HAL
open science

On the scarring of eigenstates in some arithmetic hyperbolic manifolds

Tristan Poullaouec

To cite this version:

Tristan Poullaouec. On the scarring of eigenstates in some arithmetic hyperbolic manifolds. 2005. hal-00001491v4

HAL Id: hal-00001491

https://hal.science/hal-00001491v4

Preprint submitted on 8 Dec 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ccsd-00001491, version 4 - 8 Dec 2005

On the scarring of eigenstates for some arithmetic hyperbolic manifolds

Tristan POULLAOUEC

December 8, 2005

Abstract

We shall deal here with the conjecture of *Quantum Unique Ergodicity*. In [10], Rudnick and Sarnak showed that there is no *strong scarring* on closed geodesics for compact arithmetic congruence surfaces derived from a quaternion division algebra (see Introduction and theorem 1.1).

We extend this theorem to a class (K_2^S) of Riemannian manifolds $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$ that are again derived from quaternion division algebras, and show that there is no strong scarring on closed geodesics or on $\Gamma_{_R}$ - closed imbedded totally geodesic surfaces (see theorem 4.1).

1 Introduction

• A topic of great interest in quantum mechanics, and especially in quantum chaos, is the study of the limit of the quantized systems when $\hbar \longrightarrow 0$, which is called the *semi-classical limit*. The underlying purpose is to identify in the quantized system the influence of the classical dynamic (its chaotic nature for instance). Let M be a Riemannian manifold of negative sectional curvature in dimension two or three, and Δ the Laplace-Beltrami operator on M. It is well known (see [1], [4] or [5]) that the geodesic flow on the unitary sphere tangent bundle T^1M is ergodic and chaotic. After quantization, the wave functions of the stationary Schrödinger equation

$$\frac{-\hbar^2}{2m}\,\Delta\,\phi + V(q)\phi = E\,\phi \tag{1.1}$$

are the $\mathscr{L}^2$ eigenfunctions or *eigenmodes* of $\frac{-\hbar^2}{2m}\,\Delta + V(q)$. The potential $V(q)$ is in fact related to the curvature of M. We suppose that this operator has a discrete spectrum $(\lambda_k)_{_{k\in\mathbb{N}}}$ with $\lambda_k \longrightarrow +\infty$ as $k \longrightarrow +\infty$, which is true at least in the compact case. We denote by $(\phi_k)_{_{k\in\mathbb{N}}}$ the associated normalized eigenfunctions and by $(\mu_k)_{_{k\in\mathbb{N}}}$ the corresponding probability measures that are given by

$$\mathrm{d}\mu_k(q) = \left|\phi_k(q)\right|^2 \mathrm{dvol}(q) \tag{1.2}$$

They are actually the probability of presence of a particle in the state ϕ_k at q, and the semi-classical limit is the limit at large energies, that is when $k \longrightarrow +\infty$. The Quantum Unique Ergodicity Conjecture (see [11]) states :

Conjecture *Let M be a Riemannian manifold of dimension two or three and of sectional curvature $K<0$. Then* $\mathrm{d}\mu_k \underset{k\to\infty}{\longrightarrow} \dfrac{\mathrm{dvol}}{\mathrm{vol}(M)}$.

For a compact surface M whose geodesic flow is ergodic, this result was established in [13] for a subsequence of full density of (μ_k). We shall consider more specifically quotient manifolds $M=\Gamma\backslash\mathbb{H}^n$ (with $n=2$ or 3), where Γ is a freely acting discrete subgroup of $\mathrm{Is}(\mathbb{H}^n)$. In fact, all Riemann surfaces apart from S^2, $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{T}^2$ are of such a type (see [3]). A first step towards the conjecture was taken in [10] with the following result :

Definition *A probability measure ν on M is called a* quantum limit *if there exists a sequence $\big(\phi_j\big)_{j\in\mathbb{N}}$ of normalized eigenfunctions of Δ in $\mathscr{L}^2(M)$ such that the measures $|\phi_j(z)|^2\,\mathrm{dvol}$ converge weakly towards $\mathrm{d}\nu$.*

Theorem 1.1 *Let $M=\Gamma\backslash\mathbb{H}^2$ be an arithmetic congruence surface derived from a quaternion algebra and ν a quantum limit on X. If $\sigma=\mathrm{singsupp}\ \nu$ is contained in the union of a finite number of isolated points and closed geodesics, then $\sigma=\emptyset$.*

In other words, there is no *strong scarring* (cf. [11]) of eigenmodes on closed geodesics of the Riemann surface M. In this theorem, Γ is a congruence subgroup of a discrete group derived from an indefinite quaternion division algebra. Actually, we restrict ourselves here to *arithmetic quantum limits*, that are quantum limits arising from a sequence of joint eigenfunctions of the Laplacian and all Hecke operators. It is expected that the spectrum of the Laplacian on M is essentially simple, which justifies this restriction. A recent result from [6] establishes the Quantum Unique Ergodicity Conjecture for compact arithmetic congruence surfaces :

Theorem 1.2 *Let $M=\Gamma\backslash\mathbb{H}^2$ with Γ a congruence lattice over $\mathbb{Q}$. Then for compact M the only arithmetic quantum limit is the normalized volume dvol. For M not compact any arithmetic quantum limit is of the form c dvol with $0\leqslant c\leqslant 1$.*

- We briefly recall some useful notions and results from [10] §2.1.

A correspondence $\mathscr{C}$ of order r on a Riemannian manifold X is a mapping from X to $X^r/\mathfrak{S}_r$ such that $\mathscr{C}(x)=(S_1(x),\dots,S_r(x))$ with $S_k\in\mathrm{Is}(X)$ for all $k=1\dots r$. Here, $\mathfrak{S}_r$ is the symmetric group of order r. We say that such a correspondence $\mathscr{C}$ *separates* a subset Λ of X if $\exists\, z\in X-\Lambda$ such that $\exists!\, k\in\{1,\dots,r\}$, $S_k(z)\in\Lambda$. We shall denote by $T_{\mathscr{C}}$ the associated operator of $\mathscr{L}^2(X)$ defined by $T_{\mathscr{C}}(f):x\longmapsto\sum_{k=1}^r f\big(S_k(x)\big)$.

Proposition 1.1 *Let $\Lambda\subset X$ be a closed subset of zero volume and $\mathscr{C}$ be a correspondence on X that separates Λ. Let $(\phi_j)_{j\in\mathbb{N}}$ be a sequence of normalized $\mathscr{L}^2$ - eigenfunctions of $T_{\mathscr{C}}$ such that $\mathrm{d}\nu=\lim\limits_{j\to\infty}|\phi_j(z)|^2\mathrm{dvol}(z)$ exists. Then* $\mathrm{singsupp}\ \nu\neq\Lambda$.

Keep in mind that the singular support of a probablity measure on X is a closed set.

As we are interested in arithmetic quantum limits, we shall apply this proposition to joint eigenfunctions of the Laplacian Δ and all Hecke operators T_C.

2 Complements of hyperbolic geometry and algebra

2.1 Hyperbolic geometry in dimension three

We take the upper half-space $\mathbb{H}^3 = \left\{ (x,y,t) \in \mathbb{R}^3 \ / \ t > 0 \right\}$ as a model of hyperbolic space in dimension three. We set $\mathbf{j} = 1 \wedge \mathbf{i} \in \mathbb{R}^3$ and shall identify in the sequel the space $\mathbb{H}^3 = \left\{ z + t\,\mathbf{j} \ / \ z \in \mathbb{C},\ t > 0 \right\}$ with the subset $\mathbb{R} \oplus \mathbb{R}\, i \oplus \mathbb{R}_+ \mathbf{j}$ of the algebra of quaternions of Hamilton $\mathbb{H} = \mathbb{R}[1, i, \mathbf{j}, \mathbf{k}]$. This space is provided with the Riemannian hyperbolic metric $\mathrm{d}s^2 = t^{-2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2\right)$ having a constant sectional curvature $K = -1$. The Laplace-Beltrami operator is $\Delta = t^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} \right) - t\,\frac{\partial}{\partial t}$ and the volume form is given by $\mathrm{dvol} = t^{-3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ (cf. [9] §4.6).

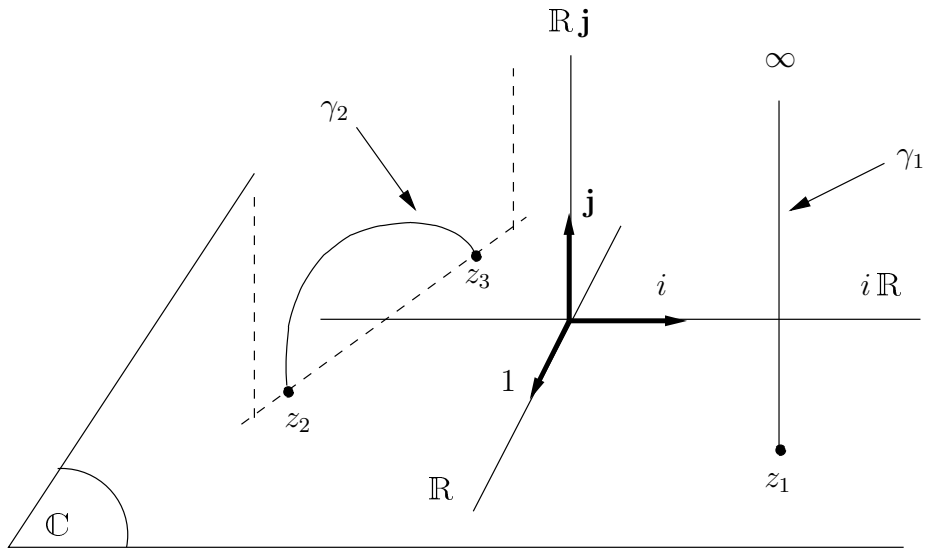

Figure 1: Geodesics of $\mathbb{H}^3$

The geodesics of $\mathbb{H}^3$ are the half-circles centered on $\mathbb{C}$ located in vertical planes and the half-lines orthogonal to $\mathbb{C}$. They are uniquely defined by their ending points in $\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{C} \cup \infty$, two distinct points that are the roots of a unique proportionality class of non-degenerate complex binary quadratic form : as in dimension two, this association is a bijection (cf. [10] §2.3). Moreover the imbedded totally geodesic submanifolds (abbreviated *itgs*) of $\mathbb{H}^3$ are the half-planes orthogonal to $\mathbb{C}$ and the half-spheres centered on $\mathbb{C}$.

We know (cf. [9] §4.4) that $\mathrm{Is}(\mathbb{H}^3)$, the group of isometries of $\mathbb{H}^3$, consists of the extensions to $\mathbb{H}^3$ of the Möbius transformations of $\mathbb{C}$

$$\mathrm{M}(\mathbb{C}) = \left\{ z \longmapsto \frac{az+b}{cz+d}\ ,\ z \longmapsto \frac{a\overline{z}+b}{c\overline{z}+d} \ \Big/ \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}) \right\}$$

and that the subgroup of orientation preserving isometries $\mathrm{Is}^+(\mathbb{H}^3)$ consists of the extensions of the complex fractional linear transformations and is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$.

We shall make the identification implicitely in the sequel. We have the following action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{H}^3 \subset \mathbb{H}$, called the *Poincaré extension*

$$\forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \quad \forall\, x \in \mathbb{H}^3 \qquad \gamma \cdot x = (ax+b).(cx+d)^{-1} \tag{2.1}$$

whence

$$\forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) \quad \text{with } \; ad - bc = n \neq 0 \qquad \forall\, (z,t) \in \mathbb{H}^3$$

if $c = 0$:

$$\gamma \,.\, \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} \dfrac{az+b}{d} \\ \left|\dfrac{a}{d}\right| t \end{pmatrix} \tag{2.2}$$

and if $c \neq 0$:

$$\gamma \,.\, \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} \dfrac{a}{c} - \dfrac{n}{c}\, \dfrac{\overline{cz+d}}{|cz+d|^2 + |c|^2 t^2} \\ \dfrac{|n|t}{|cz+d|^2 + |c|^2 t^2} \end{pmatrix} \tag{2.3}$$

The elements of $\mathrm{Is}^+(\mathbb{H}^3)$ are, exactly like those of $\mathrm{Is}^+(\mathbb{H}^2)$, characterized by their fixed points in $\mathbb{H}^3 \cup \partial\mathbb{H}^3 = \mathbb{H}^3 \cup \mathbb{C} \cup \infty$ (cf. [9] §4.7). Indeed, for $ad - bc \neq 0$ and $c \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} z \\ t \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} a + d = \mathrm{Tr}(\gamma) = 2\,\mathrm{Re}(cz+d) \\ |cz+d|^2 + |c|^2 t^2 = 1 \end{cases}$$

and for $c = 0$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} z \\ t \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} (a-d)z + b = 1 \\ |a| = |d| = 1 \end{cases}$$

In this case, we have $ad = 1$ and, setting $a = re^{i\theta}$ with $r = |a| > 0$ and $\theta \in \mathbb{R}$, we get $\mathrm{Tr}(\gamma)^2 = a^2 + 2 + \frac{1}{a^2} = 2 + \left(r^2 + \frac{1}{r^2}\right)\cos(2\theta) + i\left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)$.

- If $\mathrm{Tr}(\gamma)^2 \in [0,4[$, the isometry γ is called *elliptic* (we follow here the classical terminology). For $c \neq 0$, we set $cz + d = Z = \frac{\mathrm{Tr}(\gamma)}{2} + i|c|y$ with $y \in \mathbb{R}$ according to the fixed point condition, so that

  $$z = \left(\frac{\mathrm{Tr}(\gamma)}{2c} - \frac{d}{c}\right) + i\frac{c}{|c|}\, y \qquad \text{with} \qquad y \in \mathbb{R}$$

  and the forementionned condition writes

  $$4 = \mathrm{Tr}(\gamma)^2 + 4\,|c|^2\,(y^2 + t^2)$$

  As a consequence, we have a whole geodesic of fixed points in $\mathbb{H}^3$, the half-circle of radius $\frac{\sqrt{4 - \mathrm{Tr}(\gamma)^2}}{2|c|}$ centered on $\frac{\mathrm{Tr}(\gamma)}{2c} - \frac{d}{c}$.

For $c = 0$, we have $r = 1$ and $\theta \not\equiv 0\,[\pi]$ since $\mathrm{Tr}(\gamma)^2 \in [0, 4[$, so that $|a| = |d| = 1$ and $a \neq d = \overline{a}$. We obtain once again a whole geodesic of fixed points in $\mathbb{H}^3$, the half-line $\left]\frac{1-b}{d-a}, \infty\right)$.

- If $\mathrm{Tr}(\gamma)^2 = 4$ and $\gamma \neq \mathrm{I}_2$, the isometry γ is called *parabolic* and it has a fixed attractive point in $\mathbb{C}$, which is $z = \frac{\mathrm{Tr}(\gamma)}{2c} - \frac{d}{c}$ for $c \neq 0$ and ∞ for $c = 0$.

- If $\mathrm{Tr}(\gamma)^2 \notin [0, 4[$, the isometry γ is called *hyperbolic* and it has two fixed points in $\mathbb{C} \cup \infty$, one attractive and the other repulsive. The geodesic L connecting these points is called the *axis* of γ, which leaves it invariant and acts on it as a translation of the curvilinear abscisse.

2.2 Algebraic complements

2.2.1 Number theory

For all notions of number theory, we refer to [8] chapter 7, §2 and §3.

• Let $\mathbb{K}$ be a number field and $\mathbb{L}$ a finite extension, $\mathscr{O}_{\mathbb{K}}$ and $\mathscr{O}_{\mathbb{L}}$ their respective rings of algebraic integers. For any prime ideal $\mathfrak{B}$ of $\mathscr{O}_{\mathbb{L}}$, the prime ideal $P = \mathfrak{B} \cap \mathscr{O}_{\mathbb{K}}$ of $\mathscr{O}_{\mathbb{K}}$ is called the *underlying ideal to* $\mathfrak{B}$. Moreover, the quantity

$$f_{\mathbb{L}\,/\,\mathbb{K}}(\mathfrak{B}) \overset{def}{=} \left[\, \mathscr{O}_{\mathbb{L}}/\mathfrak{B} \,:\, \mathscr{O}_{\mathbb{K}}/P \,\right] \leqslant \left[\, \mathbb{L} \,:\, \mathbb{K} \,\right] < \infty$$

is called the *degree* of $\mathfrak{B}$ over $\mathbb{K}$. Now take a prime ideal P of $\mathscr{O}_{\mathbb{K}}$: we have $P\mathscr{O}_{\mathbb{L}} = {\mathfrak{B}_1}^{e_1} \ldots {\mathfrak{B}_s}^{e_s}$ with $\mathfrak{B}_i$ a prime ideal of $\mathscr{O}_{\mathbb{L}}$ of degree $f_i = f_{\mathbb{L}/\mathbb{K}}(\mathfrak{B}_i)$ for all $i = 1 \ldots s$. The integers e_i are called *ramification indices* and verify

$$\sum_{i=1}^{s} e_i f_i = \left[\, \mathbb{L} \,:\, \mathbb{K} \,\right] \tag{2.4}$$

If $e_i > 1$ for some $i \in \{1, \ldots, s\}$, the ideal P is *ramified*. There is only a finite number of such ones in $\mathscr{O}_{\mathbb{K}}$.

Let $\mathbb{K}$ be a number field, P a prime ideal of $\mathscr{O}_{\mathbb{K}}$, $p\mathbb{Z} = P \cap \mathbb{Z}$ the underlying prime ideal and $f = f_{\mathbb{K}/\mathbb{Q}}(P)$ its degree over $\mathbb{Q}$. The *norm* of the prime ideal P is

$$\mathrm{N}(P) \overset{def}{=} |\mathscr{O}_{\mathbb{K}}/P| = p^f$$

A set A of prime ideals of $\mathscr{O}_{\mathbb{K}}$ is *regular* with *density* a in the set of all prime ideals of $\mathscr{O}_{\mathbb{K}}$ if

$$\sum_{P \in A} \mathrm{N}(P)^{-s} \underset{s \to 1^+}{\sim} a \log \frac{1}{s-1} \tag{2.5}$$

Let $\mathbb{L}/\mathbb{K}$ be a finite extension with normal closure $\mathbb{M}/\mathbb{K}$ and Galois group $G = \mathrm{Gal}\big(\mathbb{M}/\mathbb{K}\big)$. We know that the set of the prime ideals P of $\mathscr{O}_{\mathbb{K}}$ satisfying $P\mathscr{O}_{\mathbb{L}} = \mathfrak{B}_1 \ldots \mathfrak{B}_r$ with $\mathfrak{B}_i$ a prime ideal of degree $f_{\mathbb{L}/\mathbb{K}}(\mathfrak{B}_i) = f_i$ fixed for all $i = 1 \ldots r$ is regular and its density is the relative frequence in G of the elements of G that, in the left translation representation considered as a permutation group of the set G, are the products of r disjoints cycles of length $f_1, \ldots, f_r$ (cf. [8] proposition 7.15).

