On the scarring of eigenstates for some arithmetic hyperbolic manifolds in dimension 2 and 3

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Abstract

In this paper, we shall deal with the so-called conjecture of Quantum Unique Ergodicity. In [9], Rudnick and Sarnak showed that there is no *strong scarring* (see definition page 2) on closed geodesics for compact arithmetic congruence surfaces derived from a quaternion division algebra (see Introduction and Theorem 0.1).

First we extend this Theorem to the congruence surface $X = \Gamma(2)\backslash\mathbb{H}^2$ (it is not compact but has finite measure), where $\Gamma(2)$ is the kernel of the projection of $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{SL}(2,\mathbb{Z}_2)$. Then, after some algebraic and geometric preliminaries – and the establishment of useful technical Lemmas – we extend Theorem 0.1 to a class of Riemannian manifolds $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$, the so-called (K_2^S) class, that are again derived from quaternion division algebras. We show that there is no strong scarring on closed geodesics or on $\Gamma_{_R}$ - closed imbedded totally geodesic surfaces.

Introduction

A topic of great interest in quantum mechanics is the study of the limit of the quantized systems when $\hbar \longrightarrow 0$, which we call the *semi-classical limit* of quantum mechanics. The underlying purpose is to relate classical dynamics to the quantum one, in order to understand better the quantum mechanics. This is the path we follow to study quantum chaos : we consider a classicaly chaotic system, and we try to identify in the quantized system the influence of the chaotic nature of the classical dynamics.

For M a Riemannian manifold of negative sectional curvature in dimension 2 or 3, it is well known (see [1], [4], [5]) that the geodesic flow on the unitary sphere tangent bundle T^1M is ergodic and chaotic. After quantization, the wave functions of the stationary Schrödinger equation

$$\frac{-\hbar^2}{2m}\,\Delta\,\phi + V(q)\phi = E\,\phi \quad \text{with } E \text{ a constant} \tag{0.1}$$

are the $\mathscr{L}^2$ eigenfunctions or *eigenmodes* of $\frac{-\hbar^2}{2m}\,\Delta + V(q)$, where Δ is the Laplace-Beltrami operator on M. The potential $V(q)$ is in fact related to the curvature of M. We suppose that this operator has a discrete spectrum $(\lambda_k)_{k\in\mathbb{N}}$ with $\lambda_k \underset{k\to\infty}{\longrightarrow} \infty$, which is true at least in the compact case.

We denote by $(\phi_k)_{k\in\mathbb{N}}$ the associated eigenfunctions – we assume that they are normalized, so that $\|\phi_k\|_2 = 1$ – and by $(\mu_k)_{k\in\mathbb{N}}$ the corresponding probability measures given by

$$\mathrm{d}\mu_k(q) = \left|\phi_k(q)\right|^2 \mathrm{dvol}(q) \tag{0.2}$$

The measure μ_k is actually the probability of presence of a particle in the state ϕ_k at q. Moreover the semi-classical limit is the limit at large energies *i.e.* when $k \longrightarrow \infty$. We shall now state the so-called Quantum Unique Ergodicity Conjecture (see [10]) :

Conjecture *Let M be a Riemannian manifold of dimension 2 or 3 and of sectional curvature $K < 0$. Then* $\mathrm{d}\mu_k \underset{k\to\infty}{\longrightarrow} \dfrac{\mathrm{dvol}}{\mathrm{vol}(M)}$.

For a compact surface M whose geodesic flow is ergodic, this result was established in [12] for a subsequence of full density of (μ_k). More precisely, we shall consider quotient manifolds $M = \Gamma\backslash\mathbb{H}^n$ ($n = 2$ or 3), where Γ is a freely acting discrete subgroup of $\mathrm{Is}(\mathbb{H}^n)$, endowed with the projection of the canonical Poincaré metric on $\mathbb{H}^n$. In particular (see [3]), all Riemann surfaces are of such a type, apart from S^2, $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{T}^2$. A first step towards this conjecture was realized in [9] with the following Theorem :

Theorem 0.1 *Let $X = \Gamma\backslash\mathbb{H}^2$ be an arithmetic congruence surface derived from a quaternion algebra and ν a quantum limit on X. If $\sigma = \mathrm{singsupp}\ \nu$ is contained in the union of a finite number of isolated points and closed geodesics, then $\sigma = \emptyset$.*

In other words, there is no *strong scarring* (cf. [10]) of eigenmodes on closed geodesics. In this Theorem, Γ is a congruence subgroup of a discrete group derived from an indefinite quaternion division algebra. By a *quantum limit* ν we mean a probability measure on M such that there exists a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ of $(\mu_k)_{k\in\mathbb{N}}$ with $\mu_{k_j} \underset{j\to\infty}{\longrightarrow} \nu$.

In [9], a congruence subgroup is implicitely used to get a free action on $\mathbb{H}^2$. Thus, the canonical projection $\mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2$ is a covering ; otherwise branching points would appear, at which the projection of the canonical Poincaré metric would be singular.

In this work, we shall extend Theorem 0.1 to the particular arithmetic congruence surface $X = \Gamma(2)\backslash\mathbb{H}^2$, where $\Gamma(2)$ is the congruence group $\Gamma(2) = \{\,\gamma \in \mathrm{SL}(2,\mathbb{Z}) \;/\; \gamma \equiv I_2\,[2]\,\}$. This space is not compact anymore, but it has finite measure. The extension is straight forward, we just adapt the technics and ideas of [9]. Note that, $\mathrm{SL}(2,\mathbb{Z})$ being derived from the matrix algebra $\mathrm{M}(2,\mathbb{Z})$, Theorem 0.1 does not apply to the surface X.

Then we deal with the case of $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$, where $\Gamma_{_R}$ is a discrete group derived from a class of division quaternions algebras explicitly defined in sections 3.2 and 4. We show that a non empty set Λ contained in a finite union of isolated points, closed geodesics and $\Gamma_{_R}$ - closed (see section 4 for definition) imbedded totally geodesic surfaces of $X_{_R}$ (in this last case, we shall assume that $\mathrm{area}(\Lambda) \neq 0$) cannot be the singular support of a quantum limit on $X_{_R}$.

Before giving the proof, we shall recall some useful points of algebra and geometry (cf. section 2 and 3), and establish some arithmetical and geometrical Lemmas (cf. section 3 and 4). Contrary to the previous case, the proof is not a straight adaptation of [9] because the algebraic formalism of binary quadratic forms used there does not apply to points or to the imbedded surfaces anymore.

1 The case $\Gamma(2)\backslash\mathbb{H}^2$

Let us recall that

$$\Gamma(2) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \ / \ \gamma \equiv I\,[2] \right\} \tag{1.1}$$

We define the same way $\Gamma(N)$ for $N \geqslant 2$. The result obtained is the following :

Theorem 1.1 *If the singular support σ of a quantum limit on $X = \Gamma(2)\backslash\mathbb{H}^2$ is contained in a finite union of isolated points and closed geodesics of X, then $\sigma = \emptyset$.*

1.1 Correspondences and separation

Definition *A correspondence $\mathscr{C}$ of order r on a Riemannian manifold X is a mapping from X to $X^r/\mathfrak{S}_r$ such that $\mathscr{C}(x) = (S_1(x), \ldots, S_r(x))$ with $S_k \in Is(X)$ $k = 1 \ldots r$. Here, $\mathfrak{S}_r$ is the symmetric group of order r.*

We shall denote by $T_{\mathscr{C}}$ the associated operator of $\mathscr{L}^2(X)$ defined by $T_{\mathscr{C}}(f) : x \longmapsto \sum_{k=1}^{r} f\big(S_k(x)\big)$.

Definition *Let Λ be a subset of X. We say that such a correspondence $\mathscr{C}$* separates *Λ if $\exists\, z \in X - \Lambda$ such that $\exists !\, k \in \{1, \ldots, r\}$, $S_k(z) \in \Lambda$.*

Then we have the following Proposition (proved in [9])

Proposition 1.1 *Let $\Lambda \subset X$ be a closed subset of zero volume and $\mathscr{C}$ be a correspondence on X that separates Λ. Let $(\phi_j)_{j\in\mathbb{N}}$ be a sequence of eigenfunctions of $T_{\mathscr{C}}$ such that $\forall\, j \in \mathbb{N}$, $\|\phi_j\|_2 = 1$ and that $\mathrm{d}\nu = \lim_{j\to\infty} |\phi_j(z)|^2 \mathrm{dvol}(z)$ exists. Then* singsupp $\nu \neq \Lambda$.

Keep in mind than given any measure ν on X, its singular support is a closed subset of X.

1.2 $\mathbb{H}^2$, the modular group and operators

In the sequel, we shall set $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and deal with the hyperbolic space $X = \Gamma(2)\backslash\mathbb{H}^2$.

1.2.1 Some hyperbolic geometry

• The space $\mathbb{H}^2 = \{\, z \in \mathbb{C} \,/\, \mathrm{Im}(z) > 0 \,\}$ is provided with Poincaré hyperbolic metric $\mathrm{d}s = |\mathrm{d}z|/\mathrm{Im}(z)$, which becomes in cartesian coordinates $\mathrm{d}s^2 = \big(\mathrm{d}x^2 + \mathrm{d}y^2\big)/y^2$. As shown on figure 1, the geodesics of $\mathbb{H}^2$ are the half-circles centered on the real axis (like γ who is connecting a to b) and the vertical straight half-lines (like γ_2 who is connecting c to infinity).

The metric $\mathrm{d}s$ induces on $\mathbb{H}^2$ the volume form $\mathrm{d}\sigma = \mathrm{d}x\,\mathrm{d}y/y^2$. Its curvature is $K = -1$, and the Laplace-Beltrami operator on $\mathbb{H}^2$ is given by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

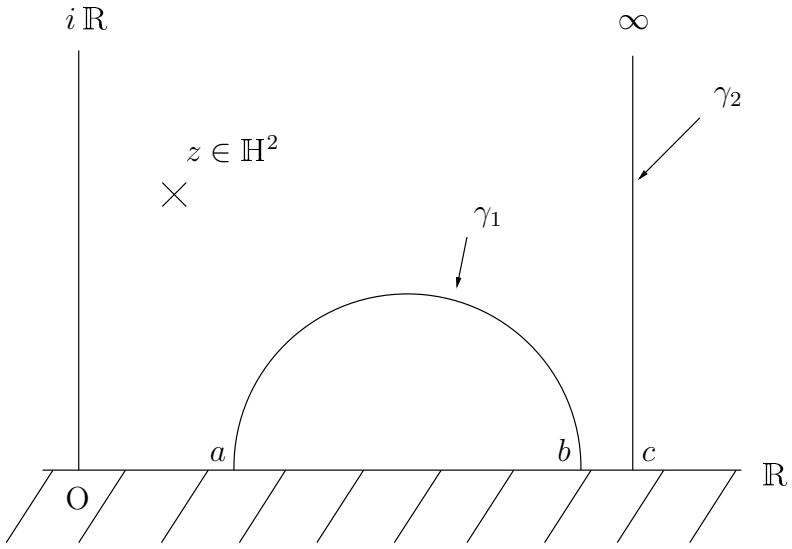

Figure 1: $\mathbb{H}^2$ and its geodesics

Definition *A measure ν on X is called a* quantum limit *if there exists a sequence $\left(\phi_j\right)_{j\in\mathbb{N}}$ of eigenfunctions of Δ in $\mathscr{L}^2(X)$, normalized by the condition $\left\|\phi_j\right\|_2 = 1$, such that the measures $\left|\phi_j(z)\right|^2 \mathrm{d}z$ converge weakly towards $\mathrm{d}\nu$.*

• We know that $\mathrm{Is}(\mathbb{H}^2)$, the group isometries of $\mathbb{H}^2$, consists of the real linear fractional transformations and fractional reflections. Moreover, $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ can be identified with $\mathrm{Is}^+(\mathbb{H}^2)$, the subgroup of isometries preserving the orientation, by the action

$$\left|\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R})\times\mathbb{H}^2 & \longrightarrow & \mathbb{H}^2 \\ (\gamma, z) & \longmapsto & \gamma\cdot z \end{array}\right. \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot z = \frac{az+b}{cz+d} \tag{1.2}$$

The elements of $\mathrm{Is}^+(\mathbb{H}^2)$ are characterized by their fixed points in $\mathbb{H}^2 \cup \partial\mathbb{H}^2 = \mathbb{H}^2 \cup \mathbb{R} \cup \infty$. Moreover $\gamma\cdot z = z \iff cz^2+(d-a)z-b$: the discriminant of this equation is $(d-a)^2+4bc = (d+a)^2-4$, so that

- if $\mathrm{Tr}(\gamma)^2 \in [0,4[$, the isometry γ has a single fixed point (which is a center) in $\mathbb{H}^2$ and is called *elliptic* (we follow the classical terminology, see [8]).
- if $\mathrm{Tr}(\gamma)^2 = 4$ and $\gamma \neq \pm I$, the isometry γ has a single fixed point (which is attractive) in $\mathbb{R}\cup\infty$ and is called *parabolic*.
- if $\mathrm{Tr}(\gamma)^2 \notin [0,4]$, the isometry γ has a two distinct fixed points in $\mathbb{R}\cup\infty$, one attractive and the other repulsive, and is called *hyperbolic*. The geodesic L connecting this fixed points is called the *axis* of γ. The isometry γ leaves L invariant and acts on it as a translation of the curvilinear abscisse.

• Finally we recall the notion of binary quadratic form and its use in geometry (cf. [9]). Set $M = [a,b,c]$, with a, b, $c \in \mathbb{R}$, the real binary quadratic form on $\mathbb{C}^2$ defined by

$$\forall\,(x,y) \in \mathbb{C}^2 \qquad M(x,y) = ax^2 + 2bxy + cy^2$$

We shall identify M with its real symmetric matrix in the canonical basis, and we set $M\circ g = M[g] = {}^t gMg$ for $g \in \mathrm{M}(2,\mathbb{R})$. Only non degenerate forms will be considered in the sequel *i.e.* we suppose that $\det M \neq 0$.

Let M be such a form : it has two distinct roots $(x : y)$ in $\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{C} \cup \infty$, the equation $M(x, y) = ax^2 + 2bxy + cy^2 = 0$ having discriminant $b^2 - ac = -\det M \neq 0$.

- if $\det M > 0$, then M is anisotropic on $\mathbb{R}$ and $M(z, 1) = 0$ has a single root $z_M \in \mathbb{H}^2$
- otherwise, M being isotropic on $\mathbb{R}$, it has two distinct roots in $\mathbb{P}^1(\mathbb{R}) \simeq \mathbb{R} \cup \infty$ and we call γ_M the geodesic connecting them.

Therefore we associate to each non degenerate binary quadratic form M a point z_M or a geodesic γ_M of $\mathbb{H}^2$. Conversely, to each point or geodesic of $\mathbb{H}^2$, we associate a single proportionality class of non degenerate real binary quadratic forms.

In this way, the action of $g \in \mathrm{SL}(2, \mathbb{R})$ on $\mathbb{H}^2$ translates into an action $M \longmapsto M[g^{-1}]$ on the binary quadratic forms.

1.2.2 The quotient space $X = \Gamma(2)\backslash\mathbb{H}^2$

• Contrary to Γ, its subgroup $\Gamma(2)$ contains no elliptic element (this is a particular case of a general property of the congruence subgroups $\Gamma(N)$ for $N \geqslant 2$ exposed in [6]) : it acts therefore freely on $\mathbb{H}^2$ (see [8]) and $X = \Gamma(2)\backslash\mathbb{H}^2$ is a Riemannian manifold of Gaussian curvature $K = -1$. We know (cf. [6]) that

$$\Gamma(2)\backslash\mathrm{SL}(2, \mathbb{Z}) = \{\, 1,\, \sigma,\, \tau,\, \tau\sigma,\, \tau\,\sigma\tau,\, \tau\,\sigma\,\tau\,\sigma \,\} \tag{1.3}$$

with σ and τ representing the classes modulo $\Gamma(2)$ of the transformations $S : z \longmapsto -1/z$ and $T : z \longmapsto z + 1$. They verify $\sigma^2 = \tau^2 = 1$ and $(\sigma\,\tau)^3 = 1$.

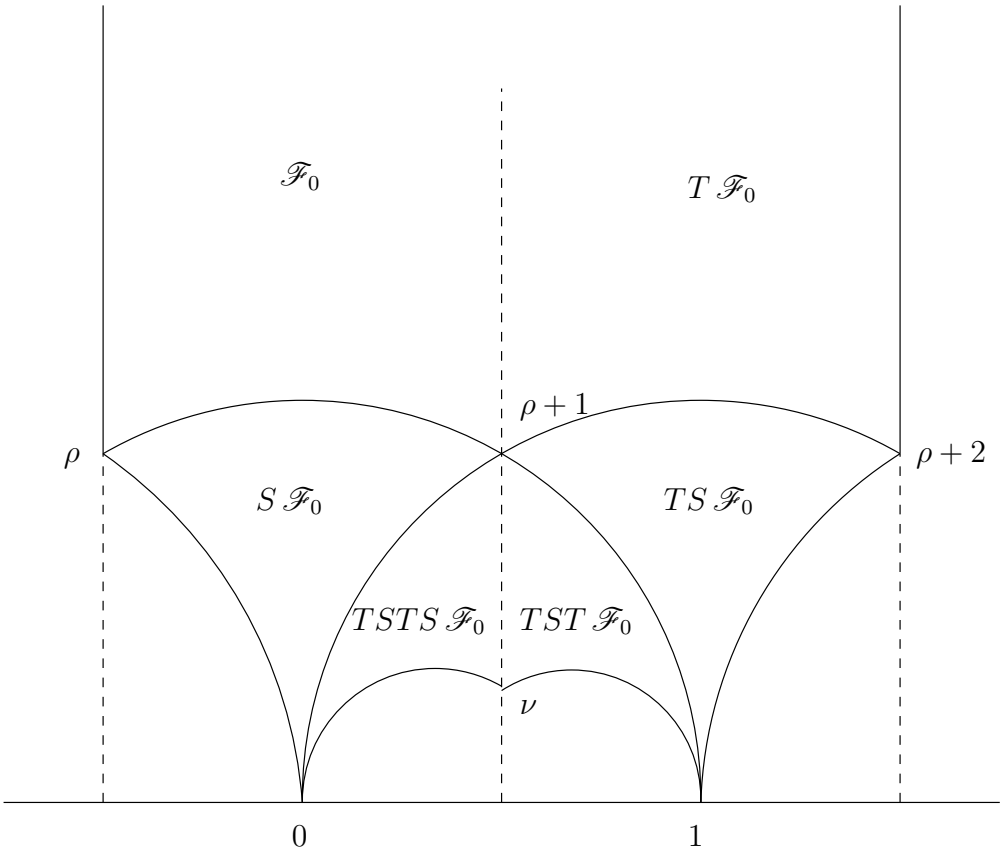

Figure 2: A fundamental domain for $\Gamma(2)\backslash\mathbb{H}^2$

The discrete group $\Gamma(2)$ acts moreover discontinuously on $\mathbb{H}^2$ (cf. [8]) ; knowing that Γ admits the geodesic triangle $\mathscr{F}_0 = (\rho, \rho+1, \infty)$ as a fundamental domain (where $\rho = e^{\frac{2i\pi}{3}}$), we deduce from (1.3) that $\mathscr{F} = \mathscr{F}_0 \cup S\mathscr{F}_0 \cup T\mathscr{F}_0 \cup TS\mathscr{F}_0 \cup TST\mathscr{F}_0 \cup TSTS\mathscr{F}_0$ is a fundamental domain for the group $\Gamma(2)$ (cf. figure 2).

On this figure, we have $\nu = \dfrac{1}{1-\rho} = \dfrac{1}{2} + \dfrac{i}{2\sqrt{3}}$.

• Let us recall that the geodesics of $\Gamma(2)\backslash\mathbb{H}^2$ are the projections of those of $\mathbb{H}^2$.

Definition *A closed geodesic of a metric space (X,d) is the image of a periodic geodesic line $\lambda : \mathbb{R} \longrightarrow X$.*

Proposition 1.2 *Let $\mathscr{L}$ be a closed geodesic of $X = \Gamma(2)\backslash\mathbb{H}^2$. There exists a hyperbolic transformation $\gamma \in \Gamma(2)$ whose axis L projects onto $\mathscr{L}$.*

The proof (see [8]) uses only the discontinuity of the action of $\Gamma(2)$ on $\mathbb{H}^2$. In particular, there exists a compact segment l of the geodesic L such that $L = \bigcup_{n\in\mathbb{Z}} \gamma^n \cdot l$ and $\mathscr{L} = \pi(L) = \pi(l)$, where π is the canonical projection $\pi : \mathbb{H}^2 \longrightarrow \Gamma(2)\backslash\mathbb{H}^2$.

Lemma 1.1 *Let F and G be two closed geodesics of X. Then $F = G$ or $F \cap G$ is finite.*

Proof : we shall use Proposition 1.2. Let γ_1 and $\gamma_2 \in \Gamma(2)$ be two hyperbolic transformations whose axis L_1 and L_2 project onto F and G respectively. Let l_1 and l_2 be compact segments of L_1 and L_2 such that $L_1 = \bigcup_{n\in\mathbb{Z}} \gamma_1^n \cdot l_1$ and $L_2 = \bigcup_{n\in\mathbb{Z}} \gamma_2^n \cdot l_2$. Then

$$F \cap G = \pi(l_1) \cap \pi(l_2) = \pi\left[\left(\cup_{\gamma\in\Gamma(2)}\, \gamma\cdot l_1\right) \cap \left(\cup_{\gamma'\in\Gamma(2)}\, \gamma'\cdot l_2\right)\right] = \pi\left[\cup_{\gamma\in\Gamma_2}\left(l_1 \cap \gamma\cdot l_2\right)\right]$$

because for $p \in \gamma\cdot l_1 \cap \gamma'\cdot l_2$, $\gamma^{-1}p \in l_1 \cap (\gamma^{-1}\gamma')\cdot l_2$ and $\pi(\gamma^{-1}\cdot p) = \pi(p)$.

Since the segments l_1 and l_2 are compact subsets of $\mathbb{H}^2$, we deduce from the discontinuity of the action of $\Gamma(2)$ on $\mathbb{H}^2$ that the set $\Gamma_0 = \left\{\, \gamma \in \Gamma(2) \;/\; l_1 \cap \gamma\cdot l_2 \neq \emptyset \,\right\}$ is finite and that

$$F \cap G = \pi\left(l_1 \cap \Gamma_0 \cdot l_2\right) \tag{1.4}$$

Let us assume that $F \cap G$ is infinite : $l_1 \cap \Gamma_0 \cdot l_2$ is infinite so that, Γ_0 being finite, there exists $\gamma \in \Gamma_0$ such that $l_1 \cap \gamma\cdot l_2$ is infinite. Hence $L_1 \cap \gamma\cdot L_2$ is infinite, L_1 and $\gamma\cdot L_2$ being two geodesics of $\mathbb{H}^2$ *i.e.* half-circles or half-lines : as a consequence, $L_1 = \gamma\cdot L_2$ and $F = G$.

1.2.3 Modular correspondences

• Let us set $P(n) = \left\{M \in \mathrm{M}(2,\mathbb{Z}) \;/\; \det M = n\right\}$ and $R(n) = \left\{M \in P(n) \;/\; M \equiv I\,[2]\right\} = \left\{M \in \mathrm{M}(2,\mathbb{Z}) \;/\; \det(M) = n \text{ and } M \equiv I\,[2]\right\}$ for $n \in \mathbb{N}$. Note that $\Gamma(2) = R(1)$ and $\Gamma = P(1)$. As is well-known, we have

Lemma $$P(1)\backslash P(n) \approx \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{M}(2,\mathbb{Z}) \;\Big/\; ad = n,\ a \geqslant 1,\ 0 \leqslant b < d \right\}$$

Using the six representants for the cosets of $R(1)\backslash P(1)$ (cf. previous section), we show that

Lemma 1.2 $R(1)\backslash R(n) \approx \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in R(n) \;\Big/\; a \geqslant 1,\, 0 \leqslant b < d \right\}$

Obviously, $\mathbb{R}(n) \neq \emptyset$ if and only if n is odd. We denote by $R^{pr}(n)$ the set of the elements of $R(n)$ that are not integer multiples of matrices in $\mathrm{M}(2,\mathbb{Z})$: these are the *primitive* elements. We can now define the modular correspondence of order $n \in \mathbb{N}$ for n odd.

Definition $\mathscr{C}_n : \left| \begin{array}{ccc} X & \longrightarrow & X^r/\mathfrak{S}_r \\ \Gamma(2)\,x & \longmapsto & \{\Gamma(2)\,\alpha_1 x, \ldots, \Gamma(2)\,\alpha_r x\} \end{array} \right.$

where $R(1)\backslash R(n) = \{\Gamma(2)\,\alpha_1, \ldots, \Gamma(2)\,\alpha_r\}$. Let T_n be the associated operator, which we call a *modular operator*

$$T_n(f)(\Gamma(2)z) = \sum_{\delta \in R(1)\backslash R(n)} f(\Gamma(2)\delta z) \tag{1.5}$$

• Because $R(1)\backslash R(n)$ is finite, the modular operators are defined on $\mathscr{L}^2(X)$: if $f \in \mathscr{L}^2(X)$ and $\alpha \in \mathrm{Is}(X)$, then $f \circ \alpha \in \mathscr{L}^2(X)$. Moreover, $\|f \circ \alpha\|_2 = \|f\|_2$: they are bounded operators on $\mathscr{L}^2(X)$. They also operators satisfy the classical properties of the modular operators :

- T_n is self adjoint and commutes with the Laplace-Beltrami operator Δ.
- $T_n\, T_m = \sum_{d/(n,m)} d T_{nm/d^2}$
- $T_n = \sum_{t^2/n} C_{n/t^2}$ where $C_n f(\Gamma(2)z) = \sum_{\delta \in R(1)\backslash R^{pr}(n)} f(\Gamma(2)\delta z)$

Nota Bene : in the sequel, we shall only consider operators $T_p = C_p$ where p is an odd prime. We shall also note $\mathscr{C}_p = C_p$ for simplicity's sake. Let $\mathscr{P}$ be the set of all prime numbers.

