyclicity of A is 1. Then for n big enough $A^{\otimes n} = Q$, where Q is defined by

$$\forall (i, j) \in I_k^2, Q_{ij} := \bigoplus_{l \in \mathcal{G}^c(A)} A_{il}^+ \otimes A_{lj}^+.$$

Remarks 2.4.

1. When $\rho(A) \neq 0$, we use that $\rho(A - \rho(A)O) = 0$, where O is the $k \times k$ matrix with all coefficients 1.
2. For any $(i, j) \in I_k^2$, Q_{ij} is the maximum of the weight of paths from i to j that crosses $\mathcal{G}^c(A)$.

3 Results and applications

The aim of this article is to prove the two following theorems :

Theorem 3.1. *For any $k \geq 1$, the complement set of*

$$\{(A, B) \in (\mathbb{R}^{k \times k})^2 \mid \exists A_1, \dots, A_n \in \{A, B\}, rk(A_1 \otimes \dots \otimes A_n) = 1\}$$

is included in a finite union of hyperplanes of $(\mathbb{R}^{k \times k})^2$.

Theorem 3.2. *Let μ be a probability on $\mathbb{R}^{k \times k}$ whose support is not included in a finite union of affine hyperplanes of $\mathbb{R}^{k \times k}$. Then there exists $n \in \mathbb{N}$ such that :*

$$\mu^{\otimes n} (\{(A_i)_{i \in I_n} \in (\mathbb{R}^{k \times k})^n \mid rk(A_1 \otimes \dots \otimes A_n) = 1\}) > 0.$$

Remarks 3.1.

- 1 For sake of simplicity, the theorems have been stated and will be proved for matrices with finite coefficients. If we want to allow $-\infty$, we have to take a condition of fixed structure, irreducibility and aperiodicity, as in [Mai97] : the place of the $-\infty$ are not random and there is a fixed power of the matrices that has no coefficient equal to $-\infty$.

The set of matrices with given coefficients equal to $-\infty$ is a vector space with dimension less than k^2 . Under the irreducibility and aperiodicity assumptions, the same proofs can be conducted in that spaces and lead to the same theorems with each occurrence of $\mathbb{R}^{k \times k}$ replaced by that spaces.

On the opposite in a fixed structure without aperiodicity or without irreducibility there will allways be $-\infty$ coefficient in every product, so memory loss property is impossible.

- 2 If $k = 2$, the hyperplanes of theorem 3.1 are explicit : if $A_{11} \neq A_{22}$, $B_{11} \neq B_{22}$ and $A_{21} - A_{12} \neq B_{12} - B_{21}$, then there is a word $A_1 \dots A_n$ in A and B such that $rk(A_1 \otimes \dots \otimes A_n) = 1$. Moreover the product can be a power of A , B or $A \otimes B$.

If $k \neq 2$, we explicitly construct products of rank 1 as powers of $A^{\otimes n} \otimes B \otimes A^{\otimes p}$ for some n and p but there are too many hyperplanes to write them down with formulas.

Corollary 3.1. *For any measure μ on $\mathbb{R}^{k \times k}$ that gives 0 measure to affine hyperplanes*

$$\mu \otimes \mu \left((\mathbb{R}^{k \times k})^2 \setminus \left\{ (A, B) \in (\mathbb{R}^{k \times k})^2 \mid \exists A_1, \dots, A_n \in \{A, B\}, rk(A_1 \otimes \dots \otimes A_n) = 1 \right\} \right) = 0.$$

Applications

Let us give some notations to state results. Let $(\Omega, \mathcal{F}, \theta, P)$ be a measurable dynamical system. That is \mathcal{F} is a σ -algebra of subsets of Ω , P is a probability on \mathcal{F} and θ is an \mathcal{F} -measurable application from Ω to itself that preserves P . Let A be a measurable function from Ω to $\mathbb{R}_{\max}^{k \times k}$. We will write $A(n)$ for $A \circ \theta^n$.

We investigate the behaviour of the sequence $x(n, x_0)$ defined by $x(0, x_0) = x_0$ and $x(n+1, x_0) = A(n) \otimes x(n, x_0)$.

Remark 3.2. This explains the name memory loss : $rk(A(n) \otimes \dots \otimes (1)) = 1$ if and only if for any couple (i, j) , $x(n, x_0)_i - x(n, x_0)_j$ does not depend on the initial condition x_0 .