Let $\mathbb{K}$ be a number field and take $a \in \mathscr{O}_{\mathbb{K}} \backslash \{1\}$ square-free. The field $\mathbb{L} = \mathbb{K}(\sqrt{a})$ is a quadratic (hence Galois) extension of $\mathbb{K}$, and its Galois group is $G = \{\mathrm{Id}, \tau\}$ where $\tau^2 = \mathrm{Id}$. Moreover $\mathscr{O}_{\mathbb{L}} = \mathscr{O}_{\mathbb{K}}[\alpha] \simeq \mathscr{O}_{\mathbb{K}}[X]/\pi_\alpha$ with $\alpha = (1+\sqrt{a})/2$ and $\pi_\alpha = X^2 - X + \frac{1-a}{4}$ if $a \equiv 1\,[4]$, $\alpha = \sqrt{a}$ and $\pi_\alpha = X^2 - a$ otherwise. For a prime ideal P of $\mathscr{O}_{\mathbb{K}}$, we have $\mathscr{O}_{\mathbb{L}}/P\mathscr{O}_{\mathbb{L}} = \mathscr{O}_{\mathbb{K}/(P)}[X]/\pi_\alpha$ and $P\mathscr{O}_{\mathbb{L}} = \mathfrak{B}_1{}^{e_1} \dots \mathfrak{B}_s{}^{e_s}$ with $\sum_i e_i f_i = \left[\mathbb{L} : \mathbb{K}\right] = 2$, so that only three situations occur :

i) $P\mathscr{O}_{\mathbb{L}} = R$, prime in $\mathscr{O}_{\mathbb{L}}$, is *inert* in iff a is not a square modulo P (*density* 1/2)

ii) $P\mathscr{O}_{\mathbb{L}} = R\overline{R}$, with R prime in $\mathscr{O}_{\mathbb{L}}$, *splits* iff a is a square modulo P (*density* 1/2)

iii) $P\mathscr{O}_{\mathbb{L}} = R^2$, with R prime in $\mathscr{O}_{\mathbb{L}}$, is *ramified* iff $a \in P$. (*density* 0)

Proposition 2.1 *Let $d \in \mathbb{Z}\backslash\{1\}$ be square-free, $\mathbb{K} = \mathbb{Q}\big(\sqrt{d}\big)$ a quadratic extension and $a \in \mathscr{O}_{\mathbb{K}}$ that is not a square. There exists a regular subset $\mathscr{C} \subset \mathscr{P}$ of density greater than 1/4 such that for any prime ideal P of $\mathscr{O}_{\mathbb{K}}$ satisfying $P \cap \mathbb{Z} = p\mathbb{Z}$ with $p \in \mathscr{C}$, a is not a square modulo P.*

Proof : let us denote by A the set $\{P$ primes of $\mathscr{O}_{\mathbb{K}}$ / a is not a square modulo $P\}$, that has density 1/2, and by B the set $\{\, p\mathbb{Z} = P \cap \mathbb{Z} \;/\; P \in A \,\}$ of underlying ideals of $\mathbb{Z}$. We shall divide B into three subsets $B = B_1 \cup B_2 \cup B_3$, respectively the sets of ideals that are inert, ramified or that split.

If $P \cap \mathbb{Z} = p\mathbb{Z} \in B_1$: $p\mathscr{O}_{\mathbb{K}} = P \in A$ so that $\mathrm{N}(P) = p^2$.

If $P \cap \mathbb{Z} = p\mathbb{Z} \in B_2 \cup B_3$: $p\mathscr{O}_{\mathbb{K}} = P^2$ or $P\overline{P}$ with $P \in A$, hence $\mathrm{N}(P) = p$.

An ideal $p\mathbb{Z}$ of B either splits ($p\mathbb{Z} \in B_3$) and there are two prime ideals of A above it, or it is inert or ramified ($p\mathbb{Z} \in B_1 \cup B_2$) and there is only one above it. Therefore

$$\sum_{P \in A} \mathrm{N}(P)^{-s} = \sum_{p\mathbb{Z} \in B_1} p^{-2s} + \sum_{p\mathbb{Z} \in B_2} p^{-s} + 2 \sum_{p\mathbb{Z} \in B_3} p^{-s} \Longrightarrow \sum_{P \in A} \mathrm{N}(P)^{-s} \leqslant 2 \sum_{p\mathbb{Z} \in B} p^{-s}$$

Because A has density 1/2, B contains a subset of density greater than 1/4.

• Let $p \in \mathscr{P}$ and $\mathbb{K}$ be a field number. Elements $a_1, a_2, \dots, a_n$ of $\mathbb{K}$ are called *p-independent* if, as soon as $a_1{}^{x_1} a_2{}^{x_2} \dots a_n{}^{x_n}$ (with $x_i \in \mathbb{Z}$ for all i) is a pth power in $\mathbb{K}$, then $x_i \equiv 0\,[p]$ for all $i = 1 \dots r$.

Theorem 2.1 *Let $a_1, \dots, a_n \in \mathbb{Z}$ be 2-independent integers and $z_1, \dots, z_n \in \{\pm 1\}$ fixed. There exists infinitely many $p \in \mathscr{P}$ such that $\left(\frac{a_i}{p}\right) = z_i$ for all $i = 1 \dots n$.*

This follows from the application of theorem 7.13 from [8] to quadratic characters.

Proposition 2.2 *Let $a_1, \dots, a_n \in \mathbb{Z}\backslash\mathbb{N}$. There exists infinitely many primes p such that $\left(\frac{a_i}{p}\right) = -1$ for all $i = 1 \dots n$.*

Proof : let $a_1, \dots, a_n \in \mathbb{Z}\backslash\mathbb{N}$. If they are 2-independent, the result is straight forward by application of theorem 2.1. Otherwise, let us take a maximal 2-independent subfamily,

that we can suppose to be $a_1, \dots, a_m$ with $1 \leqslant m < n$, after a relabelling. According to theorem 2.1, there exists infinitely many primes p such that

$$\forall\, i = 1 \dots m \quad \left(\frac{a_i}{p}\right) = -1$$

Let p be such a prime satisfying : $\forall\, i = m+1 \dots n,\ a_i \not\equiv 0\,[p]$, and let $1 \leqslant j \leqslant n-m$. Because the selected 2-independent family is maximal, the elements $a_1, \dots,\ a_m$ and a_{m+j} are not 2-independent. After simplification

$$\exists\, q \in \mathbb{Z} \quad \exists\, x_1, \dots,\ x_m \in \{0,1\} \qquad {a_1}^{x_1} \dots {a_m}^{x_m} a_{m+j} = q^2 > 0$$

As $a_i < 0$ for all i, then $\displaystyle\sum_{i=1}^{m} x_i$ is necessarily odd and

$$a_{m+j} = q^2 \prod_{i=1}^{m} a_i^{-x_i} \implies \left(\frac{a_{m+j}}{p}\right) = \prod_{i=1}^{m} \left(\frac{a_i}{p}\right)^{x_i} = (-1)^{\sum x_i} = -1$$

We can take any j with $1 \leqslant j \leqslant n-m$, which ends the proof.

2.2.2 Quaternion algebras

We first refer to [2] §1 and §2 in this section.

• Let $a \neq 1$ and b be square-free integers. The *quaternion algebra of type* (a,b) *on* $\mathbb{Q}$ is the $\mathbb{Q}$-algebra $\mathfrak{A} = \left(\frac{a,b}{\mathbb{Q}}\right) = \mathbb{Q}[1, \omega, \Omega, \omega\Omega]$, where $\omega^2 = a$, $\Omega^2 = b$ and $\omega\Omega + \Omega\omega = 0$.

The center of $\mathfrak{A}$ is $\mathbb{Q}$. Then $\mathbb{F} = \{\, q + r\,\omega \ /\ q,\ r \in \mathbb{Q}\,\}$ is a subfield of $\mathfrak{A}$ identified with $\mathbb{Q}(\sqrt{a})$. We shall write any element of $\mathfrak{A}$ as $\alpha = \xi + \eta\,\Omega$ with $\xi, \eta =\in \mathbb{F}$. We define $\overline{\alpha} = \overline{\xi}^{\mathbb{F}} - \eta\,\Omega$ the *conjugate* of α, $\mathrm{Tr}(\alpha) = \alpha + \overline{\alpha} = \mathrm{Tr}(\xi) \in \mathbb{Q}$ the *trace* of α and $\mathrm{N}(\alpha) = \alpha\overline{\alpha} = \xi\,\overline{\xi}^{\mathbb{F}} - b\,\eta\,\overline{\eta}^{\mathbb{F}} \in \mathbb{Q}$ its *norm*. Note that $\overline{\alpha_1.\alpha_2} = \overline{\alpha_2}.\overline{\alpha_1}$ for all $\alpha_1,\ \alpha_2 \in \mathfrak{A}$.

We call a quaternion algebra *definite* or *indefinite* whether its norm is definite ($a < 0$ and $b < 0$) or indefinite ($a > 0$ or $b > 0$) as a quaternary quadratic form on $\mathbb{R}$. Keep in mind that $\alpha \in \mathfrak{A}$ is a zero divisor if and only if $\alpha \neq 0$ and $\mathrm{N}(\alpha) = 0$.

Theorem $\mathfrak{A}$ *has zero divisors* $\iff \mathfrak{A} \simeq \mathrm{M}(2, \mathbb{Q})$

In this case, we shall speak of *matrix algebra*. Otherwise, we have a *division algebra*. In any case, the mapping

$$\varphi\ :\ \alpha = \xi + \eta\,\Omega \longmapsto \begin{pmatrix} \xi & \eta \\ b\,\overline{\eta}^{\mathbb{F}} & \overline{\xi}^{\mathbb{F}} \end{pmatrix} \tag{2.6}$$

provides the identification of $\mathfrak{A} \otimes \mathbb{F}$ with $\mathrm{M}(2, \mathbb{F})$ and leaves the trace and the norm (or determinant) invariant.

Proposition 2.3 *Let* $b \in \mathscr{P}$. *If* $\mathfrak{A} = \left(\frac{a,b}{\mathbb{Q}}\right)$ *is a matrix algebra, then* $\left(\frac{a}{b}\right) = 1$.

Proof : for a zero divisor $\alpha = x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega \in \mathfrak{A}$, we have $\alpha \neq 0$ and $\mathrm{N}(\alpha) = ({x_0}^2 - a{x_1}^2) - b({x_2}^2 - a{x_3}^2) = ({x_0}^2 - b{x_2}^2) - a({x_1}^2 - b{x_3}^2) = 0$. After the multiplication by the least common multiple of the denominators of the x_i, we obtain

$$ {y_0}^2 - a{y_1}^2 = b\left({y_2}^2 - a{y_3}^2\right) \qquad \text{where} \quad y_i \in \mathbb{Z} \ \text{ for all } \ i = 0\ldots 3 $$

(i) If b does not divide y_1 then $a \equiv \left(\frac{y_0}{y_1}\right)^2 \ [b]$ is a square modulo b.

(ii) If b divides y_1, this prime divides ${y_0}^2$ hence y_0 too. By noting $y'_0 = y_0/b$ and $y'_1 = y_1/b$, we get

$$ {y_2}^2 - a{y_3}^2 = b\big({y'_0}^2 - a{y'_1}^2\big) \qquad \text{where} \quad y'_i \in \mathbb{Z} \ \text{ for all } \ i = 0\ldots 3 $$

an equation in y_2, y_3 of the same type as before. After a finite number of simplifications by b, we are again in case (i), unless $y_1 = y_3 = 0$. In this last case, we have ${y_0}^2 = b{y_2}^2 \neq 0$ since $\alpha \neq 0$, which cannot happen for $b \in \mathscr{P}$.

Definition *An* order $\mathfrak{I}$ *in* $\mathfrak{A}$ *is a subring of* $\mathfrak{A}$ *such that* $1 \in \mathfrak{I}$, $N(\alpha)$ *and* $Tr(\alpha) \in \mathbb{Z}$ *for all* $\alpha \in \mathfrak{I}$, *and* $\mathfrak{I}$ *has four linearly independent generators over* $\mathbb{Q}$.

Thus, it is a free $\mathbb{Z}$-module of rank four in $\mathfrak{A}$ that is besides stable under the conjugation. For example, $\mathfrak{I}_0 = \mathscr{O}_{\mathbb{F}} \oplus \mathscr{O}_{\mathbb{F}}\,\Omega = \left\{\, \xi + \eta\,\Omega \ / \ \xi,\, \eta \in \mathscr{O}_{\mathbb{F}} \,\right\}$ is an order in the quaternion algebra $\mathfrak{A}$, and $\mathrm{M}(2,\mathbb{Z})$ is an order in the matrix algebra $\mathrm{M}(2,\mathbb{Q})$. Given $n \in \mathbb{Z}$, we define $\mathfrak{I}(n) = \left\{\, \alpha \in \mathfrak{I} \ / \ \mathrm{N}(\alpha) = n \,\right\}$ and $\mathfrak{I}^{pr}(n) = \mathfrak{I}(n) \cap \mathfrak{I}^{pr}$ the subset of primitive elements, the ones that cannot be divided in $\mathfrak{I}$ by a non unit integer.

Proposition 2.4 *Let* $\mathfrak{I}$ *be an order. Then :* $\exists\, D,\, D' \in \mathbb{Z}\backslash\{0\},\ D'\mathfrak{I} \subset D\mathfrak{I}_0 \subset \mathfrak{I}$.

Proof : since $\mathfrak{I}$ and $\mathfrak{I}_0$ are two free $\mathbb{Z}$-modules of rank four in $\mathfrak{A}$, their $\mathbb{Z}$-bases are two $\mathbb{Q}$-bases of $\mathfrak{A}$. Let $M \in \mathrm{GL}(4,\mathbb{Q})$ be a transition matrix from a basis of $\mathfrak{I}$ to a basis of $\mathfrak{I}_0$. Two integers D and D' such that $DM \in \mathrm{M}(4,\mathbb{Z})$ and $D'D^{-1}M^{-1} \in \mathrm{M}(4,\mathbb{Z})$ satisfy the above property. Moreover, $\mathscr{O}_{\mathbb{F}} \simeq \mathbb{Z}^2$ and $\mathfrak{I} \simeq \mathbb{Z}^4$ is countable.

Proposition 2.5 *Let* $p \in \mathscr{P}$ *be a prime such that* $ord_p(2abDD') = 0$ *and* $\left(\frac{a}{p}\right) = -1$. *For all* $\alpha = \xi + \eta\,\Omega \in \mathfrak{I}^{pr}$ *such that* $N(\alpha) \equiv 0\,[p]$, *we have* $ord_p\, N(\xi) = ord_p\, N(\eta) = 0$. *In particular,* $\xi\,\eta \neq 0$ *for such an* α.

Proof : let p and α satisfy the above assumptions ; by proposition 2.4, we have $D'\,\alpha = D\,\alpha_1$ with $\alpha_1 = \xi_1 + \eta_1\,\Omega \in \mathfrak{I}_0$. Besides, $\mathrm{ord}_p(DD') = 0$ so that $\mathrm{ord}_p\,\mathrm{N}(\xi) = \mathrm{ord}_p\,\mathrm{N}(\xi_1) \geqslant 0$ and $\mathrm{ord}_p\,\mathrm{N}(\eta) = \mathrm{ord}_p\,\mathrm{N}(\eta_1) \geqslant 0$. Then

$$ \mathrm{N}(D'\alpha) = \mathrm{N}\big[D(\xi_1 + \eta_1\,\Omega)\big] = D^2\big[\mathrm{N}(\xi_1) - b\,\mathrm{N}(\eta_1)\big] \equiv 0\,[p] $$

and $\mathrm{N}(\xi_1) - b\,\mathrm{N}(\eta_1) \equiv 0\,[p]$ whence $\mathrm{ord}_p\,\mathrm{N}(\xi_1) > 0 \iff \mathrm{ord}_p\,\mathrm{N}(\eta_1) > 0$ since $\mathrm{ord}_p(b) = 0$. Let us assume that $\mathrm{ord}_p\,\mathrm{N}(\xi) = \mathrm{ord}_p\,\mathrm{N}(\xi_1) > 0$ and $\mathrm{ord}_p\,\mathrm{N}(\eta) = \mathrm{ord}_p\,\mathrm{N}(\eta_1) > 0$.

$\underline{a \not\equiv 1[4]}$: therefore, $\mathscr{O}_{\mathbb{F}} = \mathbb{Z}[\sqrt{a}]$ and $\xi_1 = b_0 + b_1\sqrt{a}$ with $b_0,\, b_1 \in \mathbb{Z}$. Relation $\mathrm{ord}_p\,\mathrm{N}(\xi_1) > 0$ can be expressed as $p \,/\, {b_0}^2 - a{b_1}^2$. If $p \,/\, b_1$, then $p \,/\, b_0$ and $\xi_1 = p\,\xi_2$ with $\xi_2 \in \mathscr{O}_{\mathbb{F}}$. Otherwise $a \equiv \left(\frac{b_0}{b_1}\right)^2 [p]$ and $\left(\frac{a}{p}\right) = 1$, a contradiction.

$\underline{a \equiv 1[4]}$: in this case, $\mathscr{O}_{\mathbb{F}} = \mathbb{Z}\left[\frac{1+\sqrt{a}}{2}\right]$ and $\xi_1 = \left(b_0 + \frac{b_1}{2}\right) + \frac{b_1\sqrt{a}}{2}$ with $b_0, b_1 \in \mathbb{Z}$. We have $p \,/\, \mathrm{N}(2\xi_1)$ that provides $p \,/\, (2b_0 + b_1)^2 - a{b_1}^2$. If $p \,/\, b_1$, then $p \,/\, 2b_0 + b_1$ and $p \,/\, b_0$, so that $\xi_1 = p\,\xi_2$ with $\xi_2 \in \mathscr{O}_{\mathbb{F}}$. Otherwise, $a \equiv \left(\frac{2b_0}{b_1} + 1\right)^2 [p]$ and $\left(\frac{a}{p}\right) = 1$, a contradiction.

As a consequence : $\exists\, \xi_2 \in \mathscr{O}_{\mathbb{F}},\ \xi_1 = p\,\xi_2$. In the same way, we shall show that : $\exists\, \eta_2 \in \mathscr{O}_{\mathbb{F}},\ \eta_1 = p\,\eta_2$. Thus we have $D'\alpha = D\big(\xi_1 + \eta_1\,\Omega\big) = pD\big(\xi_2 + \eta_2\,\Omega\big) = p\,D\alpha_2$ with $\alpha_2 \in \mathfrak{I}_0$. Because $p \wedge D' = 1$, there exists $x,\, y \in \mathbb{Z}$ such that $xD' + yp = 1$ whence

$$\alpha = xD'\alpha + py\alpha = p(xD\alpha_2 + y\alpha) \tag{2.7}$$

Since $D\alpha_2 \in D\mathfrak{I}_0 \subset \mathfrak{I}$, then $\alpha \in p\mathfrak{I}$ is not primitive. This contradiction ends the proof.

• Let $\mathfrak{A}$ be an indefinite quaternion algebra of type (a, b) on $\mathbb{Q}$ (that is $a > 0$ or $b > 0$). We shall consider in the sequel orders of type (q_1, q_2) in $\mathfrak{A}$, which are principal and can be used to define modular correspondences (for more details, see [2] §3). For $q_1 = 1$, these orders are simply the maximal ones, and we know that each order is contained in a maximal one.