Lemma 1.3 *Let $p \in \mathscr{P}$ and $R(1)\backslash R(p) = \{\Gamma(2)\,\alpha_1, \ldots, \Gamma(2)\,\alpha_n\}$.*
Then $\forall\, i,\ \exists !\, j$ such that $\alpha_j\,\alpha_i \in p\,\Gamma(2)$ and $\forall\, k \neq j,\ \alpha_k\,\alpha_i \in R^{pr}(p^2)$.

Proof : we just use the fact that for any odd m, the set $R(m)$ is stable under passage to the (transposite of the) comatrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R(m) \implies \mathrm{Com}(\gamma) = m\,\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in R(m) \tag{1.6}$$

which corresponds actually to the passage to the inverse in $\mathrm{Is}^+(\mathbb{H}^2) \simeq \mathrm{PGL}^+(2,\mathbb{Z})$. Let us take $i \in \{1, \ldots, n\}$: we have $\mathrm{Com}(\alpha_i) \in R(p)$ and $\mathrm{Com}(\alpha_i)\,\alpha_i = \det(\alpha_i)\,I_2 = p\,I_2$. Since $\Gamma(2)\,\mathrm{Com}(\alpha_i)$ is a coset of $R(1)\backslash R(p)$, there exists $j \in \{1, \ldots, n\}$ such that $\Gamma(2)\,\mathrm{Com}(\alpha_i) = \Gamma(2)\,\alpha_j$. Therefore $\Gamma(2)\,\alpha_j\,\alpha_i = p\,\Gamma(2)$ and $\alpha_j\,\alpha_i \in p\,\Gamma(2)$.

For all $k \neq j$, we have $\Gamma(2)\,\alpha_k \neq \Gamma(2)\,\alpha_j$ whence $\Gamma(2)\,\alpha_k\,\alpha_i \neq \Gamma(2)\,\alpha_j\,\alpha_i = p\,\Gamma(2)$ so that $\alpha_k\,\alpha_i \in R^{pr}(p^2)$: this ends the proof of the Lemma. We deduce easily the self adjointness of T_p from this Lemma.

We shall use in section 3.2 an extension of this Lemma to an order of a quaternion algebra (cf. Proposition 3.7).

1.3 Results on binary quadratic forms

In this section, we adapt the Lemmas 2.2 and 2.3 of [9] to the discrete group $\Gamma(2)$.

Definition $M = [a, b, c]$ *is a* $\mathbb{Q}$-form *if there exists* $\lambda \in \mathbb{R}\backslash\{0\}$ *such that* $(\lambda a,\ \lambda b,\ \lambda c) \in \mathbb{Q}^3$.

Proposition 1.3 *Let* M *and* M' *be two binary quadratic forms and* p_1, p_2 *and* p_3 *be three distinct primes for which either :*

- $\exists\, \alpha_i \in \ R(p_i)$ *such that* $M[\alpha_i] = \lambda_i M' \quad \forall\, i = 1 \ldots 3.$
- $\exists\, \alpha_i \in R^{pr}({p_i}^2)$ *such that* $M[\alpha_i] = \lambda_i M' \quad \forall\, i = 1 \ldots 3.$

Then M *and* M' *are both* $\mathbb{Q}$*-forms.*

Proof : let us set $M = [A, B, C]$ and $M' = [A', B', C']$. M and M' being non degenerate, $M[\alpha_1] = \lambda_1 M' \Rightarrow \lambda_1 \neq 0$. As $\alpha_1 \in \mathrm{GL}(2, \mathbb{Q})$, M is a $\mathbb{Q}$-form if and only if M' also is a $\mathbb{Q}$-form. Similarly, M and M' are simultaneously isotropic or anisotropic over , $\mathbb{Q}$.

If M is isotropic, it splits over $\mathbb{Q}$: it is proportional to $M_0[\gamma]$, where $M_0(x, y) = xy$ and $\gamma \in \mathrm{GL}(2, \mathbb{Q})$. Thus, M and M' are both $\mathbb{Q}$-forms. Henceforth we assume that both M and M' are anisotropic over $\mathbb{Q}$. Let us fix $n \in \{1, 2\}$ and assume

$$\forall\, i,\ M[\alpha_i] = \lambda_i M' \quad \text{with} \ \ \alpha_i \in R^{pr}({p_i}^n) \tag{1.7}$$

Taking the determinant of both sides of (1.7), we get ${p_i}^{2n} \det M = {\lambda_i}^2 \det M'$, so that

$$\lambda_i = \varepsilon_i \kappa {p_i}^n \quad \text{with} \ \ \varepsilon_i = \pm 1 \ \text{ and } \ \kappa = \sqrt{\frac{\det M}{\det M'}}$$

Set $\alpha_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$; as $M[\alpha_i] = {}^t\alpha_i M \alpha_i$, equation (1.7) can be written as

$$\forall\, i \quad \lambda_i M' = \begin{pmatrix} A{a_i}^2 + 2\,B a_i c_i + C{c_i}^2 & A a_i b_i + C c_i d_i + B(a_i d_i + b_i c_i) \\ A a_i b_i + C c_i d_i + B(a_i d_i + b_i c_i) & A{b_i}^2 + 2\,B b_i d_i + C{d_i}^2 \end{pmatrix} \tag{1.8}$$

• The identification of the lower right terms in relation (1.8) gives

$$\lambda_i C' = \varepsilon_i \kappa {p_i}^n C' = A{b_i}^2 + 2\,B b_i d_i + C{d_i}^2 \quad \text{for} \ \ i = 1 \ldots 3 \tag{1.9}$$

that we will write as

$$\begin{pmatrix} {b_1}^2 & b_1 d_1 & {d_1}^2 \\ {b_2}^2 & b_2 d_2 & {d_2}^2 \\ {b_3}^2 & b_3 d_3 & {d_3}^2 \end{pmatrix} \begin{pmatrix} A \\ 2\,B \\ C \end{pmatrix} = \kappa C' \begin{pmatrix} \varepsilon_1 {p_1}^n \\ \varepsilon_2 {p_2}^n \\ \varepsilon_3 {p_3}^n \end{pmatrix}$$

that is

$$\Psi \begin{pmatrix} A \\ 2\,B \\ C \end{pmatrix} = \kappa C' N \quad \text{with} \ \ \Psi \in \mathrm{M}(3, \mathbb{Z}),\ \det \Psi = \prod_{1 \leqslant i < j \leqslant 3} (b_i d_j - b_j d_i) \ \text{ and } \ N \in \mathbb{Z}^3$$

If Ψ is invertible, $\Psi^{-1} N \in \mathbb{Q}^3$ so that M and M' are both $\mathbb{Q}$-forms. Otherwise $\det \Psi = 0$: we may assume without loss of generality that $b_1 d_2 = b_2 d_1$. There exists

$\mu \in \mathbb{Q}\backslash\{0\}$, $\mu(b_1, d_1) = (b_2, d_2)$ because $\det(\alpha_i) \neq 0$ implies $(b_i, d_i) \neq (0, 0)$. Substituting into relation (1.9), we find

$$\kappa\varepsilon_2 {p_2}^n C' = \mu^2 \kappa\varepsilon_1 {p_1}^n C' = M(b_2, d_2) \neq 0 \quad \text{from the anisotropy of } M \text{ over } \mathbb{Q}$$

so that ${p_2}^n = \mu^2 {p_1}^n$. If $n = 1$, $\mu^2 = p_2/p_1$ is a square of $\mathbb{Q}$, which cannot happen for two distinct primes p_1 and p_2. As a consequence $n = 2$, $\mu = \pm p_2/p_1$ and $p_2(b_1, d_1) = \pm p_1(b_2, d_2)$, and we deduce from the Gauss Theorem that

$$b_1 \equiv d_1 \equiv 0\,[p_1] \quad \text{and} \quad b_2 \equiv d_2 \equiv 0\,[p_2] \tag{1.10}$$

• The identification of the upper left terms in (1.8) provides

$$\lambda_i A' = \varepsilon_i \kappa {p_i}^n A' = A{a_i}^2 + 2\,B a_i c_i + C{c_i}^2 \ \text{ for } i = 1 \ldots 3 \tag{1.11}$$

that we will write

$$\begin{pmatrix} {a_1}^2 & a_1c_1 & {c_1}^2 \\ {a_2}^2 & a_2c_2 & {c_2}^2 \\ {a_3}^2 & a_3c_3 & {c_3}^2 \end{pmatrix} \begin{pmatrix} A \\ 2\,B \\ C \end{pmatrix} = \kappa A' \begin{pmatrix} \varepsilon_1 {p_1}^n \\ \varepsilon_2 {p_2}^n \\ \varepsilon_3 {p_3}^n \end{pmatrix}$$

that is

$$\Phi \begin{pmatrix} A \\ 2\,B \\ C \end{pmatrix} = \kappa A' \tilde{N} \quad \text{with } \Phi \in \mathrm{M}(3, \mathbb{Z}),\ \det\Phi = \prod_{1 \leqslant i < j \leqslant 3} (a_i c_j - a_j c_i) \ \text{ and } \tilde{N} \in \mathbb{Z}^3$$

If Φ is invertible, we show as before that M and M' are both $\mathbb{Q}$-forms. Otherwise, $\det\Phi = 0$ so that $\exists\, i \neq j \in \{1, 2, 3\}$, $\exists\, \mu' \in \mathbb{Q}\backslash\{0\}$, $\mu'(a_i, c_i) = (a_j, c_j)$. Substituting into relation (1.11), we get $\mu' = \pm p_j/p_i$ because $A' = {\lambda_1}^{-1} M(a_1, c_1) \neq 0$ by anisotropy of M over $\mathbb{Q}$. Hence :

$$a_i \equiv c_i \equiv 0\,[p_i] \quad \text{and} \quad a_j \equiv c_j \equiv 0\,[p_j] \tag{1.12}$$

Since $\{i, j\} \subset \{1, 2, 3\}$, either $i \in \{1, 2\}$ or $j \in \{1, 2\}$. We may assume that $i = 1$. From the relations (1.10) and (1.12) we deduce that $\alpha_1 \in p_1\Gamma(2)$, which contradicts the assumption $\alpha_1 \in R^{pr}({p_1}^n)$. This ends the proof of the Proposition.

Proposition 1.4 *Let M, M_1, …, M_r be binary quadratic forms representing points or closed geodesics of $\Gamma(2)\backslash\mathbb{H}^2$. Then there exists infinitely many primes p such that*

$$\forall\, \alpha \in R(p) \cup R^{pr}(p^2)\ \ \forall\, j = 1 \ldots r \quad M[\alpha] \neq \lambda_j M_j \tag{1.13}$$

Proof : by Proposition 1.3, if the forms M and M' are not $\mathbb{Q}$-forms, for all primes p except at most 4, we have $\forall\, \alpha \in R(p) \cup R^{pr}(p^2)$, $M[\alpha] \neq \lambda M'$. Hence, we just have to give the proof for $\mathbb{Q}$-forms. We shall set $M = [A, B, C]$ with A, B and $C \in \mathbb{Z}$, and $\forall\, i = 1 \ldots r$, $M_i = [A_i, B_i, C_i]$ with A_i, B_i and $C_i \in \mathbb{Z}$.

Let $p \in \mathscr{P}$, $j \in \{1, \ldots r\}$, $n = 1$ or 2 and $\alpha \in R^{pr}(p^n)$ such that $M[\alpha] = \lambda_j M_j$. Let us note that $\lambda_j \in \mathbb{Q}\backslash\{0\}$ because $\alpha \in \mathrm{GL}(2, \mathbb{Q})$ and $M \neq 0$. By taking the determinants of both sides in relation $M[\alpha] = \lambda_j M_j$, we get

$$\lambda_j = p^n \kappa_j \qquad \text{with} \quad \kappa_j = \pm\sqrt{\det M\ \det {M_j}^{-1}} \in \mathbb{Q}\backslash\{0\}$$

We restrict ourselves in the sequel to primes p such that $\forall\, j,\ \mathrm{ord}_p(\kappa_j) = 0$. As $\alpha \in \mathrm{GL}(2,\mathbb{Q})$, the relation $M[\alpha] = \lambda_j M_j$ implies that M and M_j are simultaneously isotropic or anisotropic over $\mathbb{Q}$. Thus we just have to investigate both cases.

• *Anisotropic Case :* the quadratic form M has for discriminant $B^2 - AC = d(M) \in \mathbb{Z}$, so that M is anisotropic over a field $\mathbb{K}$ if and only if $d(M)$ is not a square in $\mathbb{K}$. As a consequence $d(M)$ is not a square in $\mathbb{Q}$; let us consider the primes p satisfying

$$\forall\, j = 1 \dots r \quad \mathrm{ord}_p(\kappa_j) = 0 \qquad \text{and} \qquad \left(\frac{d(M)}{p}\right) = -1 \tag{1.14}$$

They form (cf. [11]) a set of Dirichlet density $1/2$ in $\mathscr{P}$, therefore infinite. For such primes p, the form M is still anisotropic over $\mathbb{Q}_p$.

If $\exists\, \alpha \in R^{pr}(p^n),\ \exists\, j \in \{1 \dots r\},\ M[\alpha] = \lambda_j M_j$, then for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{cases} Aa^2 + 2\,Bac + Cc^2 = \lambda_j A_j = \kappa_j p^n A_j = M(a,c) \\ Ab^2 + 2\,Bbd + Cd^2 = \lambda_j C_j = \kappa_j p^n C_j = M(b,d) \end{cases}$$

and, because $A_j,\ C_j \in \mathbb{Z}$ and $\mathrm{ord}_p(\kappa_j) = 0$, we get $M(\overline{a},\overline{c}) \equiv M(\overline{b},\overline{d}) \equiv 0\,[p]$. Therefore, by the anisotropy of M on $\mathbb{Q}_p$, $\alpha \in pR(1)$ is not primitive, which contradicts the choice of α. Thus in the anisotropic case, for all $p \in \mathscr{P}$ satisfying the relation (1.14),

$$\forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, j = 1 \dots r \qquad M[\alpha] \neq \lambda_j M_j$$

• *Isotropic Case :* we consider non degenerate quadratic forms with integer coefficients. Thus they split over $\mathbb{Q}$ as a product of two linear forms that are independant by non-degeneracy. Let M be such a form : it has two roots $z_1,\ z_2 \in \mathbb{Q} \cup \infty$; let γ_M be the geodesic of $\mathbb{H}^2$ connecting these two points. We will show that γ_M cannot be a closed geodesic of X.

We look for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ such that $\gamma(z_1) = z_1$ and $\gamma(z_2) = z_2$.

If $z_1 = \infty$ and $z_2 \in \mathbb{Q}$: $\gamma(\infty) = \infty \implies c = 0 \implies ad = 1$. As a and d are integers, $a = d = \pm 1$ and γ is not hyperbolic.

If $z_1 \neq z_2 \in \mathbb{Q}$: we have $c \neq 0$ and $\gamma(x) = x \iff cx^2 + (d-a)x - b = 0$. The two fixed points of γ are then

$$z_{1,2} = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}$$

Thus $z_1,\ z_2 \in \mathbb{Q} \implies \sqrt{(a+d)^2 - 4} \in \mathbb{Q} \implies \exists\, y \in \mathbb{Q},\ y^2 = (a+d)^2 - 4 \in \mathbb{Z}$ so that

$$\exists\, y \in \mathbb{Z} \qquad (a + d + y)(a + d - y) = 4$$

The integers $a + d + y$ and $a + d - y$ being of same parity, necessarily $a + d + y = a + d - y = a + d = \pm 2$: γ cannot be hyperbolic.

Hence, γ_M is not the axis of any hyperbolic element of $\Gamma(2)$: owing to Proposition 1.2, we can state that the binary quadratic $\mathbb{Q}$-forms isotropic on $\mathbb{Q}$ are not associated with closed geodesics of $X = \Gamma(2)\backslash\mathbb{H}^2$. This ends the proof of the Proposition.

1.4 Separation of points and geodesics

Proposition 1.5 *Let Λ be a non-empty set contained in a finite union of closed geodesics of $X = \Gamma(2)\backslash\mathbb{H}^2$. There exists a modular correspondence $\mathscr{C}_p$ separating Λ.*

Proof : given two points $z,\, z' \in \mathbb{H}^2$ and $M,\, M'$ the associated binary quadratic forms, we have $z = z' \iff \exists\, \lambda \in \mathbb{R},\ M' = \lambda M$. The same goes for two geodesics. We recall briefly the proof presented in [9].

• Λ *is finite :* let us write $\Lambda = \{z_1, \ldots, z_l\}$. We take $\tilde{z}_1, \ldots, \tilde{z}_l$ liftings to $\mathbb{H}^2$ and $M_1, \ldots, M_l$ the associated binary quadratic forms. Proposition 1.4 applied to them gives

$$\exists\, p \in \mathscr{P} \quad \forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, j = 1 \ldots l \qquad \alpha \cdot \tilde{z}_1 \neq \tilde{z}_j \tag{1.15}$$

We set $R(1)\backslash R(p) = \{\Gamma(2)\,\alpha_1, \ldots, \Gamma(2)\,\alpha_n\}$ and $w = \Gamma(2)\,\alpha_1\tilde{z}_1 \in \mathbb{H}^2$. Then, according to relation (1.15), we have $\forall\, j = 1 \ldots l,\ w \neq \Gamma(2)\tilde{z}_j$ so that $w \notin \Lambda$. Consider the correspondence $\mathscr{C}_p(w) = \big\{\, \Gamma(2)\,\alpha_1\alpha_1\tilde{z}_1, \ldots, \Gamma(2)\,\alpha_n\alpha_1\tilde{z}_1 \,\big\}$. From Lemma 1.3, we deduce

$$\mathscr{C}_p(w) = \big\{z_1\big\} \cup B \quad \text{where} \quad B \subset \big\{\, \Gamma(2)\,\alpha\,\tilde{z}_1,\ \alpha \in R^{pr}(p^2) \,\big\} = \mathscr{C}_{p^2}(z_1)$$

As $\mathscr{C}_{p^2}(z_1) \cap \Lambda = \emptyset$ by relation (1.15), then $\mathscr{C}_p(w) \cap \Lambda = \{z_1\}$. Hence, for this choice of w, we have shown that $\mathscr{C}_p$ separates Λ in the sense of section 1.1.

• Λ *is infinite :* $\Lambda \subset F_1 \cup \cdots \cup F_r$, where the F_i are closed geodesics. Besides, at least one of the sets $\Lambda \cap F_i,\ 1 \leqslant i \leqslant r$ is infinite : we may assume $\Lambda \cap F_1$ is infinite. Let $\tilde{F}_1, \ldots, \tilde{F}_r$ be liftings of those geodesics to $\mathbb{H}^2$. As before, we have according to Proposition 1.4

$$\exists\, p \in \mathscr{P} \quad \forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, j = 1 \ldots r \qquad \alpha \cdot \tilde{F}_1 \neq \tilde{F}_j \tag{1.16}$$

From Lemma 1.1, $F \cap G$ is finite for two distinct closed geodesics F and G of $X = \Gamma(2)\backslash\mathbb{H}^2$. Hence, $\mu_1 = \mathscr{C}_p(F_1) \cap (F_1 \cup \cdots \cup F_r)$ and $\mu_2 = \mathscr{C}_{p^2}(F_1) \cap (F_1 \cup \cdots \cup F_r)$ are finite subsets of X according to relation (1.16). As a consequence

$$\nu_1 = \big\{z \in X \,/\, \mathscr{C}_p(z) \cap \mu_1 \neq \emptyset\big\} \quad \text{and} \quad \nu_2 = \big\{z \in X \,/\, \mathscr{C}_{p^2}(z) \cap \mu_2 \neq \emptyset\big\}$$

are finite subsets of X too and, $\Lambda \cap F_1$ being infinite, there exists $z \in \Lambda \cap F_1\backslash(\nu_1 \cap \nu_2)$. Let $\tilde{z}$ be a lifting of z to $\mathbb{H}^2$ and $w = \Gamma(2)\,\alpha_1\tilde{z} \in \mathscr{C}_p(z) \subset \mathscr{C}_p(F_1)$. Because $z \notin \nu_1$, we have $\mathscr{C}_p(z) \cap \mu_1 = \emptyset$ so that $\mathscr{C}_p(z) \cap \Lambda = \emptyset$ (cf. $z \in F_1$) and $w \notin \Lambda$.

As before, $\mathscr{C}_p(w) = \{z\} \cup B'$ where $B' \subset \mathscr{C}_{p^2}(z)$. Finally $z \notin \nu_2$ implies $\mathscr{C}_{p^2}(z) \cap \mu_2 = \emptyset$, so that $\mathscr{C}_{p^2}(z) \cap \Lambda = \emptyset$ and $\mathscr{C}_p(w) \cap \Lambda = \{z\}$: thus $\mathscr{C}_p$ separates Λ. This ends the proof.

Conclusion : proof of Theorem 1.1

Let Λ be a non-empty closed set contained in such a finite union of closed geodesics of X, and ν a quantum limit on X. Let $(\phi_j)_{j\in\mathbb{N}}$ be the associated sequence of eigenfunctions of Δ. We make the assumption – as in [9], where an orthonormal basis of common eigenfunctions of the operators Δ and $(T_n)_{n\in\mathbb{N}}$ is considered – that they're also eigenfunctions of the modular operators (that commute with Δ) as "all evidence points to the spectrum of Δ being simple ...". We shall make a similar assumption in section 9.

So our quantum limit is associated to a sequence of eigenfunctions of T_C, where $\mathscr{C} = \mathscr{C}_p$ is the modular correspondence separating Λ given by Proposition 1.5 : owing to Proposition 1.1, we deduce that singsupp $\nu \neq \Lambda$. This proves Theorem 1.1.

2 Hyperbolic Geometry in dimension 3

2.1 Definition

• We take as model of hyperbolic space the upper half-space $\mathbb{H}^3 = \{ (x, y, t) \in \mathbb{R}^3 \ / \ t > 0 \}$. Using the classical notation $\mathbf{j} = 1 \wedge \mathbf{i} \in \mathbb{R}^3$ we get

$$\mathbb{H}^3 = \{ z + t\,\mathbf{j} \ / \ z \in \mathbb{C},\ t > 0 \}$$

Therefore, we shall identify $\mathbb{H}^3$ with the subset $\mathbb{R} \oplus \mathbb{R}\, i \oplus \mathbb{R}_+ \mathbf{j}$ of the algebra of quaternions of Hamilton $\mathbb{H} = \mathbb{R}[1, i, \mathbf{j}, \mathbf{k}]$. The space $\mathbb{H}^3$ is provided with the Riemannian hyperbolic metric

$$\mathrm{d}s^2 = \frac{|\mathrm{d}z|^2 + \mathrm{d}t^2}{t^2} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2}{t^2}$$

It is of constant sectional curvature $K = -1$ and defines the volume form $\mathrm{d}vol = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z/t^3$. Finally the associated Laplace-Beltrami operator is

$$\Delta = t^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} \right) - t\,\frac{\partial}{\partial t}$$

• We deduce easily the geodesics of $\mathbb{H}^3$ from the ones of $\mathbb{H}^2$. They are the half-circles centered on $\mathbb{C}$ and the half-lines orthogonal to $\mathbb{C}$.

Moreover they are uniquely defined by their ending points, two distinct points of $\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{C} \cup \infty$ that are the roots of a unique proportionality class of non-degenerate binary quadratic form with coefficient in $\mathbb{C}$: as in dimension 2, it is a bijection.

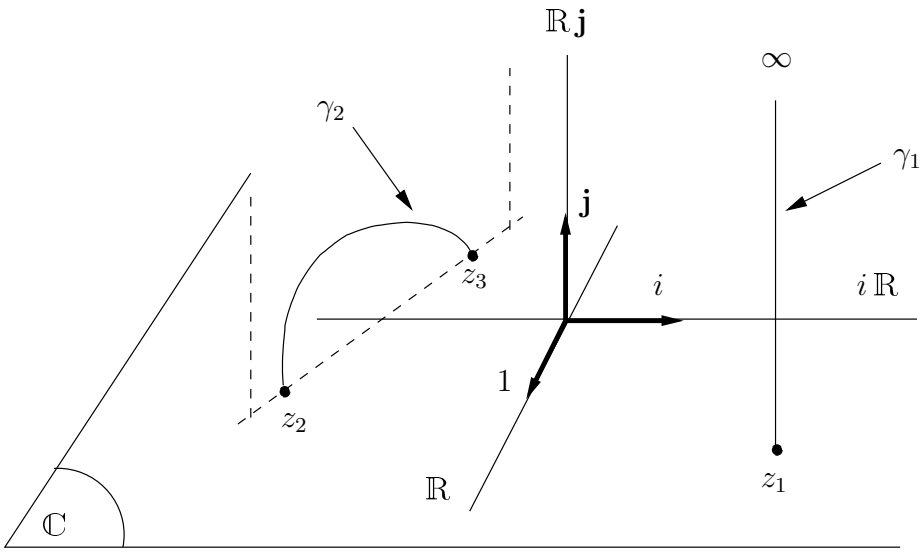

Figure 3: Geodesics of $\mathbb{H}^3$

Let us now turn to the imbedded totally geodesic submanifolds (abbreviated *itgs*) of $\mathbb{H}^3$. By definition, an imbedded submanifold S in a manifold X is called *totally geodesic* if and only if for all $(M, \vec{u}) \in TS$, the geodesic of X tangent to $\vec{u}$ at M is contained in S.

