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TRANSIENT ANALYSIS OF STOCHASTIC FLUID MODELS

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Transient Analysis of Stochastic Fluid Models

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Thème 1 — Réseaux et systèmes
Projet Model

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Abstract: We analyze the transient behavior of stochastic fluid flow models in which the input and output rates are controlled by a finite homogeneous Markov process. Such models are used in asynchronous transfer mode (ATM) to evaluate the performance of fast packet switching and in manufacturing systems for the performance of producers and consumers coupled by a buffer. The transient analysis of such models have already been considered in earlier works and solutions have been obtained by the use of Laplace transform. We derive in this paper a new transient solution only based on recurrence relations. We show that this solution is particularly interesting for its numerical properties. The limiting behavior of the solution is also considered. We empirically show that the algorithm for computing the transient solution can be stopped when some stationary behavior is detected.

Key-words: ATM, fluid models, Markov process, transient analysis.

(Résumé : tsvp)

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Analyse transitoire de modèles stochastiques fluides

Résumé : On analyse le comportement transitoire de modèles stochastiques fluides dont les taux d'entrée et de sortie sont contrôlés par un processus de Markov fini et homogène. De tels modèles sont utilisés dans l'ATM (mode de transfert asynchrone) pour évaluer les performances des réseaux haut débit et en productique pour les performances de systèmes producteurs et consommateurs couplés par un tampon. L'analyse transitoire de tels modèles a déjà été considérée lors de précédents travaux et des solutions ont été obtenues en utilisant la transformée de Laplace. Dans cet article, on obtient une nouvelle solution transitoire basée uniquement sur des relations de récurrence. On montre que cette solution est particulièrement intéressante pour ses propriétés numériques. On considère aussi le comportement limite de cette solution. On montre de manière empirique que l'algorithme calculant la solution transitoire peut être arrêté quand un certain comportement stationnaire est détecté.

Mots-clé : ATM, modèles fluides, processus de Markov, analyse transitoire.

1 Introduction

A stochastic fluid flow model describes the behavior of a fluid level in a storage device. The input and output rates are supposed to be controlled by a finite homogeneous Markov process. Such models are used in asynchronous transfer mode (ATM) to evaluate the performance of fast packet switching and in manufacturing systems for the performance of producers and consumers coupled by a buffer. There is a large number of papers dealing with the analysis of stochastic fluid flow models. Most of these papers consider such models in stationary regime. Anick et al. [1] and Kosten [2] analyzed the fluid model for several on-off input sources controlled by a two-state homogeneous Markov process. Mitra [3] and [4] generalizes this model by considering multiple on-off inputs and outputs. In [5] Stern and Elwalid considered such models for separable Markov modulated rate process which lead to a solution of the equilibrium equations expressed as a sum of terms in Kronecker product form. In [6] Igelnik et al. derive a new approach, based on the use of interpolating polynomials, for the computation of the buffer overflow probability. An extensive list of references can be found in [7] and [8].

For what concerns the transient analysis stochastic fluid flow models controlled by a finite Markov process. Narayanan and Kulkarni [9] derive explicit expressions for the Laplace transform of the joint distribution of the first time the buffer becomes empty and the state of the Markov process at that time. Guillemin et al. consider the unbuffered model in [10] and obtain a method to compute transient characteristics, such as the congestion period, with an unbounded number of exponential on-off sources. These results have been extended by Dupuis et al. in [11] to the case where the off periods are phase-type.

The Laplace transform has been largely used to evaluate the transient behavior of fluid flow models. In [12] Ren and Kobayashi studied the transient distribution of the buffer content for exponential on-off sources of a single type. The same authors deal with the case of multiple types of inputs in [13] These studies have been extended to the Markov modulated input rate model by Tanaka et al. in [14].

In this paper, we consider a general stochastic fluid flow model in which the buffer is infinite and the input and output rates are controlled by a finite homogeneous Markov process. For this model we derive a new transient solution for the distribution of the buffer content. This solution do not make use of any transform, it is only based on simple recurrence relations which are particularly interesting for their numerical properties. The algorithm implementing this solution is very accurate since it uses essentially non negative numbers bounded by one and it gives results with an error tolerance that can be specified in advance. Furthermore, by considering the limiting behavior of the solution, we empirically show that the algorithm can be stopped when some stationary behavior is detected.

The remainder of the paper is organized as follows. We describe in the following section the model and we present our new transient solution with some of its properties. In Section 3 we

describe the algorithm implementing the solution. We empirically show in Section 4, through numerical examples, that the computation time of the solution can be considerably reduced by considering the limiting behavior of the solution. Section 5 is devoted to some conclusions.

2 A New Transient Solution

We describe in this section a general fluid model with an infinite buffer for which the input and output rates are controlled by a homogeneous Markov process $X = \{X_s, s \geq 0\}$ on the finite state space S with infinitesimal generator A and initial probability distribution α. The number of states is denoted by $|S|$. The amount of fluid in the buffer at time t is denoted by Q_t and we suppose that $Q_0 = 0$. The pair (X_t, Q_t) forms a Markov process having a pair of discrete and continuous states. Let ρ_i be the input rate and c_i be the output rate when the Markov process X is in state i. We denote by d_i the effective input rate of state i, that is $d_i = \rho_i - c_i$. Let $m+1$, $m < |S|$, be the number of distinct values among all the effective rates d_i. These $m+1$ distinct effective rates are denoted by $r_0, r_1, \ldots, r_m$ and ordered as follows

$$r_m > r_{m-1} > \ldots > r_u > 0 \geq r_{u-1} > \ldots > r_1 > r_0,$$

where u is the index of the smallest positive effective rate. The state space S of the process X can then be divided into $m+1$ disjoint subsets $B_m, B_{m-1}, \ldots, B_0$ where B_i is composed by the states i of S having the same effective rate r_i, that is $B_i = \{j \in S / d_j = r_i\}$. We will denote by $|B_i|$ the cardinal of subset B_i.