Let $\mathfrak{I}$ be such an order of type (q_1, q_2), which we identify implicitly with its image $R = \varphi(\mathfrak{I})$ in $\mathrm{M}(2, \mathbb{F})$. We shall also denote by $\alpha = \xi + \eta\,\Omega$ any element of R. Via Poincaré extension, the set $R^* = \big\{\alpha \in R \,/\, \det(\alpha) \neq 0\big\}$ is identified with a subgroup of $\mathrm{Is}^+(\mathbb{H}^3)$. For $n \wedge q_1 q_2 = 1$, we define the infinite sets

$$R(n) = \{\alpha \in R \,/\, \mathrm{N}(\alpha) = n\} \quad \text{and} \quad R^{pr}(n) = \{\alpha \in R \,/\, \alpha \text{ primitive and } \mathrm{N}(\alpha) = n\}$$

For example, let $\mathfrak{I}$ be a maximal order containing $\mathfrak{I}_0$ and assume that $b > 0$: Pell-Fermat theorem applied to the equation ${x_0}^2 - b{x_2}^2 = 1$ insures that $\mathfrak{I}(1)$ is infinite. Let $\Gamma_{_R} = R(1)$ be the discrete subgroup of $\mathrm{SL}(2, \mathbb{C})$ induced by R. We shall denote $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$ the quotient space and $\pi_{_R} \ :\ \mathbb{H}^3 \longrightarrow \Gamma_{_R}\backslash\mathbb{H}^3$ the canonical projection. To insure that $X_{_R}$ inherits the Riemannian structure of $\mathbb{H}^3$, the group $\Gamma_{_R}$ must not contain any elliptic element (cf. [9] §8.1).

Proposition 2.6 *Let us assume that $b \in \mathscr{P}\backslash\{3\}$. If $\Gamma_{_R}$ contains an elliptic element, one of the integers a, $-a$ or $-3a$ is a square modulo b.*

Proof : we shall assume that $a \not\equiv 0\,[b]$, otherwise a is a square modulo b. If $\Gamma_{_R}$ contains an elliptic element, there exists $\alpha = \xi + \eta\,\Omega \in \mathfrak{I}$ such that $|\mathrm{Tr}(\alpha)| < 2$ and $\mathrm{N}(\alpha) = 1$. Since $\mathfrak{I}$ is an order, $\mathrm{Tr}(\alpha) = \mathrm{Tr}(\xi) \in \big\{-1, 0, +1\big\}$. Set $\xi = x + y\sqrt{a}$ and $\eta = z + t\sqrt{a} \in \mathbb{F} = \mathbb{Q}[\sqrt{a}]$. Then $\mathrm{N}(\alpha) = x^2 - a\,y^2 - b(z^2 - a\,t^2) = 1$ and $\mathrm{Tr}(\alpha) = 2x \in \big\{-1, 0, +1\big\}$.

$\underline{x = 0}$: thus $-ay^2 = 1 + b(z^2 - at^2)$ and after multiplication by the least common multiple of the denominators of y, z and t, we get $-ay'^2 = E^2 + b(z'^2 - at'^2)$ with integers y', z' and t' such that $y' \wedge z' \wedge t' = 1$. If $y' \not\equiv 0\,[b]$, $-a \equiv E^2/y'^2\,[b]$ is a square modulo b. If $y' \equiv 0\,[b]$, b divides E and $z'^2 - at'^2 \equiv 0\,[b]$ whence $t' \not\equiv 0\,[b]$ because $y' \wedge z' \wedge t' = 1$. Therefore $a \equiv z'^2/t'^2\,[b]$ is a square modulo b.

$\underline{x = \pm 1/2}$: then $-ay^2 = 3/4 + b(z^2 - at^2)$ and after multiplication by the least common multiple of the denominators of y, z and t, we get $-ay'^2 = 3E^2 + b(z'^2 - at'^2)$ with integers y', z' and t' such that $y' \wedge z' \wedge t' = 1$. Proceeding as in the case $x = 0$, we deduce that one of the integers $-3a$ or a is a square modulo b, which ends the proof of the proposition.

For all $n \in \mathbb{N}$ such that $n \wedge q_1 q_2 = 1$, the set $R(1)\backslash R(n) = \{\Gamma_R\, \alpha_1, \ldots, \Gamma_R\, \alpha_r\}$ is finite (cf. [2] §7) and we may define the *modular correspondance* of order n on the quotient space $X_R = \Gamma_R \backslash \mathbb{H}^3$ by

$$\mathscr{C}_n : \left| \begin{array}{ccc} X_R & \longrightarrow & {X_R}^r/\mathfrak{S}_r \\ \Gamma_R\, x & \longmapsto & \{\Gamma_R\, \alpha_1\, x, \ldots, \Gamma_R\, \alpha_r\, x\} \end{array} \right.$$

The associated modulor operators T_n of $\mathscr{L}^2(X_R)$ are defined by

$$\forall\, f \in \mathscr{L}^2(X_R) \quad \forall\, z \in \mathbb{H}^3 \qquad T_n\,(f)(\Gamma_R z) = \sum_{\alpha \in R(1)\backslash R(n)} f(\Gamma_R\, \alpha\, z) \tag{2.8}$$

They are bounded linear operators on $\mathscr{L}^2(X_R)$ and satisfy the classical properties of the modular operators (self adjointness, commutation with the Laplace-Beltrami operator Δ, composition ...). We shall only consider in the sequel modular correspondences $\mathscr{C}_p$ and modular operators T_p for $p \in \mathscr{P} \setminus \{2\}$.

Proposition 2.7 *For $p \in \mathscr{P}$ such that $p \wedge q_1 q_2 = 1$, we set $|R(1)\backslash R(p)| = m$ and denote $R(1)\backslash R(p) = \big\{\Gamma_R\, \alpha_1, \ldots, \Gamma_R\, \alpha_m\big\}$. Then we have : $\forall\, i, \exists\,!\, j$ such that $\alpha_j\, \alpha_i \in p\, R(1)$ and $\forall\, k \neq j,\ \alpha_k\, \alpha_i \in R^{pr}(p^2)$.*

Proof : for all $n \in \mathbb{N}$, as $\mathfrak{I}$ is stable under the conjugation, the set $R(n)$ is stable under the passage to the transpose of the comatrix, which corresponds to the inversion in $\mathrm{Is}^+(\mathbb{H}^3) \simeq \mathrm{PSL}(2, \mathbb{C})$

$$\varphi(\xi + \eta\, \Omega) = \begin{pmatrix} \xi & \eta \\ b\overline{\eta}^{\mathbb{F}} & \overline{\xi}^{\mathbb{F}} \end{pmatrix} \in R(n) \Longrightarrow \varphi(\overline{\xi + \eta\, \Omega}) = \begin{pmatrix} \overline{\xi}^{\mathbb{F}} & -\eta \\ -b\overline{\eta}^{\mathbb{F}} & \xi \end{pmatrix} \in R(n)$$

Take $i \in \{1, \ldots, m\}$: there is $j \in \{1, \ldots, m\}$ such that $\Gamma_R\, {}^t\mathrm{Com}(\alpha_i) = \Gamma_R\, \alpha_j$. Therefore $\Gamma_R\, \alpha_j\, \alpha_i = p\, \Gamma_R$ and $\Gamma_R\, \alpha_k\, \alpha_i \neq \Gamma_R\, \alpha_j\, \alpha_i = p\, \Gamma_R$ for all $k \neq j$.

Remark : the self-adjointness of the modular operators can be quite easily deduced from this proposition.

3 The quotient space $X_R = \Gamma_R \backslash \mathbb{H}^3$

3.1 The class of $\big(K_2\big)$ - manifolds

In an indefinite division algebra $\mathfrak{A} = \left(\frac{a,b}{\mathbb{Q}}\right)$ with $a < 0$ and $b > 0$ square-free integers, we consider an order $\mathfrak{I}$ of type (q_1, q_2) such that Γ_R acts freely on $\mathbb{H}^3$. This way, we get a class $\big(K_2\big)$ of Riemannian manifolds $X_R = \Gamma_R \backslash \mathbb{H}^3$ that have a sectional curvature

$K=-1$, when provided with the metric induced by $\mathbb{H}^3$. This class $\big(K_2\big)$ is far from being empty. Indeed, take $a\in\mathbb{Z}_-\backslash\{-1,-3\}$ square-free and $b\in\mathscr{P}\backslash\{3\}$ such that a is not a square modulo b : $\mathfrak{A}$ is an indefinite division quaternion algebra according to proposition 2.3. For the action of $\Gamma_{_R}$ to be free, we just have to impose that -1 and -3 are squares modulo b, by proposition 2.6. Given a fixed, theorem 2.1 shows that these three conditions modulo b are simultaneously satisfied by infinitely many primes b, because the negative integers a, -1 and -3 are 2-independent.

Definition *The* class $\big(K_2^S\big)$ *is the infinite set of* $\big(K_2\big)$ *- manifolds such that*

$$a\in\mathbb{Z}_-,\ b\in\mathscr{P},\ \left(\frac{a}{b}\right)=-1,\ \left(\frac{-1}{b}\right)=1,\ \left(\frac{-3}{b}\right)=1 \tag{3.1}$$

For example, we can take $a=-2$, $b=13$ and $\mathfrak{I}$ a maximal order containing $\mathfrak{I}_0=\mathscr{O}_{\mathbb{F}}\oplus\mathscr{O}_{\mathbb{F}}\,\Omega$. In the sequel of this work, we shall consider manifolds $X_{_R}$ of the class $\big(K_2^S\big)$. The conjugation in $\mathbb{F}\simeq\mathbb{Q}\big(\sqrt{a}\big)$ coincides with the complex conjugation because $a<0$.

Lemma 3.1 *Let* $\xi\in\mathbb{F}\backslash\{0\}$. *Then* $ord_b\,|\xi|^2$ *is even.*

Proof : let ξ be such an element, that we will write $\xi=pq^{-1}\big(x+y\sqrt{a}\big)q$ with $q,\,p,\,x,\,y\in\mathbb{Z}$, $x\wedge y=1$ and $p\wedge q=1$. Assume that $|x+y\sqrt{a}|^2=x^2-a\,y^2\equiv 0\,[b]$. If b divides y, then b divides x, which contradicts $x\wedge y=1$. Therefore $y\not\equiv 0\,[b]$ so that $a\equiv\left(\frac{x}{y}\right)^2\,[b]$ and $\left(\frac{a}{b}\right)=1$, a contradiction with relation (3.1). Hence $|x+y\sqrt{a}|^2\not\equiv 0\,[b]$ and, as $|\xi|^2=p^2\,q^{-2}\,|x+y\sqrt{a}|^2$, we deduce that $\mathrm{ord}_b\,|\xi|^2=2\left[\mathrm{ord}_b\,(p)-\mathrm{ord}_b\,(q)\right]$ is even.

Proposition 3.1 *Let* $X_{_R}$ *be a* $\big(K_2^S\big)$ *- manifold. Then* $\Gamma_{_R}$ *has no parabolic element.*

Proof : let $\gamma=\xi+\eta\,\Omega$ in $\Gamma_{_R}$ be parabolic. By taking its opposite $-\gamma$ if necessary, we may assume that $\xi=1+x\sqrt{a}$ and $\eta=y+z\sqrt{a}$ with $x,\,y$ and $z\in\mathbb{Q}$. We have then $\mathrm{N}(\gamma)-1=0=|\xi|^2-b\,|\eta|^2-1=-ax^2-b\,(y^2-az^2)$. Let us multiply x, y and z by the least common multiple of their denominators and divide the obtained integers by their greatest common divisor. We get $aX^2+b\,(Y^2-aZ^2)=0$ with $X,\,Y,\,Z\in\mathbb{Z}$ and $X\wedge Y\wedge Z=1$. Because $b\wedge a=1$, b divides X. Setting $X_0=X/b$, we get after simplification $ab{X_0}^2+Y^2-aZ^2=0$. If b divides Z, b divides Y too, a contradiction with relation $X\wedge Y\wedge Z=1$. Therefore $Z\not\equiv 0\,[b]$ and $a\equiv\left(\frac{Y}{Z}\right)^2\,[b]$ whence $\left(\frac{a}{b}\right)=1$, a contradiction with relation (3.1).

Proposition 3.2 *Let* $\mathscr{L}$ *be a closed geodesic of* $X_{_R}=\Gamma_{_R}\backslash\mathbb{H}^3$. *There exists an hyperbolic transformation* $\gamma\in\Gamma_{_R}$ *whose axis* $L\subset\mathbb{H}^3$ *projects onto* $\mathscr{L}$ *in* $X_{_R}$.

The proof (see [9] §9.6 for a general form) uses only the discontinuity of the action of $\Gamma_{_R}$ on $\mathbb{H}^3$. In particular, there exists a compact segment l of the geodesic L such that $L=\bigcup_{n\in\mathbb{Z}}\gamma^n\cdot l$ and $\mathscr{L}=\pi_{_R}(L)=\pi_{_R}(l)$. Using this Proposition, we show :

Lemma 3.2 *Let* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *be two closed geodesics of* $X_{_R}=\Gamma_{_R}\backslash\mathbb{H}^3$. *Then* $\mathscr{L}_1=\mathscr{L}_2$ *or* $\mathscr{L}_1\cap\mathscr{L}_2$ *is finite.*

Proof : let γ_1 and $\gamma_2 \in \Gamma_R$ be hyperbolic transformations whose axis L_1 and L_2 project onto $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. Let l_1 and l_2 be compact segments of L_1 and L_2 such that $L_1 = \bigcup_{n\in\mathbb{Z}} {\gamma_1}^n \cdot l_1$ and $L_2 = \bigcup_{n\in\mathbb{Z}} {\gamma_2}^n \cdot l_2$. Then

$$\mathscr{L}_1 \cap \mathscr{L}_2 = \pi_R \left[\left(\cup_{\gamma \in \Gamma_R} \gamma \cdot l_1 \right) \cap \left(\cup_{\gamma' \in \Gamma_R} \gamma' \cdot l_2 \right) \right] = \pi_R \left[\cup_{\gamma \in \Gamma_R} \left(l_1 \cap \gamma \cdot l_2 \right) \right]$$

Assume that $\mathscr{L}_1 \cap \mathscr{L}_2$ is infinite. The group Γ_R acts discontinuously on $\mathbb{H}^3$ and l_1 and l_2 are compact subsets, so that $\Gamma_0 = \left\{ \gamma \in \Gamma_R \ / \ l_1 \cap \gamma \cdot l_2 \neq \emptyset \right\}$ is finite. Since $\mathscr{L}_1 \cap \mathscr{L}_2 = \pi_R \left(l_1 \cap \Gamma_0 \cdot l_2 \right)$ is infinite, there exists $\gamma \in \Gamma_0$ such that $l_1 \cap \gamma \cdot l_2$ hence $L_1 \cap \gamma \cdot L_2$ is infinite. L_1 and $\gamma \cdot L_2$ being two geodesics of $\mathbb{H}^3$ *i.e.* half-circles or half-lines, we deduce that $L_1 = \gamma \cdot L_2$ and $\mathscr{L}_1 = \mathscr{L}_2$.

3.2 Imbedded surfaces in $\left(K_2^S\right)$ - manifolds

The itgs of X_R are quite simply the images under π_R of those of $\mathbb{H}^3$, that are the half-spheres centered on $\mathbb{C}$ and the half-planes orthogonal to $\mathbb{C}$.

3.2.1 Traces of itgs

Definition *Let $\mathscr{S}$ be an itgs of $\mathbb{H}^3$. Its* trace $\mathscr{C}$ on $\mathbb{C}$ *is the set of its limit points in $\mathbb{C}$, that is $\mathscr{C} = \overline{\mathscr{S}} \cap \mathbb{C}$.*

The trace of an itgs of $\mathbb{H}^3$ is then either a circle or straight line of $\mathbb{C}$ – *i.e.* a circle of $\mathbb{P}^1(\mathbb{C})$. Moreover, each itgs its uniquely defined by its trace on $\mathbb{C}$, whence

Proposition *There is a bijection between the itgs of $\mathbb{H}^3$ and the circles of $\mathbb{P}^1(\mathbb{C})$.*

As a consequence, the action of an isometry on an itgs in $\mathbb{H}^3$ is entirely determined by the former's action on the latter's trace in $\mathbb{C}$, which is much more easy to deal with. In particular

Proposition 3.3 *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two itgs of $\mathbb{H}^3$ of traces $\mathscr{C}_1$ and $\mathscr{C}_2$. Then*

$$\forall \gamma \in \mathrm{SL}(2,\mathbb{C}) \qquad \gamma \cdot \mathscr{S}_1 = \mathscr{S}_2 \iff \gamma(\mathscr{C}_1) = \mathscr{C}_2$$

3.2.2 About closed itgs

The *closed* itgs of X_R are the compact ones. The lifting to $\mathbb{H}^3$ of a closed itgs of X_R is called *closed for* Γ_R. Then :

Proposition 3.4 *Let $\mathscr{S}$ be a closed itgs of X_R and S be a lifting to $\mathbb{H}^3$. There exists a group $\Gamma_0 \subset \Gamma_R$ and a compact subset $\mathscr{F} \subset S$ with non zero area such that*

$$\gamma \in \Gamma_0 \iff \gamma \cdot S = S \quad \textit{and} \quad S = \Gamma_0 \cdot \mathscr{F} = \bigcup_{\gamma \in \Gamma_0} \gamma \cdot \mathscr{F}$$

There exists moreover $\gamma = \xi + \eta\,\Omega \in \Gamma_0$ hyperbolic.

Proof : the projection $\mathscr{S} = \pi_{_R}(S)$ being compact, there exists $\mathscr{F} \subset S$ compact and $\Gamma' \subset \Gamma_{_R}$ such that $S = \Gamma' \cdot \mathscr{F}$. Since $\Gamma_{_R}$ is countable, the set $\mathscr{F}$ has non zero area (in S). Let us set $\Gamma_0 = \big\{ \gamma \in \Gamma_{_R} \ / \ \gamma \cdot \mathscr{F} \subset S \big\} \supset \Gamma'$. For $\gamma \in \Gamma_0$, the set $\gamma \cdot S \cap S \supset \gamma \cdot \mathscr{F}$ has non zero area. As the itgs $\gamma \cdot S$ and S are half-planes or half-spheres, $\gamma \cdot S = S$. Conversely, the relation $\gamma \cdot S = S$ implies that $\gamma \cdot \mathscr{F} \subset \gamma \cdot S = S$ and $\gamma \in \Gamma_0$. Hence, $\Gamma_0 = \big\{ \gamma \in \Gamma_{_R} \ / \ \gamma \cdot S = S \big\}$ is obviously a group. From the definition of the class $\big(K_2^S\big)$ and proposition 3.1, we know that $\Gamma_{_R}$ contains only hyperbolic elements except for $\{\pm \mathrm{Id}\}$. Now $\Gamma_0 \neq \{\pm \mathrm{Id}\}$ otherwise $S = \mathscr{F}$ would be a compact itgs in $\mathbb{H}^3$, and we can find a hyperbolic element $\gamma = \xi + \eta\,\Omega \in \Gamma_0$. Then $b\,|\eta|^2 = |\xi|^2 - \mathrm{N}(\gamma) = |\xi|^2 - 1 \geqslant \mathrm{Re}^2(\xi) - 1 > 0$.