In $\mathbb{H}^3$, the geodesics being the half-lines and the half-circles orthogonal to $\mathbb{C}$, the itgs are the half-planes orthogonal to $\mathbb{C}$ and the half-spheres centered on $\mathbb{C}$.

2.2 Isometries of $\mathbb{H}^3$

We know (cf. [8] for results of inversive geometry) that $\mathrm{Is}(\mathbb{H}^3)$ consists of extensions to $\mathbb{R}^3$ of the Möbius transformations of $\mathbb{C}$

$$M(\mathbb{C}) = \left\{ z \longmapsto \frac{az+b}{cz+d}\ ,\ z \longmapsto \frac{a\overline{z}+b}{c\overline{z}+d} \ \Big/ \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}) \right\}$$

and $\mathrm{Is}^+(\mathbb{H}^3)$, whose elements are extensions of the complex fractional linear transformations, is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$: we shall make the identification implicitely in the sequel. More precisely (always identifying $\mathbb{H}^3$ with a subset of $\mathbb{H}$) we have the following action

$$\forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \quad \forall\, x \in \mathbb{H}^3 \qquad \gamma \cdot x = (ax+b).(cx+d)^{-1} \tag{2.1}$$

This extension is also known as the Poincaré extension. The action on $\mathbb{H}^3$ of an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) \quad \text{with } \ ad-bc = n \neq 0$$

is for $c = 0$

$$\gamma \,.\, \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} \dfrac{az+b}{d} \\ \left|\dfrac{a}{d}\right| t \end{pmatrix} \tag{2.2}$$

and for $c \neq 0$

$$\gamma \,.\, \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} \dfrac{a}{c} - \dfrac{n}{c}\dfrac{\overline{cz+d}}{|cz+d|^2+|c|^2t^2} \\ \dfrac{|n|t}{|cz+d|^2+|c|^2t^2} \end{pmatrix} \tag{2.3}$$

Finally, the classification of the elements of $\mathrm{Is}^+(\mathbb{H}^2)$ extends to $\mathrm{Is}^+(\mathbb{H}^3)$: they are elliptic, parabolic or hyperbolic according to their trace. The notion of axis of a hyperbolic element is the same. The main difference is that an elliptic element does not have a unique fixed point in $\mathbb{H}^3$ but a whole geodesic of them. Indeed, for $c \neq 0$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} z \\ t \end{pmatrix} \iff \begin{cases} a+d = \mathrm{Tr}(\gamma) = 2\,\mathrm{Re}(cz+d) \\ |cz+d|^2 + |c|^2t^2 = 1 \end{cases}$$

and there exists solutions in $\mathbb{H}^3$ (in fact an euclidean ellipse) iff $\mathrm{Tr}(\gamma)^2 \in [0,4[$: we are in the elliptic case. For $c = 0$,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} z \\ t \end{pmatrix} \iff \begin{cases} (a-d)z + b = 1 \\ |a| = |d| = 1 \end{cases}$$

and there exists one solution in $\mathbb{H}^3$ for $a \neq d \in S^1$,whence $\mathrm{Tr}(\gamma)^2 = a^2 + 2 + a^{-2} \in [0,4[$.

In the parabolic case – $\mathrm{Tr}(\gamma)^2 = 4$ with $\gamma \neq \pm I_2$ – or in the hyperbolic one – $\mathrm{Tr}(\gamma)^2 \notin [0,4]$, the resolution of the equation $\gamma \cdot z = z$ on $\mathbb{C}$ leads to the existence of a fixed attractive point or two fixed (attractive and repulsive) points in $\mathbb{C}$ respectively. In both cases, the value of $c \in \mathbb{C}$ does not have any importance.

3 Algebraic complements

3.1 Number theory

We base on [7] in this section.

3.1.1 Extensions and prime ideals

• Let $\mathbb{K}$ be a number field and $\mathbb{L}$ a finite extension. We denote by $\mathscr{O}_{\mathbb{K}}$ and $\mathscr{O}_{\mathbb{L}}$ their respective rings of algebraic integers. Let $\mathfrak{B}$ be a prime ideal of $\mathscr{O}_{\mathbb{L}}$. The prime ideal $P = \mathfrak{B} \cap \mathscr{O}_{\mathbb{K}}$ of $\mathscr{O}_{\mathbb{K}}$ is called the *underlying ideal to* $\mathfrak{B}$. Moreover, $\big[\, \mathscr{O}_{\mathbb{L}}/\mathfrak{B} \,:\, \mathscr{O}_{\mathbb{K}}/P \,\big] \leqslant \big[\, \mathbb{L} \,:\, \mathbb{K} \,\big] < \infty$; we denote this quantity by $f_{\mathbb{L}\,/\,\mathbb{K}}(\mathfrak{B})$: it is the *degree* of $\mathfrak{B}$ over $\mathbb{K}$.

Conversely, let us take a prime ideal P of $\mathscr{O}_{\mathbb{K}}$. We have $P\mathscr{O}_{\mathbb{L}} = {\mathfrak{B}_1}^{e_1} \dots {\mathfrak{B}_s}^{e_s}$ where the $\mathfrak{B}_i$ are prime ideals of $\mathscr{O}_{\mathbb{L}}$. The quantities e_i are called *ramification indices*, and by setting $\forall\, i,\ f_i = f_{\mathbb{L}/\mathbb{K}}(\mathfrak{B}_i)$, we get

$$\sum_{i=1}^{s} e_i f_i = \Big[\, \mathbb{L} \,:\, \mathbb{K} \,\Big] \tag{3.1}$$

If $\exists\, i$ with $e_i > 1$, we say that P is *ramified*. There is only a finite number of such ones.

• Let $\mathbb{K}$ be a number field, P a prime ideal of $\mathscr{O}_{\mathbb{K}}$, $p\mathbb{Z} = P \cap \mathbb{Z}$ the underlying prime ideal and $f = f_{\mathbb{K}/\mathbb{Q}}(P)$ its degree over $\mathbb{Q}$. We define the *norm* of the ideal P by

$$\mathrm{N}(P) \overset{def}{=} |\mathscr{O}_{\mathbb{K}}/P| = p^f$$

Given a set A of prime ideals of $\mathscr{O}_{\mathbb{K}}$, we shall say that A is *regular* with *density* a in the set of all prime ideals of $\mathscr{O}_{\mathbb{K}}$ if

$$\sum_{p \in A} \mathrm{N}(P)^{-s} \underset{s \to 1^+}{\sim} a \log \frac{1}{s-1} \tag{3.2}$$

• Finally, let us take a finite extension $\mathbb{L}/\mathbb{K}$ with normal closure $\mathbb{M}/\mathbb{K}$ and Galois group $G = \mathrm{Gal}\big(\mathbb{M}/\mathbb{K}\big)$. The set of the prime ideals P of $\mathscr{O}_{\mathbb{K}}$ satisfying $P\mathscr{O}_{\mathbb{L}} = \mathfrak{B}_1 \dots \mathfrak{B}_r$ with $\forall\, i = 1 \dots r,\ \mathfrak{B}_i$ prime ideal and $f_{\mathbb{L}/\mathbb{K}}(\mathfrak{B}_i) = f_i$ fixed, is regular and its density is the relative frequence in G of the elements of G that, in the left translation representation considered as a permutation group of the set G, are the products of r disjoints cycles of length $f_1, \dots, f_r$.

3.1.2 Application to quadratic extensions

• Let $\mathbb{K}$ be a number field and take $a \in \mathscr{O}_{\mathbb{K}} \backslash \{1\}$ square-free. The field $\mathbb{L} = \mathbb{K}\,(\sqrt{a}) \simeq \mathbb{K}[X]/(X^2 - a)$ is a quadratic (hence Galois) extension of $\mathbb{K}$, and its Galois group is $G = \{\mathrm{Id}, \tau\}$ where $\tau^2 = \mathrm{Id}$. Moreover $\mathscr{O}_{\mathbb{L}} = \mathscr{O}_{\mathbb{K}}[\alpha]$, with $\alpha = (1+\sqrt{a})/2$ if $a \equiv 1\,[4]$ and $\alpha = \sqrt{a}$ otherwise (we just use that an algebraic integer of $\mathbb{L}$ must have trace and norm in $\mathscr{O}_{\mathbb{K}}$).

For a prime ideal P of $\mathscr{O}_{\mathbb{K}}$, we have $\mathbb{L}/(P) = \mathbb{K}_{/(P)}[X]/(X^2 - a)$ and $P\mathscr{O}_{\mathbb{L}} = {\mathfrak{B}_1}^{e_1} \dots {\mathfrak{B}_s}^{e_s}$ with $\sum_i e_i f_i = \big[\, \mathbb{L} \,:\, \mathbb{K} \,\big] = 2$, so that only three situations occur :

i) $P\mathscr{O}_{\mathbb{L}} = R$, prime in $\mathscr{O}_{\mathbb{L}}$, is *inert* in iff a is not a square modulo P (*density* 1/2)

ii) $P\mathscr{O}_{\mathbb{L}} = R\overline{R}$, with R prime in $\mathscr{O}_{\mathbb{L}}$, *splits* iff a is a square modulo P (*density* 1/2)

iii) $P\mathscr{O}_{\mathbb{L}} = R^2$, with R prime in $\mathscr{O}_{\mathbb{L}}$, is *ramified* iff $a \in P$. (*density* 0)

• From now on, $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}\backslash\{1\}$ is square free, and we consider $a \in \mathscr{O}_{\mathbb{K}}$ that is not a square. Let us denote by A the set $\{P \text{ primes of } \mathscr{O}_{\mathbb{K}} \;/\; a \text{ is not a square modulo } P\}$, that is of density 1/2, and by B the set $\{\, p\mathbb{Z} = P \cap \mathbb{Z} \;/\; P \in A \,\}$ of underlying ideals of $\mathbb{Z}$.

We shall divide B into three subsets $B = B_1 \cup B_2 \cup B_3$, respectively the sets of ideals that are inert, ramified or split.

$$\begin{array}{lll} \text{If } P \cap \mathbb{Z} = p\mathbb{Z} \in B_1 & : & p\mathscr{O}_{\mathbb{K}} = P \in A \text{ so that } \mathrm{N}(P) = p^2 \text{ because } f_{\mathbb{K}/\mathbb{Q}}(P) = 2. \\ \text{If } P \cap \mathbb{Z} = p\mathbb{Z} \in B_2 \cup B_3 : & & p\mathscr{O}_{\mathbb{K}} = P^2 \text{ or } P\overline{P} \text{ with } P \in A, \text{ hence } \mathrm{N}(P) = p. \end{array}$$

An ideal $p\mathbb{Z}$ of B either splits ($p\mathbb{Z} \in B_3$) and there are two prime ideals of A above it, or it is inert or ramified ($p\mathbb{Z} \in B_1 \cup B_2$) and there is only one above it. Therefore

$$\sum_{P\in A} \mathrm{N}(P)^{-s} = \sum_{p\mathbb{Z}\in B_1} p^{-2s} + \sum_{p\mathbb{Z}\in B_2} p^{-s} + 2\sum_{p\mathbb{Z}\in B_3} p^{-s} \Longrightarrow \sum_{P\in A} \mathrm{N}(P)^{-s} \leqslant 2\sum_{p\mathbb{Z}\in B} p^{-s}$$

Since A is of density $1/2$, B contains a subset of density greater than $1/4$. This proves

Proposition 3.1 *Let $d \in \mathbb{Z}\backslash\{1\}$ be square-free, $\mathbb{K} = \mathbb{Q}\big(\sqrt{d}\big)$ a quadratic extension and $a \in \mathscr{O}_{\mathbb{K}}$ that is not a square. There exists a regular subset $\mathscr{C} \subset \mathscr{P}$ of density greater than $1/4$ such that for any prime ideal P of $\mathscr{O}_{\mathbb{K}}$ satisfying $P\cap\mathbb{Z} = p\mathbb{Z}$ with $p \in \mathscr{C}$, a is not a square modulo P.*

3.1.3 A result on Legendre character

Definition *Let $p \in \mathscr{P}$ and $\mathbb{K}$ be a field number. Elements $a_1, a_2, \ldots, a_n$ of $\mathbb{K}$ are called* p-independent *if, as soon as ${a_1}^{x_1}{a_2}^{x_2}\ldots{a_n}^{x_n}$ (with $x_i \in \mathbb{Z}$ for all i) is a p^{th} power in $\mathbb{K}$, then $x_i \equiv 0\,[p]$ for all $i = 1\ldots r$.*

Applied to quadratic characters, a Theorem from [7] states

Theorem 3.1 *Let $a_1, \ldots, a_n \in \mathbb{Z}$ be 2-independent integers and $z_1, \ldots, z_n \in \{\pm 1\}$ fixed. There exists infinitely many $p \in \mathscr{P}$ such that $\forall\, i = 1\ldots n$, $\left(\frac{a_i}{p}\right) = z_i$.*

Proposition 3.2 *Let $a_1, \ldots, a_n \in \mathbb{Z}\backslash\mathbb{N}$. There exists infinitely many primes p such that $\forall\, i = 1\ldots n$, $\left(\frac{a_i}{p}\right) = -1$.*

Proof : let $a_1, \ldots, a_n \in \mathbb{Z}\backslash\mathbb{N}$. If they are 2-independent, the result is immediate by application of Theorem 3.1. Otherwise, let us take a maximal 2-independent subfamily, that we may take up to a relabelling to be $a_1, \ldots, a_m$ with $1 \leqslant m < n$. According to Theorem 3.1, there exists infinitely many primes p such that

$$\forall\, i = 1\ldots m \quad \left(\frac{a_i}{p}\right) = -1$$

Let p be such a prime satisfying : $\forall\, i = m+1\ldots n$, $a_i \not\equiv 0\,[p]$, and let $1 \leqslant j \leqslant n-m$. Because the selected 2-independent family is maximal, the elements $a_1, \ldots, a_m$ and a_{m+j} are not 2-independent and, after simplification

$$\exists\, q \in \mathbb{Z} \quad \exists\, x_1, \ldots, x_m \in \big\{0, 1\big\} \qquad {a_1}^{x_1}\ldots{a_m}^{x_m}a_{m+j} = q^2 > 0$$

As $a_i < 0$ for all i, then $\displaystyle\sum_{i=1}^{m} x_i$ is necessarily odd and

$$a_{m+j} = q^2\prod_{i=1}^{m} a_i^{-x_i} \implies \left(\frac{a_{m+j}}{p}\right) = \prod_{i=1}^{m}\left(\frac{a_i}{p}\right)^{x_i} = (-1)^{\sum x_i} = -1$$

We can take any j with $1 \leqslant j \leqslant n-m$, which ends the proof.

3.2 Quaternion algebras

3.2.1 Definition

We base on [2] in this section.

Definition *Let* $a,\ b \in \mathbb{Q}$. *We call* quaternion algebra of type (a,b) on $\mathbb{Q}$ *the* $\mathbb{Q}$*-algebra* $\mathfrak{A} = \mathbb{Q}[1,\omega,\Omega,\omega\Omega]$, *with the multiplication table* $\omega^2 = a$, $\Omega^2 = b$ *and* $\omega\Omega+\Omega\omega = 0$.

Such an algebra will be denoted by $\mathfrak{A} = \left(\frac{a,b}{\mathbb{Q}}\right)$. We may assume without loss of generality that a and b are square free integers. We shall also take $a \neq 1$ in the sequel.

• The center of $\mathfrak{A}$ is $\mathbb{Q}$. $\mathbb{F} = \{\, q + r\,\omega \;/\; q,\ r \in \mathbb{Q} \,\}$ is a subfield of $\mathfrak{A}$ isomorphic to $\mathbb{Q}(\sqrt{a})$. We shall identify $\mathbb{F}$ to $\mathbb{Q}(\sqrt{a})$, and write any element of $\mathfrak{A}$ as $\alpha = x_0 + x_1\,\omega + x_2\,\Omega + x_3\,\omega\Omega = \xi + \eta\,\Omega$, with $\xi = x_0 + x_1\,\omega,\ \eta = x_2 + x_3\,\omega \in \mathbb{F}$. Note that $\forall\,\xi \in \mathbb{F},\ \xi\,\Omega = \Omega\,\overline{\xi}^{\mathbb{F}}$. We define :

- $\overline{\alpha} = x_0 - x_1\,\omega - x_2\,\Omega - x_3\,\omega\Omega = \overline{\xi}^{\mathbb{F}} - \eta\,\Omega$ the *conjugate* of α.
- $\mathrm{Tr}(\alpha) = \alpha + \overline{\alpha} = \mathrm{Tr}(\xi) = 2x_0 \in \mathbb{Q}$ the *trace* of α.
- $\mathrm{N}(\alpha) = \alpha\overline{\alpha} = \xi\,\overline{\xi}^{\mathbb{F}} - b\,\eta\,\overline{\eta}^{\mathbb{F}} = ({x_0}^2 - a{x_1}^2) - b({x_2}^2 - a{x_3}^2) \in \mathbb{Q}$ its *norm*.

Proposition $\forall\,\alpha_1,\ \alpha_2 \in \mathfrak{A} \qquad \overline{\alpha_1.\alpha_2} = \overline{\alpha_2}.\overline{\alpha_1}$.

Theorem $\mathfrak{A}$ *has zero divisors* $\iff \mathfrak{A} \simeq \mathrm{M}(2,\mathbb{Q})$

In this case we shall speak of *matrix algebra*. Otherwise, we have a *division algebra*, as the one considered here. But, after extending the scalars to $\mathbb{F}$, we have zero divisors and the mapping

$$\varphi \;:\; \alpha = \xi + \eta\,\Omega \longmapsto \begin{pmatrix} \xi & \eta \\ b\,\overline{\eta}^{\mathbb{F}} & \overline{\xi}^{\mathbb{F}} \end{pmatrix} \tag{3.3}$$

provides the identification of $\mathfrak{A} \otimes \mathbb{F}$ with $\mathrm{M}(2,\mathbb{F})$. Let us note that φ leaves the trace and the norm (as they are defined on $\mathfrak{A}$) invariant, the norm on $\mathrm{Im}(\varphi) \subset \mathrm{M}(2,\mathbb{F})$ being quite simply the determinant.

Finally, we will call a quaternion algebra *definite* or *indefinite* whether its norm is definite ($a < 0$ and $b < 0$) or indefinite ($a > 0$ or $b > 0$) as a quaternary quadratic form on $\mathbb{R}$.

• Going back to matrix algebras, we have the following characterization :

Proposition 3.3 *Let* $b \in \mathscr{P}$. *If* $\mathfrak{A} = \left(\frac{a,b}{\mathbb{Q}}\right)$ *is a matrix algebra, then* $\left(\frac{a}{b}\right) = 1$.

Proof : given $\alpha = x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega \in \mathfrak{A}$, its norm is

$$\mathrm{N}(\alpha) = ({x_0}^2 - a{x_1}^2) - b({x_2}^2 - a{x_3}^2) = ({x_0}^2 - b{x_2}^2) - a({x_1}^2 - b{x_3}^2) \tag{3.4}$$

If $\alpha \in \mathfrak{A}$ is a divisor of 0, we have $\alpha \neq 0$ and $\mathrm{N}(\alpha) = 0$. Hence, by relation (3.4),

$${x_0}^2 - a{x_1}^2 = b\left({x_2}^2 - a{x_3}^2\right) \qquad \text{with} \qquad \forall\, i = 0\ldots3, \quad x_i \in \mathbb{Q}$$

After the multiplication by the least common multiple of the denominators of the x_i, we obtain

$${y_0}^2 - a{y_1}^2 = b\left({y_2}^2 - a{y_3}^2\right) \qquad \text{where} \qquad \forall\, i = 0\ldots3, \quad y_i \in \mathbb{Z}$$

Let us deal with this equation :

(i) If b does not divide y_1 : as ${y_0}^2 - a{y_1}^2 \equiv 0\big[b\big]$, then $a \equiv \left(\dfrac{y_0}{y_1}\right)^2 \big[b\big]$ and a is a square modulo b.

(ii) If b divides y_1 : then b divides ${y_0}^2$ so that, b being prime, b divides y_0. By noting $y'_0 = y_0/b$ and $y'_1 = y_1/b$, we get
$$ {y_2}^2 - a{y_3}^2 = b\big({y'_0}^2 - a{y'_1}^2\big) $$

We find in case (ii) an equation in $y_2,\ y_3$ of the same type as before. Therefore after a finite number of simplifications by b, we are in case (i), unless $y_1 = y_3 = 0$. In this last case, we have ${y_0}^2 = b{y_2}^2 \neq 0$ as $\alpha \neq 0$, so that $b \in \mathbb{Z}$ is a square in $\mathbb{Q}$ *i.e.* in $\mathbb{Z}$, which cannot happen for $b \in \mathscr{P}$. So we have shown

$$ \text{For } b \in \mathscr{P} \qquad \mathfrak{A} \text{ matrix algebra } \implies \Big(a \text{ is a square modulo } b\Big) $$

3.2.2 Definition of a discrete group of isometries associated to $\mathfrak{A}$

Definition *An* order $\mathfrak{I}$ *in* $\mathfrak{A}$ *is a subring of* $\mathfrak{A}$ *such that*

1) $1 \in \mathfrak{I}$.

2) $\alpha \in \mathfrak{I} \implies N(\alpha)$ & $Tr(\alpha) \in \mathbb{Z}$.

3) $\mathfrak{I}$ *has four linearly independent generators over* $\mathbb{Q}$.

Thus, $\mathfrak{I}$ is a free $\mathbb{Z}$-module of rank four in $\mathfrak{A}$, that is besides stable under the conjugation. For example, $\mathfrak{I}_0 = \mathscr{O}_{\mathbb{F}} \oplus \mathscr{O}_{\mathbb{F}}\,\Omega = \big\{\,\xi + \eta\,\Omega \;/\; \xi,\ \eta \in \mathscr{O}_{\mathbb{F}}\,\big\}$ is a particular order for any quaternion algebra $\mathfrak{A} = \left(\frac{a,b}{\mathbb{Q}}\right)$, and $\mathrm{M}(2,\mathbb{Z})$ is an order for the matrix algebra $\mathfrak{A} = \mathrm{M}(2,\mathbb{Q})$.

Proposition 3.4 *Let* $\mathfrak{I}$ *be an order. Then* $\exists\, D,\ D' \in \mathbb{Z}\backslash\{0\}\quad D'\mathfrak{I} \subset D\mathfrak{I}_0 \subset \mathfrak{I}$.

Proof : the orders $\mathfrak{I}$ and $\mathfrak{I}_0$ being two free $\mathbb{Z}$-modules of rank four in $\mathfrak{A}$, their $\mathbb{Z}$-bases are two $\mathbb{Q}$-bases of $\mathfrak{A}$. Let us call $M \in \mathrm{GL}(4,\mathbb{Q})$ any transition matrix from a basis of $\mathfrak{I}$ to a basis of $\mathfrak{I}_0$. We just have to take two integers D and D' such that $DM \in \mathrm{M}(4,\mathbb{Z})$ and $D'D^{-1}M^{-1} \in \mathrm{M}(4,\mathbb{Z})$ to satisfy the above property. Moreover, $\mathscr{O}_{\mathbb{F}} \simeq \mathbb{Z}^2$ and $\mathfrak{I} \simeq \mathbb{Z}^4$ is countable.

• Given $n \in \mathbb{Z}$, we define $\mathfrak{I}(n) = \big\{\,\alpha \in \mathfrak{I} \;/\; \mathrm{N}(\alpha) = n\,\big\}$ and $\mathfrak{I}^{pr}(n) = \mathfrak{I}(n) \cap \mathfrak{I}^{pr}$ the subset of primitive elements, where the primitive elements are the ones that cannot be divided in $\mathfrak{I}$ by a non trivial integer.

Proposition 3.5 *Let* $p \in \mathscr{P}$ *be a prime such that* $ord_p(2abDD') = 0$ *and* $\left(\frac{a}{p}\right) = -1$.
$\forall\, \alpha = \xi + \eta\,\Omega \in \mathfrak{I}^{pr},\ N(\alpha) \equiv 0\,[p] \implies ord_p\, N(\xi) = ord_p\, N(\eta) = 0$.
In particular, $\xi\,\eta \neq 0$ *for such an* α.