Together with theorem 3.2, theorem [Mai97, 6.15] gives :

Corollary 3.2. *Let μ be a probability on $\mathbb{R}^{k \times k}$ whose support is not in a finite union of affine hyperplanes. Let P be the product probability on $\Omega = (\mathbb{R}^{k \times k})^{\mathbb{N}}$ with marginal μ . Then, for any $i, j \in I_k$, the sequences $x(n, x_0)_i - x(n, x_0)_j$ and $x(n+1, x_0)_i - x(n, x_0)_j$ converge in total variation, uniformly in x_0 .*

Remark 3.3. Under the conditions of the theorem, the sequence $A(-1) \otimes \dots \otimes A(-n)$ contracts the $(\max, +)$ projective space in a finite number of steps. That behaviour looks like that of classic products of random matrices : in [GR85] Y. Guivarc'h and A. Raugi proved that products of i.i.d. invertible matrices contract the projective space under generic hypothesis.

In the classic case, É. Le Page used the contraction to prove a *CLT* and other limit theorems (cf. [LP82]). Following the same way, we are currently writing a proof of these theorems for what we called $x(n, x_0)$. This proof will require the memory loss property.

Together with theorem 3.1, a particular case of theorem [GH00, 4.1] gives :

Corollary 3.3. *For any t -uple of matrices $M = (A_1 \dots A_t)$ and any $p \in \{p \in \mathbb{R}^t \mid \sum p_s = 1\}$, we call $L_M(p)$ the Lyapunov exponent of i.i.d random variables $A(n)$ such that*

$$\forall n \in \mathbb{N}, \forall i \in I_t, P(A(n) = A_i) = p_i.$$

For any M outside a finite union of hyperplanes of $(\mathbb{R}^{k \times k})^t$, $L_M(p)$ is an analytical function of p on the domain $\{p \in \mathbb{R}^t \mid \sum p_s = 1, p > 0\}$.

4 Proofs

4.1 Proof of theorems 3.1 and 3.2.

Both theorems follow from the next lemma which will be proved in section 4.3.

Lemma 4.1. *For any couple $(A, B) \in (\mathbb{R}^{k \times k})^2$ outside a finite union of hyperplanes, there exist two integers m and n such that the critical graph of the matrix $A^{\otimes m} \otimes B \otimes A^{\otimes n}$ is a loop. As a consequence that matrix is **scs1-cyc1**.*

Theorem 3.1 directly follows from lemma 4.1 and the following :

Lemma 4.2. *Every matrix **scs1-cyc1** $A \in \mathbb{R}^{k \times k}$ has a power with rank 1 .*

Proof. Let O be the $k \times k$ matrix with all coefficient 1. From proposition 2.3, when n is big enough, the column vectors of $A^{\otimes n} - n\rho(A)O = (A - \rho(A)O)^{\otimes n}$ are eigenvectors of $A - \rho(A)O$. Moreover this last matrix has spectral radius zero so it follows from proposition 2.2 that all this vectors are proportional. \square

Notation We will denote a n -uple of matrices by $(A^i)_{i \in I_n}$ instead of $(A_i)_{i \in I_n}$ to use indexes for coefficients of matrices.

Theorem 3.2, will be deduced from lemma 4.1 and the following :

Lemma 4.3. *If $\mathcal{G}^c(A)$ has only one vertex, then there is a neighbourhood V of A and an integer n such that :*

$$\forall (A^i)_{i \in I_n} \in V^n, rk(A^1 \otimes \dots \otimes A^n) = 1.$$

Proof of theorem 3.2 :

Every hyperplane of lemma 4.1 is the kernel of a linear form f^α on $(\mathbb{R}^{k \times k})^2$. This linear form can be written $f_1^\alpha + f_2^\alpha$ where f_1^α depends only on the first matrix and f_2^α on the second one. The support of μ is not included in the union of the $\ker f_1^\alpha$. Therefore there exists a matrix A in the support of μ such that $\forall \alpha, f_1^\alpha(A) \neq 0$. For any α , the set $\{B \in \mathbb{R}^{k \times k} \mid f_2^\alpha(B) = -f_1^\alpha(A)\}$ is an affine hyperplane of $\mathbb{R}^{k \times k}$ or the emptyset. Therefore there exists a B

in the support of μ such that B is not in $\cup_{\alpha} \{B \in \mathbb{R}^{k \times k} \mid f_2^{\alpha}(B) = -f_1^{\alpha}(A)\}$. Finally $(A, B) \notin \cup_{\alpha} \ker f^{\alpha}$.