For a fixed $t > 0$, the random variable Q_t takes its values in the interval $[0, r_m t]$. For $t > 0$, the distribution of Q_t has $m-u+1$ jumps at positive values and one jump at point 0 corresponding to the case where the buffer is empty at time t. The jumps at the $m-u+1$ positive values correspond to the case where the Markov process X remains during the whole interval $[0, t]$ in the different subsets $B_u, B_{u+1}, \ldots, B_m$, provided that the initial probabilities of these subsets are positive. These jumps probabilities are then given, for $j = u, u+1, \ldots, m$ by

$$\Pr\{X_t = i, Q_t = r_j t\} = \alpha_{B_j} e^{A_{B_j B_j} t} \mathbf{1}_i \text{ if } i \in B_j,$$

where $A_{B_j B_j}$ is the sub-infinitesimal generator of dimension $|B_j|$ obtained from A by considering only the internal transitions of the subset B_j and α_{B_j} is the subvector of dimension $|B_j|$ obtained from the row vector α by considering the initial probabilities of the subset B_i. The vector $\mathbf{1}_{(i)}$ is the column vector whose ith entry is 1 and the others 0, its dimension being given by the context ($|B_j|$ in this relation).

The jump at point 0 is not so easy to obtain since the process X can eventually visit all the subsets B_i before that the buffer becomes empty at time t.

Let $F_i(t,x) = \Pr\{X_t = i, Q_t > x\}$. We then have the following partial differential equation, see for instance [14],

$$\frac{\partial F_i(t,x)}{\partial t} = -d_i \frac{\partial F_i(t,x)}{\partial x} + \sum_{r \in S} F_r(t,x) A(r,i). \tag{1}$$

We denote by P the transition probability matrix of the uniformized Markov chain with respect to the uniformization rate λ which verifies $\lambda \geq \max(-A(i,i), i \in S)$. The matrix P is then related to A by $P = I + A/\lambda$, where I denotes the identity matrix. In the following, to simplify notation, we will consider X as the uniformized process. For every $i, j = 0, \ldots, m$, we denote by $P_{B_i B_j}$ the submatrix of P containing the transition probabilities from states of B_i to states of B_j.

The main result of this paper, which is the distribution of the pair (X_t, Q_t) is given by the following theorem.

Theorem 2.1 *For every $i \in S$, we have*

$$F_i(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} b_i^{(j)}(n,k), \tag{2}$$

where $x_j = \dfrac{x - r_{j-1}^+ t}{(r_j - r_{j-1}^+)t}$ *if* $x \in [r_{j-1}^+ t, r_j t)$, *for* $j = u, u+1, \ldots, m$, *with* $r_{j-1}^+ = 0$ *for* $j = u$ *and* $r_{j-1}^+ = r_{j-1}$ *for* $j > u$. *The coefficients* $b_i^{(j)}(n,k)$ *are given by the following recursive expressions on the row vectors* $b_{B_l}^{(j)}(n,k) = \left(b_i^{(j)}(n,k)\right)_{i \in B_l}$ *for* $0 \leq l \leq m$ *and* $u \leq j \leq m$.

for $j \leq l \leq m$ *:*

$$\text{for } n \geq 0: \quad b_{B_l}^{(u)}(n,0) = (\alpha P^n)_{B_l} \quad \text{and} \quad b_{B_l}^{(j)}(n,0) = b_{B_l}^{(j-1)}(n,n) \text{ for } j > u$$

$$\text{for } 1 \leq k \leq n: \quad b_{B_l}^{(j)}(n,k) = \frac{r_l - r_j}{r_l - r_{j-1}^+} b_{B_l}^{(j)}(n,k-1) + \frac{r_j - r_{j-1}^+}{r_l - r_{j-1}^+} \sum_{i=0}^{m} b_{B_i}^{(j)}(n-1,k-1) P_{B_i B_l}$$

for $0 \leq l \leq j-1$ *:*

$$\text{for } n \geq 0: \quad b_{B_l}^{(m)}(n,n) = 0_{B_l} \quad \text{and} \quad b_{B_l}^{(j)}(n,n) = b_{B_l}^{(j+1)}(n,0) \text{ for } j < m$$

$$\text{for } 0 \leq k \leq n-1: \quad b_{B_l}^{(j)}(n,k) = \frac{r_{j-1}^+ - r_l}{r_j - r_l} b_{B_l}^{(j)}(n,k+1) + \frac{r_j - r_{j-1}^+}{r_j - r_l} \sum_{i=0}^{m} b_{B_i}^{(j)}(n-1,k) P_{B_i B_l}.$$

Proof. See Appendix A.