Proof : let $p \in \mathscr{P}$ and $\alpha = \xi + \eta\,\Omega \in \mathfrak{I}$ satisfy the above assumptions ; thus we have $\mathrm{N}(\alpha) = \xi\,\overline{\xi}^{\mathbb{F}} - b\,\eta\,\overline{\eta}^{\mathbb{F}} = \mathrm{N}(\xi) - b\,\mathrm{N}(\eta) \equiv 0\,[p]$. As $D'\mathfrak{I} \subset D\mathfrak{I}_0 \subset \mathfrak{I}$, then $D'\xi = D\xi_1$ and $D'\eta = D\eta_1$ with $\xi_1,\ \eta_1 \in \mathscr{O}_{\mathbb{F}}$. Besides, $\mathrm{ord}_p(DD') = 0$ so that $\mathrm{ord}_p\,\mathrm{N}(\xi) = \mathrm{ord}_p\,\mathrm{N}(\xi_1) \geqslant 0$ and $\mathrm{ord}_p\,\mathrm{N}(\eta) = \mathrm{ord}_p\,\mathrm{N}(\eta_1) \geqslant 0$ (we follow the classical convention $\mathrm{ord}_p(0) = +\infty$). By taking the norm, we get
$$ \mathrm{N}(D'\alpha) = \mathrm{N}\big[D(\xi_1 + \eta_1\,\Omega)\big] = D^2\big[\mathrm{N}(\xi_1) - b\,\mathrm{N}(\eta_1)\big] \equiv 0\,[p] $$

hence, as $\mathrm{ord}_p(D) = 0$, $\mathrm{N}(\xi_1) - b\,\mathrm{N}(\eta_1) \equiv 0\,[p]$ and $\mathrm{ord}_p(b) = 0$ implies

$$\mathrm{ord}_p\,\mathrm{N}(\xi_1) > 0 \iff \mathrm{ord}_p\,\mathrm{N}(\eta_1) > 0$$

Let us assume that

$$\mathrm{ord}_p\,\mathrm{N}(\xi) = \mathrm{ord}_p\,\mathrm{N}(\xi_1) > 0 \tag{3.5}$$

$$\mathrm{ord}_p\,\mathrm{N}(\eta) = \mathrm{ord}_p\,\mathrm{N}(\eta_1) > 0 \tag{3.6}$$

$\underline{a \not\equiv 1[4]}$: as a consequence, $\mathscr{O}_{\mathbb{F}} = \mathbb{Z}[\sqrt{a}]$ and $\xi_1 = b_0 + b_1\sqrt{a}$ with $b_0,\, b_1 \in \mathbb{Z}$. Relation (3.5) can be expressed as $p \,/\, {b_0}^2 - a{b_1}^2$.

. If $p \,/\, b_1$, then $p \,/\, b_0$ and $\xi_1 = p\,\xi_2$ with $\xi_2 \in \mathscr{O}_{\mathbb{F}}$.

. Otherwise $a \equiv \left(\dfrac{b_0}{b_1}\right)^2 [p]$ and $\left(\dfrac{a}{p}\right) = 1$, contradicting the assumption.

$\underline{a \equiv 1[4]}$: in this case, $\mathscr{O}_{\mathbb{F}} = \mathbb{Z}\left[\dfrac{1+\sqrt{a}}{2}\right]$ and $\xi_1 = \left(b_0 + \dfrac{b_1}{2}\right) + \dfrac{b_1\sqrt{a}}{2}$ with $b_0,\, b_1 \in \mathbb{Z}$.

We have $p \,/\, \mathrm{N}(2\xi_1)$ thus $p \,/\, (2b_0 + b_1)^2 - a{b_1}^2$.

. If $p \,/\, b_1$, then $p \,/\, 2b_0 + b_1$ and $p \,/\, b_0$, so that $\xi_1 = p\,\xi_2$ with $\xi_2 \in \mathscr{O}_{\mathbb{F}}$.

. Otherwise, $a \equiv \left(\dfrac{2b_0}{b_1} + 1\right)^2 [p]$ and $\left(\dfrac{a}{p}\right) = 1$, contradicting the assumption.

Therefore, relation (3.5) leads to $\exists\, \xi_2 \in \mathscr{O}_{\mathbb{F}},\ \xi_1 = p\,\xi_2$. In the same way, relation (3.6) leads to $\exists\, \eta_2 \in \mathscr{O}_{\mathbb{F}},\ \eta_1 = p\,\eta_2$. Thus $D'\alpha = D\big(\xi_1 + \eta_1\,\Omega\big) = pD\big(\xi_2 + \eta_2\,\Omega\big) = p\,D\alpha_2$ where $\alpha_2 \in \mathfrak{I}_0$. Because $p \wedge D' = 1$, there exists $x,\, y \in \mathbb{Z}$ such that $xD' + yp = 1$ (Bézout) and

$$\alpha = xD'\alpha + py\alpha = p(xD\alpha_2 + y\alpha). \tag{3.7}$$

As $\alpha_2 \in \mathfrak{I}_0$ and $D\mathfrak{I}_0 \subset \mathfrak{I}$, we have $D\alpha_2 \in \mathfrak{I}$ and $\alpha \in p\mathfrak{I} : \alpha$ is not primitive. This ends the proof of the Proposition.

• Let $\mathfrak{A}$ be an indefinite quaternion algebra of type (a, b) on $\mathbb{Q}$; we shall consider in the sequel orders of type (q_1, q_2) in $\mathfrak{A}$; they are principal and we can use them to define modular correspondences (cf. [2] §3). For $q_1 = 1$, these orders are simply the maximal ones.

Let $\mathfrak{I}$ be an order of type (q_1, q_2) and $R = \varphi(\mathfrak{I})$ its image in $\mathrm{M}(2, \mathbb{F})$, where φ is defined by relation (3.3). We shall implicitely identify $\mathfrak{I}$ with its image by φ and denote by $\alpha = \xi + \eta\,\Omega$ any element of R. Via Poincaré extension, the set $R^* = \big\{\alpha \in R \,\big/\, \det(\alpha) \neq 0\big\}$ is identified with a subgroup of $\mathrm{Is}^+(\mathbb{H}^3)$. For $n \wedge q_1 q_2 = 1$, we define

$$R(n) = \{\ \alpha \in R\ /\ \mathrm{N}(\alpha) = n\} \tag{3.8}$$

$$R^{pr}(n) = \{\ \alpha \in R\ /\ \alpha \text{ primitive and } \mathrm{N}(\alpha) = n\} \tag{3.9}$$

which are infinite subsets of $\mathrm{M}(2, \mathbb{F})$. For example, let $\mathfrak{I}$ be a maximal order containing $\mathfrak{I}_0$: Pell-Fermat Theorem applied to the equation $\mathrm{N}(\alpha) = {x_0}^2 - a{x_1}^2 - b{x_2}^2 + ab{x_3}^2 = 1$ implies that $\mathfrak{I}(1)$ is infinite because at least one of the square-free integers a, b is positive.

Let $\Gamma_{_R} = R(1)$ be the discrete subgroup of $\mathrm{SL}(2, \mathbb{C})$ induced by R. We shall denote the quotient space by $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$ and the canonical projection by $\pi_{_R} : \mathbb{H}^3 \longrightarrow \Gamma_{_R}\backslash\mathbb{H}^3$.

To insure that $X_{_R}$ inherits the Riemannian structure of $\mathbb{H}^3$, the group $\Gamma_{_R}$ must not contain any elliptic element.

Proposition 3.6 *Let us assume that $b \in \mathscr{P}\backslash\{3\}$. If $\Gamma_{_R}$ contains an elliptic element, one of the integers a, $-a$ or $-3a$ is a square modulo b.*

Proff : if $\Gamma_{_R}$ contains an elliptic element, there exists $\alpha = \xi + \eta\,\Omega \in \mathfrak{I}$ such that $|\mathrm{Tr}(\alpha)| < 2$ and $\mathrm{N}(\alpha) = \xi\,\overline{\xi}^{\mathbb{F}} - b\,\eta\,\overline{\eta}^{\mathbb{F}} = 1$. Since $\mathfrak{I}$ is an order, we have $\mathrm{Tr}(\alpha) = \mathrm{Tr}(\xi) \in \big\{-1, 0, +1\big\}$. Set $\xi = x + y\sqrt{a}$ and $\eta = z + t\sqrt{a} \in \mathbb{F}$, where x, y, z and $t \in \mathbb{Q}$. Then $\mathrm{N}(\alpha) = x^2 - a\,y^2 - b(z^2 - a\,t^2) = 1$ and $\mathrm{Tr}(\alpha) = 2x \in \big\{-1, 0, +1\big\}$. We shall assume that $a \not\equiv 0\,[b]$ (otherwise a would automatically be a square modulo b).

If $x = 0$: then $-ay^2 = 1 + b(z^2 - at^2)$ and after multiplication by the least common multiple of the denominators of y, z and t, we get $-ay'^2 = E^2 + b(z'^2 - at'^2)$ with integers y', z' and t' such that $y' \wedge z' \wedge t' = 1$.

. If $y' \not\equiv 0\,[b]$, $-a \equiv E^2/y'^2\,[b]$ and $-a$ is a square modulo b.

. If $y' \equiv 0\,[b]$, b divides E and $z'^2 - at'^2 \equiv 0\,[b]$ where $z' \not\equiv 0\,[b]$ and $t' \not\equiv 0\,[b]$ because $a \not\equiv 0\,[b]$ and $y' \wedge z' \wedge t' = 1$. Therefore $a \equiv z'^2/t'^2\,[b]$ and a is a square modulo b.

If $x = \pm 1/2$: then $-ay^2 = 3/4 + b(z^2 - at^2)$ and after multiplication by the least common multiple of the denominators of y, z and t, we get $-ay'^2 = 3E^2 + b(z'^2 - at'^2)$ with integers y', z' and t' such that $y' \wedge z' \wedge t' = 1$. As in the case $x = 0$, we deduce that

. If $y' \not\equiv 0\,[b]$, $-3a \equiv 9E^2/y'^2\,[b]$ and $-3a$ is a square modulo b.

. If $y' \equiv 0\,[b]$, b divides E because $|b| \neq 3$, and $z'^2 - at'^2 \equiv 0\,[b]$ where $z' \not\equiv 0\,[b]$ and $t' \not\equiv 0\,[b]$ because $a \not\equiv 0\,[b]$ and $y' \wedge z' \wedge t' = 1$. Thus $a \equiv z'^2/t'^2\,[b]$ and a is a square modulo b, which ends the proof of the Proposition.

• For all $n \in \mathbb{N}$ such that $n \wedge q_1 q_2 = 1$, $R(1)\backslash R(n)$ is finite (cf. [2] §7) : we may define (as in section 1.2.3) the modular correspondences on the quotient space $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$ by

$$T_n\,f(z) = \sum_{\alpha \in R(1)\backslash R(n)} f(\alpha \cdot z) \qquad\qquad C_n\,f(z) = \sum_{\alpha \in R(1)\backslash R^{pr}(n)} f(\alpha \cdot z) \tag{3.10}$$

We call these operators *modular operators*. They possess the classical properties of the modular operators we have seen in dimension 2 (cf. section 1.2.3) and they are bounded linear operators of $\mathscr{L}^2(X_{_R})$. Moreover, the proof of Lemma 1.3 can be adapted to an order R : for any $n \in \mathbb{N}$, the set $R(n)$ is stable under the passage to the comatrix, which corresponds to the conjugation of the underlying element of $\mathfrak{I}$

$$\varphi(\xi + \eta\,\Omega) = \begin{pmatrix} \xi & \eta \\ b\overline{\eta}^{\mathbb{F}} & \overline{\xi}^{\mathbb{F}} \end{pmatrix} \in R(n) \implies \varphi(\overline{\xi}^{\mathbb{F}} - \eta\,\Omega) = \begin{pmatrix} \overline{\xi}^{\mathbb{F}} & -\eta \\ -b\overline{\eta}^{\mathbb{F}} & \xi \end{pmatrix} \in R(n) \tag{3.11}$$

Adapting the proof of Lemma 1.3), we show that for $p \in \mathscr{P}$ such that $p \wedge q_1 q_2 = 1$ (only such p will be considered thereafter)

Proposition 3.7 *$|R(1)\backslash R(p)| = m$. We shall denote $R(1)\backslash R(p) = \big\{R(1)\,\sigma_1, \ldots, R(1)\,\sigma_m\big\}$. $\forall\, i, \exists\,!\, j$ such that $\sigma_j\sigma_i \in p\,R(1)$ and $\forall\, k \neq j$, $\sigma_k\sigma_i \in R^{pr}(p^2)$.*

4 The quotient space $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$

4.1 Definition of the studied class

We take an indefinite division algebra $\mathfrak{A} = \left(\frac{a,b}{\mathbb{Q}}\right)$ with $a < 0$ and $b > 0$, and consider an order $\mathfrak{I}$ of type (q_1, q_2), e.g. a maximal order containing the order $\mathfrak{I}_0$, such that $\Gamma_{_R}$ acts freely on $\mathbb{H}^3$. This way, we get a class $\big(K_2\big)$ of quotient manifolds $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$ that are Riemannian and of sectional curvature $K = -1$ when provided with the metric induced by $\mathbb{H}^3$.

This class $\big(K_2\big)$ is far from being empty. Indeed, take $a \in \mathbb{Z}_-\backslash\{-1, -3\}$ and $b \in \mathscr{P}\backslash\{3\}$ such that a is not a square modulo b : $\mathfrak{A}$ is an indefinite division quaternion algebra according to Proposition 3.3. For the action of $\Gamma_{_R}$ to be free, we just have to impose that -1 and -3 are squares modulo b, by Proposition 3.6. Given a fixed, Theorem 3.1 shows that these three conditions modulo b are simultaneously satisfied by infinitely many primes b, because the negative integers a, -1 and -3 are 2-independent.

Definition *The infinite set of* $\big(K_2\big)$ *- manifolds satisfying the conditions*

$$a \in \mathbb{Z}_-,\ b \in \mathscr{P},\ \left(\frac{a}{b}\right) = -1,\ \left(\frac{-1}{b}\right) = 1,\ \left(\frac{-3}{b}\right) = 1 \tag{4.1}$$

is called the class $\big(K_2^S\big)$.

For example, we can take $a = -2$ and $b = 13$.

4.2 Properties of $\big(K_2^S\big)$ - manifolds

• In the sequel of this work, we shall consider manifolds $X_{_R}$ of the class $\big(K_2^S\big)$. The conjugation in $\mathbb{F} \simeq \mathbb{Q}\big(\sqrt{a}\big)$ coincides with the complex conjugation because $a < 0$.

Lemma 4.1 *Let* $\xi \in \mathbb{F}\backslash\{0\}$. *Then* $ord_b\,|\xi|^2$ *is even.*

Proof : let $\xi = x' + y'\sqrt{a}$ be any element in $\mathbb{F}$. We can write

$$\xi = \frac{p\left(x + y\sqrt{a}\right)}{q}$$

with $q,\ p,\ x,\ y \in \mathbb{Z},\ x \wedge y = 1$ and $p \wedge q = 1$. Assume that $|x + y\sqrt{a}|^2 = x^2 - a\,y^2 \equiv 0\,[b]$. If b divides y, then b divides x, which contradicts $x \wedge y = 1$. Therefore $y \not\equiv 0\,[b]$ so that

$$a \equiv \left(\frac{x}{y}\right)^2\ [b] \quad i.e \quad \left(\frac{a}{b}\right) = 1$$

and $X_{_R}$ cannot be of class $\big(K_2^S\big)$. Hence $|x + y\sqrt{a}|^2 \not\equiv 0\,[b]$ and, as $|\xi|^2 = p^2\,q^{-2}\,|x + y\sqrt{a}|^2$, we deduce that $\mathrm{ord}_b\,|\xi|^2 = 2\left[\mathrm{ord}_b\,(p) - \mathrm{ord}_b\,(q)\right]$ is even. This ends the proof of the Lemma.

Proposition 4.1 *Let* $X_{_R}$ *be a* $\big(K_2^S\big)$ *- manifold. Then* $\Gamma_{_R}$ *has no parabolic element.*

Proof : let us take a parabolic element $\gamma = \xi + \eta\,\Omega$ in $\Gamma_{_R}$. By taking its opposite $-\gamma$ if necessary, we may always assume that $\xi = 1 + x\sqrt{a}$ and $\eta = y + z\sqrt{a}$ with x, y and $z \in \mathbb{Q}$. As $\mathrm{N}(\gamma) = 1$, we have

$$0 = |\xi|^2 - b\,|\eta|^2 - 1 = -ax^2 - b\,(y^2 - az^2)$$

Let us multiply x, y and z by the least common multiple of their denominators and divide the obtained integers by their greatest common divisor ; we get

$$aX^2 + b\,(Y^2 - aZ^2) = 0 \quad \text{with} \quad X,\ Y,\ Z \in \mathbb{Z} \quad \text{and} \quad X \wedge Y \wedge Z = 1 \tag{4.2}$$

Because $b \wedge a = 1$, b divides X. Setting $X_0 = X/b$, we get after simplification

$$ab{X_0}^2 + Y^2 - aZ^2 = 0 \tag{4.3}$$

If b divides Z, b divides Y too, which contradicts the relation $X \wedge Y \wedge Z = 1$. Therefore $Z \not\equiv 0\,[b]$ and we get by reduction of (4.3) modulo b

$$a \equiv \left(\frac{Y}{Z}\right)^2\ [b] \quad \textit{i.e.} \quad \left(\frac{a}{b}\right) = 1$$

As a consequence, $X_{_R}$ cannot be of class $\left(K_2^S\right)$.

• Two geometrical properties of the space $\Gamma(2)\backslash\mathbb{H}^2$ stated in section 1.2.2 extend directly to the space $X_{_R}$, because $\Gamma_{_R}$ acts freely and discontinuously on $\mathbb{H}^3$ cf. [8]. We have thus

Proposition 4.2 *Let $\mathscr{L}$ be a closed geodesic of $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$. There exists an hyperbolic transformation $\gamma \in \Gamma_{_R}$ whose axis $L \subset \mathbb{H}^3$ projects onto $\mathscr{L}$ in $X_{_R}$.*

In particular, there exists a compact portion l of L such that $\mathscr{L} = \pi_{_R}(L) = \pi_{_R}(l) = \pi_R(l)$.

Lemma 4.2 *Let F and G be two closed geodesics of $X_{_R}$. Then $F = G$ or $F \cap G$ is finite.*

• We have similar properties for closed itgs, which are compact itgs of $X_{_R}$.

Definition *An itgs S of $\mathbb{H}^3$ is* closed for $\Gamma_{_R}$ *if its projection $\mathscr{S}$ in $X_{_R}$ is closed, that is*

$$\exists\, \mathscr{F} \subset S \ \textit{ compact } \ \exists\, \Gamma' \subset \Gamma_{_R} \qquad S = \bigcup_{\gamma\in\Gamma'} \gamma \cdot \mathscr{F} = \Gamma'.\mathscr{F} \tag{4.4}$$

Thus $\mathscr{S} = \pi_{_R}(S) = \pi_{_R}(\mathscr{F})$. As the group $\Gamma_{_R}$ is countable and the complete space S has non empty interior, then $\mathscr{F}$ has non empty interior by Baire's Lemma. More precisely

Proposition 4.3 *Let $\mathscr{S}$ be a closed itgs of $X_{_R}$ and S be a lifting to $\mathbb{H}^3$. There exists a group $\Gamma_0 \subset \Gamma_{_R}$ and a compact subset $\mathscr{F} \subset S$ with non empty interior such that*

$$\gamma \in \Gamma_0 \iff \gamma \cdot S = S \quad \textit{and} \quad S = \bigcup_{\gamma\in\Gamma_0} \gamma \cdot \mathscr{F} = \gamma_0 \cdot \mathscr{F}$$

There exists moreover $\gamma = \xi + \eta\,\Omega \in \Gamma_0$ hyperbolic, with and $\eta \neq 0$.

Proof : let Γ_0 denote the set

$$\Gamma_0 = \left\{\, \gamma \in \Gamma_{_R} \ /\ \gamma.\mathscr{F} \subset S \,\right\} \supset \Gamma' \tag{4.5}$$

For $\gamma \in \Gamma_0$, the set $\gamma \cdot S \cap S \supset \gamma\mathscr{F}$ has non zero area. As $\gamma \cdot S$ and S are both itgs of $\mathbb{H}^3$ *i.e.* half-planes or half-spheres, $\gamma \cdot S = S$. Conversely, the relation $\gamma \cdot S = S$ implies that $\gamma.\mathscr{F} \subset \gamma S = S$ and $\gamma \in \Gamma_0$. Hence, $\Gamma_0 = \left\{\, \gamma \in \Gamma_{_R} \,/\, \gamma.S = S \,\right\}$. That Γ_0 is a group is obvious.

From the definition of the class $\big(K_2^S\big)$ and Proposition 4.1, we know that the groups Γ_0 and $\Gamma_{_R}$ contain only hyperbolic elements except for $\{\pm\mathrm{Id}\}$. Now $\Gamma_0 \neq \pm\mathrm{Id}$ otherwise $S = \mathscr{F}$ would be a compact itgs in $\mathbb{H}^3$, and we can find a hyperbolic element $\gamma = \xi + \eta\,\Omega \in \Gamma_0$. Then $|\xi|^2 \geqslant \mathrm{Re}^2(\xi) > 1$ and $b\,|\eta|^2 = |\xi|^2 - \mathrm{N}(\gamma) = |\xi|^2 - 1 > 0$, so that $\eta \neq 0$.

Lemma 4.3 *Let S_1 and S_2 be two distinct closed itgs of $X_{_R}$. Their intersection $S_1 \cap S_2$ is either the empty set or a closed geodesic of $X_{_R}$.*

Proof : in a Riemannian manifold, two distinct itgs intersect transversally, because an itgs is entirely defined by a point and the tangent space at this point. Then their intersection has dimension 1 if it is not empty.

Since S_1 and S_2 are closed itgs of $X_{_R}$, they are compact. Hence, $L = S_1 \cap S_2$ is a compact subset of $X_{_R}$. If $L \neq \emptyset$, it is a complete geodesic because for $(M, \vec{u}) \in TL$, the geodesic of $X_{_R}$ tangent to $\vec{u}$ at M is contained in both S_1 and S_2. As L is compact, it is a closed geodesic of $X_{_R}$.

4.3 The type S^o and $\Gamma_{_R}$ - closed itgs

• The half-sphere $S^o = S\big(\mathrm{O}, 1/\sqrt{b}\big)$ is invariant under the action of all the isometries induced by $\mathfrak{A} \otimes \mathbb{R}$: given $\gamma = \xi + \eta\,\Omega \in \mathfrak{A} \otimes \mathbb{R}$ such that $\mathrm{N}(\gamma) = |\xi|^2 - b\,|\eta|^2 \neq 0$, we have

$$\forall\,\theta \in \mathbb{R} \qquad \big|\gamma(b^{-1/2}\,e^{i\theta})\big| = \left|\frac{\xi e^{i\theta} b^{-1/2} + \eta}{b\,\overline{\eta} e^{i\theta} b^{-1/2} + \overline{\xi}}\right| = \frac{1}{\sqrt{b}}\left|\frac{\xi\,e^{i\theta} + \eta\,\sqrt{b}}{\overline{\eta}\,\sqrt{b} + \overline{\xi}\,e^{-i\theta}}\right| = \frac{1}{\sqrt{b}} \tag{4.6}$$

whence $\gamma \cdot (S^o \cap \mathbb{C}) = S^o \cap \mathbb{C}$ and $\gamma \cdot S^o = S^o$. We shall also denote by S^o the projection of this half-sphere in $X_{_R}$. Unfortunately, we shall see in Appendix A that S^o is the only itgs of $\mathbb{H}^3$ that is closed for $\Gamma_{_R}$: indeed, the subgroup Γ_0 of elements of $\Gamma_{_R}$ leaving an itgs S invariant is generically a one-parameter group (cf. $\mathfrak{I}$ is a $\mathbb{Z}$-modulus of rank 4) so that $\Gamma_0 \backslash S$ cannot be compact.

• Lacking of closed itgs in $X_{_R}$, we shall use instead the weaker notion of $\Gamma_{_R}$ - *closed itgs* and look for modular correspondances separating points, closed geodesics or $\Gamma_{_R}$ - closed itgs.

Definition *An itgs S of $\mathbb{H}^3$ is called* $\Gamma_{_R}$ - closed *if there exists $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}$ hyperbolic such that $\gamma \cdot S = S$. Its projection $\mathscr{S}$ in $X_{_R}$ is also called a* $\Gamma_{_R}$ - closed itgs.

There are infinitely many of them, as we shall see in Proposition 8.3 section 8.

Lemma 4.4 *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be to distinct $\Gamma_{_R}$ - closed itgs of $X_{_R}$. Then area$(\mathscr{S}_1 \cap \mathscr{S}_2) = 0$.*

Proof : it is a straightforward corollary of the first part of the proof of Lemma 4.3, since every 1-dimensional set has zero area. Just be aware that the notion of area is here inherent to the manifold Σ_1 (for example), which is provided with the Riemannian metric induced by $\mathbb{H}^3$. That will be – and has been – always the case : we only mention the area of subsets of two-dimensional imbedded manifolds of $X_{_R}$.

Definition *We say that $\Lambda \subset X_{_R}$ has* type (S^o) *if Λ is contained in a finite union of $\Gamma_{_R}$ - closed itgs and area(Λ) = area$(\Lambda \cap S^o) \neq 0$.*

5 Statement of the result in dimension 3

Theorem 5.1 *Let $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$ be a 3-dimensional manifold, $\Gamma_{_R}$ being a discrete subgroup of $Is^+(\mathbb{H}^3)$ derived from an indefinite quaternion algebra $\mathfrak{A} = \left(\frac{a,b}{\mathbb{Q}}\right)$. We shall assume moreover that $X_{_R}$ is a manifold of class $\left(K_2^S\right)$.*

Let $\Lambda \subset X_{_R}$ be a non empty set contained in a finite union of isolated points, closed geodesics and $\Gamma_{_R}$ - closed itgs of $X_{_R}$ (we assume in that area$(\Lambda) \neq 0$ in this last case) that has not type (S^o). Then Λ cannot be the singular support of a quantum limit on $X_{_R}$.