By lemma 4.1 there exists m and n such that $\mathcal{G}^c(A^{\otimes m} \otimes B \otimes A^{\otimes n})$ has only one vertex. By lemma 4.3 there exists a neighbourhood V of $A^{\otimes m} \otimes B \otimes A^{\otimes n}$ and an integer N such that every matrix in $V^{\otimes N}$ has rank 1. Let $V_1 \times V_2$ be a neighbourhood of (A, B) such that $V_1^{\otimes m} \otimes V_2 \otimes V_1^{\otimes n} \subset V$. The matrices in $(V_1^{\otimes m} \otimes V_2 \otimes V_1^{\otimes n})^{\otimes N}$ have rank 1. As A and B are in the support of μ , $\mu(V_1) > 0$ and $\mu(V_2) > 0$, so

$$\mu^{\otimes(n+m+1)N} \left((V_1^{\otimes m} \otimes V_2 \otimes V_1^{\otimes n})^{\otimes N} \right) \geq \mu(V_1)^{(n+m)N} \mu(V_2)^N > 0.$$

This concludes the proof of theorem 3.2. The end of this section will be devoted to the proof of lemma 4.3 by putting together the proofs of the results we used in lemma 4.2.

To understand the powers of A , we used that their coefficients are the weights of paths on $\mathcal{G}(A)$. We want to do the same for products of two matrices, which means the edges weights can be different at every step.

For any finite sequence of matrices $({}^i A)_{i \in I_n}$ we give the following definitions :

Definition 4.1. From now on, \mathcal{G} will be the complete directed graph with vertices the elements of I_k .

- The weight of a path $pth = (i_j)_{j \in I_{n+1}}$ on \mathcal{G} (with respect to $({}^i A)_{i \in I_n}$) is $w(pth) := \sum_{j \in I_n}^j A_{i_j i_{j+1}}$
- A path is maximising if its weight is maximal among the weights of paths with the same origin, the same end, and the same length.

With such definitions $(i_j)_{j \in I_{n+1}}$ is maximising if and only if its weight is $({}^1 A \otimes \dots \otimes {}^n A)_{i_1 i_{n+1}}$.

Proof of lemma 4.3 :

As $\mathcal{G}^c(A)$ has only one vertex there exists $l \in I_k$ such that : $\forall c \in \mathcal{C}(A) \setminus \{(l, l)\}, A_{ll} > aw(A, c)$. So there exists an $\varepsilon > 0$ such that

$$\forall c \in \mathcal{C}(A) \setminus \{(l, l)\}, A_{ll} - aw(A, c) > 3\varepsilon.$$

Let V be the open ball with center A and radius ε for infinity norm and M be the maximum of the infinity norm on V .

Let us notice that every matrix $B \in V$ has the same critical graph as A . Let \tilde{B} be the matrix with spectral radius 0 defined by $\tilde{B}_{ij} = B_{ij} - B_{ll}$. Then $\|\tilde{B} - \tilde{A}\|_{\infty} < 2\varepsilon$.

From now on, $(^iA)_{i \in I_n}$ will be in V^n and the weights of paths will always be with respect to $(^iA)_{i \in I_n}$.

Let $pth = (i_j)_{j \in I_{n+1}}$ be a path of length n that does not cross l . It can split into a path of length less than k and minimal cycles. As a minimal cycle that is not (l, l) and whose length is t has weight $w(c) \leq aw(\tilde{A}, c)t + 2t\varepsilon < -t\varepsilon$,

$$w(pth) < -(n - k)\varepsilon + 2kM.$$

But for any $i, j \in I_k$,

$$\left(\tilde{A} \otimes \cdots \otimes \tilde{A} \right)_{ij} \geq w((i, l, \dots, l, j)) > -2M,$$

so there exists an N such that every maximising path of length $n \geq N$ crosses l .