Lemma 3.3 *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two distinct closed itgs of $X_{_R}$. Their intersection $\mathscr{S}_1 \cap \mathscr{S}_2$ is either the empty set or a closed geodesic of $X_{_R}$.*

Proof : in a Riemannian manifold, two distinct itgs intersect transversally, because an itgs is entirely defined by a point and the tangent space at this point. Then their intersection has dimension one if it is not empty. Since $\mathscr{S}_1$ and $\mathscr{S}_2$ are closed itgs of $X_{_R}$, they are compact. Hence, $\mathscr{L} = \mathscr{S}_1 \cap \mathscr{S}_2$ is a compact subset of $X_{_R}$. If $\mathscr{L} \neq \emptyset$, it is a complete geodesic because for $(M, \vec{u}) \in T\mathscr{L}$, the geodesic of $X_{_R}$ tangent to $\vec{u}$ at M is contained in both $\mathscr{S}_1$ and $\mathscr{S}_2$. As $\mathscr{L}$ is compact, it is then a closed geodesic of $X_{_R}$.

The half-sphere $S^o = S\big(\mathrm{O}, 1/\sqrt{b}\big)$ is invariant under the action of all the isometries induced by $\mathfrak{A} \otimes \mathbb{R}$: indeed, given $\gamma = \xi + \eta\,\Omega \in \mathfrak{A} \otimes \mathbb{R}$ such that $\mathrm{N}(\gamma) = |\xi|^2 - b\,|\eta|^2 \neq 0$, we have

$$\forall\, \theta \in \mathbb{R} \qquad \big|\gamma(b^{-1/2}\, e^{i\theta})\big| = \left| \frac{\xi e^{i\theta} b^{-1/2} + \eta}{b\,\overline{\eta} e^{i\theta} b^{-1/2} + \overline{\xi}} \right| = \frac{1}{\sqrt{b}} \left| \frac{\xi\, e^{i\theta} + \eta\,\sqrt{b}}{\overline{\eta}\,\sqrt{b} + \overline{\xi}\, e^{-i\theta}} \right| = \frac{1}{\sqrt{b}} \tag{3.2}$$

whence $\gamma \cdot (S^o \cap \mathbb{C}) = S^o \cap \mathbb{C}$ and $\gamma \cdot S^o = S^o$. We also denote by S^o its projection in $X_{_R}$. Unfortunately, as we shall see in Appendix A, it is the only closed itgs of $X_{_R}$: indeed, the subgroup $\Gamma_0 \subset \Gamma_{_R}$ leaving an itgs S invariant is generically a one-parameter group (cf. $\mathfrak{I}$ is a modulus of rank four) so that $\Gamma_0 \backslash S$ cannot be compact.

3.2.3 Notion of $\Gamma_{_R}$ - closed itgs

Lacking of closed itgs in our space $X_{_R}$, we shall use instead the follwing weaker notion, defined by analogy with the closed geodesics of $X_{_R}$, whose liftings to $\mathbb{H}^3$ are invariant under a hyperbolic element of $\Gamma_{_R}$:

Definition *An itgs S of $\mathbb{H}^3$ is called* $\Gamma_{_R}$ - closed *if there exists $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}$ hyperbolic such that $\gamma \cdot S = S$. Its projection $\mathscr{S}$ in $X_{_R}$ is also called* $\Gamma_{_R}$ - closed.

The closed itgs are $\Gamma_{_R}$ - closed, but the converse is false. Indeed, we shall show in Appendix B the existence of infinitely many $\Gamma_{_R}$ - closed itgs in $\mathbb{H}^3$, then verify that their projections define infinitely many distinct itgs of $X_{_R}$. Therefore, there are infinitely many $\Gamma_{_R}$ - closed itgs in $X_{_R}$.

Lemma 3.4 *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be to distinct Γ_R - closed itgs of X_R. Then we have area $(\mathscr{S}_1 \cap \mathscr{S}_2) = 0$.*

Proof : it is a straight forward corollay of the first part of the proof of lemma 3.3 since every one-dimensional set has zero area. Just be aware that the notion of area is here inherent to the manifold $\mathscr{S}_1$ provided with the Riemannian metric induced by $\mathbb{H}^3$: we can indeed only mention the area of subsets of two-dimensional imbedded manifolds of X_R.

Definition *We say that $\Lambda \subset X_R$ is of* type (S^o) *if Λ is contained in a finite union of Γ_R - closed itgs and area $(\Lambda) =$ area $(\Lambda \cap S^o) \neq 0$.*

3.3 Isometries and itgs of $\mathbb{H}^3$

For the sequel, we shall need to exploit relations such as $\gamma \cdot \mathscr{S} = \mathscr{S}'$ for an isometry γ and two itgs $\mathscr{S}$ and $\mathscr{S}'$ of $\mathbb{H}^3$, at least to characterize Γ_R - closed itgs. All the half-planes and the half-spheres considered in the sequel are implicitely itgs of $\mathbb{H}^3$.

3.3.1 Of the half-planes

Proposition 3.5 *Let $\mathscr{P}$ be a half-plane, $\mathscr{D}$ its trace and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$ with $a + d \neq 0$ and $c \neq 0$. Then*

$$\gamma \cdot \mathscr{P} = \mathscr{P} \iff \begin{cases} \frac{a}{c} \in \mathscr{D},\ \frac{-d}{c} \in \mathscr{D} \ : \ \mathscr{D} \text{ is given by relation } (3.3) \\ (a+d)^2 \in \mathbb{R} \end{cases}$$

Proof : by proposition 3.3, we have $\gamma \cdot \mathscr{P} = \mathscr{P} \iff \gamma(\mathscr{D}) = \mathscr{D}$. If $\gamma(\mathscr{D}) = \mathscr{D}$, then $\gamma(\infty) = \frac{a}{c} \in \mathscr{D}$ and $\gamma^{-1}(\infty) = -\frac{d}{c} \neq \frac{a}{c} \in \mathscr{D}$. As a consequence,

$$z \in \mathscr{D} \iff \frac{z - \gamma^{-1}(\infty)}{\gamma(\infty) - \gamma^{-1}(\infty)} = \frac{cz + d}{a + d} \in \mathbb{R} \iff \frac{cz - a}{a + d} \in \mathbb{R}$$

whence

$$\mathscr{D} = \left\{ z \in \mathbb{C} \ \Big/ \ \mathrm{Im}\left(\frac{cz}{a+d}\right) = \mathrm{Im}\left(\frac{a}{a+d}\right) \right\} \tag{3.3}$$

Moreover $\frac{a}{c} \in \mathscr{D} = \gamma(\mathscr{D})$ so that

$$\begin{aligned} \gamma\left(\frac{a}{c}\right) = \frac{a^2 + bc}{c(a+d)} \in \mathscr{D} &\iff \frac{a^2 + bc - a(a+d)}{(a+d)^2} \in \mathbb{R} \\ &\iff \left(\frac{bc - ad}{(a+d)^2}\right) \in \mathbb{R} \\ &\iff (a+d)^2 \in \mathbb{R} \end{aligned}$$

Reciprocally, assume that $\frac{a}{c} = \gamma(\infty) \in \mathscr{D}$, $-\frac{d}{c} = \gamma^{-1}(\infty) \in \mathscr{D}$ and $(a+d)^2 \in \mathbb{R}$: then relation (3.3) still holds and $\gamma(\frac{a}{c}) \in \mathscr{D}$. Moreover $\gamma\left(\frac{a}{c}\right) = \frac{a^2+bc}{c(a+d)} \neq \frac{a}{c}$ since

$bc = ad - 1 \neq ad$. Therefore, the isometry γ takes the three distinct points ∞ , $\frac{a}{c}$, $-\frac{d}{c}$ of $\mathscr{D}$ into $\frac{a}{c}$, $\gamma(\frac{a}{c})$ and ∞, which also are distinct points of $\mathscr{D}$: any circle of $\mathbb{P}^1(\mathbb{C})$ being uniquely defined by three points, we have indeed $\gamma(\mathscr{D}) = \mathscr{D}$, which ends the proof.

Keep in mind that for any hyperbolic element $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}$, we have $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\xi) \neq 0$ and $\eta \neq 0$ (cf. proof of proposition 3.4) : those elements, which are the only interesting ones for us, will satisfy the hypothesis of the above proposition.

3.3.2 Of the half-spheres

Proposition 3.6 *Let the half-spheres $\mathscr{S}_1 = S(a_1, r_1)$ and $\mathscr{S}_2 = S(a_2, r_2)$ be itgs of $\mathbb{H}^3$, $N \in \mathbb{Z}$ and $\alpha = \xi + \eta\,\Omega \in R^{pr}(N)$ with $\eta \neq 0$. Then $\alpha(\mathscr{S}_1) = \mathscr{S}_2$ iff*

$$\exists\,\varepsilon = \pm 1 \qquad \left\{\begin{array}{rcll} b\big(r_1\,\overline{\eta}\,a_2 - \varepsilon\,r_2\,\eta\,\overline{a_1}\big) &=& \big(r_1 + \varepsilon r_2\big)\xi & (3.6)\\ b^2\,{r_1}^2\,|\eta|^2 - |\xi + b\,\eta\,\overline{a_1}|^2 &=& N\varepsilon\,\dfrac{r_1}{r_2} & (3.7)\\ |\xi|^2 - b|\eta|^2 &=& N & (3.8)\end{array}\right.$$

Proof : by proposition 3.3, we have $\alpha(\mathscr{S}_1) = \mathscr{S}_2 \iff \alpha(\mathscr{C}_1) = \alpha(\mathscr{C}_2)$, where $\mathscr{C}_1 = C(a_1, r_1)$ and $\mathscr{C}_2 = C(a_2, r_2)$ are the traces of $\mathscr{C}_1$ and $\mathscr{S}_2$. Moreover,

$$\forall\, z \in \mathbb{C}, \quad \alpha(z) = \frac{\xi z + \eta}{b\overline{\eta} z + \overline{\xi}} = \frac{\xi}{b\overline{\eta}} + \frac{b|\eta|^2 - |\xi|^2}{b\overline{\eta}\left(b\overline{\eta} z + \overline{\xi}\right)} = \frac{\xi}{b\overline{\eta}} - \frac{N}{b\overline{\eta}\left(b\overline{\eta} z + \overline{\xi}\right)}$$

so that

$$\forall\, z \in \mathbb{C}, \quad \alpha(z) = \frac{\xi}{b\overline{\eta}} + \frac{k}{z - \zeta} \qquad \text{where} \quad k = -\frac{N}{b^2\overline{\eta}^2} \quad \text{and} \quad \zeta = -\overline{\xi}/b\overline{\eta} \tag{3.4}$$

<u>*Direct implication :*</u> assume that $\alpha(\mathscr{C}_1) = \mathscr{C}_2$. Note that $\zeta \notin \mathscr{C}_1$ otherwise $\alpha(\zeta) = \infty \in \mathscr{C}_2$, a contradiction. Relation (3.4) implies that $\alpha = \alpha_R \circ \alpha_I$, where $\alpha_I : z \longmapsto \zeta + |k|/(\overline{z} - \overline{\zeta})$ is an inversion of center ζ and α_R is an orientation reversing euclidean isometry of $\mathbb{C}$.

First we assume that $\zeta \neq a_1$. Let $\mathrm{Inv}_{\mathscr{C}_1}$ be the inversion of circle $\mathscr{C}_1$ and $\hat{\zeta} = \mathrm{Inv}_{\mathscr{C}_1}(\zeta) = a_1 + {r_1}^2/(\overline{\zeta} - \overline{a_1})$. All the circles passing through ζ and $\hat{\zeta}$ are orthogonal to $\mathscr{C}_1$ because they are invariant under $\mathrm{Inv}_{\mathscr{C}_1}$. Let $\mathscr{C}_0$ be such a circle : $\alpha_I(\zeta) = \infty$ so that $\alpha_I \cdot \mathscr{C}_0$ is a line orthogonal to $\alpha_I \cdot \mathscr{C}_1 = \mathscr{C}'$, hence a diameter. Moreover it contains the point $\alpha_I(\hat{\zeta})$. As a consequence $\alpha_I(\hat{\zeta})$, the intersection of all the diameters of $\mathscr{C}'$, is the center a' of $\mathscr{C}'$ and $\alpha(\hat{\zeta}) = \alpha_R \circ \alpha_I(\hat{\zeta}) = a_2$ is the center of $\mathscr{C}_2$. We set $\big(\zeta, a_1\big) \cap \mathscr{C}_1 = \big\{A_1, B_1\big\}$ and $\big(\zeta, a_1\big) \cap \mathscr{C}' = \big\{A', B'\big\}$, taking here as a convention that the points A_1 and ζ are on the same side of a_1 on the line (ζ, a_1) whereas B_1 and ζ are on opposite sides. We have therefore $A' = \alpha_I(B_1)$, $B' = \alpha_I(A_1)$ and

$$A_1 - \zeta = \frac{a_1 - \zeta}{|a_1 - \zeta|}\big(|a_1 - \zeta| - r_1\big), \quad B_1 - \zeta = \frac{a_1 - \zeta}{|a_1 - \zeta|}\big(|a_1 - \zeta| + r_1\big)$$

whence

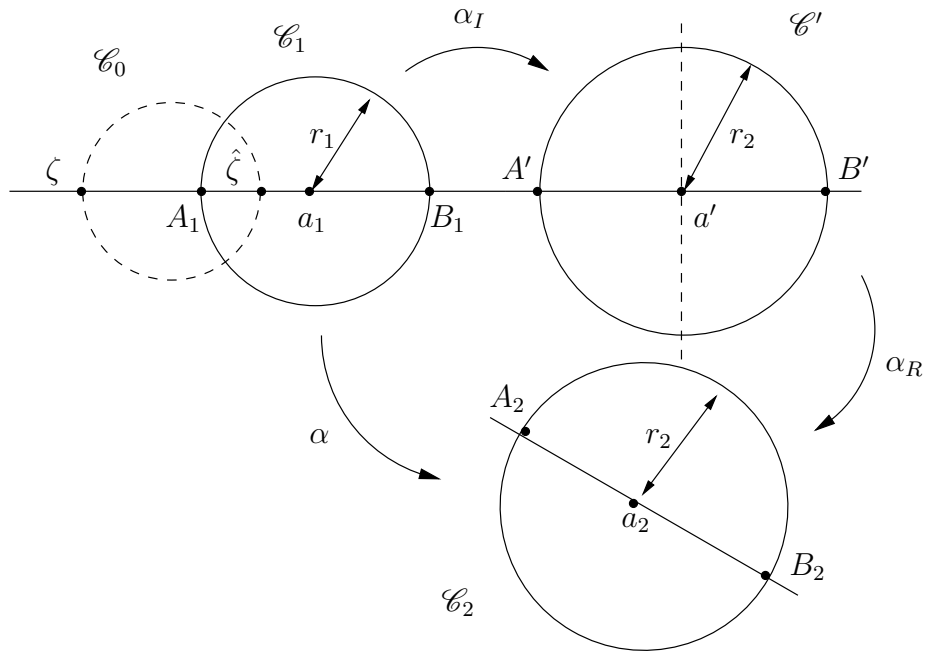

Figure 2: Action of α on $\mathscr{C}_1$

$$A' = \zeta + \frac{|a_1-\zeta|}{\overline{a_1}-\overline{\zeta}}\,\frac{|k|}{|a_1-\zeta|+r_1} \quad \text{and} \quad B' = \zeta + \frac{|a_1-\zeta|}{\overline{a_1}-\overline{\zeta}}\,\frac{|k|}{|a_1-\zeta|-r_1}$$

Therefore $2r_2 = |A'-B'| = 2|k|r_1/\big||a_1-\zeta|^2 - {r_1}^2\big|$. We set $\varepsilon = 1$ if ζ is inside of $\mathscr{C}_1$ and $\varepsilon = -1$ otherwise, so that

$$\frac{r_2}{r_1} = \frac{\varepsilon|k|}{{r_1}^2 - |a_1-\zeta|^2} \tag{3.5}$$

As $\hat{\zeta}-\zeta = a_1-\zeta+{r_1}^2/\big(\overline{\zeta}-\overline{a_1}\big) = \big({r_1}^2-|a_1-\zeta|^2\big)/\big(\overline{\zeta}-\overline{a_1}\big) = \varepsilon|k|r_1/r_2\big(\overline{\zeta}-\overline{a_1}\big)$, we deduce from relation (3.4) that $\alpha(\hat{\zeta}) = \xi/b\overline{\eta} + \varepsilon\big(\overline{\zeta}-\overline{a_1}\big)\,k\,r_2/|k|\,r_1$. Besides, $|k| = N/b^2|\eta|^2$ so that $k/|k| = -|\eta|^2/\overline{\eta}^2 = -\eta/\overline{\eta}$ and

$$\alpha(\hat{\zeta}) = \xi/b\overline{\eta} - \varepsilon\,\eta\,r_2\big(\overline{\zeta}-\overline{a_1}\big)/\,\overline{\eta}\,r_1 = a_2$$

This is equivalent to

$$b\big(r_1\overline{\eta}a_2 - \varepsilon\,r_2\eta\overline{a_1}\big) = \big(r_1+\varepsilon r_2\big)\xi \tag{3.6}$$

Injecting $\zeta = -\overline{\xi}/b\overline{\eta}$ and $|\,k| = N/b^2|\eta|^2$ into the relation (3.5), we get

$$b^2{r_1}^2|\eta|^2 - |\xi + b\eta\overline{a_1}|^2 = N\varepsilon\,\frac{r_1}{r_2} \tag{3.7}$$

At last, $\mathrm{N}(\alpha) = N$ so that

$$|\xi|^2 - b|\eta|^2 = N \tag{3.8}$$

In the case $\zeta = a_1 = -\overline{\xi}/b\overline{\eta}$, we have $\varepsilon = 1$, $\hat{\zeta} = \infty$ and $\alpha(\hat{\zeta}) = \xi/b\overline{\eta} = a_2$ so that the relation (3.6) is still satisfied. Moreover, $\mathscr{C}' = C(a_1, r_2)$: $\alpha_I \cdot C(a_1, r_1) = C(a_1, r_2)$ whence $|k| = r_1 r_2 = N/b^2|\eta|^2$ and relation (3.7) holds. We still have relation (3.8).

Backward implication : assume that the three relations hold and keep the previous notations. We deduce from relation (3.7) that

$$|\zeta - a_1|^2 - {r_1}^2 = |\overline{\xi}/b\,\overline{\eta} + a_1|^2 - {r_1}^2 \neq 0$$

whence $\zeta \notin \mathscr{C}_1$. Therefore, $\alpha_I(\mathscr{C}_1) = \mathscr{C}'$ is a circle and $\alpha(\mathscr{C}_1) = \alpha_R(\mathscr{C}') = \mathscr{C}'' = C(a'', r'')$. We still have, by relation (3.7), $\varepsilon = 1$ iff ζ is inside of $\mathscr{C}_1$. Applying the direct implication to $\mathscr{C}_1$ and $\mathscr{C}''$, we deduce from relation (3.5)

$$\frac{r''}{r_1} = \frac{\varepsilon|k|}{{r_1}^2 - |a_1 - \zeta|^2} = \frac{\varepsilon\, N}{b^2\, {r_1}^2\, |\eta|^2 - |\xi + b\,\eta\,\overline{a_1}|^2} = \frac{r_2}{r_1}$$

because $|k| = N/b^2\,|\eta|^2$. Then $r'' = r_2$. Finally, $\alpha_I(\hat{\zeta})$ is the center of $\mathscr{C}'$ so that $\alpha(\hat{\zeta}) = a''$; as relation (3.6) is equivalent to $\alpha(\hat{\zeta}) = a_2$, then $a'' = a_2$ whence $\alpha(\mathscr{C}_1) = \mathscr{C}_2$.