First we prove a separation result on such a subset Λ :

Proposition 5.1 *Let $X_{_R}$ be a manifold of class $\left(K_2^S\right)$. For all non-empty subset $\Lambda \subset X_{_R}$ contained in a finite union of isolated points, closed geodesics and $\Gamma_{_R}$ - closed itgs (with area$(\Lambda) \neq 0$ in this last case) that has not type (S^o), there exists a correspondence T separating Λ.*

We shall distinguish three cases and treat them separately in sections 6 to 8 :

1) Λ is finite
2) $\Lambda \subset F_1 \cup \cdots \cup F_r$ a finite union of closed geodesics
3) $\Lambda \subset S_1 \cup \cdots \cup S_l$ a finite union of $\Gamma_{_R}$ - closed itgs.

and complete the proof in section 9. Let us first state a proposition that will simplify the calculations in the sequel. We call *objects of the same type* of $X_{_R}$ a set of points, a set of closed geodesics or a set of $\Gamma_{_R}$ - closed itgs of $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$.

Proposition 5.2 *Let $F_1, \ldots, F_r$ be* objects of the same type *of $X_{_R}$ and $G_1, \ldots, G_r$ fixed liftings of these objects to $\mathbb{H}^3$. There exists a finite subset of prime numbers $\mathscr{F} \subset \mathscr{P}$ such that, given $p \in \mathscr{P}\backslash\mathscr{F}$, the relation*

$$\exists\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \exists\, i \in \{1 \ldots r\} \qquad \alpha \cdot G_1 = G_i$$

leads to

$$\exists\, N \in \mathscr{F} \quad \exists\, \tilde{\alpha} \in R^{pr}(N\,p) \cup R^{pr}(N^2p^2) \qquad \tilde{\alpha} \cdot G_1 = G_1 \tag{5.1}$$

Proof : let us fix $n = 1$ or 2 and assume that $\exists\, i \in \{1, \ldots, r\}$, $\exists\, p_i \in \mathscr{P}$, $\exists\, \alpha_i \in R^{pr}({p_i}^n)$, $\alpha_i \cdot G_1 = G_i$. As a consequence $G_1 = \mathrm{Com}(\alpha_i) \cdot G_i$. Take $p \neq p_i \in \mathscr{P}$ and $\alpha \in R^{pr}(p^n)$: $\alpha \cdot G_1 = G_i \implies \tilde{\alpha} \cdot G_1 = G_1$ where $\tilde{\alpha} = \mathrm{Com}(\alpha_i)\,\alpha \in R({p_i}^n p^n) = R(N^n p^n)$. This element is primitive : otherwise, $n = 2$ and $\tilde{\alpha} \in pR$ or $\tilde{\alpha} \in p_i R$ as p and p_i are both primes, whence

. if $\tilde{\alpha} \in pR$ then $\alpha_i\,\tilde{\alpha} = \alpha_i \mathrm{Com}(\alpha_i)\,\alpha = {p_i}^2\alpha \in pR$ and $\alpha \in pR$ as $\mathrm{pgcd}(p_i, p) = 1$, a contradiction with $\alpha \in R^{pr}(p^2)$.

. if $\tilde{\alpha} \in p_i R$ then $\tilde{\alpha}\,\mathrm{Com}(\alpha) = \mathrm{Com}(\alpha_i)\,\alpha\,\mathrm{Com}(\alpha) = p^2\mathrm{Com}(\alpha_i) \in p_i R$ and $\alpha_i \in p_i R$, a similar contradiction.

Proceeding the same way with all the indices $i \in \{1 \ldots r\}$, we get to relation (5.1) after exclusion of at most r values of $p \in \mathscr{P}$, the forementionned set $\mathscr{F} = \{p_1, \ldots, p_r\}$.

6 Case of the points

• Let $\Lambda = \{\tilde{x}_1, \ldots, \tilde{x}_l\}$ be a set of points of X_R and $x_i = (z_i, t_i) \in \mathbb{H}^3$ liftings of the points $\tilde{x}_i$ for $i = 1 \ldots l$. Let us apply Proposition 5.2 to those points : for $n = 1$ or 2, $N \in \mathscr{F}$ and $p \in \mathscr{P} \backslash \mathscr{F}$ a prime satisfying the assumptions of Proposition 3.5 *i.e.*

$$\mathrm{ord}_p(2abDD') = 0 \ \text{ and } \ \left(\frac{a}{p}\right) = -1 \tag{6.1}$$

we take $\alpha = \xi + \eta\,\Omega \in R^{pr}(N^n p^n)$ such that $\alpha \cdot x_1 = x_1$. In that case, $\mathrm{ord}_p\,|\xi|^2 = \mathrm{ord}_p\,|\eta|^2 = 0$; in particular $\xi \neq 0$ and $\eta \neq 0$. By relation (2.3), the action of α as an isometry of $\mathbb{H}^3$ is

$$\alpha \cdot \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} \dfrac{\xi}{b\overline{\eta}} - \dfrac{N^n p^n}{b\overline{\eta}} \dfrac{\xi + b\eta\overline{z}}{|\xi + b\eta\overline{z}|^2 + b^2|\eta|^2 t^2} \\ \dfrac{N^n p^n t}{|\xi + b\eta\overline{z}|^2 + b^2|\eta|^2 t^2} \end{pmatrix}$$

It is well defined for any $t > 0$ because $\eta \neq 0$. The relation $\alpha \cdot x_1 = x_1$ implies

$$N^n p^n = |\xi + b\,\eta\,\overline{z_1}|^2 + b^2\,|\eta|^2\,{t_1}^2 = |\xi|^2 - b\,|\eta|^2 \tag{6.2}$$

$$\overline{\eta}\,z_1 + \eta\,\overline{z_1} = 0 \tag{6.3}$$

Note that $z_1 \neq 0$, otherwise relation (6.2) gives $|\xi|^2 + b^2\,|\eta|^2\,{t_1}^2 = |\xi|^2 - b\,|\eta|^2$ so that $|\eta| = 0$, a contradiction. As a consequence, we deduce from relation (6.3) that

$$-\frac{\overline{z_1}}{z_1} = \frac{\overline{\eta}}{\eta} = \text{constant } \in \mathbb{F}$$

Fix $\eta_0 \in \mathscr{O}_{\mathbb{F}}$ such that $\overline{\eta_0}/\eta_0 = -\overline{z_1}/z_1$. We have

$$\frac{\overline{\eta}}{\eta} = \frac{\overline{\eta_0}}{\eta_0} \quad \text{whence} \quad \frac{\eta}{\eta_0} = \frac{\overline{\eta}}{\overline{\eta_0}} \in \mathbb{R} \cap \mathbb{F} = \mathbb{Q} \quad (\text{because } a < 0)$$

so that $\exists\, m \in \mathbb{Q}$, $\eta = m\,\eta_0$. As $D'\,\eta \in \mathscr{O}_{\mathbb{F}}$ by Proposition 3.4, we get by taking the norm $m^2\,D'^2\,|\eta_0|^2 \in \mathbb{Z}$, with $D'^2\,|\eta_0|^2$ fixed in $\mathbb{Z}$. Thus, there exists $E \in \mathbb{N}$ fixed (*i.e.* only depending on η_0) such that $E\,m \in \mathbb{Z}^*$. The expansion of relation (6.2) provides

$$N^n\,p^n = |\xi|^2 + 2\,b\,m\,\mathrm{Re}\,(\xi_0\,\overline{\eta_0}\,z_1) + b^2\,m^2\,|\eta_0|^2\,({t_1}^2 + |z_1|^2) = |\xi|^2 - b\,m^2\,|\eta_0|^2$$

and we get after simplification by $b\,m \neq 0$, we get

$$2\,\mathrm{Re}\,(\xi\,\overline{\eta_0}\,z_1) + m\,|\eta_0|^2\,\left[1 + b\,({t_1}^2 + |z_1|^2)\right] = 0 \tag{6.4}$$

As $D'\,\xi \in \mathscr{O}_{\mathbb{F}}$ by Proposition 3.4, we have $2\,D'\xi = X + Y\sqrt{a}$ with X, $Y \in \mathbb{Z}$. The coefficient of m in relation (6.4) being strictly positive because $b > 0$, then m is a linear function of X and Y. Hence the middle term of (6.2) is a definite positive quadratic form of the two integer variables X and Y, which we will write

$$N^n\,p^n = c_1 X^2 + c_2 XY + c_3 Y^2 \tag{6.5}$$

A priori c_1, c_2, $c_3 \in \mathbb{R}$; moreover ${c_2}^2 - 4c_1c_3 < 0$ since the quadratic form is definite positive.

• Let us suppose that for all $N \in \mathscr{F}$, there are at most two primes $p \in \mathscr{P} \backslash \mathscr{F}$ satisfying both relations (6.1) and (6.5) ; let $\Delta \in \mathbb{N}$ be the product of all those primes p. For all

$p \in \mathscr{P}\backslash\mathscr{F}$ satisfying relation (6.1) and such that $\mathrm{ord}_p(\Delta) = 0$, for all $N \in \mathscr{F}$ and for all $\alpha \in R^{pr}(N^2p^2) \cup R(Np)$, we have $\alpha \cdot x_1 \neq x_1$. We deduce from Proposition 5.2 that

$$\left| \begin{array}{l} \forall\, p \in \mathscr{P}\backslash\mathscr{F} \ \text{ such that } \ \mathrm{ord}_p(2abDD'\Delta) = 0 \ \text{ and } \left(\dfrac{a}{p}\right) = -1 \\ \forall\, \alpha \in R(p) \cup R^{pr}(p^2) \qquad \forall\, i \in \{\,1 \ldots l\,\} \qquad \alpha \cdot x_1 \neq x_i \end{array} \right. \tag{6.6}$$

We know that there are infinitely many convenient primes p ; they even form a regular subset of $\mathscr{P}$ of density 1/2.

• Now assume that for some $N \in \mathscr{F}$, equation (6.5) is solvable for at least three distinct primes p_1, p_2, p_3 satisfying relation (6.1) : we have three points $(X_i : Y_i)_{i=1,2,3} \in \mathbb{P}^1(\mathbb{Q})$ such that

$$\forall\, i = 1 \ldots 3 \qquad N^n\, {p_i}^n = c_1 {X_i}^2 + c_2 X_i Y_i + c_3 {Y_i}^2 \tag{6.7}$$

Using these three relations, we shall show that c_1, c_2, $c_3 \in \mathbb{Q}$. If $(X_i : Y_i) = (X_j : Y_j)$ for $i \neq j$, $n = 2$ and $p_j(X_i : Y_i) = p_i(X_j : Y_j)$; since $\mathrm{pgcd}(p_i, p_j) = 1$, $\alpha_i \in p_i R$, a contradiction with relation (6.1) and Proposition 3.5. Thus $(X_1 : Y_1)$, $(X_2 : Y_2)$ and $(X_3 : Y_3)$ are three distinct points of $\mathbb{P}^1(\mathbb{Q})$. By relation (6.7), we can write

$$\begin{pmatrix} {X_1}^2 & X_1Y_1 & {Y_1}^2 \\ {X_2}^2 & X_2Y_2 & {Y_2}^2 \\ {X_3}^2 & X_3Y_3 & {Y_3}^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = N^n \begin{pmatrix} {p_1}^n \\ {p_2}^n \\ {p_3}^n \end{pmatrix} \in \mathbb{Z}^3$$

The above Vandermonde matrix has a non-zero determinant $\prod_{i<j}(Y_jX_i - Y_iX_j) \in \mathbb{Q}^*$ so that c_1, c_2, $c_3 \in \mathbb{Q}$ after inversion of the linear system. Thus, relation (6.5) becomes

$$\kappa p^n = \alpha X^2 + \beta XY + \gamma Y^2 \qquad \text{with} \quad p \in \mathscr{P}\backslash\mathscr{F} \quad X,\, Y \in \mathbb{Z} \tag{6.8}$$

with κ, α, β, $\gamma \in \mathbb{Z}$. Besides, $\delta = \beta^2 - 4\alpha\gamma < 0$ as the quadratic form in the previous relation is positive definite. By reduction modulo p of relation (6.8), we finally get

$$\delta \text{ is a square modulo } p \tag{6.9}$$

Therefore, if $p \in \mathscr{P}\backslash\mathscr{F}$ satisfies

$$\mathrm{ord}_p(2abDD') = 0 \qquad \left(\frac{a}{p}\right) = -1 \qquad \text{and} \qquad \left(\frac{\delta}{p}\right) = -1 \tag{6.10}$$

we have according to relations (6.1) and (6.9)

$$\forall\, N \in \mathscr{F} \quad \forall\, \alpha \in R(Np) \cup R^{pr}(N^2p^2) \qquad \alpha.x_1 \neq x_1$$

whence, by Proposition 5.2

$$\forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i \in \{1, \ldots, l\} \qquad \alpha.x_1 \neq x_i \tag{6.11}$$

By Proposition 3.2, a and δ being strictly negative integers, there exists infinitely many primes $p \notin \mathscr{F}$ satisfying relations (6.10) and – as a consequence – (6.11). Together with relation (6.6), this leads to

Proposition 6.1 *Let $x_1, \ldots, x_l$ be points of $\mathbb{H}^3$. There exists infinitely many primes p such that*

$$\forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i \in \{1, \ldots, l\} \qquad \alpha \cdot x_1 \neq x_i \tag{6.11}$$

7 The geodesics

The proof presented in [9] still holds ; we associate to any geodesic of $\mathbb{H}^3$ a proportionality class of complex binary quadratic and, using Proposition 3.1, we obtain

Proposition 7.1 *Let $L_1, \ldots, L_r$ be geodesics of $\mathbb{H}^3$. There exists infinitely many primes p such that*

$$\forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i \in \{1 \ldots r\} \qquad \alpha \cdot L_1 \neq L_i \qquad (7.1)$$

8 Case of the Γ_R - closed Itgs

Keep in mind that the itgs of $\mathbb{H}^3$ are the half-spheres centered on $\mathbb{C}$ and the half-planes orthogonal to $\mathbb{C}$. Only such half-spheres and half-planes will be considered in the sequel of this section, without need for us to mention it.

Definition *Let $\mathscr{S}$ be an itgs of $\mathbb{H}^3$. Its* trace $\mathscr{C}$ on $\mathbb{C}$ *is the set of its limit points in $\mathbb{C}$, that is $\mathscr{C} = \overline{\mathscr{S}} \cap \mathbb{C}$ in $\mathbb{R}^3$.*

The trace of an itgs of $\mathbb{H}^3$ is then either a circle, either a straight line of $\mathbb{C}$ – *id est* a circle of $\mathbb{P}^1(\mathbb{C})$. Moreover, each itgs its uniquely defined by its trace on $\mathbb{C}$, whence

Proposition *There is a bijection between the itgs of $\mathbb{H}^3$ and the circles of $\mathbb{P}^1(\mathbb{C})$.*

As a consequence, the action of an isometry on an itgs in $\mathbb{H}^3$ is entirely determinated by the former's action on the latter's trace in $\mathbb{C}$, much more easy to deal with. In particular

Proposition 8.1 *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two itgs of $\mathbb{H}^3$ whose traces on $\mathbb{C}$ are $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively. Then*

$$\forall\, \gamma \in \mathrm{SL}(2,\mathbb{C}) \qquad \gamma \cdot \mathscr{S}_1 = \mathscr{S}_2 \iff \gamma(\mathscr{C}_1) = \mathscr{C}_2$$

To obtain a separation result on a finite set of Γ_R - closed itgs of $\mathbb{H}^3$, we shall deal with isometries leaving such an itgs invariant, and then apply Proposition 5.2. We shall prove by the way the existence of infinitely many Γ_R - closed itgs in a manifold X_R of class $\left(K_2^S\right)$.

8.1 Of the half-planes

Proposition 8.2 *Let the half-plane $\mathscr{P}$ be an itgs of $\mathbb{H}^3$, and $\mathscr{D}$ be its trace. Consider an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ with $a + d \neq 0$ and $c \neq 0$:*

$$\gamma \cdot \mathscr{P} = \mathscr{P} \iff \begin{cases} \dfrac{a}{c} \in \mathscr{D},\ \dfrac{-d}{c} \in \mathscr{D} \ : \ \mathscr{D} \text{ is given by relation } (8.1) \\ (a+d)^2 \in \mathbb{R} \end{cases}$$

Proof : we have by Proposition 8.1

$$\gamma \cdot \mathscr{P} = \mathscr{P} \iff \gamma(\mathscr{D}) = \mathscr{D}$$

If $\gamma(\mathscr{D}) = \mathscr{D}$, then $\gamma(\infty) = a/c \in \mathscr{D}$ and $\gamma^{-1}(\infty) = -d/c \neq a/c \in \mathscr{D}$; so

$$z \in \mathscr{D} \iff \exists\, \lambda \in \mathbb{R},\ z = \lambda \left(\frac{a}{c}\right) + (1-\lambda) \left(\frac{-d}{c}\right) = \frac{\lambda\, a + (\lambda - 1)\, d}{c} = \frac{\lambda\,(a+d) - d}{c}$$

$$\iff \frac{cz+d}{a+d} \in \mathbb{R} \iff \frac{cz-a}{a+d} \in \mathbb{R}$$

whence

$$\mathscr{D} = \left\{ z \in \mathbb{C} \ \Big/ \ \mathrm{Im}\left(\frac{cz}{a+d}\right) = \mathrm{Im}\left(\frac{a}{a+d}\right) = \mathrm{Im}\left(\frac{-d}{a+d}\right) \right\} \tag{8.1}$$

Moreover, $a/c \in \mathscr{D} = \gamma(\mathscr{D})$ so that

$$\gamma\left(\frac{a}{c}\right) \in \mathscr{D} \iff \frac{a^2+bc}{c(a+d)} \in \mathscr{D}$$

$$\iff \mathrm{Im}\left(\frac{a^2+bc}{(a+d)^2}\right) = \mathrm{Im}\left(\frac{a}{a+d}\right) = \mathrm{Im}\left(\frac{a^2+ad}{(a+d)^2}\right)$$

$$\iff \mathrm{Im}\left(\frac{ad-bc}{(a+d)^2}\right) = \mathrm{Im}\left(\frac{1}{(a+d)^2}\right)$$

$$\iff (a+d)^2 \in \mathbb{R}$$

Reciprocally, assume that $a/c = \gamma(\infty) \in \mathscr{D}$, $-d/c = \gamma^{-1}(\infty) \in \mathscr{D}$ and $(a+d)^2 \in \mathbb{R}$: by relation (8.1), we still have $\gamma(a/c) \in \mathscr{D}$; besides

$$\gamma\left(\frac{a}{c}\right) = \frac{a^2+bc}{c(a+d)} \neq \frac{a}{c} \quad \text{since} \quad ad - bc = 1 \neq 0$$

Therefore, the isometry γ takes the three distinct points ∞, a/c, $-d/c$ of $\mathscr{D}$ into a/c, $\gamma(a/c)$ and ∞, which also are distinct points of $\mathscr{D}$: any circle of $\mathbb{P}^1(\mathbb{C})$ being uniquely defined by three points, we have *de facto* $\gamma(\mathscr{D}) = \mathscr{D}$, which ends the proof.

Keep in mind that for any hyperbolic element $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}$, we have $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\xi) \neq 0$ and $\eta \neq 0$ (cf. proof of Proposition 4.3) : those elements, which are the only interesting ones for us, will satisfy the hypothesis of the above Proposition.

• We are now able to show easily the existence of infinitely many $\Gamma_{_R}$ - closed itgs in $\mathbb{H}^3$. Take any hyperbolic element $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}$: $z_1 = \gamma^{-1}(\infty)$ and $z_2 = \gamma(\infty)$ are two distinct points of $\mathbb{C}$, since $\mathrm{Tr}(\gamma) \neq 0$. Let us define the itgs $\mathscr{P}_\gamma$ by $\mathscr{P}_\gamma = \mathscr{D}_\gamma \oplus \mathbb{R}_+^* \mathbf{j}$, where

$$\mathscr{D}_\gamma = (z_1, z_2) = \left\{ z \in \mathbb{C} \ \Big/ \ \mathrm{Im}(b\,\overline{\eta}\, z) = \mathrm{Im}(\xi) \right\} \tag{8.2}$$

Since $\mathrm{Tr}(\gamma) \in \mathbb{Z} \subset \mathbb{R}$, we have $\gamma(\mathscr{D}_\gamma) = \mathscr{D}_\gamma$ and $\gamma \cdot \mathscr{P}_\gamma = \mathscr{P}_\gamma$ by Proposition 8.2 : the half-plane $\mathscr{P}_\gamma$ is a $\Gamma_{_R}$ - closed itgs of $\mathbb{H}^3$. Furthermore :

Proposition 8.3 *There exist infinitely many* $\Gamma_{_R}$ *- closed itgs in* $\mathbb{H}^3$, e.g. *the half-planes* $\mathscr{P}(t) = \mathscr{D}(t) \oplus \mathbb{R}_+^* \mathbf{j} = \mathbb{R}\big(1 + t\sqrt{a}\big) \oplus \mathbb{R}_+^*\ \mathbf{j}$ *for* $t \in \mathbb{Z}$.

Proof : let us fix $t \in \mathbb{Z}$ and look for an element $\gamma_t = \xi + \eta\,\Omega \in \Gamma_{_R}$ having the form $\gamma_t = x + y(1 + t\sqrt{a})\,\Omega$, with $x,\, y \in \mathbb{Z}^*$; γ_t is hyperbolic as $y \neq 0$, and relation (8.2) provides

$\mathscr{D}_{\gamma_t} = \mathbb{R}\left(1 + t\sqrt{a}\right) = \mathscr{D}(t)$. For such a γ_t,

$$\mathrm{N}(\gamma_t) = 1 \iff x^2 - b\,(1 - at^2)\,y^2 = 1 \tag{8.3}$$

and, by Fermat's Theorem on the Equation of Pell, we can solve this equation for non-trivial integers x, y as soon as $d = b\,(1 - at^2)$ is not a square in $\mathbb{Z}$. But

$$b\,(1 - at^2) \text{ is a square in } \mathbb{Z} \implies 1 - at^2 \equiv 0\,[b] \implies \left(\frac{a}{b}\right) = 1$$

a contradiction with the definition of the $\left(K_2^S\right)$ - manifolds. So, for each $t \in \mathbb{Z}$, the natural integer $d = b\,(1 - at^2)$ is not a square and we can find $\gamma_t = \xi_t + \eta_t\,\Omega \in \Gamma_{_R}$ such that $\mathscr{D}_{\gamma_t} = \mathscr{D}(t) = \mathbb{R}\left(1 + t\sqrt{a}\right)$. This fullfills the proof.

Let us take for example $a = -2$ and $b = 13$:

. for $t = 0$, $d = 13$, $\gamma_0 = 649 + 180\Omega \in \Gamma_{_R}$ and $\mathscr{P}(0) = \mathbb{R} \oplus \mathbb{R}_+^*\,\mathbf{j}$.

. for $t = 1$, $d = 39$, $\gamma_1 = 25 + 4(1 + i\sqrt{2})\Omega \in \Gamma_{_R}$ and $\mathscr{P}(1) = \mathbb{R}\big(1 + i\sqrt{2}\big) \oplus \mathbb{R}_+^*\mathbf{j}$.

. for $t = 2$, $d = 117$, $\gamma_2 = 649 + 60(1 + 2i\sqrt{2})\Omega \in \Gamma_{_R}$ and $\mathscr{P}(2) = \mathbb{R}\big(1 + 2i\sqrt{2}\big) \oplus \mathbb{R}_+^*\mathbf{j}$.

We shall see other examples of $\Gamma_{_R}$ - closed itgs in Appendix B, half-planes that do not contain $\mathbb{R}_+^*\mathbf{j}$ and miscellaneous half-spheres. Moreover, we shall prove that

Proposition 8.4 *There exist infinitely many $\Gamma_{_R}$ - closed itgs in $X_{_R}$.*

• Let us finally state the main result of this section :

Proposition 8.5 *Let $\mathscr{P}_1$ be a $\Gamma_{_R}$ - closed half-plane of $\mathbb{H}^3$ and $\mathscr{F}$ a finite subset of $\mathscr{P}$. There exists infinitely many primes $p \in \mathscr{P}\backslash\mathscr{F}$ such that*

$$\forall\, N \in \mathscr{F} \quad \forall\, \alpha \in R^{pr}(Np) \cup R^{pr}(N^2p^2) \qquad \alpha \cdot \mathscr{P}_1 \neq \mathscr{P}_1 \tag{8.4}$$

Proof : let $\mathscr{D}_1$ be the trace of $\mathscr{P}_1$ on $\mathbb{C}$. As $\mathscr{P}_1$ is $\Gamma_{_R}$ - closed, there exists a hyperbolic element $\gamma_1 = \xi_1 + \eta_1\,\Omega \in \Gamma_{_R}$ such that $\gamma_1 \cdot \mathscr{P}_1 = \mathscr{P}_1$, whence $\gamma \cdot \mathscr{D}_1 = \mathscr{D}_1$. Moreover $\eta_1 \neq 0$, $\mathrm{Tr}(\gamma_1) \neq 0$ and we can apply Proposition 8.2 : we deduce from relation (8.1) that

$$\mathscr{D}_1 = \left\{\, z \in \mathbb{C} \;\middle/\; \mathrm{Im}(b\,\overline{\eta_1}\,z) = \mathrm{Im}(\xi_1) \,\right\} \tag{8.5}$$

Let us take a prime number $p \in \mathscr{P}\backslash\mathscr{F}$ satisfying the assumptions of Proposition 3.5

$$\mathrm{ord}_p(2abDD') = 0 \quad \text{ and } \quad \left(\frac{a}{p}\right) = -1$$

and assume that for $n = 1$ or 2,

$$\exists\, N \in \mathscr{F} \quad \exists\, \alpha = \xi + \eta\,\Omega \in R^{pr}(N^n p^n) \qquad \alpha(\mathscr{D}_1) = \mathscr{D}_1 \tag{8.6}$$

Then $\eta \neq 0$ and relation (8.6) implies that $\alpha^{-1}(\infty) = -\overline{\xi}/b\overline{\eta} \in \mathscr{D}_1$ which means that

$$-\mathrm{Im}\left(\frac{\overline{\eta_1\,\xi}}{\overline{\eta}}\right) = \mathrm{Im}\left(\frac{\eta_1\,\xi}{\eta}\right) = \mathrm{Im}(\xi_1) \quad \text{ and } \quad \exists\,\lambda \in \mathbb{R},\ \frac{\xi}{\eta} = \frac{\lambda + i\,\mathrm{Im}(\xi_1)}{\eta_1}$$

In fact, $\lambda \in \mathbb{Q} = \mathbb{R} \cap \mathbb{F}$ because all the complex numbers considered belong to the number

field $\mathbb{F}$. From the norm equation $\mathrm{N}(\alpha) = N^n p^n$, we deduce that

$$N^n p^n = |\xi|^2 - b\,|\eta|^2 = |\eta|^2 \left(\frac{|\xi|^2}{|\eta|^2} - b\right) = \frac{|\eta|^2}{|\eta_1|^2}\Big[\lambda^2 + \mathrm{Im}(\xi_1)^2 - b\,|\eta_1|^2\Big]$$

From $|\xi_1|^2 - b\,|\eta_1|^2 = \mathrm{N}(\gamma_1) = 1$, we get $\mathrm{Im}(\xi_1)^2 - b\,|\eta_1|^2 = 1 - \mathrm{Re}(\xi_1)^2$ so that

$$N^n p^n = \frac{|\eta|^2}{4|\eta_1|^2}\Big[4\lambda^2 + 4 - \mathrm{Tr}(\gamma_1)^2\Big] \tag{8.7}$$

By Proposition 3.5, $\mathrm{ord}_p|\eta|^2 = 0$; we shall moreover impose $\mathrm{ord}_p|\eta_1|^2 = 0$. As $\mathrm{Tr}(\gamma_1) \in \mathbb{Z}$, relation (8.7) provides $\mathrm{ord}_p(4\lambda^2) \geqslant 0$ whence $4\lambda^2 + 4 - \mathrm{Tr}(\gamma_1)^2 \equiv 0\,[p]$, so that $\mathrm{Tr}(\gamma_1)^2 - 4$ is a square modulo p.