Let $pth = (i_j)_{j \in I_{n+1}}$ be a maximising path of length $n \geq 2N + 1$. Then $(i_j)_{j \in I_{N+1}}$ is also maximising, so there is a $j_0 \leq N$ such that $i_{j_0} = l$. And $(i_j)_{n-N \leq j \leq n+1}$ is maximising for $(^jA)_{n-N \leq j \leq n+1}$, so there exists $n - N \leq j_1 \leq n + 1$ such that $i_{j_1} = l$. The path $(i_j)_{j_0 \leq j \leq j_1}$ is a cycle, so it can split into minimal cycles. As minimal cycles have a negative weight, except for (l, l) , the only subcycle of $(i_j)_{j_0 \leq j \leq j_1}$ is (l, l) . So for any j between j_0 and j_1 , $i_j = l$. Practically for $N + 1 \leq j \leq n - N$, so that

$$w(pth) = w((i_j)_{1 \leq j \leq N+1}) + w((i_j)_{n-N \leq j \leq n+1}).$$

We proved that :

$$\forall n \geq 2N+1, \forall i, j \in I_k, \left(\tilde{A} \otimes \cdots \otimes \tilde{A} \right)_{i,j} = \left(\tilde{A} \otimes \cdots \otimes \tilde{A} \right)_{il} + \left({}^{n-N}\tilde{A} \otimes \cdots \otimes \tilde{A} \right)_{lj},$$

so that $rk \left(\tilde{A} \otimes \cdots \otimes \tilde{A} \right) = 1$ and also $rk \left({}^1A \otimes \cdots \otimes {}^nA \right) = 1$. This concludes the proof of lemma 4.3.

4.2 Reduced matrix

To prove lemma 4.1, we introduce the notion of reduced matrix.

Definition 4.2. A matrix $A \in \mathbb{R}_{\max}^{k \times k}$ is reduced if :

$$\begin{cases} \forall (i, j) \in I_k^2, A_{ij} \leq 0 \\ \forall i \in I_k, \exists j \in I_k A_{ij} = 0 \end{cases} \quad (1)$$

It is strictly reduced if :

$$\begin{cases} \forall (i, j) \in I_k^2, A_{ij} \leq 0 \\ \forall i \in I_k, \exists ! j \in I_k A_{ij} = 0 \end{cases} \quad (2)$$

Proposition 4.1.

- i) The set of reduced matrices is a semigroup. The set of strictly reduced matrices too.
- ii) Reduced matrices have spectral radius zero and a critical graph with 0 on each edge.

Proof. The proof is straightforward. □

To easy the statements, we will use the following notion of genericity.

Definition 4.3. A property P of a matrix is said to be generic (or a generic matrix fulfills P) if every matrix outside a finite union of hyperplanes of $\mathbb{R}^{k \times k}$ fulfills P .

A property P of a couple of matrices is generic if every couple of matrices outside a finite union of hyperplanes of $(\mathbb{R}^{k \times k})^2$ fulfills P .

Proposition 4.2. *There is a finite number of linear forms ρ^α and V_i^α in the coefficients of A , such that for any $A \in \mathbb{R}^{k \times k}$, there exists an α such that \bar{A} defined by $\bar{A}_{ij} := A_{ij} - \rho^\alpha(A) - V_i^\alpha(A) + V_j^\alpha(A)$ is reduced. Moreover, for a generic A , \bar{A} is strictly reduced.*

To prove this proposition, we need the following definition.

Definition 4.4. For any $\xi \in I_k^I$, let A^ξ be $\{C \in \mathbb{R}^{k \times k} \mid \forall i \in I_k, C_{i\xi(i)}\} = 0$ and let φ_ξ be the linear application from $A^\xi \times \mathbb{R} \times \mathbb{R}^{k-1}$ to $\mathbb{R}^{k \times k}$ defined by :

$$\varphi_\xi(C, \rho, V)_{ij} = C_{ij} + V_i - V_j + \rho$$

where $V_k = 0$ by definition.

Proof. If V is an eigenvector associated with eigenvalue ρ , the matrix with coefficients $A_{ij} - \rho - V_i + V_j$ is reduced : it is equivalent to the relation $A \otimes V = \rho \otimes V$.