Formula (2) is particularly interesting from a computational point of view. Indeed, for every $j = u, \ldots, m$ and $x \in [r_{j-1}^+ t, r_j t)$ we have $x_j \in [0,1]$,

$$0 \leq \frac{r_l - r_j}{r_l - r_{j-1}^+} = 1 - \frac{r_j - r_{j-1}^+}{r_l - r_{j-1}^+} \leq 1 \text{ for } l = j, \ldots, m,$$

and

$$0 \leq \frac{r_{j-1}^+ - r_l}{r_j - r_l} = 1 - \frac{r_j - r_{j-1}^+}{r_j - r_l} \leq 1 \text{ for } l = 0, \ldots, j-1.$$

It is then easy to check that for every $i \in S$, $j = u, \ldots, m$, $n \geq 0$ and $k = 0, \ldots, n$ we have $b_i^{(j)}(n,k) \in [0,1]$. Moreover the error truncation of the series in (2) can be determined in advance. These properties are very important for what concerns the numerical stability of the computation.

For a given error tolerance ε, we define integer N as

$$N = \min \left\{ n \in \mathbb{N} \;\middle|\; \sum_{i=0}^{n} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \geq 1 - \varepsilon \right\}.$$

We then get, for every $i \in S$,

$$F_i(t,x) = \sum_{n=0}^{N} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} b_i^{(j)}(n,k) + e(N),$$

where the rest of the series $e(N)$ satisfies $e(N) \leq \varepsilon$.

The main computational effort is due to the computation of the $b_{B_l}^{(j)}(n,k)$ given in Theorem 2.1. To illustrate the recurrence relation, we proceed as done in [15] for the performability computation. For each $j = u, \ldots, m$, we define a partition of the state space S as

$$U_j = B_m \cup \cdots \cup B_j \text{ and } D_j = B_{j-1} \cup \cdots \cup B_0,$$

and denoting by T the transpose operator, we also define the following column vectors

$$b_{U_j}(n,k) = \left(b_{B_m}^{(j)}(n,k), \ldots, b_{B_j}^{(j)}(n,k) \right)^{\mathrm{T}} \text{ and } b_{D_j}(n,k) = \left(b_{B_{j-1}}^{(j)}(n,k), \ldots, b_{B_0}^{(j)}(n,k) \right)^{\mathrm{T}}.$$

With this notation, Fig. 1 illustrates the sequence of computations (drawn only for $n = 0,1,2,3$) that have to be done in order to evaluate the $b_{B_l}^{(j)}(n,k)$'s. The upper (resp. lower) part of cell (n,k) in triangle j contains the vector $b_{U_j}(n,k)$ (resp. $b_{D_j}(n,k)$). The computation is done in

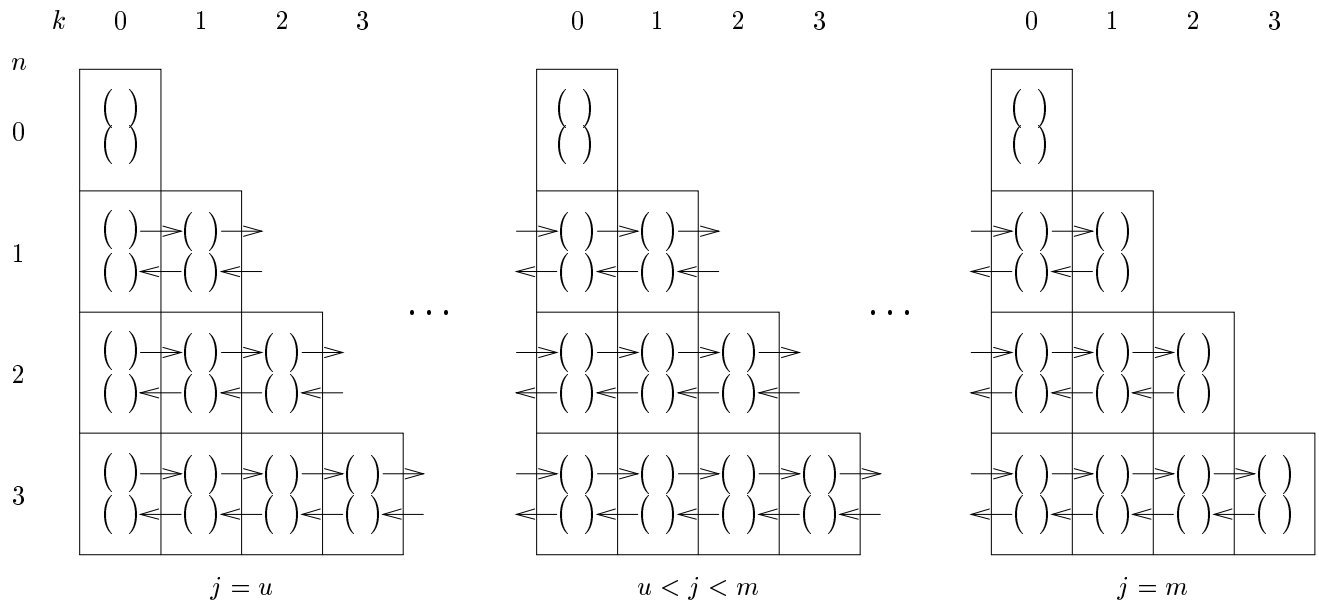

Figure 1: In cell (n,k) the vectors $b_{U_j}(n,k)$ and $b_{D_j}(n,k)$.