Assume that the integer $\mathrm{Tr}(\gamma_1)^2 - 4$ is a square : $\mathrm{Tr}(\gamma_1) = m \in \mathbb{Z}$ and $\exists\, n \in \mathbb{Z}$ such that $m^2 - 4 = n^2$, whence $m^2 - n^2 = (|m| + |n|) \times (|m| - |n|) = 4 = 2 \times 2 = 4 \times 1$. Because $|m| + |n| \geqslant |m| - |n| > 0$, we have the following alternative :

. either $|m| + |n| = |m| - |n| = 2$, so that $n = 0$ and $m = \mathrm{Tr}(\gamma_1) = \pm 2$, a contradiction with γ_1 hyperbolic.

. or $|m| + |n| = 4$ and $|m| - |n| = 1$, so that $|m| = 5/2 \in \mathbb{N}$, a contradiction.

Hence, $c = \mathrm{Tr}(\gamma_1)^2 - 4$ is not a square. Since γ_1 is hyperbolic, $c > 0$.

Finally, we see that $\forall\,(x, y) \in \mathbb{Z}^2$, $a^x c^y$ square in $\mathbb{Z} \implies a^x > 0 \implies x \equiv 0\,[2]$ as $a < 0$, whence c^y is a square in $\mathbb{Z}$ and $y \equiv 0\,[2]$ because c is not a square in $\mathbb{Z}$. Thus, a and c are 2-independent and there exists by Theorem 3.1 infinitely many primes $p \in \mathscr{P}$ such that

$$\left(\frac{a}{p}\right) = \left(\frac{\mathrm{Tr}(\gamma_1)^2 - 4}{p}\right) = -1$$

If we restrict to the primes $p \in \mathscr{P}\backslash\mathscr{F}$ such that $\mathrm{ord}_p(2abDD') = 0 = \mathrm{ord}_p|\eta_1|^2$, we obtain by relation (8.7) infinitely many primes for which relation (8.6) can't be satisfied. They satisfy therefore relation (8.4), which ends the proof of the Proposition.

8.2 Of the half-spheres

• We shall begin with a characterization of the half-spheres of $\mathbb{H}^3$ that are Γ_R - closed itgs. Let $\mathscr{S} = S(a_1, r)$ be such an itgs and $\mathscr{C} = C(a_1, r)$ be its trace on $\mathbb{C}$: there exists a hyperbolic element $\gamma \in \Gamma_R$ such that $\gamma \cdot \mathscr{S} = \mathscr{S}$, whence $\gamma(\mathscr{C}) = \mathscr{C}$. Using this relation, we obtain a system of three algebraic equations that lead to

Proposition 8.6 *Let $S(a_1, r) \neq S^o$ be a Γ_R - closed itgs : $a_1 \neq 0$. If $q = 1 + b\,(|a_1|^2 - r^2) \neq 0$, then $\zeta = \dfrac{a_1}{q} \in \mathbb{F}^*$ and $\exists\,(X, Y) \in \mathbb{Z} \times \mathbb{Q}$, $a\,(1 - 4b|\zeta|^2) = (X^2 - 4)\,Y^2 > 0$.*

Applying Proposition 8.6 to $\mathscr{S}_1$, we obtain relations on a_1 and r (the same notations q and ζ are used). Let $\mathscr{F}$ be a finite subset of $\mathscr{P}$ and $n = 1$ or 2. Then, for $N \in \mathscr{F}$ and $p \in \mathscr{P}\backslash\mathscr{F}$, consider $\alpha = \xi + \eta\,\Omega \in R^{pr}(N^n p^n)$ such that $\alpha \cdot \mathscr{S}_1 = \mathscr{S}_1$. The case $\eta = 0$ is straightforward. If $\eta \neq 0$, we proceed as in the proof of Proposition 8.6 and obtain a similar system of three algebraic equations. From this system we deduce that except for finitely many primes p : if $q \neq 0$ then $a\,(1 - 4\,b\,|\zeta|^2)$ is a square modulo p ; if $q = 0$ then $b\,|\eta_1|^2$ is a square

modulo p, where η_1 is a fixed non nul integer of $\mathbb{F}$ that depends only on a_1. Therefore, after proving that the above quantites are not squares in $\mathbb{Q}$, we deduce from Theorem 3.1 that

Proposition 8.7 *Let $\mathscr{S} = S(a_1, r) \neq S^o$ be a Γ_R-closed itgs of $\mathbb{H}^3$ and $\mathscr{F}$ be a finite subset of $\mathscr{P}$. There exists infinitely many primes $p \in \mathscr{P} \backslash \mathscr{F}$ such that*

$$\forall\, N \in \mathscr{F} \quad \forall\, \alpha \in R^{pr}(Np) \cup R^{pr}(N^2p^2) \qquad \alpha \cdot \mathscr{S} \neq \mathscr{S} \tag{8.8}$$

• First we need a technical lemma for both Propositions :

Lemma 8.1 *Let $\mathscr{C}_1 = C(a_1, r_1)$ and $\mathscr{C}_2 = C(a_2, r_2)$ be two circles of $\mathbb{C}$, $N \in \mathbb{Z}$ and $\alpha = \xi + \eta\,\Omega \in R^{pr}(N)$ with $\eta \neq 0$. If $\alpha(\mathscr{C}_1) = \mathscr{C}_2$, then*

$$\exists\, \varepsilon = \pm 1 \qquad \begin{cases} b\big(r_1\,\overline{\eta}\,a_2 - \varepsilon\, r_2\,\eta\,\overline{a_1}\big) = \big(r_1 + \varepsilon r_2\big)\xi & (8.13) \\ b^2\, r_1^2\,|\eta|^2 - |\xi + b\,\eta\,\overline{a_1}|^2 = N\varepsilon\,\dfrac{r_1}{r_2} & (8.14) \\ |\xi|^2 - b|\eta|^2 = N & (8.15) \end{cases}$$

Proof : assume the hypothesis of the Lemma. We have

$$\forall\, z \in \mathbb{C}, \quad \alpha(z) = \frac{\xi z + \eta}{b\overline{\eta} z + \overline{\xi}} = \frac{\xi}{b\overline{\eta}} + \frac{b|\eta|^2 - |\xi|^2}{b\overline{\eta}\,\big(b\overline{\eta} z + \overline{\xi}\big)} = \frac{\xi}{b\overline{\eta}} - \frac{N}{b\overline{\eta}\,\big(b\overline{\eta} z + \overline{\xi}\big)}$$

so that

$$\forall\, z \in \mathbb{C}, \quad \alpha(z) = \frac{\xi}{b\overline{\eta}} + \frac{k}{z - \zeta} \qquad \text{where} \quad k = -\frac{N}{b^2\overline{\eta}^2} \quad \text{and} \quad \zeta = -\overline{\xi}/b\overline{\eta} \tag{8.9}$$

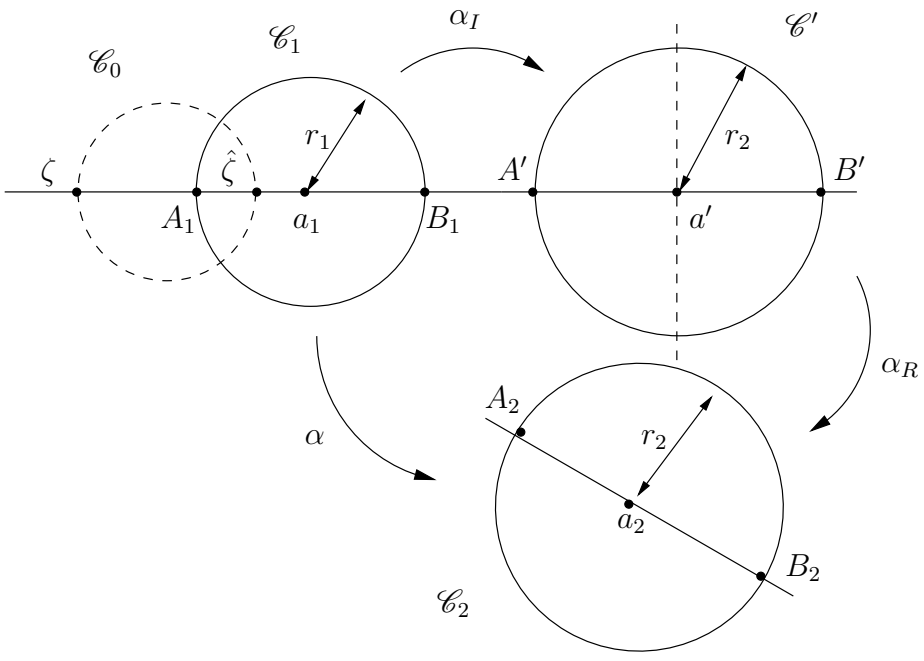

Figure 4: Action of α on $\mathscr{C}_1$

Note that $\zeta \notin \mathscr{C}_1$ otherwise $\alpha(\zeta) = \infty \in \mathscr{C}_2$, and that cannot happen. Relation (8.9) implies that $\alpha = \alpha_R \circ \alpha_I$, where $\alpha_I : z \longmapsto \zeta + |k|/(\overline{z} - \overline{\zeta})$ is an inversion of center ζ and α_R is an orientation reversing euclidean isometry of $\mathbb{C}$.

First we assume that $\zeta \neq a_1$. We denote by $\mathrm{Inv}_{\mathscr{C}_1}$ the inversion of circle $\mathscr{C}_1$ and we set $\hat{\zeta} = \mathrm{Inv}_{\mathscr{C}_1}(\zeta) = a_1 + r_1^2/(\overline{\zeta} - \overline{a_1})$. All the circles passing through ζ and $\hat{\zeta}$ are orthogonal to $\mathscr{C}_1$ because they are invariant under $\mathrm{Inv}_{\mathscr{C}_1}$. Let $\mathscr{C}_0$ be such a circle : $\alpha_I.\mathscr{C}_0$ is a line and it is orthogonal to $\alpha_I.\mathscr{C}_1 = \mathscr{C}'$, hence it is a diameter. As a consequence, $\alpha_I(\hat{\zeta}) = a'$ – the intersection of all the diameters of $\mathscr{C}'$ – is the center of $\mathscr{C}'$, and $\alpha(\hat{\zeta}) = \alpha_R \circ \alpha_I(\hat{\zeta}) = a_2$ is the center of $\mathscr{C}_2$. We set

$$\big(\zeta, a_1\big) \cap \mathscr{C}_1 = \big\{A_1, B_1\big\} \qquad \big(\zeta, a_1\big) \cap \mathscr{C}' = \big\{A', B'\big\}$$

We take here as a convention that the points A_1 and ζ are on the same side of a_1 on the line (ζ, a_1), and that B_1 and ζ are on opposite sides. We have $A' = \alpha_I(B_1)$, $B' = \alpha_I(A_1)$ and

$$A_1 - \zeta = \frac{a_1 - \zeta}{|a_1 - \zeta|}\big(|a_1 - \zeta| - r_1\big), \quad B_1 - \zeta = \frac{a_1 - \zeta}{|a_1 - \zeta|}\big(|a_1 - \zeta| + r_1\big) \tag{8.10}$$

whence

$$A' = \zeta + \frac{|a_1 - \zeta|}{\overline{a_1} - \overline{\zeta}} \frac{|k|}{|a_1 - \zeta| + r_1} \quad \text{and} \quad B' = \zeta + \frac{|a_1 - \zeta|}{\overline{a_1} - \overline{\zeta}} \frac{|k|}{|a_1 - \zeta| - r_1}$$

Therefore $2r_2 = |A' - B'| = 2|k|r_1/\,|a_1 - \zeta|^2 - {r_1}^2|$. We set $\varepsilon = 1$ if ζ is inside of $\mathscr{C}_1$ and $\varepsilon = -1$ otherwise, so that

$$\frac{r_2}{r_1} = \frac{\varepsilon|k|}{{r_1}^2 - |a_1 - \zeta|^2} \tag{8.11}$$

As $\hat{\zeta} - \zeta = a_1 - \zeta + {r_1}^2/\big(\overline{\zeta} - \overline{a_1}\big) = \big({r_1}^2 - |a_1 - \zeta|^2\big)/\big(\overline{\zeta} - \overline{a_1}\big) = \varepsilon|k|r_1/r_2\big(\overline{\zeta} - \overline{a_1}\big)$, we deduce from relation (8.9) that

$$\alpha(\hat{\zeta}) = \frac{\xi}{b\overline{\eta}} + \frac{\varepsilon\big(\overline{\zeta} - \overline{a_1}\big)\, k\, r_2}{r_1\,|k|}$$

Besides, $|k| = N/b^2|\eta|^2$ so that $k/|k| = -|\eta|^2/\overline{\eta}^2 = -\eta/\overline{\eta}$ and

$$\alpha(\hat{\zeta}) = \frac{\xi}{b\overline{\eta}} - \varepsilon\,\frac{\eta}{\overline{\eta}}\,\frac{r_2}{r_1}\,\big(\overline{\zeta} - \overline{a_1}\big) = a_2 \tag{8.12}$$

whence

$$b\big(r_1\overline{\eta}a_2 - \varepsilon\, r_2\eta\overline{a_1}\big) = \big(r_1 + \varepsilon r_2\big)\xi \tag{8.13}$$

Injecting $\zeta = -\overline{\xi}/b\overline{\eta}$ and $|\,k| = N/b^2|\eta|^2$ in relation (8.11), we get

$$b^2{r_1}^2|\eta|^2 - |\xi + b\eta\overline{a_1}|^2 = N\varepsilon\,\frac{r_1}{r_2} \tag{8.14}$$

Finally, the computation of the norm of α provides the relation

$$|\xi|^2 - b|\eta|^2 = N \tag{8.15}$$

In the case $\zeta = a_1 = -\overline{\xi}/b\overline{\eta}$, we have $\varepsilon = 1$, $\hat{\zeta} = \infty$ and $\alpha(\hat{\zeta}) = \xi/b\overline{\eta} = a_2$ so that the relation (8.13) is still satisfied. Moreover, $\mathscr{C}' = C(a_1, r_2)$: $\alpha_I.C(a_1, r_1) = C(a_1, r_2)$ so that $|k| = r_1 r_2 = N/b^2|\eta|^2$ whence relation (8.14). Hence, in each case, we obtain the system of three equations (8.13), (8.14) and (8.15), which ends the proof of the Lemma. We shall prove a converse in Appendix B.

• *Proof of Proposition 8.6 :* let us take a hyperbolic element $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}$ such that $\gamma(\mathscr{C}) = \mathscr{C} = C(a_1, r)$. If $a_1 = 0$, then $\gamma \cdot C(0, r) = C(0, r)$ *id est* $\forall\,\theta \in \mathbb{R}$, $\big|\gamma(r\,e^{i\theta})\big| = r$. Thus

$$\forall\,\theta \in \mathbb{R} \qquad \big|\xi\, r\, e^{i\theta} + \eta\big| = r\,\big|b\,\overline{\eta}\, r\, e^{i\theta} + \overline{\xi}\big| = \big|\xi\, r\, e^{i\theta} + b\, r^2\,\eta\big| \tag{8.16}$$

By taking the maxima of both sides of this equality considered as functions of $e^{i\theta} \in S^1$, we get $r\,|\xi| + |\eta| = r\,|\xi| + b\,r^2\,|\eta|$. As $\eta \neq 0$ by hyperbolicity of γ, $1 = b\,r^2$ and $r = 1/\sqrt{b}$ whence $\mathscr{S} = S^o$, a contradiction with the definition of $\mathscr{S}$. As a consequence, $a_1 \neq 0$. We shall assume in the sequel that $q = 1 + b\,(|a_1|^2 - r^2) \neq 0$. By Lemma 8.1, the relation $\gamma(\mathscr{C}) = \mathscr{C}$ leads to

$$\exists\,\varepsilon = \pm 1 \qquad \left\{ \begin{array}{rl} b\big(\overline{\eta}\,a_1 - \varepsilon\,\eta\,\overline{a_1}\big) = \big(1+\varepsilon\big)\,\xi & (8.13') \\ b^2\,r^2\,|\eta|^2 - |\xi + b\,\eta\,\overline{a_1}|^2 = \varepsilon & (8.14') \\ |\xi|^2 - b|\eta|^2 = 1 & (8.15') \end{array} \right.$$

If $\varepsilon = 1$, we deduce from (8.13') that $\xi = i\,b\,\mathrm{Im}(\overline{\eta}\,a)$ and $\mathrm{Re}(\xi) = 0$, which contradicts the hyperbolicity of γ. Therefore $\varepsilon = -1$, and relation (8.13′) implies that $\mathrm{Re}(\overline{\eta}\,a_1) = 0$. From relations (8.14') and (8.15'), we deduce that

$$|\xi + b\,\eta\,\overline{a_1}|^2 - b^2 r^2 |\eta|^2 = 1 = |\xi|^2 - b|\eta|^2$$

and

$$|\xi|^2 + 2\,b\,\mathrm{Re}(\xi\,\overline{\eta}\,a_1) + b^2\,(|a_1|^2 - r^2)\,|\eta|^2 = |\xi|^2 - b\,|\eta|^2$$

As $2\,b\,\mathrm{Re}(\xi\,\overline{\eta}\,a_1) = b\,\big(\xi\,\overline{\eta}\,a_1 + \overline{\xi}\,\eta\,\overline{a_1}\big) = b\,\big(\xi - \overline{\xi}\big)\,\overline{\eta}\,a_1$ and $\eta \neq 0$, we deduce from the above equation that

$$\big(\xi - \overline{\xi}\big)\,a_1 + \big[\,\underbrace{1 + b\,(|a_1|^2 - r^2)}_{q}\,\big]\,\eta = 0$$

De facto $\xi - \overline{\xi} \neq 0$ and $\zeta = a_1/q = \eta/(\overline{\xi} - \xi) \in \mathbb{F}^*$, that is $\eta = (\overline{\xi} - \xi)\,\zeta = -2i\mathrm{Im}(\xi)\,\zeta$. Injecting this in relation (8.15'), we obtain

$$1 = \mathrm{Re}(\xi)^2 + (1 - 4\,b\,|\zeta|^2)\,\mathrm{Im}(\xi)^2$$

Setting $2\,\mathrm{Re}(\xi) = X \in \mathbb{Z}$ and $2\,\mathrm{Im}(\xi)/\sqrt{-a} = Y^{-1} \in \mathbb{Q}^*$, we get $X^2 - a\,(1 - 4\,b\,|\zeta|^2)\,Y^{-2} = 4$ whence

$$\exists\,(X,\,Y) \in \mathbb{Z} \times \mathbb{Q} \qquad a\,(1 - 4\,b\,|\zeta|^2) = Y^2\,(X^2 - 4) > 0$$

because $Y \neq 0$ and $X^2 = \mathrm{Tr}^2(\gamma) > 4$ as γ is hyperbolic. This ends the proof.

• *Proof of Proposition 8.7 :* we fix $n = 1$ or 2 ; let us consider a prime $p \in \mathscr{P}\backslash\mathscr{F}$ such that $\mathrm{ord}_p(2abDD') = 0$ and

$$\exists\,N \in \mathscr{F} \quad \exists\,\alpha = \xi + \eta\,\Omega \in R^{pr}(N^n p^n) \qquad \alpha \cdot \mathscr{S} = \mathscr{S} = S(a_1, r) \tag{8.17}$$

Keep in mind that $\mathscr{S}$ being a Γ_R-closed itgs of $\mathbb{H}^3$, $a_1 \neq 0$ by Proposition 8.6. If $\eta = 0$, $\alpha \cdot (z,t) = (\xi z/\overline{\xi} t)$ for any $(z,t) \in \mathbb{H}^3$: the transformation α acts as an euclidean rotation of the space $\mathbb{R}^3$. The relation $\alpha \cdot \mathscr{S} = \mathscr{S}$ implies then $a_1 = \alpha \cdot a_1 = \xi\,a_1/\overline{\xi} \neq 0$ and $\xi/\overline{\xi} = 1$: $\xi \in \mathbb{R}$ and $\alpha = \xi = \pm Np \in pR$ cannot be primitive in R, a contradiction.

Since $\eta \neq 0$, we can proceed as in the proof of Proposition 8.6 : let us set $\mathscr{C} = C(a_1, r)$; relation (8.17) provides $\alpha(\mathscr{C}) = \mathscr{C}$ whence, by Lemma 8.1,

$$\exists\,\varepsilon = \pm 1 \qquad \left\{ \begin{array}{rl} b\big(\overline{\eta}\,a_1 - \varepsilon\,\eta\,\overline{a_1}\big) = \big(1+\varepsilon\big)\xi & (8.13'') \\ b^2\,r^2|\eta|^2 - |\xi + b\,\eta\,\overline{a_1}|^2 = \varepsilon\,N^n p^n & (8.14'') \\ |\xi|^2 - b|\eta|^2 = N^n p^n & (8.15'') \end{array} \right.$$

▪ If $\varepsilon = -1$, we get $\overline{\eta}\,a_1 + \eta\,\overline{a_1} = 2\,\mathrm{Re}(\eta\,\overline{a_1}) = 0$ and $\eta\overline{a_1} \in i\,\mathbb{R}\backslash\{0\}$ from relation (8.13"). We proceed as with relation (6.3) : there exists $\eta_1 \in \mathscr{O}_{\mathbb{F}}$ fixed (depending on a_1) such that

$$\exists\,\lambda \in \mathbb{Q} \qquad \eta = \lambda\,\eta_1$$

Using the relations (8.14") and (8.15"), we get $|\xi + b\,\eta\,\overline{a_1}|^2 - b^2\,r^2|\eta|^2 = |\xi|^2 - b\,|\eta|^2$ whence $2\,b\,\mathrm{Re}(\xi\,\overline{\eta}\,a_1) + b\,|\eta|^2\big[1 + b\,(|a_1|^2 - r^2)\big] = 0$ *i.e.* $2\,\mathrm{Re}(\xi\,\overline{\eta}\,a_1) + q\,|\eta|^2 = 0$ and

$$2\,\mathrm{Re}\,(\xi\,\overline{\eta_1}\,a_1) = -\lambda\,q\,|\eta_1|^2 \qquad \text{where} \qquad q = 1 + b\,(|a_1|^2 - r^2) \tag{8.18}$$

Let us assume for the moment that $q \neq 0$. By Proposition 8.6, $\zeta = a_1/q \in \mathbb{F}$ and $\xi\,\overline{\eta_1}\,\zeta \in \mathbb{F}$, so that $\exists\,\mu \in \mathbb{Q},\ 2\,i\,\mathrm{Im}(\xi\,\overline{\eta_1}\,\zeta) = \mu\,\sqrt{a}$. Therefore, $2\,\xi\,\overline{\eta_1}\,\zeta = -\lambda\,|\eta_1|^2 + \mu\,\sqrt{a}$ and $4\,|\xi\,\eta_1\,\zeta|^2 = \lambda^2|\eta_1|^4 - a\,\mu^2$. Injecting this in relation (8.15"), we get

$$\begin{aligned} 4\,N^n\,p^n\,|\eta_1|^2|\zeta|^2 &= 4\,|\xi\,\eta_1\,\zeta|^2 - 4\,b\,|\eta|^2|\eta_1|^2|\zeta|^2 \\ &= -a\,\mu^2 + \lambda^2|\eta_1|^4\big(1 - 4\,b\,|\zeta|^2\big) \end{aligned}$$

For $\lambda = l/r,\ \mu = m/r \in \mathbb{Q}$ with integers l, m, r satisfying $l \wedge m \wedge r = 1$, this relation becomes after multiplication of both sides by $a\,r^2$

$$a\,\big(1 - 4\,b\,|\zeta|^2\big)(l\,|\eta_1|^2)^2 = 4\,a\,|\eta_1|^2|\zeta|^2\,r^2\,N^n\,p^n + a^2m^2 \tag{8.19}$$

We know by Proposition 8.6 that $1 - 4\,b\,|\zeta|^2 \neq 0$. For $p \in \mathscr{P}$ such that $\mathrm{ord}_p(1 - 4\,b\,|\zeta|^2) = \mathrm{ord}_p|\zeta|^2 = \mathrm{ord}_p|\eta_1|^2 = 0$, we see from relation (8.19) that $l \equiv 0\,[p]$ if and only if $m \equiv 0\,[p]$. If $l \equiv 0\,[p]$, $(2\,r\,\overline{\eta_1}\,\zeta)\,\xi = \big(-\,l\,|\eta|^2 + m\sqrt{a}\,\big) \in p\mathscr{O}_{\mathbb{F}}$ with $r \not\equiv 0\,[p]$ because $l \wedge m \wedge r = 1$, and $r\,\eta = l\,\eta_1 \in p\mathscr{O}_{\mathbb{F}}$: thus $\alpha = \xi + \eta\,\Omega \in pR$, which contradicts the choice of α primitive. As a consequence, $l \not\equiv 0\,[p]$ and relation (8.19) implies that $a\,\big(1 - 4\,b\,|\zeta|^2\big)$ is a square modulo p.