For any cycle $c = (i_1, i_2, \dots, i_n)$ of I_k , the application that maps A on $aw(A, c) = \frac{1}{n} \sum_{j=1}^n A_{i_j i_{j+1}}$ is a linear form of A . So let us choose as ρ^α the $aw(\cdot, c)$, for $c \in \mathcal{C}(A)$. Since the length of c is always less than k , there are finitely many c .

By proposition 2.2, we can take $V_i = A_{ij}^+$ for some j . As $A_{ij}^+ = A_{ij}^{\otimes n} = \sum_{j=1}^n A_{i_j i_{j+1}}$ for some $n \leq k$ and some finite sequence of i_l , we have found the V_i^α among the applications of the form $A \mapsto \sum_{j=1}^n A_{i_j i_{j+1}}$. The set of this applications is finite.

It remains to show that for a generic A , \bar{A} is strictly reduced. Let $V_i := V_i^\alpha(A) - V_k^\alpha(A)$. Then

$$\forall A \in \mathbb{R}^{k \times k}, \exists \xi, A = \varphi_\xi(\bar{A}, \rho(A), V).$$

If \bar{A} is not strictly reduced, there exist i and $j \neq \xi(i)$, such that $\bar{A}_{ij} = 0$, so $A \in \varphi_\xi(\{(C, \rho, V) | C_{ij} = 0\})$. Since $A^\xi \times \mathbb{R} \times \mathbb{R}^{k-1}$ has dimension k^2 , $\varphi_\xi(\{(C, \rho, V) | C_{ij} = 0\})$ has dimension at most $k^2 - 1$. Therefore the set of matrices A with \bar{A} not strictly reduced is a subset of the union on ξ , i and $j \neq \xi(i)$ of these strict subspaces of $\mathbb{R}^{k \times k}$. A generic A is outside this union so generically \bar{A} is strictly reduced. \square

In what follows, we will use that the critical graph of \bar{A} has the same vertices and edges as the one of A with 0 on each edge.

4.3 Proof of lemma 4.1

Let A and B be two square matrices of size k . By proposition 4.2 there are \bar{A} and α such that :

$$\bar{A}_{ij} := A_{ij} - \rho^\alpha(A) - V_i^\alpha(A) + V_j^\alpha(A).$$

Let \bar{B} be such that :

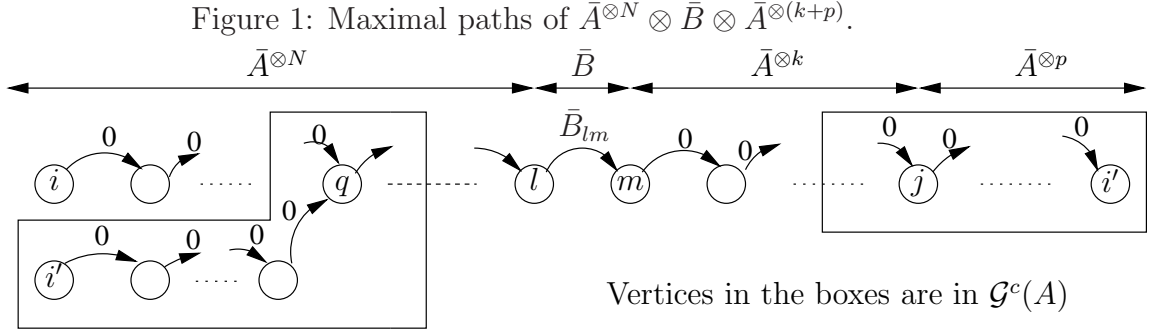
$$\bar{B}_{ij} := B_{ij} - V_i^\alpha(A) + V_j^\alpha(A).$$

Moreover we assume that \bar{A} is strictly reduced and that ξ maps i to the unique j such that $\bar{A}_{ij} = 0$. We also assume that there is only one cycle on $\mathcal{G}^c(A)$. The first assumption is generic from proposition 4.2. The second one is generic because if c_1 and c_2 are two minimal cycles on $\mathcal{G}^c(A)$, the non-zero linear form $aw(\cdot, c_1) - aw(\cdot, c_2)$ vanishes in A .