a line by line manner over all the triangles following the arrows in Fig. 1. Note that the upper part of the diagonal of each triangle of cells is reported in the upper part of the first column of the next one and the lower part of the first column each triangle of cells is reported in the lower part of the diagonal of the previous triangle of cells. The starting points are given, for $j = u, \ldots, m$ and for every $n \geq 0$, by

$$b_{U_j}(0,0) = \left(\alpha_{B_m}, \ldots, \alpha_{B_j}\right)^{\mathrm{T}}, \; b_{D_j}(0,0) = \left(0_{B_{j-1}}, \ldots, 0_{B_0}\right)^{\mathrm{T}},$$

and

$$b_{U_u}(n,0) = \left((\alpha P^n)_{B_m}, \ldots, (\alpha P^n)_{B_j}\right)^{\mathrm{T}}, \; b_{D_m}(n,n) = \left(0_{B_{m-1}}, \ldots, 0_{B_0}\right)^{\mathrm{T}}.$$

The way in which the computation of each cell (n,k) is performed is shown in Fig. 2. It is now easy to evaluate the complexity of this method. The computation of one cell consists essentially in a vector matrix product. If d denotes the maximum number of nonzero entries in each column of the matrix P, the computational complexity of a cell is $O(d|S|)$. There are $m-u+1$ triangles each containing $(N+1)(N+2)/2$ cells. The computational complexity of our method is then $O(d|S|(m-u+1)N^2/2)$. We see from Fig. 2 and Fig. 1 that it is sufficient to store 2 rows of cells in order to compute the $b_i^{(j)}(n,k)$. Thus the storage complexity of our method is $O((m-u+1)N|S|)$.

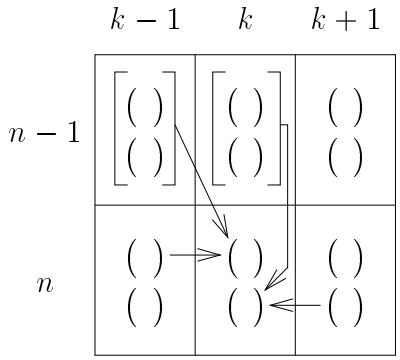

Figure 2: Computation of cell (n, k).

3 Stationarity Detection

We empirically show in this section that the algorithm described above can be stopped when the stationary behavior of the model is detected.

Let us denote by π the stationary distribution of the Markov process X. We suppose that the stability condition is satisfied, that is

$$\rho = \frac{\sum_{i \in S} \rho_i \pi_i}{\sum_{i \in S} c_i \pi_i} < 1,$$

where ρ is the traffic intensity, so that the limiting behavior exists. We also suppose for sake of simplicity that for every $i \in S$, we have $\rho_i \neq c_i$.

With these assumptions, we have for every $j = u, \ldots, m$,

$$\lim_{t \longrightarrow \infty} \Pr\{Q_t > 0\} = \rho \text{ and } \lim_{t \longrightarrow \infty} \Pr\{Q_t > r_j t\} = 0.$$

From Relation (2), we have for every $j = u, \ldots, m$, and $x \in [r_{j-1}^+ t, r_j t)$

$$\Pr\{Q_t > x\} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1 - x_j)^{n-k} b^{(j)}(n, k),$$

where x_j is as in Theorem 2.1 and

$$b^{(j)}(n, k) = \sum_{i \in S} b_i^{(j)}(n, k).$$

The following theorem gives an upper bound of the $b_i^{(j)}(n, k)$. If v and w are two vectors having the same dimension, the notation $v \leq w$ means that the inequality stands for each of their entry, that is, $v_i \leq w_i$ for every i.

Theorem 3.1 *For every $n \geq 0$, for every $j = u, \ldots, m$, for every $0 \leq k \leq n$ and for every $0 \leq l \leq m$, we have*

$$b_{B_l}^{(j)}(n,k) \leq (\alpha P^n)_{B_l}. \tag{3}$$

Proof. See Appendix B.

Using this theorem, we easily verify that $b^{(j)}(n,k) \leq 1$.

Theorem 3.2 *For every $n \geq 1$, for every $j = u, \ldots, m$ and for every $1 \leq k \leq n$, we have*

$$b_{B_l}^{(j)}(n,0) \leq b_{B_l}^{(j-1)}(n,n) \text{ for } j > u \tag{4}$$

$$b_{B_l}^{(j)}(n,k) \leq b_{B_l}^{(j)}(n,k-1) \tag{5}$$

Proof. See Appendix C.