Now we can characterize the invariance of a half-sphere under a hyperbolic element. The following result applies in particular to closed itgs of $X_{_R}$.

Proposition 3.7 *Let $S(a_1, r) \neq S^o$ be a $\Gamma_{_R}$ - closed itgs of $\mathbb{H}^3$: therefore $a_1 \neq 0$. If $q = 1 + b\,(|a_1|^2 - r^2) \neq 0$, then $\zeta = \frac{a_1}{q} \in \mathbb{F}^*$ and $\exists\,(X, Y) \in \mathbb{Z} \times \mathbb{Q}$ such that $a\,(1 - 4b|\zeta|^2) = (X^2 - 4)\,Y^2 > 0$.*

Proof : let us take a hyperbolic element $\gamma = \xi + \eta\,\Omega \in \Gamma_{_R}$ such that $\gamma(\mathscr{C}) = \mathscr{C} = C(a_1, r)$. If $a_1 = 0$, then $\gamma \cdot C(0, r) = C(0, r)$ *i.e.* $\left|\gamma(r\,e^{i\theta})\right| = r$ for all $\theta \in \mathbb{R}$. Thus

$$\forall\,\theta \in \mathbb{R} \qquad \left|\xi\, r\, e^{i\theta} + \eta\right| = r\,\left|b\,\overline{\eta}\, r\, e^{i\theta} + \overline{\xi}\right| = \left|\xi\, r\, e^{i\theta} + b\, r^2\, \eta\right| \tag{3.9}$$

By taking the maxima of both sides, we get $r\,|\xi| + |\eta| = r\,|\xi| + b\,r^2\,|\eta|$. As γ is hyperbolic, $\eta \neq 0$ and $r = 1/\sqrt{b}$ whence $\mathscr{S} = S^o$, a contradiction. Therefore $a_1 \neq 0$. We shall now assume that $q = 1 + b\,(|a_1|^2 - r^2) \neq 0$. By proposition 3.6, the relation $\gamma(\mathscr{C}) = \mathscr{C}$ leads to

$$\exists\,\varepsilon = \pm 1 \qquad \left\{\begin{array}{rl} b\big(\overline{\eta}\, a_1 - \varepsilon\,\eta\,\overline{a_1}\big) = \big(1 + \varepsilon\big)\,\xi & (3.6') \\ b^2\, r^2\, |\eta|^2 - |\xi + b\,\eta\,\overline{a_1}|^2 = \varepsilon & (3.7') \\ |\xi|^2 - b|\eta|^2 = 1 & (3.8') \end{array}\right.$$

If $\varepsilon = 1$, we deduce from (3.6') that $\mathrm{Re}(\xi) = 0$, and this contradicts the hyperbolicity of γ. Therefore $\varepsilon = -1$, and relation (3.6′) implies that $\overline{\eta}\, a_1 + \eta\,\overline{a_1} = 0$. Summing relations (3.7') and (3.8'), we get

$$2\,b\,\mathrm{Re}(\xi\,\overline{\eta}\, a_1) + b^2\,(1 + |a_1|^2 - r^2)\,|\eta|^2 = 0$$

As $\eta \neq 0$ and $2\,b\,\mathrm{Re}(\xi\,\overline{\eta}\, a_1) = b\,\big(\xi\,\overline{\eta}\, a_1 + \overline{\xi}\,\eta\,\overline{a_1}\big) = b\,\big(\xi - \overline{\xi}\big)\,\overline{\eta}\, a_1$, we have

$$\big(\xi - \overline{\xi}\big)\, a_1 + \big[\underbrace{1 + b\,(|a_1|^2 - r^2)}_{q}\big]\,\eta = 0$$

Indeed $\xi - \overline{\xi} \neq 0$ and ζ, equal to $a_1/q = \eta/(\overline{\xi} - \xi) \in \mathbb{F}^*$, satisfies $\eta = (\overline{\xi} - \xi)\,\zeta = -2i\mathrm{Im}(\xi)\,\zeta$. Injecting this into the relation (3.8'), we obtain

$$1 = \mathrm{Re}(\xi)^2 + (1 - 4\,b\,|\zeta|^2)\,\mathrm{Im}(\xi)^2$$

Finally setting finally $2\,\mathrm{Re}(\xi) = X \in \mathbb{Z}$ and $2\,\mathrm{Im}(\xi)/\sqrt{-a} = Y^{-1} \in \mathbb{Q}^*$, we get after multiplication

$$4 = X^2 - a\,(1 - 4\,b\,|\zeta|^2)\,Y^{-2}$$

whence

$$\exists\,(X,\,Y) \in \mathbb{Z} \times \mathbb{Q} \qquad a\,(1 - 4\,b\,|\zeta|^2) = Y^2\,(X^2 - 4) > 0$$

because $Y \neq 0$ and $X^2 = \mathrm{Tr}^2(\gamma) > 4$ as γ is hyperbolic. This ends the proof.

4 Separation results in $\left(K_2^S\right)$-manifolds

We shall prove the following extension of theorem 1.1 to the $\left(K_2^S\right)$-manifolds.

Theorem 4.1 *Let $X_{_R} = \Gamma_{_R}\backslash\mathbb{H}^3$ be a three-dimensional manifold, $\Gamma_{_R}$ being a discrete subgroup of $Is^+(\mathbb{H}^3)$ derived from an indefinite quaternion algebra $\mathfrak{A} = \left(\frac{a,\,b}{\mathbb{Q}}\right)$. We shall assume moreover that $X_{_R}$ is a manifold of class $\left(K_2^S\right)$.*

Let $\Lambda \subset X_{_R}$ be a non empty set contained in a finite union of $\Gamma_{_R}$ - closed itgs of $X_{_R}$ such that area$(\Lambda) \neq 0$, unless Λ is contained in a finite union of isolated points and closed geodesics, and that is not of type (S^o). Then Λ cannot be the singular support of an arithmetic quantum limit on $X_{_R}$.

4.1 Foreword

First we prove a separation result on such a subset Λ :

Proposition 4.1 *Let $X_{_R}$ be a manifold of class $\left(K_2^S\right)$. For all non-empty subset $\Lambda \subset X_{_R}$ contained in a finite union of $\Gamma_{_R}$ - closed itgs such that area$(\Lambda) \neq 0$, unless Λ is contained in a finite union of isolated points and closed geodesics, and that is not of type (S^o), there exists a correspondence $\mathscr{C}$ separating Λ.*

We shall consider a subset Λ of a finite union of *objects of the same type* of $X_{_R}$ (a set of points, a set of closed geodesics or a set of $\Gamma_{_R}$ - closed itgs) and treat the cases separately in the next sections. The following proposition will simplify the calculations and be very helpful in the sequel.

Proposition 4.2 *Let $F_1,\,\ldots,\,F_r$ be* objects of the same type *of $X_{_R}$ and $G_1,\,\ldots,\,G_r$ a choice of liftings of these objects in $\mathbb{H}^3$. There exists a finite subset $\mathscr{F} \subset \mathscr{P}$ such that, given $p \in \mathscr{P}\backslash\mathscr{F}$, the relation*

$$\exists\,\alpha \in R(p) \cup R^{pr}(p^2) \quad \exists\,i \in \{1\ldots r\} \qquad \alpha \cdot G_1 = G_i$$

leads to

$$\exists\,N \in \mathscr{F} \quad \exists\,\tilde{\alpha} \in R^{pr}(N\,p) \cup R^{pr}(N^2 p^2) \qquad \tilde{\alpha} \cdot G_1 = G_1 \tag{4.1}$$

Proof : let us fix $n = 1$ or 2 and assume that $\exists\, p_i \in \mathscr{P}$, $\exists\, \alpha_i \in R^{pr}({p_i}^n)$ such that $\alpha_i \cdot G_1 = G_i$ whence $G_1 = {}^t\mathrm{Com}(\alpha_i) \cdot G_i$, for a certain $i \in \{1 \dots r\}$. Take $p \neq p_i \in \mathscr{P}$ and $\alpha \in R^{pr}(p^n)$: then $\alpha \cdot G_1 = G_i \Longrightarrow \tilde{\alpha} \cdot G_1 = G_1$ where $\tilde{\alpha} = {}^t\mathrm{Com}(\alpha_i)\, \alpha \in R({p_i}^n p^n) = R(N^n p^n)$. This element is primitive : otherwise, we would have $\tilde{\alpha} \in pR$ or $\tilde{\alpha} \in p_i R$ since p and p_i are both primes, so that

- if $\tilde{\alpha} \in pR$ we get $\alpha_i\, \tilde{\alpha} = \alpha_i\, {}^t\mathrm{Com}(\alpha_i)\, \alpha = {p_i}^n \alpha \in pR$ and $\alpha \in pR$ as $p_i \wedge p = 1$, a contradiction with $\alpha \in R^{pr}(p^n)$.
- if $\tilde{\alpha} \in p_i R$, then $\tilde{\alpha}\, {}^t\mathrm{Com}(\alpha) = {}^t\mathrm{Com}(\alpha_i)\, \alpha\, {}^t\, \mathrm{Com}(\alpha) = p^n\, {}^t\mathrm{Com}(\alpha_i) \in p_i R$ and $\alpha_i \in p_i R$, a similar contradiction.

Proceeding the same way with all the indices $i \in \{1 \dots r\}$ and all the values of $n \in \{1, 2\}$, we get to relation (4.1) after exclusion of at most $2r$ values of $p \in \mathscr{P}$, the forementionned set $\mathscr{F}$.

4.2 Case of the points

In this section, we only consider primes p such that $\mathrm{ord}_p(2abDD') = 0$ and $\left(\frac{a}{p}\right) = -1$.

• Let $\Lambda = \left\{\tilde{x}_1, \dots, \tilde{x}_l\right\}$ be a set of points of $X_{_R}$ and their liftings $x_i = (z_i, t_i) \in \mathbb{H}^3$ for $i = 1 \dots l$, to which we apply proposition 4.2. For $n = 1$ or 2, $N \in \mathscr{F}$ and $p \in \mathscr{P} \backslash \mathscr{F}$, we take $\alpha = \xi + \eta\, \Omega \in R^{pr}(N^n p^n)$ such that $\alpha \cdot x_1 = x_1$. By proposition 2.5, we have $\xi\, \eta \neq 0$. By relation (2.3), the action of α on $\mathbb{H}^3$ is

$$\alpha \cdot \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} \dfrac{\xi}{b\overline{\eta}} - \dfrac{N^n p^n}{b\overline{\eta}} \dfrac{\xi + b\eta\overline{z}}{|\xi + b\eta\overline{z}|^2 + b^2|\eta|^2 t^2} \\ \dfrac{N^n p^n t}{|\xi + b\eta\overline{z}|^2 + b^2|\eta|^2 t^2} \end{pmatrix}$$

It is well defined for any $t > 0$ because $\eta \neq 0$. The relation $\alpha \cdot x_1 = x_1$ implies that

$$N^n p^n = |\xi + b\, \eta\, \overline{z_1}|^2 + b^2\, |\eta|^2\, {t_1}^2 = |\xi|^2 - b\, |\eta|^2 \tag{4.2}$$

and

$$\overline{\eta}\, z_1 + \eta\, \overline{z_1} = 0 \tag{4.3}$$

Note that $z_1 \neq 0$, otherwise relation (4.2) gives $|\xi|^2 + b^2\, |\eta|^2\, {t_1}^2 = |\xi|^2 - b\, |\eta|^2$ so that $|\eta| = 0$, a contradiction. Relation (4.3) implies that $-\frac{\overline{z_1}}{z_1} = \frac{\overline{\eta}}{\eta}$ is a constant. Choose a $\eta_0 \in \mathscr{O}_{\mathbb{F}}$ such that $\frac{\overline{\eta_0}}{\eta_0} = -\frac{\overline{z_1}}{z_1}$. We have $\frac{\overline{\eta}}{\eta} = \frac{\overline{\eta_0}}{\eta_0}$ whence $\frac{\eta}{\eta_0} = \frac{\overline{\eta}}{\overline{\eta_0}} \in \mathbb{R} \cap \mathbb{F} = \mathbb{Q}$ (because $a < 0$) and $\exists\, m \in \mathbb{Q}$ such that $\eta = m\, \eta_0$. The expansion of relation (4.2) provides

$$N^n\, p^n = |\xi|^2 + 2\, b\, m\, \mathrm{Re}\, (\xi_0\, \overline{\eta_0}\, z_1) + b^2\, m^2\, |\eta_0|^2\, ({t_1}^2 + |z_1|^2) = |\xi|^2 - b\, m^2\, |\eta_0|^2$$

and after division by $b\, m \neq 0$ we get

$$2\, \mathrm{Re}\, (\xi\, \overline{\eta_0}\, z_1) + m\, |\eta_0|^2 \left[1 + b\, ({t_1}^2 + |z_1|^2)\right] = 0 \tag{4.4}$$

As $D'\, \xi \in \mathscr{O}_{\mathbb{F}}$ by proposition 2.4, we have $2\, D'\xi = X + Y\sqrt{a}$ with $X, Y \in \mathbb{Z}$. Because $b > 0$, the coefficient of m in relation (4.4) is strictly positive so that m is a linear

function of X and Y. Therefore the middle term in (4.2) is a definite positive quadratic form of the two integer variables X and Y, that we will write

$$N^n\, p^n = c_1 X^2 + c_2 XY + c_3 Y^2 \tag{4.5}$$

with $(c_1,\, c_2,\, c_3) \in \mathbb{R}^3$ and ${c_2}^2 - 4c_1c_3 < 0$ (the form is definite positive)

• Let us suppose that for each $N \in \mathscr{F}$, there exist at most two primes $p \in \mathscr{P}\backslash\mathscr{F}$ satisfying this relation, and let $\Delta \in \mathbb{N}$ be the product of all those primes p : for all $p \in \mathscr{P}\backslash\mathscr{F}$ such that $\mathrm{ord}_p(\Delta) = 0$, for all $N \in \mathscr{F}$, for all $\alpha \in R^{pr}(N^2p^2) \cup R^{pr}(Np)$, we have $\alpha \cdot x_1 \neq x_1$. We deduce from proposition 4.2 that

$$\left| \begin{array}{l} \forall\, p \in \mathscr{P}\backslash\mathscr{F} \ \text{ such that } \ \mathrm{ord}_p(2abDD'\Delta) = 0 \ \text{ and } \ \left(\frac{a}{p}\right) = -1 \\ \forall\, \alpha \in R(p) \cup R^{pr}(p^2) \qquad \forall\, i \in \{\, 1 \ldots l \,\} \qquad \alpha \cdot x_1 \neq x_i \end{array} \right. \tag{4.6}$$

Now assume that for some $N \in \mathscr{F}$, equation (4.5) is solvable for at least three distinct primes $p_1,\, p_2,\, p_3$. We have three points $(X_i : Y_i)_{i=1,2,3} \in \mathbb{P}^1(\mathbb{Q})$ such that

$$\forall\, i = 1 \ldots 3 \qquad N^n\, {p_i}^n = c_1 {X_i}^2 + c_2 X_i Y_i + c_3 {Y_i}^2 \tag{4.7}$$

If $(X_i : Y_i) = (X_j : Y_j)$ for $i \neq j$, then $p_j(X_i, Y_i) = \pm p_i(X_j, Y_j)$ whence $\alpha_i \in p_i R$ since $p_i \wedge p_j = 1$, a contradiction. Thus $(X_1 : Y_1)$, $(X_2 : Y_2)$ and $(X_3 : Y_3)$ are three distinct points of $\mathbb{P}^1(\mathbb{Q})$. By relation (4.7) we can write

$$\begin{pmatrix} {X_1}^2 & X_1Y_1 & {Y_1}^2 \\ {X_2}^2 & X_2Y_2 & {Y_2}^2 \\ {X_3}^2 & X_3Y_3 & {Y_3}^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = N^n \begin{pmatrix} {p_1}^n \\ {p_2}^n \\ {p_3}^n \end{pmatrix} \in \mathbb{Z}^3$$

The determinant of the above matrix is $\prod_{i<j}(Y_jX_i - Y_iX_j) \in \mathbb{Q}^*$ so that $c_1,\, c_2,\, c_3 \in \mathbb{Q}$ after inversion of the linear system. Therefore, relation (4.5) becomes

$$\kappa p^n = \alpha X^2 + \beta XY + \gamma Y^2 \qquad \text{for} \quad p \in \mathscr{P}\backslash\mathscr{F} \quad \text{and} \quad X,\, Y \in \mathbb{Z} \tag{4.8}$$

with $(\kappa,\, \alpha,\, \beta,\, \gamma) \in \mathbb{Z}^4$ and $\delta = \beta^2 - 4\alpha\gamma < 0$, which implies that δ is a square modulo p. We deduce from proposition 4.2 that

$$\left| \begin{array}{l} \forall\, p \in \mathscr{P}\backslash\mathscr{F} \ \text{ such that } \ \mathrm{ord}_p(2abDD') = 0, \ \left(\frac{a}{p}\right) = -1 \ \text{ and } \ \left(\frac{\delta}{p}\right) = -1 \\ \forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i \in \{1, \ldots, l\} \qquad \alpha.x_1 \neq x_i \end{array} \right. \tag{4.9}$$

By proposition 2.2, a and δ being strictly negative integers, there exists infinitely many primes $p \notin \mathscr{F}$ satisfying the hypothesis in relation (4.9) and (4.6). This leads to

Proposition 4.3 *Let $x_1, \ldots, x_l$ be points of $\mathbb{H}^3$. There exists infinitely many primes p such that :* $\forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i \in \{1, \ldots, l\} \qquad \alpha \cdot x_1 \neq x_i$

4.3 The geodesics

The proof given in [10] still holds ; we associate to any geodesic of $\mathbb{H}^3$ a proportionality class of binary quadratic forms (they are complex this time). Using proposition 2.1, we obtain

Proposition 4.4 *Let $L_1, \ldots, L_r$ be geodesics of $\mathbb{H}^3$. There exists infinitely many primes p such that : $\forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i \in \{1 \ldots r\} \qquad \alpha \cdot L_1 \neq L_i$*

4.4 The Γ_R - closed Itgs

4.4.1 The half-planes

In this section too, we consider primes p such that $\mathrm{ord}_p(2abDD') = 0$ and $\left(\frac{a}{p}\right) = -1$.