By proposition 2.3 applied to $\bar{A}^{\otimes c(A)}$ there is $P = \bar{A}^{\otimes N}$ such that every great power of \bar{A} is equal to $\bar{A}^{\otimes p} \otimes P$ for some $p \leq k$. As in the proof of lemma 4.3, we investigate the paths on \mathcal{G} . The decomposition of the paths in minimal cycles shows that, up to taking a bigger N , every maximising path of length N on $\mathcal{G}(\bar{A})$ crosses $\mathcal{G}^c(A)$.

Let us search for the maximal coefficients of $\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)}$. Such a coefficient $(\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)})_{ij}$ is the weight of a path $(i_r)_{r \in I_{N+k+p+1}}$ from i to j (figure 1). We denote i_{N+1} by l and i_{N+2} by m .

If we replace every i_r with $r \geq N+3$ by $\xi^{r-N-2}(m)$, we replace edges with negative weights by edges with weight 0, so we get a path with greater



weight. Since this weight can not be strictly greater, and since \bar{A} is strictly reduced, this means that $i_r = \xi^{r-N-2}(m)$. Therefore a maximal coefficient is equal to $P_{il} + \bar{B}_{lm}$ and does not depend on p . Moreover, $(i_r)_{N+2 \leq r \leq N+k+p+2}$ crosses twice the same vertex, so it contains a cycle. Since all edges have zero weight, the cycle is on $\mathcal{G}^c(A)$. As for any $r \geq N+3$, $i_{r+1} = \xi(i_r)$, the path stays on this cycle so $j \in \mathcal{G}^c(A)$.

On the other side, $(i_r)_{r \in I_{N+1}}$ crosses $\mathcal{G}^c(A)$. Let q be the first vertex in $\mathcal{G}^c(A)$. We can find a new path such that every edge before q is in $\mathcal{G}^c(A)$: we construct it backward from q . The first vertex of this path is called i' and belongs to $\mathcal{G}^c(A)$. By construction $(\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)})_{i'j}$ is greater than $(\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)})_{ij}$ so it is maximal. Since $\mathcal{G}^c(A)$ is connected, there is a p such that $j = i'$. From now on, we will suppose this is the case.

The matrix $M := \bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes(k+p)}$ has a maximum coefficient on its diagonal. This coefficient is the spectral radius of M and the weight of every edge of $\mathcal{G}^c(M)$. Let us show that all edges with this weight have the same end.

First there is a finite set of linear forms f^β such that $A_{ij}^+ = f^\beta(A)$ for some β , because A^+ is a $(\max, +)$ polynomial in A . Second $P_{ij} = (\bar{A}^{\otimes c(A)})_{ih}^+ + (\bar{A}^{\otimes c(A)})_{hj}^+$ is also the image of \bar{A} by a linear form, and the image of A by another linear form. At each step, the linear form is chosen in a finite set, so there exists a finite set of linear forms P_{ij}^γ and V_i^γ such that for every A there is a γ such that $P = P^\gamma(A)$ and $V = V^\gamma(A)$.

We have proved that the maximal M_{ij} are equal to $P_{il} + \bar{B}_{lm}$ for some l and m . If there are two indices such that (l, m) are different, then the linear form

$$P_{i_1 l_1}^\gamma + B_{l_1 m_1}^\circ + V_{m_1}^\gamma - V_{l_1}^\gamma - P_{i_2 l_2}^\gamma - B_{l_2 m_2}^\circ - V_{m_2}^\gamma + V_{l_2}^\gamma$$

where B_{lm}° is the (l, m) -coordinate application, vanishes in (A, B) . But this

linear form is not zero, because the B component is the difference of the two coordinates applications $B_{l_1 m_1}^\circ$ and $B_{l_2 m_2}^\circ$. If no such linear form vanishes in (A, B) , then all maximal coefficients of M are equal to $P_{il} + \bar{B}_{lm}$ with the same (l, m) . Since there are finitely many γ and indices, this condition is fulfilled outside a finite union of hyperplanes.

So generically if M_{ij} is a maximal coefficient of M we have $j = \xi^{k+p}(m) = i'$. That means that all edges of $\mathcal{G}^c(M)$ end in i' . The only possible cycle with such edges is (i', i') .

To conclude we notice that $\bar{A}^{\otimes N} \otimes \bar{B} \otimes \bar{A}^{\otimes k+p}$ and $A^{\otimes N} \otimes B \otimes A^{\otimes k+p}$ have the same critical graph.

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