This theorem shows that for fixed n and $i \in S$, the sequences $b_i^{(j)}(n,k)$ and $b^{(j)}(n,k)$ are wide-sense decreasing in both j and k. It follows that $b^{(j)}(n,k)$ can be interpreted as the complementary distribution function of a discrete random variable which is the discrete version of Q_t. For this discrete random variable, the integer n represents the number of transitions over the interval $(0,t)$ for the uniformized Markov chain of X. Thus the limits $\lim_{n \longrightarrow \infty} b^{(j)}(n,k)$ exist and in this case we must have necessarily

$$\lim_{n \longrightarrow \infty} b^{(u)}(n,0) = \rho, \lim_{n \longrightarrow \infty} b^{(u)}(n,n) = 0 \text{ and } \lim_{n \longrightarrow \infty} b^{(j)}(n,0) = 0 \text{ for } j = u+1, \ldots, m.$$

The stationarity detection consists in stopping the computation of the $b_i^{(j)}(n,k)$ when the values $|b^{(u)}(n,0) - \rho|$, $b^{(u)}(n,n)$ and $b^{(j)}(n,0)$, $j > u$, are sufficiently small. This can be done as follows

Consider the integer N defined in the previous section. We define the integer N_u as

$$N_u = \min\{n \mid 1 \leq n < N \text{ and } |b^{(u)}(n,0) - \rho| \leq \varepsilon/3 \text{ and } b^{(u)}(n,n) \leq \varepsilon/3\}$$

and for $j = u+1, \ldots, m$, we define the integers N_j as

$$N_j = \min\{n \mid 1 \leq n < N \text{ and } b^{(j)}(n,0) \leq \varepsilon\}.$$

When N_j does not exist, we set $N_j = N$. If all the N_j are equal to N, we obtain the exact solution described in the previous section.

The approximation made here consists in considering that for $n \geq N_u$ and for every $k = 0, \ldots, n$, we have $|b^{(u)}(n,k) - \lim_{n \longrightarrow \infty} b^{(u)}(n,k)| \leq \varepsilon/3$ and that for every $j = u+1, \ldots, m$ and for $n \geq N_j$, we have $b^{(j)}(n,0) \leq \varepsilon$.

In practise, we often observe that for $n \geq N_j$, the sequence $b^{(j)}(n,k)$ are wide-sense monotone, so the approximation is justified.

From theorem 3.2, we can easily check that we have

$$N_m \leq N_{m-1} \leq \cdots \leq N_u,$$

so, when, for $j > u+1$, the integer N_j is reached, we stop the computation over triangle j (see Fig. 1) and we set $b_{D_{j-1}}(N_j+1, N_j+1) = 0$. The computation then continues over triangles $u, u+1, \ldots, j-1$.

We then have for $j > u+1$ and $x \in [r_{j-1}t, r_j t)$,

$$\Pr\{Q_t > x\} = \sum_{n=0}^{N_j} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} b^{(j)}(n,k) + e(N_j),$$

where the rest of series $e(N_j)$ satisfies under the approximation hypothesis

$$\begin{aligned} e(N_j) &= \sum_{n=N_j+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} b^{(j)}(n,k) \\ &\leq \sum_{n=N_j+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} b^{(j)}(n,0) \\ &\leq \varepsilon \sum_{n=N_j+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &\leq \varepsilon. \end{aligned}$$

For $j = u$ and $x \in [0, r_u t)$, we get

$$\begin{aligned} \Pr\{Q_t > x\} &= \sum_{n=0}^{N_u} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_u^k (1-x_u)^{n-k} b^{(u)}(n,k) \\ &+ \sum_{n=N_u+1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{N_u} \binom{n}{k} x_u^k (1-x_u)^{n-k} b^{(u)}(N_u,k) + e(N_u). \end{aligned}$$

The second sum which is infinite can be easily expressed as a finite one; we then obtain

$$\begin{aligned} \Pr\{Q_t > x\} &= \sum_{n=0}^{N_u} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_u^k (1-x_u)^{n-k} b^{(u)}(n,k) \\ &+ \sum_{k=0}^{N_u} e^{-\lambda t x_u} \frac{(\lambda t x_u)^k}{k!} b^{(u)}(N_u,k) \left[1 - \sum_{n=0}^{N_u-k} e^{-\lambda t(1-x_u)} \frac{(\lambda t(1-x_u))^n}{n!} \right] + e(N_u), \end{aligned}$$

where the rest of series $e(N_u)$ verifies

$$\begin{aligned} e(N_u) &= \sum_{n=N_u+1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{N_u}\binom{n}{k} x_u^k(1-x_u)^{n-k}(b^{(u)}(n,k)-b^{(u)}(N_u,k)) \\ &+ \sum_{n=N_u+1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=N_u+1}^{n}\binom{n}{k} x_u^k(1-x_u)^{n-k}b^{(u)}(n,k). \end{aligned}$$

Under our approximation hypothesis, we get for $n \geq N_u$: $|b^{(u)}(n,k)-b^{(u)}(N_u,k)| \leq 2\varepsilon/3$ for $k \leq N_u$ and $b^{(u)}(n,k) \leq \varepsilon/3$ for $k \geq N_u+1$, so finally we obtain $e(N_u) \leq \varepsilon$.

The complexity of this approximation is now a function of the truncation integers N_i. The number of cells that must be computed in triangles i is equal to $(N_i+1)(N_i+2)/2$. So as for the exact algorithm, we easily obtain the computational complexity of the approximation which is $O(d|S|\sum_{i=u}^{m-u+1} N_i^2/2)$. By comparing the computational complexities of the exact and of the approximation method, we see that, the approximation, if sufficiently accurate, must be used for large values of m. We will see in the next section that the values of the N_i can be very small with respect to N with a very high accuracy for the results obtained by the approximation.

4 Numerical Examples

We present here some numerical results to illustrate our new solution technique and the approximation based on the stationarity detection.