Proposition 4.5 *Let $\mathscr{P}_1$ be a Γ_R - closed half-plane of $\mathbb{H}^3$ and $\mathscr{F}$ a finite set of primes. There exists infinitely many primes $p \notin \mathscr{F}$ such that*

$$\forall\, N \in \mathscr{F} \quad \forall\, \alpha \in R^{pr}(Np) \cup R^{pr}(N^2p^2) \qquad \alpha \cdot \mathscr{P}_1 \neq \mathscr{P}_1$$

Proof : let $\gamma_1 = \xi_1 + \eta_1\,\Omega \in \Gamma_R$ hyperbolic be such that $\gamma_1 \cdot \mathscr{P}_1 = \mathscr{P}_1$. By proposition 3.5, the trace of $\mathscr{P}_1$ on $\mathbb{C}$ is

$$\mathscr{D}_1 = \left\{\, z \in \mathbb{C} \;\Big/\; \mathrm{Im}(b\,\overline{\eta_1}\,z) = \mathrm{Im}(\xi_1) \,\right\} \tag{4.10}$$

For $n = 1$ or 2, $p \notin \mathscr{F}$ prime and $N \in \mathscr{F}$, we take $\alpha = \xi + \eta\,\Omega \in R^{pr}(N^n p^n)$ such that $\alpha \cdot \mathscr{P}_1 = \mathscr{P}_1$. By proposition 2.5, $\xi\,\eta \neq 0$. Then $\alpha^{-1}(\infty) = -\frac{\overline{\xi}}{b\overline{\eta}} \in \mathscr{D}_1$ which means that

$$-\mathrm{Im}\left(\frac{\overline{\eta_1\,\xi}}{\overline{\eta}}\right) = \mathrm{Im}\left(\frac{\eta_1\,\xi}{\eta}\right) = \mathrm{Im}(\xi_1) \quad \text{ and } \quad \exists\,\lambda \in \mathbb{R},\ \frac{\xi}{\eta} = \frac{\lambda + i\,\mathrm{Im}(\xi_1)}{\eta_1}$$

In fact, $\lambda \in \mathbb{Q} = \mathbb{R} \cap \mathbb{F}$ because all the complex numbers considered belong to the number field $\mathbb{F}$. Moreover $N^n p^n = \mathrm{N}(\alpha) = |\xi|^2 - b\,|\eta|^2 = |\eta|^2\left(\frac{|\xi|^2}{|\eta|^2} - b\right) = \frac{|\eta|^2}{|\eta_1|^2}\,[\lambda^2 + \mathrm{Im}(\xi_1)^2 - b\,|\eta_1|^2]$. As $|\xi_1|^2 - b\,|\eta_1|^2 = \mathrm{N}(\gamma_1) = 1$, then $\mathrm{Im}(\xi_1)^2 - b\,|\eta_1|^2 = 1 - \mathrm{Re}(\xi_1)^2$ so that

$$4|\eta_1|^2 N^n p^n = |\eta|^2\Big[4\lambda^2 + 4 - \mathrm{Tr}(\gamma_1)^2\Big] \tag{4.11}$$

By proposition 2.5, $\mathrm{ord}_p|\eta|^2 = 0$. Since $D'\eta_1 \in \mathscr{O}_{\mathscr{F}}$, we have $D'^2|\eta_1|^2 \in \mathbb{Z}$ so that $\mathrm{ord}_p|\eta_1|^2 \geqslant 0$. As $\mathrm{Tr}(\gamma_1) \in \mathbb{Z}$, relation (4.11) provides $\mathrm{ord}_p(4\lambda^2) = \mathrm{ord}_p(4|\eta|^2\lambda^2) = \mathrm{ord}_p\big[|\eta|^2(\mathrm{Tr}(\gamma_1)^2 - 4) + 4|\eta_1|^2 N^n p^n\big] \geqslant 0$ so that $4\lambda^2 + 4 - \mathrm{Tr}(\gamma_1)^2 \equiv 0\,[p]$. Therefore the nonnegative integer $c = \mathrm{Tr}(\gamma_1)^2 - 4$ is a square modulo p.

Assume that the integer c is a square : we set $\mathrm{Tr}(\gamma_1) = m \in \mathbb{Z}$ and $c = m^2 - 4 = n^2$ with $n \in \mathbb{Z}$, whence $m^2 - n^2 = (|m| + |n|) \times (|m| - |n|) = 4 = 2 \times 2 = 4 \times 1$. Because $|m| + |n| \geqslant |m| - |n| > 0$, we have the following alternative : either $|m| + |n| = |m| - |n| = 2$ so that $n = 0$ and $m = \mathrm{Tr}(\gamma_1) = \pm 2$, or $|m| + |n| = 4$ and $|m| - |n| = 1$ so that $m = \mathrm{Tr}(\gamma_1) = 5/2 \in \mathbb{N}$, a contradiction in each case. Hence, $c = \mathrm{Tr}(\gamma_1)^2 - 4 > 0$ is not a square. As a consequence, the integers c, a and ac are not squares in $\mathbb{Z}$ (the two last

being strictly negative) so that a and c are 2-independent : there exists by theorem 2.1 infinitely many primes $p \notin \mathscr{F}$ such that

$$\mathrm{ord}_p(2abDD') = 0 \qquad \text{and} \qquad \left(\frac{a}{p}\right) = \left(\frac{c}{p}\right) = -1$$

By relation (4.11), these primes are the ones we were looking for.

4.4.2 The half-spheres

Proposition 4.6 *Let $\mathscr{S} = S(a_1, r) \neq S^o$ be a Γ_R - closed itgs of $\mathbb{H}^3$ and $\mathscr{F}$ be a finite subset of $\mathscr{P}$. There exists infinitely many primes $p \in \mathscr{P} \backslash \mathscr{F}$ such that*

$$\forall\, N \in \mathscr{F} \quad \forall\, \alpha \in R^{pr}(Np) \cup R^{pr}(N^2p^2) \qquad \alpha \cdot \mathscr{S} \neq \mathscr{S}$$

Proof : since $\mathscr{S}$ is a Γ_R - closed itgs of $\mathbb{H}^3$, $a_1 \neq 0$ by proposition 3.7. For $n = 1$ or 2, $p \in \mathscr{P} \backslash \mathscr{F}$ such that $\mathrm{ord}_p(2abDD') = 0$ and $N \in \mathscr{F}$, we take $\alpha = \xi + \eta\,\Omega \in R^{pr}(N^n p^n)$ such that $\alpha \cdot \mathscr{S} = \mathscr{S} = S(a_1, r)$. If $\eta = 0$, then $\alpha \cdot (z, t) = (\xi z/\overline{\xi}, t)$ for any $(z, t) \in \mathbb{H}^3$ and the transformation α is as an euclidean rotation of the space $\mathbb{R}^3$. The relation $\alpha \cdot \mathscr{S} = \mathscr{S}$ implies then $a_1 = \alpha \cdot a_1 = \xi\, a_1/\overline{\xi} \neq 0$ whence $\xi = \overline{\xi} \in \mathbb{R}$ and $\alpha = \xi = \pm Np \in pR$, a contradiction. As a consequence, $\eta \neq 0$ and we can proceed as in the proof of proposition 3.7 : by proposition 3.6,

$$\exists\, \varepsilon = \pm 1 \qquad \begin{cases} b\big(\overline{\eta}\, a_1 - \varepsilon\, \eta\, \overline{a_1}\big) = \big(1 + \varepsilon\big)\xi & (3.6'') \\ b^2\, r^2|\eta|^2 - |\xi + b\, \eta\, \overline{a_1}|^2 = \varepsilon\, N^n p^n & (3.7'') \\ |\xi|^2 - b|\eta|^2 = N^n p^n & (3.8'') \end{cases}$$

<u>*If* $\varepsilon = -1$</u> : we get $\overline{\eta}\, a_1 + \eta\, \overline{a_1} = 2\,\mathrm{Re}(\eta\, \overline{a_1}) = 0$ and $\eta \overline{a_1} \in i\,\mathbb{R}^*$ from relation (3.6"). We proceed as with relation (4.3) : fix $\eta_1 \in \mathscr{O}_{\mathbb{F}}$ such that $\eta_1 \overline{a_1} \in \mathbb{R}^*$: then $\frac{\eta}{\eta_1} \in \mathbb{R} \cap \mathbb{F} = \mathbb{Q}$ so that $\exists\, \lambda \in \mathbb{Q}$, $\eta = \lambda\, \eta_1$. Using the relations (3.7") and (3.8"), we get $|\xi + b\, \eta\, \overline{a_1}|^2 - b^2\, r^2|\eta|^2 = |\xi|^2 - b\,|\eta|^2$ whence $2\, b\, \mathrm{Re}(\xi\, \overline{\eta}\, a_1) + b\,|\eta|^2\big[1 + b\,(|a_1|^2 - r^2)\big] = 0$ and

$$2\,\mathrm{Re}\,(\xi\, \overline{\eta_1}\, a_1) = -\lambda\, q\, |\eta_1|^2 \qquad \text{where} \qquad q = 1 + b\,(|a_1|^2 - r^2) \tag{4.12}$$

Let us assume for the moment that $q \neq 0$. By proposition 3.7, $\zeta = \frac{a_1}{q} \in \mathbb{F}$ so that $2\, i\, \mathrm{Im}(\xi\, \overline{\eta_1}\, \zeta) = \mu\, \sqrt{a}$ for $\mu \in \mathbb{Q}$. Therefore, $2\, \xi\, \overline{\eta_1}\, \zeta = -\lambda\, |\eta_1|^2 + \mu\, \sqrt{a}$ and $4\,|\xi\, \eta_1\, \zeta|^2 = \lambda^2|\eta_1|^4 - a\, \mu^2$. Injecting this in relation (3.8"), we get $4\, N^n\, p^n\, |\eta_1|^2|\zeta|^2 = 4\,|\xi\, \eta_1\, \zeta|^2 - 4\, b\,|\eta|^2|\eta_1|^2|\zeta|^2 = -a\, \mu^2 + \lambda^2|\eta_1|^4\big(1 - 4\, b\,|\zeta|^2\big)$. For integers l, m, r relatively prime such that $\lambda = l/r$ and $\mu = m/r \in \mathbb{Q}$, this relation becomes

$$a\,\big(1 - 4\, b\,|\zeta|^2\big)(l\,|\eta_1|^2)^2 = 4\, a\,|\eta_1|^2|\zeta|^2\, r^2\, N^n\, p^n + a^2 m^2 \tag{4.13}$$

after multiplication of both sides by $a\, r^2$. According to proposition 3.7, $1 - 4\, b\,|\zeta|^2 \neq 0$. For $p \in \mathscr{P}$ such that $\mathrm{ord}_p(1 - 4\, b\,|\zeta|^2) = \mathrm{ord}_p|\zeta|^2 = \mathrm{ord}_p|\eta_1|^2 = 0$, we deduce from relation (4.13) that $l \equiv 0\,[p]$ if and only if $m \equiv 0\,[p]$. If $l \equiv m \equiv 0\,[p]$, then $r \not\equiv 0\,[p]$ since $l \wedge m \wedge r = 1$ and

$$(2\, r\, \overline{\eta_1}\, \zeta)\, \xi = \big(-\, l\,|\eta|^2 + m\sqrt{a}\,\big) \in p\mathscr{O}_{\mathbb{F}} \tag{4.14}$$

Moreover

$$r\,\eta = l\,\eta_1 \in p\mathscr{O}_{\mathbb{F}} \tag{4.15}$$

We have $D'\mathfrak{I} \subset D\mathfrak{I}_0 \subset \mathfrak{I}$ by proposition 2.4 : then there exists $\alpha_1 = \xi_1 + \eta_1\,\Omega \in \mathfrak{I}_0$ such that $D'\alpha = D\alpha_1$. As $p \wedge D = 1$, we deduce from relations (4.14) and (4.15) that $\alpha_1 = p\,\alpha_2$, where $\alpha_2 = \xi_2 + \eta_2\,\Omega \in \mathfrak{I}_0 = \mathscr{O}_{\mathbb{F}} \oplus \mathscr{O}_{\mathbb{F}}\,\Omega$. Finally $p \wedge D' = 1$ implies that $xp + yD' = 1$ holds for two integers $x,\ y \in \mathbb{Z}$ (theorem of Bézout). We have then $\alpha = xp\alpha + yD'\alpha = xp\alpha + yD\alpha_1 = p\big(x\alpha + yD\alpha_2\big) \in p\mathfrak{I}$ since $D\mathfrak{I}_0 \subset \mathfrak{I}$, a contradicition as α is primitive. As a consequence, $l \not\equiv 0\,[p]$ and relation (4.13) implies that $a\left(1 - 4\,b\,|\zeta|^2\right)$ is a square modulo p.

If $q = 0$, we deduce from relation (4.12) that $\xi\,\overline{\eta_1}\,a_1 \in i\mathbb{R}$ whence $\xi \in \mathbb{R}$ since $\overline{\eta_1}\,a_1 \in i\mathbb{R}^*$. Then $\xi = \mu \in \mathbb{Q}$ and relation (3.8") becomes $N^n p^n = \mu^2 - b\,|\eta_1|^2\,\lambda^2$. As in the case $q \neq 0$, we are lead to

$$r^2 N^n p^n = m^2 - b\,|\eta_1|^2\,l^2 \tag{4.16}$$

for integers $l,\, m,\, r$ relatively prime. Let $p \in \mathscr{P}$ such that $\mathrm{ord}_p|\eta_1|^2 = 0$: we have $l \equiv 0\,[p] \implies m \equiv 0\,[p] \implies \alpha \in pR$, a contradiction. Thus $l \not\equiv 0\,[p]$ and the integer $b\,|\eta_1|^2$ is a square modulo p by relation (4.16).

<u>*If* $\varepsilon = 1$</u> : relation (3.6") leads to $\xi = i\,b\,\mathrm{Im}(\overline{\eta}\,a_1) \in i\mathbb{R}$ and $\xi + b\,\eta\,\overline{a_1} = b\,\mathrm{Re}(\overline{\eta}\,a_1) \in \mathbb{R}$. Using relations (3.7") and (3.8"), we get $N^n p^n = b^2\,{r_1}^2\,|\eta|^2 - b^2\,\mathrm{Re}^2(\eta\,\overline{a_1}) = b^2\,\mathrm{Im}^2(\eta\,\overline{a_1}) - b\,|\eta|^2$, so that $b^2\,|\eta|^2\big(|a_1|^2 - {r_1}^2\big) = b\,|\eta|^2 \neq 0$. Hence $b\left(|a_1|^2 - {r_1}^2\right) = 1$ and $q = 2$: by proposition $a_1 = 2\,\zeta \in \mathbb{F}^*$. We can set $\eta\,\overline{a_1} = X + Y\sqrt{a}$ with $X,\, Y \in \mathbb{Q}$. After multiplication of both sides by $|a_1|^2$, relation (3.8") becomes

$$|a_1|^2\,N^n p^n = -a\,b^2\,|a_1|^2\,Y^2 - b\left(X^2 - a\,Y^2\right) = b\left[a\left(1 - b\,|a_1|^2\right)Y^2 - X^2\right]$$

Let $p \in \mathscr{P}$ such that $\mathrm{ord}_p|a_1|^2 = 0 = \mathrm{ord}_p b$: as before, if $Y \equiv 0\,[p]$ then $X \equiv 0\,[p]$ whence $\alpha \in pR$, a contradiction. Thus $Y \not\equiv 0\,[p]$ and $a\left(1 - b\,|a_1|^2\right) = a\left(1 - 4\,b\,|\zeta|^2\right)$ is a square modulo p.

<u>*End of the proof*</u> : if $q = 0$, we have $\eta_1 \in \mathbb{F}^*$ and $\mathrm{ord}_b(|\eta_1|^2)$ is even by lemma 3.1, so that $b\,|\eta_1|^2$ is not a square. Therefore, there exists infinitely many primes p such that $b\,|\eta_1|^2$ is not a square modulo p (cf. section 2.2.1).

If $q \neq 0$, then $a\left(1 - 4\,b\,|\zeta|^2\right) = (X^2 - 4)\,Y^2 > 0$ with $(X, Y) \in \mathbb{Z} \times \mathbb{Q}$, according to proposition 3.7. As we saw in the proof of proposition 4.5, the nonnegative integer $X^2 - 4$ cannot be a square. Hence, $a\left(1 - 4\,b\,|\zeta|^2\right)$ is not a square in $\mathbb{Q}$, and there exists once again infinitely many primes p such that $a\left(1 - 4\,b\,|\zeta|^2\right)$ is not a square modulo p.

In each case, after the exclusion of the prime factors of a finite set of rational numbers, we still have infinitely many primes p for which :

$$\forall\, N \in \mathscr{F} \quad \forall\, \alpha \in R^{pr}(N^n p^n) \qquad \alpha \cdot \mathscr{S} \neq \mathscr{S}$$

4.4.3 Synthesis

Let $\mathscr{S}_1 \neq S^o, \ldots, \mathscr{S}_l$ be Γ_R - closed itgs of $\mathbb{H}^3$, to which we apply proposition 4.2. We know by propositions 4.5 and 4.6 that there are infinitely many primes $p \in \mathscr{P} \backslash \mathscr{F}$ such that : $\forall N \in \mathscr{F}$, $\forall \alpha = \xi + \eta\,\Omega \in R^{pr}(Np) \cup R^{pr}(N^n p^n)$, $\alpha \cdot \mathscr{S}_1 \neq \mathscr{S}_1$ whence, by proposition 4.2

Proposition 4.7 *Let $\mathscr{S}_1 \neq S^o, \ldots, \mathscr{S}_l$ be Γ_R - closed itgs of $\mathbb{H}^3$. There exists infinitely many primes $p \in \mathscr{P}$ such that*

$$\forall\, \alpha \in R(p) \cup R^{pr}(p^2) \quad \forall\, i \in \{\, 1 \ldots l \,\} \qquad \alpha \cdot \mathscr{S}_1 \neq \mathscr{S}_i$$

4.5 Conclusion

<u>*Proof of proposition 4.1*</u> : let X_R be a manifold of class (K_2^S), $\Lambda \subset X_R$ a non-empty set that is not of type (S^o) such that $\Lambda \subset z_1 \cup \ldots \cup z_l \cup L_1 \cup \ldots \cup L_r \cup \Sigma_1 \cup \ldots \cup \Sigma_s$ where the z_i are points, the L_j are closed geodesics and the Σ_k are Γ_R - closed itgs of X_R. We assume moreover that area$(\Lambda) \neq 0$, unless Λ is contained in a finite union of isolated points and closed geodesics. We look for a modular correspondence $\mathscr{C}_p$ separating Λ ; to this end, we shall adapt the method of [10] §2.4.

. Λ *is finite* : using proposition 4.3, we finish the proof as in [10].

. Λ *is infinite and contained in a finite union of closed geodesics* : using proposition 4.4, we finish the proof as in [10].