If $q = 0$, we deduce from relation (8.18) that $\mathrm{Re}(\xi\,\overline{\eta_1}\,a_1) = 0 = -\mathrm{Im}(\xi)\,\mathrm{Im}(\overline{\eta_1}\,a_1)$, and $\mathrm{Im}(\xi) = 0$, because $\overline{\eta_1}\,a_1 \in i\mathbb{R}\backslash\{0\}$. Then $\xi = \mu \in \mathbb{Q}$ and $\eta = \lambda\,\eta_1$, so that $N^np^n = \mu^2 - b\,|\eta_1|^2\,\lambda^2$ by relation (8.15"), whence

$$\exists\,(l, m, r) \in \mathbb{Z}^3, \quad l \wedge m \wedge r = 1 \quad \text{and} \quad r^2N^np^n = m^2 - b\,|\eta_1|^2\,l^2 \tag{8.20}$$

Let $p \in \mathscr{P}$ such that $\mathrm{ord}_p|\eta_1|^2 = 0$: we have $l \equiv 0\,[p] \iff m \equiv 0\,[p] \iff \alpha \in pR$ as in the case $q \neq 0$, a contradiction with the primitivity of α. Thus $l \not\equiv 0\,[p]$ and relation (8.20) implies that the integer $b\,|\eta_1|^2$ is a square modulo p.

▪ If $\varepsilon = 1$, relation (8.13") leads to $\xi = b\,(\overline{\eta}\,a_1 - \eta\,\overline{a_1})/2 = i\,b\,\mathrm{Im}(\overline{\eta}\,a_1) \in i\mathbb{R}$ and $\xi + b\,\eta\,\overline{a_1} = b\,(\overline{\eta}\,a_1 + \eta\,\overline{a_1})/2 = b\,\mathrm{Re}(\overline{\eta}\,a_1) \in \mathbb{R}$. Using relations (8.14") and (8.15"), we get $N^np^n = b^2\,{r_1}^2\,|\eta|^2 - b^2\,\mathrm{Re}^2(\eta\,\overline{a_1}) = b^2\,\mathrm{Im}^2(\eta\,\overline{a_1}) - b\,|\eta|^2$, so that $b^2\,|\eta|^2\big({r_1}^2 - |a_1|^2\big) = -b\,|\eta|^2 \neq 0$. Hence

$$b\,\big(|a_1|^2 - {r_1}^2\big) = 1 \quad \text{and} \quad q = 2$$

So $a_1 = 2\,\zeta \in \mathbb{F}^*$ (cf. Proposition 8.6). We can set $\eta\,\overline{a_1} = X + Y\,\sqrt{a}$ with $X, Y \in \mathbb{Q}$. After multiplication of both sides by $|a_1|^2$, relation (8.15") becomes

$$|a_1|^2\,N^np^n = -a\,b^2\,|a_1|^2\,Y^2 - b\,\big(X^2 - a\,Y^2\big) = b\,\Big[a\,\big(1 - b\,|a_1|^2\big)\,Y^2 - X^2\Big]$$

Let $p \in \mathscr{P}$ such that $\mathrm{ord}_p|a_1|^2 = 0 = \mathrm{ord}_p(1 - b\,|a_1|^2)$: as before, $a\,\big(1 - b\,|a_1|^2\big) = a\,\big(1 - 4\,b\,|\zeta|^2\big)$ is a square modulo p.

▪ We can now end the proof of the Proposition : if $q = 0$, $\eta_1 \in \mathbb{F}^*$ and $\mathrm{ord}_b(|\eta_1|^2)$ is even by Lemma 4.1, so that $b\,|\eta_1|^2$ is not a square in $\mathbb{Q}$. Therefore, there exists infinitely many primes p such that $b\,|\eta_1|^2$ is not a square modulo p (cf. section 3.1.2).

If $q \neq 0$, we know that $a\left(1 - 4\,b\,|\zeta|^2\right) = (X^2 - 4)\,Y^2 > 0$ with $(X, Y) \in \mathbb{Z} \times \mathbb{Q}$, from Proposition 8.6. As we saw in the proof of Proposition 8.5, $X^2 - 4$ is a square in $\mathbb{Z}$ if and only if $X = \pm 2$ *i.e.* $X^2 - 4 = 0$, which cannot happen here. Hence, $a\left(1 - 4\,b\,|\zeta|^2\right)$ is not a square in $\mathbb{Q}$, and there exists once again infinitely many primes p such that $a\left(1 - 4\,b\,|\zeta|^2\right)$ is not a square modulo p.

As a consequence, after the exclusion of the prime factors of a finite set of rational numbers, we still have infinitely many primes p for which relation (8.17) cannot be satisfied, whence $\forall\, N \in \mathscr{F}$, $\forall\, \alpha \in R^{pr}(N^n p^n)$, $\alpha \cdot \mathscr{S} \neq \mathscr{S}$. This ends the proof of the Proposition.

8.3 Synthesis

Let us take $\mathscr{S}_1, \ldots, \mathscr{S}_l$ $\Gamma_{_R}$ - closed itgs of $\mathbb{H}^3$, with $\mathscr{S}_1 \neq S^o$, to which we apply Proposition 5.2. Let $\mathscr{F}$ be the finite subset of $\mathscr{P}$ given by this Proposition ; we know by Propositions 8.5 and 8.7 that there are infinitely many primes $p \in \mathscr{P} \backslash \mathscr{F}$ such that

$$\forall\, N \in \mathscr{F} \quad \forall\, \alpha = \xi + \eta\,\Omega \in R^{pr}(Np) \cup R^{pr}(N^n p^n) \qquad \alpha \cdot \mathscr{S}_1 \neq \mathscr{S}_1 \tag{8.21}$$

whence, by Proposition 5.2

Proposition 8.8 *Let $\mathscr{S}_1, \ldots, \mathscr{S}_l$ be $\Gamma_{_R}$ - closed itgs of $\mathbb{H}^3$, with $\mathscr{S}_1 \neq S^o$. There exists infinitely many primes $p \in \mathscr{P}$ such that*

$$\forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i \in \{\,1 \ldots l\,\} \qquad \alpha \cdot \mathscr{S}_1 \neq \mathscr{S}_i \tag{8.22}$$

9 Conclusion

Proof of Proposition 5.1

We consider a manifold $X_{_R}$ of class (K_2^S) and $\Lambda \subset X_{_R}$ a non-empty set that has not type (S^o) such that

$$\Lambda \subset z_1 \cup \ldots \cup z_l \cup L_1 \cup \ldots \cup L_r \cup \Sigma_1 \cup \ldots \cup \Sigma_s$$

where the z_i are points, the L_j are closed geodesics and the Σ_k are $\Gamma_{_R}$ - closed itgs of $X_{_R}$. We assume moreover that $\mathrm{area}(\Lambda) \neq 0$ if $s \geqslant 1$. We look for a modular correspondence $\mathscr{C}_p$ separating Λ (see section 1.1 for definitions) ; to this end, we shall adapt the method borrowed from [9] and exposed in section 1.4.

• Λ *is finite* : we have $\Lambda = \{z_1, \ldots, z_l\}$. Let us take by $\tilde{z}_1, \ldots, \tilde{z}_l$ liftings of these points of $X_{_R}$ to $\mathbb{H}^3$. By Proposition 6.1,

$$\exists\, p \in \mathscr{P} \quad \forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i = 1 \ldots l \qquad \alpha\,\tilde{z}_1 \neq \tilde{z}_i \tag{9.1}$$

As in section 1.4, we show that there exists a prime p such that the modular correspondence $\mathscr{C}_p$ separates Λ.

• Λ *is infinite and contained in a finite union of closed geodesics* : then we shall write $\Lambda \subset L_1 \cup \ldots \cup L_r$, where the L_j are closed geodesics, and assume that $\Lambda \cap L_1$ is infinite. Let the geodesics $\tilde{L}_1, \ldots, \tilde{L}_r$ be liftings to $\mathbb{H}^3$ of the L_j. By Proposition 7.1,

$$\exists\, p \in \mathscr{P} \quad \forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, j = 1 \ldots r \qquad \alpha \cdot \tilde{L}_1 \neq \tilde{L}_j \tag{9.2}$$

As in section 1.4, we deduce that there exists a prime p such that $\mathscr{C}_p$ separates Λ.

• Λ *is contained in a finite union of* $\Gamma_{_R}$ *- closed itgs and area(*Λ*)* $\neq 0$: as Λ has not type (S^o), we can write $\Lambda \subset \Sigma_1 \cup \ldots \cup \Sigma_s$ where the Σ_k are $\Gamma_{_R}$ - closed itgs, $\Sigma_1 \neq S^o$ and area$(\Sigma_1 \cap \Lambda) \neq 0$. Let the itgs $\mathscr{S}_1, \ldots, \mathscr{S}_s$ be liftings of the Σ_k to $\mathbb{H}^3$: $\mathscr{S}_1 \neq S^o$ and we deduce from Proposition 8.8 that

$$\exists\, p \in \mathscr{P} \quad \forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, k = 1 \ldots r \qquad \alpha \cdot \mathscr{S}_1 \neq \mathscr{S}_k \tag{9.3}$$

Let $p \in \mathscr{P}$ be such a prime : $\mathscr{C}_p(\Sigma_1)$ and $\mathscr{C}_{p^2}(\Sigma_1)$ consist of $\Gamma_{_R}$ - closed itgs all distinct from the Σ_k. By Lemma 4.3, the sets

$$\mu_1 = \mathscr{C}_p(\Sigma_1) \cap (\Sigma_1 \cup \ldots \cup \Sigma_l) \quad \text{ and } \quad \mu_2 = \mathscr{C}_{p^2}(\Sigma_1) \cap (\Sigma_1 \cup \ldots \cup \Sigma_l)$$

have zero area. As a consequence,

$$\nu_1 = \left\{ z \in X_{_R} \,/\, \mathscr{C}_p(z) \cap \mu_1 \neq \emptyset \right\} \quad \text{and} \quad \nu_2 = \left\{ z \in X_{_R} \,/\, \mathscr{C}_{p^2}(z) \cap \mu_2 \neq \emptyset \right\}$$

have zero area, so that there exists $z \in \Lambda \cap \Sigma_1 \backslash (\nu_1 \cup \nu_2)$. Let $\tilde{z}$ be a lifting of z to $\mathbb{H}^3$ and $w = \Gamma_{_R} \alpha_1 \tilde{z} \in \mathscr{C}_p(z) \subset \mathscr{C}_p(\Sigma_1)$. Proceeding as in section 1.4 (infinite case), we show that $w \notin \Lambda$ and $\mathscr{C}_p(w) \cap \Lambda = \{z\}$: the modular correspondence $\mathscr{C}_p$ separates Λ. This ends the proof of Proposition 5.1.

Proof of Theorem 5.1

Let $\Lambda \subset X_{_R}$ be a set satisfying the statement of Proposition 5.1 : there is a modular correspondence $\mathscr{C}$ separating Λ. Let ν be a quantum limit on $X_{_R}$ which is – we make the same assumption as in section 1.4 – associated to a sequence of eigenfunctions of Δ and $(T_n)_{n \in \mathbb{N}}$, hence of $T = T_C$. From Proposition 1.1, we deduce that singsupp $\nu \neq \Lambda$, which finally proves Theorem 5.1.

A Of closed itgs in X_R

Let $\mathscr{S} \neq S^o$ be an itgs of $\mathbb{H}^3$ closed for Γ_R : by Proposition 4.3, there exists a compact subset $\mathscr{F} \subset \mathscr{S}$ and a group $\Gamma_0 \subset \Gamma_R$ such that $\mathscr{S} = \Gamma_0 \cdot \mathscr{F}$, *id est*

$$\forall\, x \in \mathscr{S} \quad \exists\, \gamma \in \Gamma_0 \qquad \gamma \cdot x \in \mathscr{F} \tag{1.1}$$

We set $t_0 = \inf \left\{ t \,/\, (z,t) \in \mathscr{F} \right\} = \min \left\{ t \,/\, (z,t) \in \mathscr{F} \right\} > 0$ since $\mathscr{F}$ is compact in $\mathbb{H}^3$.

A.1 The half-spheres

We shall write $\mathscr{S} = S(a_1, r)$ with $a_1 \in \mathbb{C}$ and $r > 0$. By Proposition 8.6, $a_1 \neq 0$; we fix

$$(z,t) = \left(a_1 \left[1 + \frac{\sqrt{r^2 - t^2}}{|a_1|} \right], t \right) \in \mathscr{S} \quad \text{with} \quad 0 < t < t_0$$

Let $\gamma = \xi + \eta\,\Omega \in \Gamma_0$. If $\eta \neq 0$, γ is hyperbolic and we deduce from relation (8.13') – see the proof of Proposition 8.6 – that $\overline{\eta} a_1 \in i\mathbb{R}^*$; moreover

$$\gamma \cdot (z,t) = (\tilde{z}, \tilde{t}) = \left(*\,,\, \frac{t}{|\xi + b\eta\overline{z}|^2 + b^2|\eta|^2 t^2} \right) \tag{1.2}$$

by relation (2.3). As $\eta\overline{z} \in i\mathbb{R}$, $|\xi + b\eta\overline{z}|^2 \geqslant \mathrm{Re}(\xi)^2 > 1$ by hyperbolicity of γ : hence $\tilde{t} < t < t_0$ and $\gamma \cdot (z,t) = (\tilde{z}, \tilde{t}) \notin \mathscr{F}$. If $\eta = 0$, $\gamma = \pm I_2$ and $\gamma \cdot (z,t) = (z,t) \notin \mathscr{F}$. Therefore

$$\exists\, x \in \mathscr{S} \quad \forall\, \gamma \in \Gamma_0 \qquad \gamma \cdot x \notin \mathscr{F}$$

a contradiction with relation (1.1).

A.2 The half-planes

We shall write $\mathscr{S} = \mathscr{D} \oplus \mathbb{R}_+^* \mathbf{j}$. Let $\gamma_0 = \xi_0 + \eta_0\,\Omega \in \Gamma_0$ with $\eta_0 \neq 0$: by relation (8.2) we have

$$\mathscr{D} = \left\{ z \in \mathbb{C} \,\middle/\, \mathrm{Im}(b\,\overline{\eta_0}\, z) = \mathrm{Im}(\xi_0) \right\} = \left(\frac{-\overline{\xi_0}}{b\overline{\eta_0}} \,;\, \frac{\xi_0}{b\overline{\eta_0}} \right)$$

We set

$$\forall\, \lambda \in \mathbb{R} \qquad z_\lambda = \frac{-\overline{\xi_0}}{b\overline{\eta_0}} + \frac{\lambda}{b\overline{\eta_0}} \in \mathscr{D}$$

Let $\gamma = \xi + \eta\,\Omega \in \Gamma_0$. If $\eta \neq 0$,

$$\xi + b\eta\overline{z_\lambda} = \xi - \frac{\xi_0 \eta}{\eta_0} + \frac{\lambda\eta}{\eta_0} = \eta \left[\frac{\xi}{\eta} - \frac{\xi_0}{\eta_0} + \frac{\lambda}{\eta_0} \right] \tag{1.3}$$

Since $\gamma \in \mathfrak{I}$, $D'^2|\eta|^2 \in \mathbb{N}$ whence $D'^2|\eta|^2 \geqslant 1$: as a consequence,

$$\mathrm{N}(\gamma) = |\xi|^2 - b|\eta|^2 = 1 \implies \left| \frac{\xi}{\eta} \right|^2 = b + \frac{1}{|\eta^2|} \leqslant b + D'^2$$

and

$$\forall\, \gamma \in \Gamma_R \qquad \left| \frac{\xi}{\eta} - \frac{\xi_0}{\eta_0} \right|^2 \leqslant 2\left(b + D'^2\right) \tag{1.4}$$

Fix $\lambda > \left[\sqrt{2\,(b + D'^2)} + D' \right] |\eta_0|$ and $t \in]0, t_0[$; from relations (1.4) and (1.3), we deduce that $|\xi + b\eta\overline{z_\lambda}|^2 > D'^2|\eta|^2 > 1$: $\gamma \cdot (z_\lambda, t) \notin \mathscr{F}$ by relation (1.2) . If $\eta = 0$, $\gamma \cdot (z_\lambda, t) = (z_\lambda, t) \notin \mathscr{F}$ and we get the same contradiction with relation (1.1) as before.

A.3 Synthesis

So we have just shown that for any (K_2^S)-manifold $X_{_R}$, there is only one possible closed itgs, the projection of $S^o = S(0, 1/\sqrt{b})$ in $X_{_R}$. We shall prove now that this itgs is actually closed : $\Gamma_{_R}$ acts on the hyperbolic surface S^o – which is equivalent to $\mathbb{H}^2$ – as a subgroup of its direct isometries ; we proceed exactly the same way as in [2] to prove that $\Gamma_{_R}\backslash S^o$ is compact for a maximal order R in an indefinite division quaternion algebra over $\mathbb{Q}$.

• Set $\mathfrak{I} = \mathbb{Z}[i_1, i_2, i_3, i_4]$ and $\mathfrak{J} = \mathbb{R}[i_1, i_2, i_3, i_4] = \mathfrak{I} \otimes \mathbb{R} = \mathfrak{A} \otimes \mathbb{R}$; for the mapping φ defined by relation (3.3), we have $\Gamma_{_R} \overset{\varphi}{\simeq} \mathfrak{I}(1)$ and $G_{_R} \overset{\varphi}{\simeq} \mathfrak{J}(1)$, where

$$G_{_R} = \left\{ \begin{pmatrix} \xi & \eta \\ b\overline{\eta} & \overline{\xi} \end{pmatrix} \in \mathrm{M}(2, \mathbb{C}) \;\Big/\; |\xi|^2 - b\,|\eta|^2 = 1 \right\} \tag{1.5}$$

is the group of isometries induced by $\mathfrak{A} \otimes \mathbb{R}$. We already know from relation (4.6) that $\forall\, \gamma \in \Gamma_{_R},\ \gamma \cdot S^o = S^o$ whence $\gamma \cdot \left(0,\, 1/\sqrt{b}\right) \in S^o$. Further :

Lemma 1.1 *The mapping*

$$\psi \,:\, \left| \begin{array}{ccc} G_{_R} & \longrightarrow & S^o \\ \gamma & \longmapsto & \gamma \cdot \left(0,\, 1/\sqrt{b}\right) \end{array} \right.$$

defines a continuous surjection. Set $\mathscr{M}_c = \left\{ \xi + \eta\,\Omega \in G_{_R} \;/\; |\xi| \leqslant c,\ |\eta| \leqslant c \right\}$ *: for every* $c > 0$*, the set* $\psi(\mathscr{M}_c)$ *is compact.*

Proof : For $\gamma = \xi + \eta\,\Omega \in G_{_R}$ such that $\eta \neq 0$, we have $|\xi|^2 = 1 + b\,|\eta|^2$ so that

$$\gamma \cdot \begin{pmatrix} 0 \\ \\ 1/\sqrt{b} \end{pmatrix} = \begin{pmatrix} \dfrac{\xi}{b\overline{\eta}} - \dfrac{1}{b\overline{\eta}}\,\dfrac{\xi}{|\xi|^2 + b|\eta|^2} \\ \\ \dfrac{1}{\sqrt{b}\,(|\xi|^2 + b|\eta|^2)} \end{pmatrix} = \begin{pmatrix} \dfrac{\xi}{b\overline{\eta}}\left(1 - \dfrac{1}{1 + 2\,b\,|\eta|^2}\right) \\ \\ \dfrac{1}{\sqrt{b}\,(1 + 2\,b\,|\eta|^2)} \end{pmatrix}$$

if $\eta \neq 0$, whence

$$\forall\, \gamma \in G_{_R} \qquad \psi(\gamma) = \left(\frac{2\,\xi\,\eta}{1 + 2\,b\,|\eta|^2},\, \frac{1}{\sqrt{b}\,(1 + 2\,b\,|\eta|^2)} \right) \in S^o$$

If $\eta = 0$, $\psi(\gamma) = (0, 1/\sqrt{b})$ and we find again the same expression. We obtain this way all the points of S^o when $(\,|\eta|,\, \mathrm{Arg}\,\xi\eta\,)$ runs $[0, +\infty[\times]-\pi, \pi]$ and ψ is a surjection. From its above expression, ψ is moreover continuous on $G_{_R}$. For all $c > 0$, $\mathscr{M}_c$ is a compact subset of $\mathrm{M}(2, \mathbb{C})$ so that $\psi(\mathscr{M}_c)$ is compact.

Note that $\forall\, \gamma_1,\, \gamma_2 \in G_{_R},\ \psi(\gamma_1.\gamma_2) = \gamma_1 \cdot \psi(\gamma_2)$ from the definition of ψ.

• Let us consider elements $\xi = x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4$ with $\forall\, j,\, x_j \in \mathbb{R}$ such that $\mathrm{N}(\xi) = 1$, and set $M_c = \left\{ \xi \in \mathfrak{A} \otimes \mathbb{R} \;/\; \forall\, j,\, |x_j| \leqslant c \right\} \cap \left\{ \xi \;:\; \mathrm{N}(\xi) = 1 \right\}$ for $c > 0$. We have (see [2])

Lemma 1.2 *Let* $\mathfrak{A}$ *be an indefinite division algebra over* $\mathbb{Q}$ *and* $\mathfrak{I} = \mathbb{Z}[i_1, i_2, i_3, i_4]$ *a maximal order of* $\mathfrak{A}$*. There exists* $c > 0$ *fixed such that*

$$\forall\, \xi \in \mathfrak{A} \otimes \mathbb{R} \;\; \textit{with} \;\; N(\xi) = 1 \quad \exists\, \varepsilon \in \mathfrak{I}(1) \qquad \varepsilon\xi = \eta \in M_c$$

By application of the mapping φ, we obtain from the above relation

$$\exists\, c>0 \qquad \forall\, \gamma \in G_{_R} \qquad \exists \gamma_0 \in \Gamma_{_R} \qquad \gamma_0\gamma = \gamma_1 \in \mathscr{M}_c$$

so that

$$\exists\, c>0 \qquad \forall\, \gamma \in G_{_R} \qquad \exists\, \gamma_0 \in \Gamma_{_R} \qquad \gamma_0 \cdot \psi(\gamma) = \psi(\gamma_0.\gamma) \in \psi(\mathscr{M}_c)$$

By Proposition 1.1

$$\forall\, x \in S^o \qquad \exists\, \gamma \in G_{_R} \qquad x = \psi(\gamma)$$

and the set $\mathscr{F} = \psi(\mathscr{M}_c)$ is a compact subset of S^o such that

$$\forall\, x \in S^o \qquad \exists\, \gamma_0 \in \Gamma_{_R} \qquad \gamma_0 \cdot x \in \mathscr{F} \qquad i.e. \qquad S^o = \Gamma_{_R}\mathscr{F}$$

which leads to

Proposition 1.1 *Let $X_{_R}$ be a (K_2^S) - manifold. The half-sphere $S^o = S(0, 1/\sqrt{b})$ is the only*

itgs of $\mathbb{H}^3$ closed for $\Gamma_{_R}$; its projection in $X_{_R}$ is then compact.

A.4 Complement : the case $a > 0$

• One could think that the arbitrary choice of $a < 0$ is the cause of the lack of closed itgs in the (K_2^S) - manifolds. Thus, we define a class (K_1^S) of quotients manifolds $X_{_R}$ by taking

$$a \in \mathbb{Z} \qquad b \in \mathscr{P} \qquad \left(\frac{a}{b}\right) = -1 \qquad \left(\frac{-1}{b}\right) = 1 \qquad \left(\frac{-3}{b}\right) = 1 \tag{1.6}$$

For example, $a = 2$ and $b = 13$ zre convenient, since we just have to take the opposite of a to get from the (K_2^S) - manifolds to the (K_1^S) ones. We have the same properties as in section 4 – they are consequences of relation (1.6) – except that the conjugation in $\mathbb{F}$ does not coincide with the complex conjugation anymore. For the definition of the $\Gamma_{_R}$ - closed itgs, we shall moreover impose that the considered hyperbolic elements $\gamma = \xi + \eta\,\Omega$ satisfy $\eta \neq 0$.