We consider m statistically independent and identical on-off sources. For each source, we assume that the on periods and the off periods form an alternating renewal process and their durations are exponentially distributed with mean β^{-1} and γ^{-1} respectively. When a source is in the state on, it generates packets (or cells in the ATM terminology) at rate θ. We denote by C the multiplexer's output link capacity. Let X_s be the number of sources in the state on at time s. The process $X = \{X_s, s \geq 0\}$ is then a homogeneous Markov process over the state space $S = \{0, 1, \ldots, m\}$. Its infinitesimal generator A is a tridiagonal matrix whose entries are $A(i,i-1) = i\beta$ for $i = 1, \ldots, m$, $A(i,i+1) = (m-i)\gamma$ for $i = 0, \ldots, m-1$, and so $A(i,i) = -i\beta - (m-i)\gamma$ for $i = 0, \ldots, m$. For each $i \in S$, we have $\rho_i = i\theta$ and $c_i = C$. The traffic intensity ρ is then

$$\rho = \frac{m\theta\gamma}{C(\beta+\gamma)}.$$

We fix $\theta = 1$, $\beta = 1$ and $C = 0.8$. This gives $u = 1$ and so the number of triangles that we have to consider is equal to m. We consider various values of the number m of sources and of the off rate γ or of the traffic intensity ρ. The error tolerance is fixed to $\varepsilon = 10^{-5}$. The figures

3 to 6 have been obtained using the exact algorithm and figure 7 has been obtained using the approximation method detecting the stationary behavior of the model.

Figure 3 shows the complementary distribution of the buffer content at time t for various values of t. There are 2 input sources, the traffic intensity is $\rho = 5/6$ and both sources are initially in the off state. It can be noted that both distributions for $t = 100$ and $t = 200$ are very near from each other, which means that the stationary regime seems to be reached.

Figure 4 shows the complementary emptiness function $\Pr(Q_t > 0)$ for 2, 5 and 10 sources when the traffic intensity is $\rho = 5/6$ and all the input sources are initially in the off state. It can also be noted the convergence of the curves to the traffic intensity ρ.

Figure 5 is particularly interesting from a numerical point of view. The value of the time is fixed to $t = 1$, the number of input sources is $m = 2$ and the traffic intensity is $\rho = 5/6$. This figure shows the complementary distribution of Q_1 for different initial probability distributions, which correspond to the case where all the input sources are off, i.e. $X_0 = 0$, the case where the input sources are in stationary regime, i.e. the distribution of X_0 is π, and the case where all the input sources are on, i.e. $X_0 = 2$. When $X_0 = 0$, the distribution has only one jump at point 0 and it is not differentiable at point $x = r_1 t = 0.2$. When the distribution of X_0 is π, we observe the three discontinuities at points $x = 0$, $x = r_1 t = 0.2$ and $x = r_2 t = 1.2$. These two last discontinuities are easy to determine. For instance, we have $\Pr\{Q_1 = 0.2\} = \pi_1 e^{-(\beta+\gamma)t} = 4e^{-1.5}/9$. The computation of $\Pr\{Q_1 > 0.2 - 10^{-16}\} - \Pr\{Q_1 > 0.2\}$ using our algorithm, gives exactly this result, the precision obtained is greater than 10^{-10}. The same observation holds at point $x = 1.2$. Finally when $X_0 = 2$, we observe the two jumps at points $x = 0$ and $x = r_2 t = 1.2$. As before, it is easy to check that, at point $x = r_2 t = 1.2$, the result obtained using our algorithm is highly accurate. We also observe that the distribution is not differentiable at point $x = r_1 t = 0.2$.

Figure 6 shows the complementary distribution of Q_{100} for various values of the traffic intensity ρ, including values greater than 1. The number of input sources is $m = 2$ and both sources are initially off. For instance, we have $\Pr\{Q_{100} > 45\} = 0.0001$ for $\rho = 1.25$.

Figure 7 shows the complementary distribution of Q_t for various values of t. The traffic intensity is $\rho = 5/6$, the number of input sources is $m = 50$ and all the sources are initially off. This figure has been obtained by using the approximation method based on the stationarity detection. For $t = 10$, we obtained for the different truncation steps $N = 598$, $N_i = 51 - i$ for $i = 5, \ldots, 50$, $N_4 = 374$ and $N_3 = N_2 = N_1 = N$. This shows the important gain in computational complexity obtained by the approximation method. To evaluate the accuracy of the approximation method, we have executed the exact algorithm with the same input parameters. We have observed that the greatest difference between the results of the two algorithms is equal to 2.2×10^{-6}. This shows that our approximation method is highly accurate even for small values of t. For $t > 10$ we obtain, as expected, an accuracy still higher than for $t = 10$.