. Λ *is contained in a finite union of Γ_R - closed itgs and area(Λ) $\neq 0$* : since it is not of type (S^o), we may write $\Lambda \subset \Sigma_1 \cup \ldots \cup \Sigma_s$ where the Σ_k are Γ_R - closed itgs, $\Sigma_1 \neq S^o$ and area$(\Sigma_1 \cap \Lambda) \neq 0$. Let $\mathscr{S}_1, \ldots, \mathscr{S}_s$ be liftings of these itgs Σ_k in $\mathbb{H}^3$. Then $\mathscr{S}_1 \neq S^o$ and there exists by proposition 4.7 a prime p such that : $\forall\, \alpha \in R(p) \cup R^{pr}(p^2)$, $\forall\, k = 1 \ldots r$, $\alpha \cdot \mathscr{S}_1 \neq \mathscr{S}_k$. Therefore $\mathscr{C}_p(\Sigma_1)$ and $\mathscr{C}_{p^2}(\Sigma_1)$ consist of Γ_R - closed itgs all distinct from the Σ_k. By lemma 3.4, the sets $\mu_1 = \mathscr{C}_p(\Sigma_1) \cap (\Sigma_1 \cup \ldots \cup \Sigma_l)$ and $\mu_2 = \mathscr{C}_{p^2}(\Sigma_1) \cap (\Sigma_1 \cup \ldots \cup \Sigma_l)$ have zero area. The same goes for $\nu_1 = \left\{\, z \in X_R \,/\, \mathscr{C}_p(z) \cap \mu_1 \neq \emptyset \,\right\}$ and $\nu_2 = \left\{\, z \in X_R \,/\, \mathscr{C}_{p^2}(z) \cap \mu_2 \neq \emptyset \,\right\}$, so that there exists $z \in \Lambda \cap \Sigma_1 \backslash (\nu_1 \cup \nu_2)$. We finish the proof as in [10].

<u>*Proof of theorem 4.1*</u> : let $\Lambda \subset X_R$ be a set satisfying the statement of proposition 4.1. There is a modular correspondence $\mathscr{C}$ separating Λ. Let ν be an arithmetic quantum limit on X_R : it is associated to a sequence of eigenfunctions of the Laplacian Δ and the Hecke operators $(T_n)_{n \in \mathbb{N}}$, hence of $T = T_C$. Using proposition 1.1, we deduce that singsupp $\nu \neq \Lambda$, which proves theorem 4.1.

A Of closed itgs in X_R

Let $\mathscr{S} \neq S^o$ be a itgs of $\mathbb{H}^3$ closed for Γ_R : by proposition 3.4, there exists a compact subset $\mathscr{F} \subset \mathscr{S}$ and a group $\Gamma_0 \subset \Gamma_R$ such that $\mathscr{S} = \Gamma_0 \cdot \mathscr{F}$, *i.e.*

$$\forall\, x \in \mathscr{S} \quad \exists\, \gamma \in \Gamma_0 \qquad \gamma \cdot x \in \mathscr{F} \tag{1.1}$$

We set $t_0 = \inf \left\{ t \,/\, (z,t) \in \mathscr{F} \right\} = \min \left\{ t \,/\, (z,t) \in \mathscr{F} \right\} > 0$ because $\mathscr{F}$ is compact.

A.1 The half-spheres

We shall write $\mathscr{S} = S(a_1, r)$ with $a_1 \in \mathbb{C}^*$ (cf. proposition 3.7) and $r > 0$. For $0 < t < t_0$, we fix $x = (z,t) = \left(a_1 \left[1 + \frac{\sqrt{r^2 - t^2}}{|a_1|} \right], t \right) \in \mathscr{S} \backslash \mathscr{F}$. Let $\gamma = \xi + \eta\,\Omega \in \Gamma_0$. If $\eta \neq 0$, γ is hyperbolic and $\overline{\eta} a_1 \in i\mathbb{R}^*$ (cf. proof of proposition 3.7). Moreover

$$\gamma \cdot x = (\tilde{z}, \tilde{t}) = \left(*\,,\ \frac{t}{|\xi + b\eta\overline{z}|^2 + b^2|\eta|^2 t^2} \right) \tag{1.2}$$

by relation (2.3). As $\eta\overline{z} \in i\mathbb{R}$, we have $|\xi + b\eta\overline{z}|^2 \geqslant \mathrm{Re}(\xi)^2 > 1$ by hyperbolicity of γ : hence $\tilde{t} < t < t_0$ and $\gamma \cdot x = (\tilde{z}, \tilde{t}) \notin \mathscr{F}$. If $\eta = 0$, then $\gamma = \pm I_2$ and $\gamma \cdot x = x \notin \mathscr{F}$. Therefore $\gamma \cdot x \notin \mathscr{F}$ for all $\gamma \in \Gamma_0$, a contradiction with relation (1.1).

A.2 The half-planes

We shall write $\mathscr{S} = \mathscr{D} \oplus \mathbb{R}_+^* \mathbf{j}$. Let $\gamma_0 = \xi_0 + \eta_0\,\Omega \in \Gamma_0$ with $\eta_0 \neq 0$: by relation (3.3) we have $\mathscr{D} = \left\{ z \in \mathbb{C} \,\middle/\, \mathrm{Im}(b\,\overline{\eta_0}\,z) = \mathrm{Im}(\xi_0) \right\} = \left(\frac{-\overline{\xi_0}}{b\overline{\eta_0}} \,;\, \frac{\xi_0}{b\overline{\eta_0}} \right)$. For $\lambda \in \mathbb{R}$, we set $z_\lambda = \frac{-\overline{\xi_0}}{b\overline{\eta_0}} + \frac{\lambda}{b\overline{\eta_0}} \in \mathscr{D}$. Let $\gamma = \xi + \eta\,\Omega \in \Gamma_0$. If $\eta \neq 0$,

$$\xi + b\eta\overline{z_\lambda} = \xi - \frac{\xi_0 \eta}{\eta_0} + \frac{\lambda\eta}{\eta_0} = \eta \left[\frac{\xi}{\eta} - \frac{\xi_0}{\eta_0} + \frac{\lambda}{\eta_0} \right] \tag{1.3}$$

As $\gamma \in \mathfrak{I}$, then $D'^2|\eta|^2 \in \mathbb{N}$ whence $D'^2|\eta|^2 \geqslant 1$. As $\mathrm{N}(\gamma) = |\xi|^2 - b|\eta|^2 = 1$, we have $\left| \frac{\xi}{\eta} \right|^2 = b + \frac{1}{|\eta^2|} \leqslant b + D'^2$ and

$$\forall\, \gamma \in \Gamma_R \ \text{ with } \ \eta \neq 0 \qquad \left| \frac{\xi}{\eta} - \frac{\xi_0}{\eta_0} \right|^2 \leqslant 4\left(b + D'^2\right) \tag{1.4}$$

We fix $\lambda > \left[2\sqrt{(b + D'^2)} + D' \right] |\eta_0|$, $t \in]0, t_0[$ and $x = (z_\lambda, t) \in \mathscr{S} \backslash \mathscr{F}$. From relations (1.4) and (1.3), we deduce that $|\xi + b\eta\overline{z_\lambda}|^2 > D'^2|\eta|^2 > 1$, whence $\gamma \cdot x \notin \mathscr{F}$ by relation (1.2). If $\eta = 0$, then $\gamma = \pm I_2$ and $\gamma \cdot x = x \notin \mathscr{F}$. Thus we get the same contradiction with relation (1.1) as before.

A.3 Synthesis

We have shown that any (K_2^S) - manifold X_R contains at most one closed itgs, the projection of $S^o = S(0, 1/\sqrt{b})$ in X_R. Now, Γ_R acts on the hyperbolic surface S^o – which is equivalent to $\mathbb{H}^2$ – as a subgroup of its orientation preserving isometries ; we proceed

exactly the same way as in [2] §4 to prove that $\Gamma_{_R}\backslash S^o$ is compact for a maximal order R, hence for any order (cf. if $R \subset R'$ maximal order, then $\left[\Gamma_{_R} : \Gamma_{_{R'}}\right] < \infty$).

Set $\mathfrak{I} = \mathbb{Z}\left[i_1, i_2, i_3, i_4\right]$ and $\mathfrak{J} = \mathbb{R}\left[i_1, i_2, i_3, i_4\right] = \mathfrak{I} \otimes \mathbb{R} = \mathfrak{A} \otimes \mathbb{R}$. The images of the sets $\mathfrak{I}(1)$ and $\mathfrak{J}(1)$ under the mapping φ defined in relation (2.6) are $\Gamma_{_R}$ and

$$G_{_R} = \left\{ \left(\begin{array}{cc} \xi & \eta \\ b\overline{\eta} & \overline{\xi} \end{array} \right) \in \mathrm{M}(2, \mathbb{C}) \Big/ \left|\xi\right|^2 - b\left|\eta\right|^2 = 1 \right\} \tag{1.5}$$

the group of isometries induced by $\mathfrak{J} = \mathfrak{A} \otimes \mathbb{R}$. We already know from relation (3.2) that $\gamma \cdot \left(0,\, 1/\sqrt{b}\right) \in S^o$ for all $\gamma \in \Gamma_{_R}$. Furthermore :

Lemma 1.1 *The mapping* $\psi \,:\, \gamma \longmapsto \gamma \cdot \left(0,\, 1/\sqrt{b}\right)$ *defines a continuous surjection from* $G_{_R}$ *to* S^o*. Set* $\mathscr{M}_c = \left\{ \xi + \eta\,\Omega \in G_{_R} \,/\, |\xi| \leqslant c,\ |\eta| \leqslant c \right\}$ *: for every* $c > 0$*, the set* $\psi(\mathscr{M}_c)$ *is compact.*

Proof : let $\gamma = \xi + \eta\,\Omega \in G_{_R}$; if $\eta \neq 0$, we have

$$\gamma \cdot \left(\begin{array}{c} 0 \\ 1/\sqrt{b} \end{array} \right) = \left(\begin{array}{c} \dfrac{\xi}{b\overline{\eta}} - \dfrac{1}{b\overline{\eta}}\,\dfrac{\xi}{|\xi|^2 + b|\eta|^2} \\ \dfrac{1}{\sqrt{b}\,\left(|\xi|^2 + b|\eta|^2\right)} \end{array} \right) = \left(\begin{array}{c} \dfrac{\xi}{b\overline{\eta}}\left(1 - \dfrac{1}{1 + 2\,b\,|\eta|^2}\right) \\ \dfrac{1}{\sqrt{b}\,\left(1 + 2\,b\,|\eta|^2\right)} \end{array} \right)$$

since $|\xi|^2 = 1 + b\,|\eta|^2$, so that

$$\forall\, \gamma \in G_{_R} \qquad \psi(\gamma) = \left(\frac{2\,\xi\,\eta}{1 + 2\,b\,|\eta|^2},\, \frac{1}{\sqrt{b}\,\left(1 + 2\,b\,|\eta|^2\right)} \right) \in S^o$$

If $\eta = 0$, then $\psi(\gamma) = (0, 1/\sqrt{b})$ as $\gamma = \pm I_2$, a particular case of the above expression. We get this way all the points of S^o when $(\,|\eta|,\, \mathrm{Arg}\,\xi\eta\,)$ runs $[0, +\infty[\times]-\pi, \pi]$ and ψ is a surjection. From its expression, ψ is moreover continuous on $G_{_R}$: as $\mathscr{M}_c$ is a compact subset of $\mathrm{M}(2, \mathbb{C})$ for all $c > 0$, then $\psi(\mathscr{M}_c)$ is compact. We can also note that from the definition of ψ, we have

$$\forall\, \gamma_1,\, \gamma_2 \in G_{_R} \quad \psi(\gamma_1.\gamma_2) = \gamma_1 \cdot \psi(\gamma_2)$$

Let us set $M_c = \left\{ \alpha = x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 \in \mathfrak{J} = \mathfrak{A} \otimes \mathbb{R} \,/\, \forall\, j = 1 \ldots 4,\ |x_j| \leqslant c \right\}$ $\cap \left\{ \alpha \in \mathfrak{J} \,:\, \mathrm{N}(\alpha) = 1 \right\}$ for $c > 0$. We have (see [2] §4)

Lemma 1.2 *Let* $\mathfrak{A}$ *be an indefinite division algebra over* $\mathbb{Q}$ *and* $\mathfrak{I} = \mathbb{Z}\left[i_1, i_2, i_3, i_4\right]$ *a maximal order of* $\mathfrak{A}$*. There exists* $c > 0$ *fixed such that*

$$\forall\, \alpha \in \mathfrak{A} \otimes \mathbb{R} \ \ \textit{with}\ \ N(\alpha) = 1 \quad \exists\, \varepsilon \in \mathfrak{I}(1) \qquad \varepsilon\alpha \in M_c$$

After application of the mapping φ, the above relation becomes

$$\exists\, c' = c\sqrt{1 - a} > 0 \quad \forall\, \gamma \in G_{_R} \quad \exists\, \gamma_0 \in \Gamma_{_R} \qquad \gamma_0\gamma \in \mathscr{M}_{c'}$$

so that : $\exists\, c' > 0,\ \forall\, \gamma \in G_{_R},\ \exists\, \gamma_0 \in \Gamma_{_R},\ \gamma_0 \cdot \psi(\gamma) = \psi(\gamma_0.\gamma) \in \psi(\mathscr{M}_{c'})$. By lemma 1.1, we know that : $\forall\, x \in S^o,\ \exists\, \gamma \in G_{_R},\ x = \psi(\gamma)$. We deduce that the set $\mathscr{F} = \psi(\mathscr{M}_{c'})$ is a compact subset of S^o such that

$$\forall\, x \in S^o \quad \exists\, \gamma_0 \in \Gamma_{_R} \qquad \gamma_0 \cdot x \in \mathscr{F}$$

which means that $S^o = \Gamma_{_R} \cdot \mathscr{F}$. As a consequence, the itgs S^o is closed for $\Gamma_{_R}$, whence

Proposition 1.1 *Let $X_{_R}$ be a (K_2^S) - manifold. The half-sphere $S^o = S(0, 1/\sqrt{b})$ is the only itgs of $\mathbb{H}^3$ closed for $\Gamma_{_R}$; its projection in $X_{_R}$ is then compact.*

A.4 Complement : the case $a > 0$

To see if the choice of $a < 0$ is the cause of the lack of closed itgs in the (K_2^S) - manifolds, we define as in section 3.1 a class (K_1^S) of quotient manifolds $X_{_R}$ by taking

$$a \in \mathbb{N} \qquad b \in \mathscr{P} \qquad \left(\frac{a}{b}\right) = -1 \qquad \left(\frac{-1}{b}\right) = 1 \qquad \left(\frac{-3}{b}\right) = 1 \tag{1.6}$$

For example, $a = 2$ and $b = 13$ are appropriate ; actually, we just have to take the opposite of a to get from the (K_2^S) - manifolds to the (K_1^S) ones. The properties seen in section 3 still hold, except that the conjugation in $\mathbb{F}$ does not coincide with the complex conjugation anymore. In the definition of the $\Gamma_{_R}$ - closed itgs, we impose moreover that the considered hyperbolic elements $\gamma = \xi + \eta\,\Omega$ satisfy $\eta \neq 0$. This time, the itgs $P^o = \mathbb{R} \oplus \mathbb{R}_+^* \mathbf{j} \simeq \mathbb{H}^2$ is left invariant under the action of all the isometries induced by $\mathfrak{A} \otimes \mathbb{R}$ *i.e.* the group $\mathrm{SL}(2, \mathbb{R})$ since $\sqrt{a} \in \mathbb{R}$. The image of $(x, t) \in \mathbb{R} \times \mathbb{R}_+^* \simeq P^o$ under the action of

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

is given by

$$\gamma \cdot \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{t} \end{pmatrix} \in P^o = \left| \begin{array}{ll} \begin{pmatrix} \dfrac{ax+b}{d} \\ \dfrac{a\,t}{d} \end{pmatrix} & \text{if } c = 0 \\[2ex] \begin{pmatrix} \dfrac{a}{c} - \dfrac{1}{c}\,\dfrac{cx+d}{(cx+d)^2 + c^2t^2} \\ \dfrac{t}{(cx+d)^2 + c^2t^2} \end{pmatrix} & \text{if } c \neq 0 \end{array} \right.$$

If $c = 0$, we have

$$\tilde{x} + i\tilde{t} = \frac{a(x+it) + b}{d} = \gamma(x+it)$$

If $c \neq 0$, we get

$$\begin{aligned}
\tilde{x}+i\tilde{t} &= \frac{a}{c}-\frac{c(x-it)+d}{c\,|c(x+it)+d|^2} \;=\; \frac{a}{c}-\frac{1}{c\,[c(x+it)+d]} \\
&= \frac{a\,[c(x+it)+d]-1}{c\,[c(x+it)+d]} \;=\; \frac{ac\,(x+it)+bc}{c\,[c(x+it)+d]} \quad \text{since} \quad ad-1=bc \\
&= \frac{a\,(x+it)+b}{c(x+it)+d} \;=\; \gamma(x+it)
\end{aligned}$$

and we recognize in each case the fractional linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^2$. Therefore the discrete group $\Gamma_{_R} \subset \mathrm{SL}(2,\mathbb{R})$ has the same action on P^o and on $\mathbb{H}^2$: we deduce that the quotient $\Gamma_{_R}\backslash P^o$ is compact and that the itgs P^o is closed for $\Gamma_{_R}$ for any order R (cf. [2] §4).

Let us verify that it is the only one in $\mathbb{H}^3$. For $x=(z,t)\in\mathbb{H}^3$, we define $f(x) \overset{def}{=} \frac{\mathrm{Im}(z)}{t}$. Let $\gamma=\xi+\eta\,\Omega\in\Gamma_{_R}$: we have

$$\gamma\cdot x \;=\; \begin{pmatrix}\tilde{z}\\ \tilde{t}\end{pmatrix} \;=\; \left|\begin{array}{ll}
\begin{pmatrix}\xi^2 z\\ \xi^2 t\end{pmatrix} & \text{if } \eta=0 \\[2ex]
\begin{pmatrix} \dfrac{\xi}{b\overline{\eta}^{\mathbb{F}}}-\dfrac{1}{b\overline{\eta}^{\mathbb{F}}}\,\dfrac{\overline{\xi}^{\mathbb{F}}+b\overline{\eta}^{\mathbb{F}}\overline{z}}{|\overline{\xi}^{\mathbb{F}}+b\overline{\eta}^{\mathbb{F}}z|^2+b(\overline{\eta}^{\mathbb{F}})^2t^2} \\[2ex] \dfrac{t}{|\overline{\xi}^{\mathbb{F}}+b\overline{\eta}^{\mathbb{F}}z|^2+b(\overline{\eta}^{\mathbb{F}})^2t^2}\end{pmatrix} & \text{if } \eta\neq 0
\end{array}\right.$$

Since $\xi,\ \eta\in\mathbb{F}\subset\mathbb{R}$, we easily compute in each case that $f(\gamma\cdot x)=f(x)$ for all $\gamma\in\Gamma_{_R}$ and $x\in\mathbb{H}^3$. Let the itgs $\mathscr{S}\neq P^o$ be closed for $\Gamma_{_R}$: there exists a compact subset $\mathscr{F}\subset\mathscr{S}$ and a group $\Gamma_0\subset\Gamma_{_R}$ such that $\mathscr{S}=\Gamma_0\cdot\mathscr{F}$ *i.e.*

$$\forall\, x\in\mathscr{S} \qquad \exists\,\gamma\in\Gamma_0 \qquad \gamma\cdot x\in\mathscr{F}$$

Since the function f is continous on $\mathbb{H}^3$ and invariant under $\Gamma_{_R}$, this function is bounded on the compact set $\mathscr{F}$, hence on $\mathscr{S}$ according to previous relation. We can now get to the desired contradiction :

- If $\mathscr{S}$ is a half-plane, its trace is a line $\mathscr{D}\neq\mathbb{R}$ since $\mathscr{S}\neq P^o$. Fix $z_0\in\mathscr{D}\backslash\mathbb{R}$ and set $x_t=(z_0,t)\in\mathscr{S}$ for all $t>0$. Then $|f(x_t)|\longrightarrow\infty$ as $t\to 0$, a contradiction with relation.
- If $\mathscr{S}=S(a,r)$ is a half-sphere, we have $a+ir\notin\mathbb{R}$ or $a-ir\notin\mathbb{R}$ since $r\neq 0$: we may assume without loss of generality that $a+ir\notin\mathbb{R}$. Set $x_t=(a+i\sqrt{r^2-t^2},t)$ for $t\in]0,r]$: then $f(x_t)\sim|\mathrm{Im}(a)+r|\,/t\longrightarrow\infty$ as $t\to 0$, a contradiction again. Hence :

Proposition 1.2 *Let $X_{_R}$ be a (K_1^S)-manifold. The half-plane $P^o=\mathbb{R}\oplus\mathbb{R}_+^*\boldsymbol{j}$ is the only itgs of $\mathbb{H}^3$ closed for $\Gamma_{_R}$; its projection in $X_{_R}$ is thus compact.*

Therefore, the lack of closed itgs is not specific to the (K_2^S) - manifolds, as we could have thought *a priori*. Moreover, we may verify using relation (4.10) that P^o is the only Γ_R - closed half-plane in any (K_1^S) - manifold, that we also have restrictions on the closed geodesics (they link two points of $\mathbb{P}^1(\mathbb{R})$) and on the Γ_R - closed half-spheres, which justifies *a posteriori* our choice to deal with the (K_2^S) - manifolds rather than with the (K_1^S) - ones.