• This time, the itgs that is left invariant under the action of all the isometries induced by $\mathfrak{A} \otimes \mathbb{R}$ is $P^o = \mathbb{R} \oplus \mathbb{R}_+^*\,\mathbf{j} \simeq \mathbb{H}^2$. Indeed, $\mathfrak{A} \otimes \mathbb{R}$ induces the group $\mathrm{SL}(2,\mathbb{R})$ since $\sqrt{a} \in \mathbb{R}$. The action of

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

on P^o is given by

$$\forall\, (x,t) \in \mathbb{R} \times \mathbb{R}_+^* \qquad \gamma \cdot \begin{pmatrix} x \\ \\ t \end{pmatrix} = \begin{pmatrix} \dfrac{ax+b}{d} \\ \\ \dfrac{a\,t}{d} \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \\ \tilde{t} \end{pmatrix} \in P^o$$

if $c = 0$. Then

$$\tilde{x} + i\tilde{t} = \frac{a(x+it)+b}{d} = \alpha(x+it)$$

and we recognize the fractional linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^2$. If $c \neq 0$

$$\forall\, (x,t) \in \mathbb{R} \times \mathbb{R}_+^* \qquad \gamma \cdot \begin{pmatrix} x \\ \\ t \end{pmatrix} = \begin{pmatrix} \dfrac{a}{c} - \dfrac{1}{c}\,\dfrac{cx+d}{(cx+d)^2 + c^2t^2} \\ \\ \dfrac{t}{(cx+d)^2 + c^2t^2} \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \\ \tilde{t} \end{pmatrix} \in P^o$$

whence

$$\begin{aligned}\tilde{x}+i\tilde{t} &= \frac{a}{c}-\frac{c(x-it)+d}{c\,|c(x+it)+d|^2} \;=\; \frac{a}{c}-\frac{1}{c\,[c(x+it)+d]}\\ &= \frac{a\,[c(x+it)+d]-1}{c\,[c(x+it)+d]} \;=\; \frac{ac\,(x+it)+bc}{c\,[c(x+it)+d]} \quad \text{since} \quad ad-1=bc\\ &= \frac{a\,(x+it)+b}{c(x+it)+d} \;=\; \alpha(x+it)\end{aligned}$$

and we recognize once again the fractional linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^2$. Therefore the discrete group $\Gamma_{_R} \subset \mathrm{SL}(2,\mathbb{R})$ has the same action on P^o and on $\mathbb{H}^2$: we deduce [2]) that, for a maximal order R in an indefinite division quaternion algebra $\mathfrak{A}$ over $\mathbb{Q}$, the quotient $\Gamma_{_R}\backslash P^o$ is compact : the itgs P^o is closed for $\Gamma_{_R}$. Let us verify that it is the only one $\mathbb{H}^3$.

• Let $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}$ and $x=(z,t)\in\mathbb{H}^3$:

$$\gamma\cdot\begin{pmatrix} z\\ t\end{pmatrix} = \begin{pmatrix} \xi^2 z\\ \xi^2 t\end{pmatrix} = \begin{pmatrix} \tilde{z}\\ \tilde{t}\end{pmatrix}$$

if $\eta = 0$, and

$$\gamma\cdot\begin{pmatrix} z\\ \\ t\end{pmatrix} = \begin{pmatrix} \dfrac{\xi}{b\overline{\eta}^{\mathbb{F}}}-\dfrac{1}{b\overline{\eta}^{\mathbb{F}}}\,\dfrac{\overline{\xi}^{\mathbb{F}}+b\overline{\eta}^{\mathbb{F}}\overline{z}}{|\overline{\xi}^{\mathbb{F}}+b\overline{\eta}^{\mathbb{F}}z|^2+b(\overline{\eta}^{\mathbb{F}})^2t^2}\\ \\ \dfrac{t}{|\overline{\xi}^{\mathbb{F}}+b\overline{\eta}^{\mathbb{F}}z|^2+b(\overline{\eta}^{\mathbb{F}})^2t^2}\end{pmatrix}\begin{pmatrix} \tilde{z}\\ \\ \tilde{t}\end{pmatrix}$$

if $\eta \neq 0$. Since $\xi,\ \eta \in \mathbb{F} \subset \mathbb{R}$, we easily compute that in each case

$$\frac{\mathrm{Im}(\tilde{z})}{\tilde{t}} = \frac{\mathrm{Im}(z)}{t} \overset{def}{=} f(x) \tag{1.7}$$

Let us consider an itgs $\mathscr{S} \neq P^o$ that is closed for $\Gamma_{_R}$: there exists a compact subset $\mathscr{F} \subset \mathscr{S}$ and a group $\Gamma_0 \subset \Gamma_{_R}$ such that $\mathscr{S} = \Gamma_0\cdot\mathscr{F}$ *id est*

$$\forall\, x\in\mathscr{S} \qquad \exists\,\gamma\in\Gamma_0 \qquad \gamma\cdot x\in\mathscr{F}$$

As $\mathscr{F}$ is a compact set of $\mathbb{H}^3$ and f is continous on $\mathbb{H}^3$, this function is bounded on $\mathscr{F}$; we deduce from both previous relations that

$$\exists\, M>0 \qquad \forall x\in\mathscr{S} \qquad |f(x)|\leqslant M \tag{1.8}$$

We can now get to the desired contradiction :

. If $\mathscr{S}$ is a half-plane – whose trace is denoted by $\mathscr{D}$ – different from P^o, there exists $z_0\in\mathscr{D}\backslash\mathbb{R}$: for all $t>0$, $x_t=(z_0,t)\in\mathscr{S}$ and $|f(x_t)|\longrightarrow\infty$ as $t\to 0$, a contradiction with relation (1.8).

. If $\mathscr{S}=S(a,r)$ is a half-sphere, we have $r\neq 0$ and $a+ir\notin\mathbb{R}$ or $a-ir\notin\mathbb{R}$: we may assume *wlog* that $a+ir\notin\mathbb{R}$. Set $x_t=(a+i\sqrt{r^2-t^2},t)$ for $t\in]0,r]$: then $f(x_t)\sim|\mathrm{Im}(a)+r|\,/t\longrightarrow\infty$ as $t\to 0$, a contradiction again.

Hence we have established

Proposition 1.2 *Let $X_{_R}$ be a (K_1^S) - manifold. The half-plane $P^o = \mathbb{R} \oplus \mathbb{R}_+^* \boldsymbol{j}$ is the only closed itgs of $\mathbb{H}^3$ for $\Gamma_{_R}$; its projection in $X_{_R}$ is thus compact.*

De facto, the lack of closed itgs is not specific to the (K_2^S) - manifolds, as we could have thought *a priori*. Moreover, for any (K_1^S) - manifold, we see from (8.5) that there is only one $\Gamma_{_R}$ - closed half-plane, P^o. The closed geodesics are particular too, since they link two points of $\mathbb{P}^1(\mathbb{R})$. We shall also have restrictions on the $\Gamma_{_R}$ - closed half-spheres. All those considerations justify *a posteriori* our choice to deal with the (K_2^S) - manifolds rather with the (K_1^S) - ones.

B Of $\Gamma_{_R}$ - closed itgs

B.1 Proof of Proposition 8.4

Let us consider the infinite family $\big(\mathscr{P}(t)\big)_{t\in\mathbb{Z}}$ of $\Gamma_{_R}$ - closed half-planes of $\mathbb{H}^3$ given by Proposition 8.3. To prove Proposition 8.4, we must verify that the set of their projections in $X_{_R}$ is still infinite.

Let us take t_1 and $t_2 \in \mathbb{N}$; the half-planes $\mathscr{P}(t_1)$ and $\mathscr{P}(t_2)$ have for traces on $\mathbb{C}$ respectively $\mathscr{D}_1 = (1 + t_1\sqrt{a})\,\mathbb{R}$ and $\mathscr{D}_2 = (1 + t_2\sqrt{a})\,\mathbb{R}$. A circle of $\mathbb{P}^1(\mathbb{C})$ being entirely defined by three distinct points, we have for $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}\backslash\{\pm\mathrm{Id}\}$ (so that $\eta \neq 0$)

$$\gamma\cdot\mathscr{P}(t_1) = \mathscr{P}(t_2) \iff \gamma(\mathscr{D}_1) = \mathscr{D}_2 \iff \begin{cases} \gamma(\infty) = \dfrac{\xi}{b\,\overline{\eta}} \in \mathscr{D}_2 & (1)\\[2mm] \gamma(0) = \dfrac{\eta}{\overline{\xi}} \in \mathscr{D}_2 & (2)\\[2mm] \gamma(1 + t_1\sqrt{a}) \in \mathscr{D}_2 & (3) \end{cases}$$

The relations (1) and (2) are equivalent since $\eta/\overline{\xi} = (b\,|\eta|^2/|\xi|^2) \times \xi/b\,\overline{\eta}$. By relation (1), we have $\xi\,\eta \in \mathscr{D}_2$ and

$$\exists\,\lambda \in \mathbb{Q} \qquad \xi = \lambda\,b\,\overline{\eta}\,(1 + t_2\sqrt{a})$$

Relation (3) provides

$$\begin{aligned} & \frac{\xi\,(1 + t_1\sqrt{a}) + \eta}{b\,\overline{\eta}\,(1 + t_1\sqrt{a}) + \overline{\xi}} \in \mathscr{D}_2\\ \iff & [\xi\,(1 + t_1\sqrt{a}) + \eta]\;[\xi + b\,\eta\,(1 - t_1\sqrt{a})] \in \mathscr{D}_2\\ \iff & \xi^2\,(1 + t_1\sqrt{a}) + b\,\eta^2\,(1 - t_1\sqrt{a}) \in \mathscr{D}_2 \quad \text{since } \xi\eta \in \mathscr{D}_2\\ \iff & \underbrace{\lambda^2\,b\,(1 + t_2\sqrt{a})\,(1 + t_1\sqrt{a})\,\overline{\eta}^2}_{z_1} + \underbrace{\left(\frac{1 - t_1\sqrt{a}}{1 + t_2\sqrt{a}}\right)\eta^2}_{z_2} \in \mathbb{R} \end{aligned}$$

Note that the complex numbers z_1 and $z_2 \in \mathbb{F}$ have opposite arguments : as a consequence, either they have the same module, either they are both reals. In the first case,

$$\lambda^4\,b^2\,(1 - at_2{}^2)\,(1 - at_1{}^2)\,|\eta|^4 = \frac{1 - at_1{}^2}{1 - at_2{}^2}\,|\eta|^4 \quad \textit{id est} \quad b\,\lambda^2\,(1 - at_2{}^2) = 1$$

whence $\mathrm{ord}_b(1-at_2{}^2) = \mathrm{ord}_b|1+t_2\sqrt{a}|^2$ is odd, a contradiction with Lemma 4.1. Thus, z_1 and z_2 are real hence rational numbers – even rational numbers since $\mathbb{F}\cap\mathbb{R} = \mathbb{Q}$ – and

$$\exists\,\mu\in\mathbb{Q}\qquad \eta^2 = \mu\,(1+t_1\sqrt{a})\,(1+t_2\sqrt{a})$$

By taking the square of the module, we get $|\eta|^4 = (|\eta|^2)^2 = \mu^2\,(1-at_1{}^2)\,(1-at_2{}^2)$ with $|\eta|^2\in\mathbb{Q}$, so that $(1-at_1{}^2)\,(1-at_2{}^2)$ is a square in $\mathbb{Q}$. We have hence proved that

$$\Big(\exists\,\gamma\in\Gamma_{_R}\quad \gamma\cdot\mathscr{P}(t_1)=\mathscr{P}(t_2)\Big)\implies (1-at_1{}^2)\,(1-at_2{}^2)\text{ is a square in }\mathbb{N}$$

The first condition means that the itgs $\mathscr{P}(t_1)$ and $\mathscr{P}(t_2)$ have the same projection in $X_{_R}$.

To fullfill the proof of Proposition 8.4, we have to find an infinite subset $\mathscr{I}\subset\mathbb{N}$ such that

$$\forall\,t_1\neq t_2\in\mathscr{I}\qquad (1-at_1{}^2)\,(1-at_2{}^2)\text{ is not a square in }\mathbb{N}\tag{2.1}$$

and it seems reasonable to think that it is possible for every negative integer a. Take $a=-2$ for instance : we verify that all numbers between 0 and 24000 statisfy this relation, except from 2, 11, 12, 70, 109, 225, 408, 524, 1015, 1079, 1746, 2378, 2765, 4120, 5859, 8030, 10681, 13860, 16647, 17615 and 21994. More generally, a conjectural Theorem states

Conjecture *Let A, B, C be integers relatively primes such that A is positive, $A+B$ and C are not both even and B^2-4AC is not a perfect square. Then there are infinitely many primes of the form An^2+Bn+C with $n\in\mathbb{Z}$.*

So, according to this highly probable Conjecture, there are for all negative integer a infinitely many primes of the form $1-at^2$ with $t\in\mathbb{N}$, whence the existence of an infinite set of integers $\mathscr{I}$ satisfying relation (2.1).

B.2 Of families of $\Gamma_{_R}$ - closed half-planes

We have seen in Proposition 8.3 the existence of infinitely many $\Gamma_{_R}$ - closed half-planes in $\mathbb{H}^3$, denoted by $\mathscr{P}(t) = \mathbb{R}\big(1+t\sqrt{a}\big)\oplus\mathbb{R}_+^*\,\mathbf{j}$ for $t\in\mathbb{Z}$. They all contain the half-line $\mathbb{R}_+^*\,\mathbf{j}$. But it is easy to find other $\Gamma_{_R}$ - closed half-planes : indeed, for $\gamma=\xi+\eta\,\Omega$ hyperbolic, we have $0\in\mathscr{D}_\gamma\iff \mathrm{Im}(\xi)=0$, by relation (8.2). Given $t\in\mathbb{Z}$, we shall then look for an hyperbolic element in $\Gamma_{_R}$ of the form $\gamma = x+y\sqrt{a}+z(1+t\sqrt{a})\,\Omega$ with x, y, $z\in\mathbb{Z}^*$. For such a γ,

$$\mathrm{N}(\gamma)=1\iff x^2-ay^2-b(1-at^2)z^2=1\tag{2.2}$$

Let us fix $u\in\mathbb{Z}$ and set $y=uz$: then $\gamma=\gamma_{t,u}=x+zu\sqrt{a}+z(1+t\sqrt{a})\,\Omega$ and

$$\mathrm{N}(\gamma_{t,u})=1\iff x^2-\big[\underbrace{au^2+b(1-at^2)}_{d}\big]z^2=1\tag{2.3}$$

As $d\equiv au^2\,[b]$ and a is not a square modulo b, since $X_{_R}$ is a manifold of class (K_2^S), d cannot be a square in $\mathbb{Z}$. Moreover, $d=b-a(bt^2-u^2)>0$ as soon as $|t|$ is big enough : in that case (cf. Pell-Fermat), we can solve the above equation for non-trivial integers x, z. We deduce finally from relation (8.2)

$$\mathscr{D}_{\gamma_{t,u}}=\mathscr{D}(u,t)=\left(\mathbb{R}+\frac{i\,\mathrm{Im}(\xi)}{b|\eta|^2}\right)\eta=\left(\mathbb{R}+\frac{u\sqrt{a}}{b(1-at^2)}\right)(1+t\sqrt{a})$$

which proves the following extension of Proposition 8.3 :

Proposition 2.1 *There are infinitely many Γ_R - closed half-planes in $\mathbb{H}^3$ that do not contain $\mathbb{R}_+^*\,\mathbf{j}$, the half-planes $\mathscr{P}(t,u) = \left(\mathbb{R} + \frac{u\sqrt{a}}{b(1-at^2)}\right)(1+t\sqrt{a}) \oplus \mathbb{R}_+^*\,\mathbf{j}$ for $t,\, u \in \mathbb{Z}$ such that $d = b - a(bt^2 - u^2) > 0$.*

For $a = -2$ and $b = 13$ again :

. $\gamma_{0,1} = 10 + 3i\sqrt{2} - 3\Omega$ leaves $\mathscr{P}(0,1) = \left(\mathbb{R} + \frac{i\sqrt{2}}{13}\right) \oplus \mathbb{R}_+^*\,\mathbf{j}$ invariant..

. $\gamma_{1,4} = 8 + 12i\sqrt{2} + 3(1+i\sqrt{2})\,\Omega$ leaves $\mathscr{P}(1,4) = \left(\mathbb{R} + \frac{4i\sqrt{2}}{39}\right)(1+i\sqrt{2}) \oplus \mathbb{R}_+^*\,\mathbf{j}$ invariant.

. $\gamma_{2,3} = 10 + 3i\sqrt{2} + (1+2i\sqrt{2})\,\Omega$ leaves $\mathscr{P}(2,3) = \left(\mathbb{R} + \frac{i\sqrt{2}}{39}\right)(1+2i\sqrt{2}) \oplus \mathbb{R}_+^*\,\mathbf{j}$ invariant.

B.3 Families of half-spheres

• We shall first complete Lemma 8.1 to obtain

Proposition 2.2 *Let $\mathscr{C}_1 = C(a_1, r_1)$ and $\mathscr{C}_2 = C(a_2, r_2)$ be two circles of $\mathbb{C}$, $N \in \mathbb{Z}$ and $\alpha = \xi + \eta\,\Omega \in R^{pr}(N)$ with $\eta \neq 0$. Then $\alpha(\mathscr{C}_1) = \mathscr{C}_2$ if and only if*

$$\exists\, \varepsilon = \pm 1 \qquad \left\{ \begin{array}{rcll} b\big(r_1\,\overline{\eta}\,a_2 - \varepsilon\, r_2\,\eta\,\overline{a_1}\big) & = & \big(r_1 + \varepsilon r_2\big)\xi & (8.13) \\ b^2\,{r_1}^2\,|\eta|^2 - |\xi + b\,\eta\,\overline{a_1}|^2 & = & N\varepsilon\,\dfrac{r_1}{r_2} & (8.14) \\ |\xi|^2 - b|\eta|^2 & = & N & (8.15) \end{array} \right.$$

Proof : we just have to prove the backward implication to fullfill the proof. Keep the notations of Lemma 8.1. Relation (8.14) provides

$$\left|\frac{\overline{\xi}}{b\,\overline{\eta}} + a_1\right|^2 - {r_1}^2 \neq 0 \qquad \text{whence} \qquad \zeta = -\frac{\overline{\xi}}{b\,\overline{\eta}} \notin \mathscr{C}_1$$

Therefore, $\alpha_I(\mathscr{C}_1) = \mathscr{C}'$ is a circle and $\alpha(\mathscr{C}_1) = \alpha_R(\mathscr{C}') = \mathscr{C}'' = C(a'', r'')$. We still have, by relation (8.14), $\varepsilon = 1$ iff ζ is inside of $\mathscr{C}_1$. Applying the Lemma to $\mathscr{C}_1$ and $\mathscr{C}''$, we deduce from relation (8.11)

$$\frac{r''}{r_1} = \frac{\varepsilon|k|}{{r_1}^2 - |a_1 - \zeta|^2} = \frac{\varepsilon\, N}{b^2\,{r_1}^2\,|\eta|^2 - |\xi + b\,\eta\,\overline{a_1}|^2} = \frac{r_2}{r_1}$$

since $|k| = N/b^2\,|\eta|^2$: then $r'' = r_2$. Finally, $\alpha_I(\hat{\zeta})$ is the center of $\mathscr{C}'$ so that $\alpha(\hat{\zeta}) = a''$; as (8.13) $\iff$ (8.12) $\iff$ $\alpha(\hat{\zeta}) = a_2$, then $a'' = a_2$ so that $\alpha(\mathscr{C}_1) = \mathscr{C}_2$.

• Now we can use of the above Proposition in our quest for Γ_R - closed half-spheres in $\mathbb{H}^3$. Let $\mathscr{S}$ be an half-sphere of $\mathbb{H}^3$, $\mathscr{C} = C(a_1, r)$ its trace on $\mathbb{C}$ and $\gamma = \xi + \eta\,\Omega \in \Gamma_R$ a hyperbolic element ; then $\gamma \cdot \mathscr{S} = \mathscr{S} \iff \gamma(\mathscr{C}) = \mathscr{C}$ and we apply Proposition 2.2 to the circle $\mathscr{C} = C(a_1, r)$. Necessarily, as we already saw in the proof of Proposition 8.6, $\varepsilon = -1$ by hyperbolicity of γ. As a consequence,

$$\gamma \cdot \mathscr{S} = \mathscr{S} \quad \iff \quad \left\{ \begin{array}{r} \overline{\eta}\,a_1 + \eta\,\overline{a_1} = 0 \\ |\xi + b\,\eta\,\overline{a_1}|^2 - b\,r^2\,|\eta|^2 = 1 \\ |\xi|^2 - b\,|\eta|^2 = 1 \end{array} \right.$$

whence

$$\gamma\cdot\mathscr{S}=\mathscr{S}\quad\Longleftrightarrow\quad\left\{\begin{array}{r}\overline{\eta}\,a_1+\eta\,\overline{a_1}=0\\ (\xi-\overline{\xi})\,a_1+\left[1+b\,(|a_1|^2-r^2)\right]\eta=0\\ |\xi|^2-b\,|\eta|^2=1\end{array}\right.\qquad(E)$$

Let us fix $a_1\in\mathbb{F}=\mathbb{Q}[\sqrt{a}]$, set $r=\sqrt{|a_1|^2+1/b}\in\mathbb{R}^*$ and consider $\mathscr{S}=S(a_1,r)$. For $\xi=X\in\mathbb{Z}$ and $\eta=a_1\,Y\sqrt{a}$ with $Y\in\mathbb{Z}$, we have $\overline{\eta}\,a_1+\eta\,\overline{a_1}=0$ and $\xi=\overline{\xi}$ (cf. $\sqrt{a}\in i\mathbb{R}$), so that

$$\gamma\cdot\mathscr{S}=\mathscr{S}\iff|\xi|^2-b\,|\eta|^2=X^2+a\,b\,|a_1|^2\,Y^2=1$$

From Lemma 4.1, we know that $\mathrm{ord}_b(|a_1|^2)$ is even as $a_1\in\mathbb{F}$, so that the natural integer $d=-a\,b\,|a_1|^2$ cannot be a square in $\mathbb{Q}$. Therefore, the previous Pell equation is solvable for non-trivial integers X and Y. We have proved

Lemma 2.1 *For all $a_1\in\mathbb{F}=\mathbb{Q}[\sqrt{a}]$, the half-sphere $S\left(a_1,\sqrt{|a_1|^2+\frac{1}{b}}\right)$ is $\Gamma_{_R}$ - closed.*

Here are some examples for $a=-2$ and $b=13$:

. $\gamma=64425+1666i\sqrt{2}(2+5i\sqrt{2})\,\Omega$ leaves $S\left(2+5i\sqrt{2},\sqrt{54+\frac{1}{13}}\right)$ invariant.

. $\gamma=96747+2318i\sqrt{2}(7+3i\sqrt{2})\,\Omega$ leaves $S\left(7+3i\sqrt{2},\sqrt{67+\frac{1}{13}}\right)$ invariant.

. $\gamma=561835+11074i\sqrt{2}(7+5i\sqrt{2})\,\Omega$ leaves $S\left(\frac{1}{5}+\frac{i\sqrt{2}}{7},\sqrt{\frac{99}{1225}+\frac{1}{13}}\right)$ invariant.

More generally, for $a_1\in\mathbb{F}^*$ and $r\in\mathbb{R}^*$ such that $r^2\in\mathbb{Q}$, the resolution of the system (E) leads to

$$\left\{\begin{array}{l}\xi=X-\frac{1}{2}\left[1+b\,(|a_1|^2-r^2)\right]Y\sqrt{a}\\ \eta=Y\,a_1\,\sqrt{a}\end{array}\right.\qquad\text{with } X,\ Y\in\mathbb{Z}$$

and the norm equation provides

$$1=X^2-\underbrace{\frac{a}{4}\left\{\left[1+b\,(|a_1|^2-r^2)\right]^2-4\,b\,|a_1|^2\right\}}_{d}Y^2$$

For $r^2\in\mathbb{Q}$ close enough to $|a_1|^2$, we have $4\,b\,|a_1|^2\geqslant\left[1+b\,(|a_1|^2-r^2)\right]^2$ and the rational number d is non negative. Assume moreover that $\mathrm{ord}_b\,|a_1|^2\geqslant 0$ and $\mathrm{ord}_b\,r^2\geqslant 0$: then $d\equiv a/4\,[b]$ is not a square modulo b for $X_{_R}$ a (K_2^S) - manifold, so that d is not a square in $\mathbb{Q}$. Hence, the above Pell equation is solvable for non-trivial $X,\ Y\in\mathbb{Z}$, which proves

Proposition 2.3 *Let $a_1\in\mathbb{F}=\mathbb{Q}[\sqrt{a}]$ and $r^2\in\mathbb{Q}$ such that $ord_b\,|a_1|^2\geqslant 0$, $ord_b\,r^2\geqslant 0$ and $4\,b\,|a_1|^2\geqslant\left[1+b\,(|a_1|^2-r^2)\right]^2$. The half-sphere $S(a_1,r)$ is a $\Gamma_{_R}$ - closed itgs of $\mathbb{H}^3$.*

Let us give some examples for $a=-2$ and $b=13$:

. $\gamma=359+168i\sqrt{2}+18i\sqrt{2}(2+3i\sqrt{2})\,\Omega$ leaves $S\left(\frac{2}{3}+i\sqrt{2},\sqrt{3}\right)$ invariant.

. $\gamma=19603-51480i\sqrt{2}+2574i\sqrt{2}(5+2i\sqrt{2})\,\Omega$ leaves $S\left(5+2i\sqrt{2},\sqrt{30}\right)$ invariant.

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