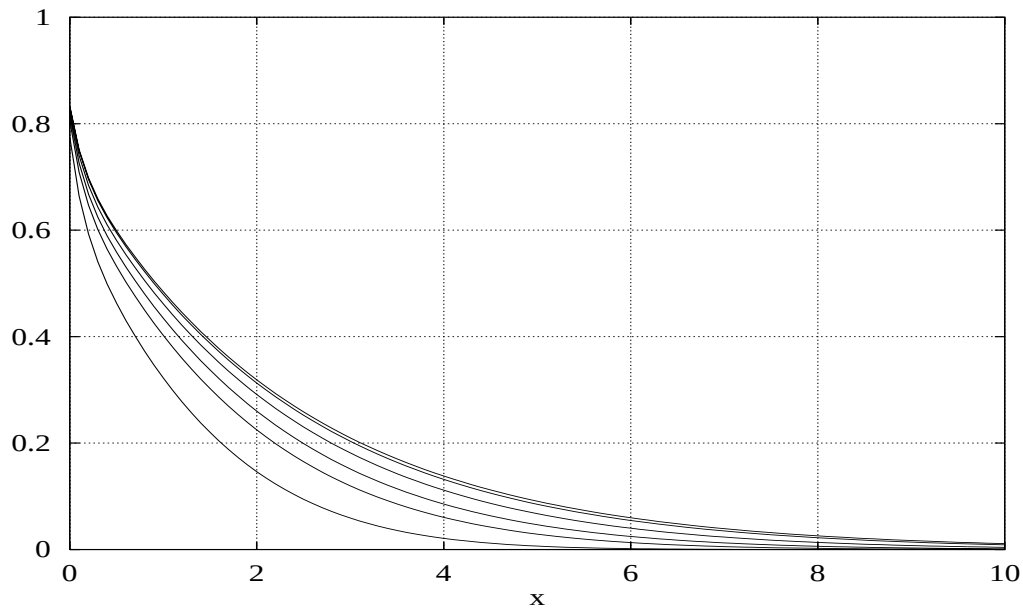

Figure 3: From bottom to the top, $\Pr\{Q_t > x\}$ versus x for $t = 10, 20, 30, 50, 100, 200$, $X_0 = 0$, $m = 2$ and $\rho = 5/6$

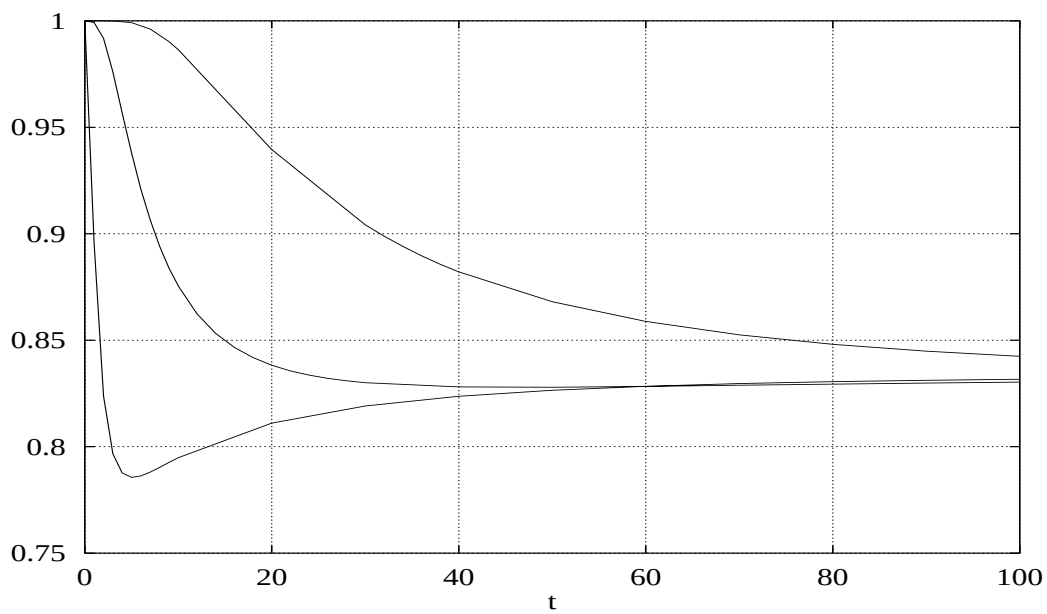

Figure 4: From bottom to the top, $\Pr\{Q_t > 0\}$ versus t for $m = 2, 5, 10$, $X_0 = m$ and $\rho = 5/6$