B Of Γ_R - closed itgs

B.1 Families of Γ_R - closed half-planes of $\mathbb{H}^3$

Take a hyperbolic element $\gamma = \xi + \eta\,\Omega \in \Gamma_R$: $z_1 = \gamma^{-1}(\infty)$ and $z_2 = \gamma(\infty)$ are distinct points of $\mathbb{C}$ since $\mathrm{Tr}(\gamma) \neq 0$. Let us define the itgs $\mathscr{P}_\gamma = \mathscr{D}_\gamma \oplus \mathbb{R}_+^*\mathbf{j}$ where

$$\mathscr{D}_\gamma = (z_1, z_2) = \left\{ z \in \mathbb{C} \Big/ \mathrm{Im}(b\,\overline{\eta}\,z) = \mathrm{Im}(\xi) \right\} \tag{2.1}$$

Since $\mathrm{Tr}(\gamma) \in \mathbb{Z} \subset \mathbb{R}$, we have $\gamma(\mathscr{D}_\gamma) = \mathscr{D}_\gamma$ and $\gamma \cdot \mathscr{P}_\gamma = \mathscr{P}_\gamma$ according to Proposition 3.5 : the half-plane $\mathscr{P}_\gamma$ is a Γ_R - closed itgs of $\mathbb{H}^3$. Let us fix $(t,\, u) \in \mathbb{Z}^2$ and look for an element in Γ_R of the form

$$\gamma = \gamma_{t,u} = \underbrace{x + yu\sqrt{a}}_{\xi} + \underbrace{y(1 + t\sqrt{a})}_{\eta}\,\Omega$$

with $x,\, y \in \mathbb{Z}^*$. As $x \neq 0$, $\gamma_{t,u}$ is hyperbolic and

$$\mathrm{N}(\gamma_{t,u}) = 1 \iff x^2 - \big[\underbrace{au^2 + b(1 - at^2)}_{d}\big]y^2 = 1 \tag{2.2}$$

By Fermat's Theorem on the Equation of Pell, we can solve this equation for non-trivial integers $x,\, y$ as soon as $d = au^2 + b\,(1 - at^2) \in \mathbb{N}$ is not a square in $\mathbb{Z}$. Let us assume the contrary.

<u>*If* $u \not\equiv 0\,[b]$ *:*</u> then $a \equiv du^{-2}\,[b]$ is a square modulo b, a contradiction.

<u>*If* $u \equiv 0\,[b]$ *:*</u> then $d \equiv 0\,[b]$ and, as d is a square, $d \equiv 0\,[b^2]$ since b is square-free, whence $1 - at^2 \equiv 0\,[b]$ and a is a square modulo b, a contradiction again.

Moreover, $d = b - a(bt^2 - u^2) > 0$ as soon as $|t|$ is big enough : in that case, we can solve the previous equation for non-trivial integers $x,\, y$. We deduce finally from relation (2.1) that

$$\mathscr{D}_{\gamma_{t,u}} = \mathscr{D}(u,t) = \left(\mathbb{R} + \frac{i\,\mathrm{Im}(\xi)}{b|\eta|^2}\right)\eta = \left(\mathbb{R} + \frac{u\sqrt{a}}{b(1 - at^2)}\right)(1 + t\sqrt{a})$$

is the trace of a Γ_R - closed itgs of $\mathbb{H}^3$, which proves the following Proposition :

Proposition 2.1 *There are infinitely many* Γ_R *- closed half-planes in* $\mathbb{H}^3$*,* e.g. *the half-planes* $\mathscr{P}(t,u) = \left(\mathbb{R} + \frac{u\sqrt{a}}{b(1-at^2)}\right)(1 + t\sqrt{a}) \oplus \mathbb{R}_+^*\mathbf{j}$ *for integers* t *and* $u \in \mathbb{Z}$ *such that* $d = b - a(bt^2 - u^2) > 0$.

Let us take for example $a=-2$ and $b=13$:

. $\gamma_{0,1}=10+3i\sqrt{2}-3\Omega$ leaves $\mathscr{P}(0,1)=\left(\mathbb{R}+\frac{i\sqrt{2}}{13}\right)\oplus\mathbb{R}_+^*\,\mathbf{j}$ invariant.

. $\gamma_{1,0}=25+4(1+i\sqrt{2})\,\Omega$ leaves $\mathscr{P}(1,0)=\mathbb{R}(1+i\sqrt{2})\oplus\mathbb{R}_+^*\,\mathbf{j}$ invariant.

. $\gamma_{2,3}=10+3i\sqrt{2}+(1+2i\sqrt{2})\,\Omega$ leaves $\mathscr{P}(2,3)=\left(\mathbb{R}+\frac{i\sqrt{2}}{39}\right)(1+2i\sqrt{2})\oplus\mathbb{R}_+^*\,\mathbf{j}$ invariant.

B.2 Families of Γ_R - closed half-spheres of $\mathbb{H}^3$

Let the half-sphere $\mathscr{S}=S(a_1,r)$ be an itgs of $\mathbb{H}^3$ and $\gamma=\xi+\eta\,\Omega$ a hyperbolic element of Γ_R. We apply Proposition 3.6 to the relation $\gamma\cdot\mathscr{S}=\mathscr{S}$: as we saw in the proof of Proposition 3.7, we have $\varepsilon=-1$ by hyperbolicity of γ. As a consequence,

$$\gamma\cdot\mathscr{S}=\mathscr{S}\iff\left\{\begin{array}{l}0=\overline{\eta}\,a_1+\eta\,\overline{a_1}\\ 1=|\xi+b\,\eta\,\overline{a_1}|^2-b\,r^2\,|\eta|^2\\ 1=|\xi|^2-b\,|\eta|^2\end{array}\right.$$

$$\iff\left\{\begin{array}{l}0=\overline{\eta}\,a_1+\eta\,\overline{a_1}\\ 0=(\xi-\overline{\xi})\,a_1+\left[1+b\,(|a_1|^2-r^2)\right]\eta\\ 1=|\xi|^2-b\,|\eta|^2\end{array}\right.\qquad(E)$$

For $a_1\in\mathbb{F}^*$ and $r\in\mathbb{R}^*$ such that $r^2\in\mathbb{Q}$, the resolution of the system (E) leads to

$$\left|\begin{array}{l}\xi=X-\frac{1}{2}\left[1+b\,(|a_1|^2-r^2)\right]Y\sqrt{a}\\ \eta=Y\,a_1\,\sqrt{a}\end{array}\right.$$

with $X,\,Y\in\mathbb{Z}$. The norm equation provides

$$1=X^2-\underbrace{\frac{a}{4}\left\{\left[1+b\,(|a_1|^2-r^2)\right]^2-4\,b\,|a_1|^2\right\}}_{d}Y^2\qquad(2.3)$$

For $r^2\in\mathbb{Q}$ close enough to $|a_1|^2$, we have $4\,b\,|a_1|^2\geqslant\left[1+b\,(|a_1|^2-r^2)\right]^2$ and the rational number d is nonnegative. Assume moreover that $\mathrm{ord}_b\,|a_1|^2\geqslant 0$ and $\mathrm{ord}_b\,r^2\geqslant 0$: since $a\equiv 4d\,[b]$ is not a square modulo b, the rational number d is not a square in $\mathbb{Q}$. Set $d=p/q$ with relatively prime integers p and q, and $D=q^2d=pq$. As the integer D is not a square, the Pell equation $x^2-Dy^2=1$ is solvable for non-trivial integers $x,\,y\in\mathbb{Z}$. Taking $X=x$ and $Y=qy$, we have non-trivial integers $X,\,Y\in\mathbb{Z}$ satisfying equation (2.3), which proves the following Proposition.

Proposition 2.2 *Let $a_1\in\mathbb{F}=\mathbb{Q}[\sqrt{a}]$ and $r^2\in\mathbb{Q}$ such that $ord_b\,|a_1|^2\geqslant 0$, $ord_b\,r^2\geqslant 0$ and $4\,b\,|a_1|^2\geqslant\left[1+b\,(|a_1|^2-r^2)\right]^2$. The half-sphere $S(a_1,r)$ is a Γ_R - closed itgs of $\mathbb{H}^3$.*

For $a=-2$ and $b=13$ again :

. $\gamma=359+168i\sqrt{2}+18i\sqrt{2}(2+3i\sqrt{2})\,\Omega$ leaves $S\left(\frac{2}{3}+i\sqrt{2},\sqrt{3}\right)$ invariant.

. $\gamma = 106133 - 69160i\sqrt{2}(7+3i\sqrt{2})\,\Omega$ leaves $S\left(7+3i\sqrt{2},8\right)$ invariant.

. $\gamma = 19603 - 51480i\sqrt{2} + 2574i\sqrt{2}(5+2i\sqrt{2})\,\Omega$ leaves $S\left(5+2i\sqrt{2},\sqrt{30}\right)$ invariant.

B.3 Projections on X_R

The Propositions 2.1 and 2.2 have shown the existence of infinitely many Γ_R - closed itgs in $\mathbb{H}^3$. Now, we have to verify that the set of their projections in X_R is still infinite, to prove that

Proposition 2.3 *There exist infinitely many Γ_R - closed itgs in X_R.*

Proof : we shall consider here the Γ_R - closed half-planes $\mathscr{P}(t,0)$ given by Proposition 2.1. Let us take t_1 and $t_2 \in \mathbb{N}$. We denote by $\mathscr{D}_1 = (1+t_1\sqrt{a})\,\mathbb{R}$ and $\mathscr{D}_2 = (1+t_2\sqrt{a})\,\mathbb{R}$ the traces of $\mathscr{P}(t_1,0)$ and $\mathscr{P}(t_2,0)$ on $\mathbb{C}$. A circle of $\mathbb{P}^1(\mathbb{C})$ being entirely defined by three distinct points, we have for $\gamma = \xi + \eta\,\Omega \in \Gamma_R\backslash\{\pm\mathrm{Id}\}$ (so that $\eta \neq 0$)

$$\gamma\cdot\mathscr{P}(t_1,0) = \mathscr{P}(t_2,0) \iff \gamma(\mathscr{D}_1) = \mathscr{D}_2 \iff \begin{cases} \gamma(\infty) = \dfrac{\xi}{b\,\overline{\eta}} \in \mathscr{D}_2 & (1) \\ \gamma(0) \;=\; \dfrac{\eta}{\overline{\xi}} \in \mathscr{D}_2 & (2) \\ \gamma(1+t_1\sqrt{a}) \in \mathscr{D}_2 & (3) \end{cases}$$

The relations (1) and (2) are equivalent since $\eta/\overline{\xi} = (b\,|\eta|^2/|\xi|^2)\times\xi/b\,\overline{\eta} \in \mathbb{R}\,\xi/b\,\overline{\eta}$. By relation (1), there exists $\lambda \in \mathbb{Q}$ such that $\xi = \lambda\,b\,\overline{\eta}\,(1+t_2\sqrt{a})$. Relation (3) provides

$$\begin{aligned} \frac{\xi\,(1+t_1\sqrt{a})+\eta}{b\,\overline{\eta}\,(1+t_1\sqrt{a})+\overline{\xi}} \in \mathscr{D}_2 &\iff [\xi\,(1+t_1\sqrt{a})+\eta]\;[\xi+b\,\eta\,(1-t_1\sqrt{a})] \in \mathscr{D}_2 \\ &\iff \xi^2\,(1+t_1\sqrt{a}) + b\,\eta^2\,(1-t_1\sqrt{a}) \in \mathscr{D}_2 \quad \text{since } \xi\eta \in \mathscr{D}_2 \\ &\iff \underbrace{\lambda^2\,b\,(1+t_2\sqrt{a})\,(1+t_1\sqrt{a})\,\overline{\eta}^2}_{z_1} + \underbrace{\left(\frac{1-t_1\sqrt{a}}{1+t_2\sqrt{a}}\right)\eta^2}_{z_2} \in \mathbb{R} \end{aligned}$$

The numbers z_1 and $z_2 \in \mathbb{F}$ having opposite arguments, either they have the same module, either they are both reals. In the first case, we get after simplification

$$b\,\lambda^2\,(1-at_2{}^2) = 1$$

whence $\mathrm{ord}_b(1-at_2{}^2) = \mathrm{ord}_b|1+t_2\sqrt{a}|^2$ is odd, a contradiction with Lemma 3.1. Thus, z_1 and z_2 are real hence rational numbers (since $\mathbb{F}\cap\mathbb{R} = \mathbb{Q}$) and there exists $\mu \in \mathbb{Q}$ such that $\eta^2 = \mu\,(1+t_1\sqrt{a})\,(1+t_2\sqrt{a})$. Taking the square of the modulus, we get $|\eta|^4 = (|\eta|^2)^2 = \mu^2\,(1-at_1{}^2)\,(1-at_2{}^2)$ with $|\eta|^2 \in \mathbb{Q}$, so that $(1-at_1{}^2)\,(1-at_2{}^2)$ is a square in $\mathbb{Q}$. We have hence proved that

$$\Big(\exists\,\gamma \in \Gamma_R,\ \gamma\cdot\mathscr{P}(t_1,0) = \mathscr{P}(t_2,0)\Big) \implies (1-at_1{}^2)\,(1-at_2{}^2) \text{ is a square in } \mathbb{N}$$

the first condition meaning that $\mathscr{P}(t_1,0)$ and $\mathscr{P}(t_2,0)$ have the same projection in X_R. To fullfill the proof of the Proposition, we just have to find an infinite subset $\mathscr{I} \subset \mathbb{N}$

such that $(1-at_1{}^2)(1-at_2{}^2)$ is not a square in $\mathbb{N}$ for all $t_1 \neq t_2 \in \mathscr{I}$, and it seems reasonable to think that it is possible for every negative integer a. Take $a=-2$ for instance : we verify that all numbers between 0 and 24000 statisfy this relation, except from 2, 11, 12, 70, 109, 225, 408, 524, 1015, 1079, 1746, 2378, 2765, 4120, 5859, 8030, 10681, 13860, 16647, 17615 and 21994. More generally, a conjectural Theorem states

Conjecture *Let A, B, C be integers relatively primes such that A is positive, $A+B$ and C are not both even and B^2-4AC is not a perfect square. Then there are infinitely many primes of the form An^2+Bn+C with $n \in \mathbb{Z}$.*

According to this Conjecture, there are for each $a \in \mathbb{Z}^-$ such infinitely many primes of the form $1-at^2$ with $t \in \mathbb{N}$: we just have to set $\mathfrak{I} = \left\{ t \in \mathbb{N} \,/\, 1-at^2 \text{ prime} \right\}$ to end the proof.

Remark : we used a very particular and limited subset of Γ_R - closed itgs of $\mathbb{H}^3$ to prove Proposition 2.3. It is clear that other itgs shall provide infinitely many Γ_R - closed itgs of X_R.

References

[1] D.V. Anosov, Geodesic Flows on Closed Riemann Manifolds with Negative Curvature, *Proceedings of the Steklov Institute of Mathematics*, **90** (1969), AMS-Providence

[2] Martin Eichler, Lectures on Modular Correspondances, *Lectures on Mathematics and Physics*, Tata Institute of fundamental research, 1957.

[3] Hershel M. Farkas & Irwin Kra, Riemann Surfaces, *Graduate Text in Mathematics* **71**, Springer-Verlag, 1980

[4] G.A. Hedlund, The Dynamics of Geodesic Flows, *Bulletin of the American Mathematical Society*, **45** (1939) n°4, 241-260

[5] H.H. Karimova, Geodesic Flows in three dimensional Spaces of Variable Spaces of Negative Curvature, *Vestnik Moskov. Univ. Serija Mat.*, **5** (1959), 3-12

[6] Lindenstrauss E. : Invariant Measures and Arithmetic Quantum Unique Ergodicity. To appear in Annals of Mathematics.

[7] Toshitsune Miyake, *Modular Forms*, Springer-Verlag, 1989

[8] Wladyslaw Narkiewicz, Elementary and Analytic Theory of Algebraic Numbers, *Springer-Verlag* (1990)

[9] John G. Ratcliffe, Foundations of Hyperbolic Manifolds, *Graduate Texts in Mathematics*, **149**, Springer-Verlag (1994)

[10] Zeév Rudnick and Peter Sarnak, The Behaviour of Eigenstates of Arithmetic Hyperbolic Manifolds, *Communications in Mathematical Physics*, **161** (1994), 195-213

[11] Peter Sarnak, Arithmetic Quantum Chaos, *First R.A. Blyth Lectures*, University of Toronto, Preprint (1993)

[12] Jean-Pierre Serre, *Cours d'Arithmétique*, Presses Universitaires de France, 1970

[13] Steven Zelditch, Uniform Distribution of Eigenfunctions on Compact Hyperbolic, *Duke Math. Journal*, **55** (1987) n°4, 919-941.

Contents

1 Introduction **1**

2 Complements of hyperbolic geometry and algebra **3**
2.1 Hyperbolic geometry in dimension three 3
2.2 Algebraic complements 5

3 The quotient space $X_R = \Gamma_R \backslash \mathbb{H}^3$ **10**
3.1 The class of (K_2) - manifolds 10
3.2 Imbedded surfaces in (K_2^S) - manifolds 12
3.3 Isometries and itgs of $\mathbb{H}^3$ 14

4 Separation results in (K_2^S)-manifolds **18**
4.1 Foreword 18
4.2 Case of the points 19
4.3 The geodesics 21
4.4 The Γ_R - closed Itgs 21
4.5 Conclusion 24

A Of closed itgs in X_R **25**

B Of Γ_R - closed itgs **29**

References **33**

Index **34**

Table of contents **34**