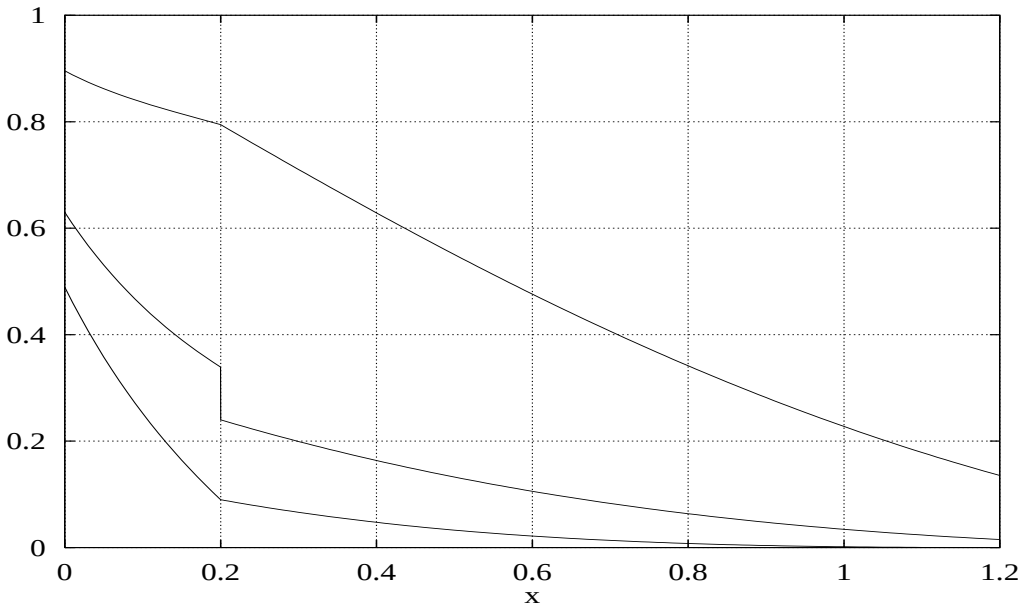

Figure 5: From bottom to the top, $\Pr\{Q_1 > x\}$ versus x for $X_0 = 0$, $X_0 \sim \pi$ and $x_0 = 2$ when $m = 2$ and $\rho = 5/6$

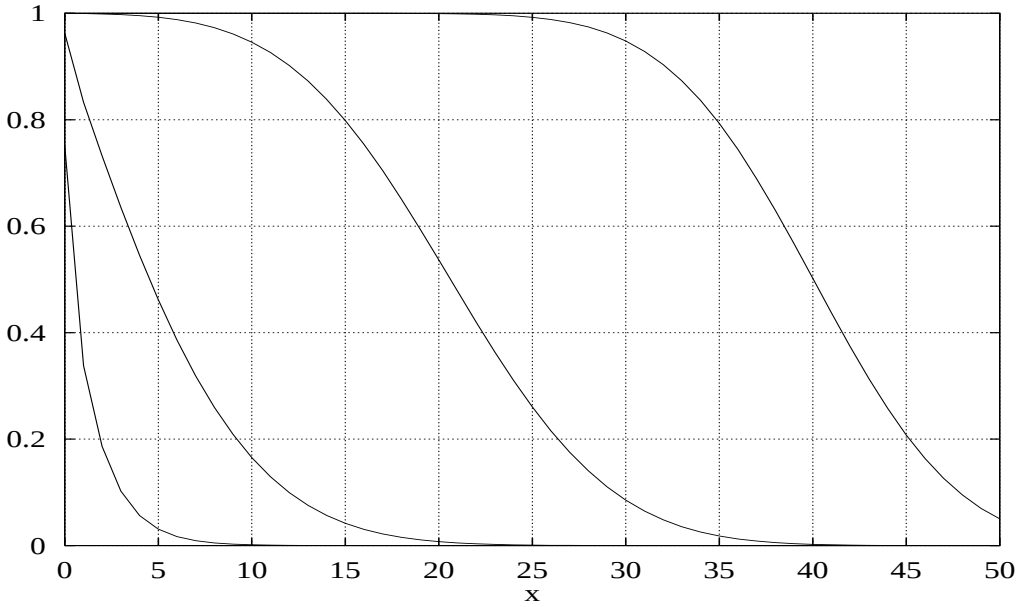

Figure 6: From bottom to the top, $\Pr\{Q_{100} > x\}$ versus x for $\rho = 0.75, 1, 1.25, 1.5$, $m = 2$ and $X_0 = 0$

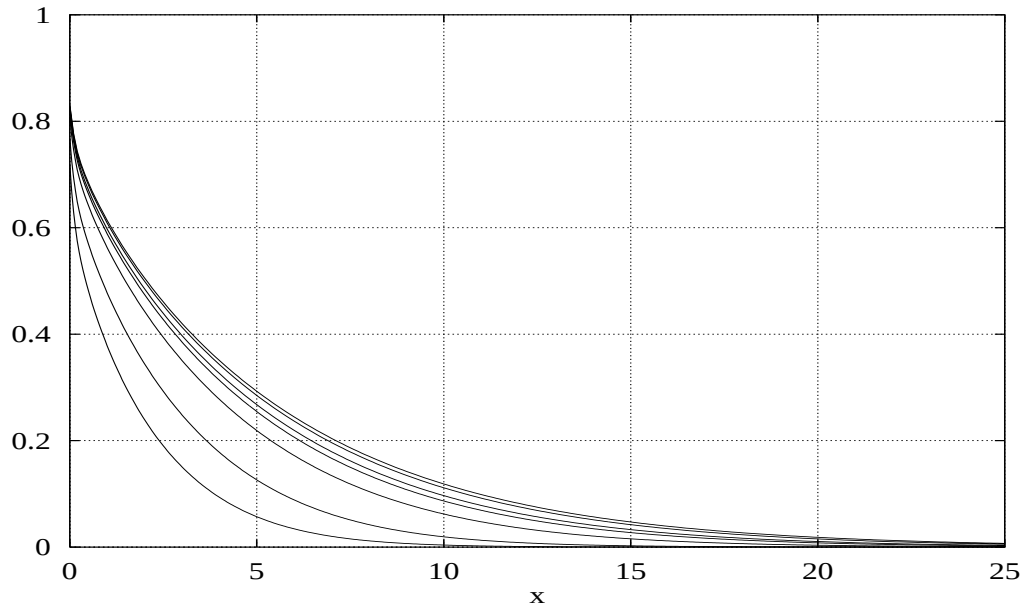

Figure 7: From bottom to the top, $\Pr\{Q_t > x\}$ versus x for $t = 10, 20, 50, 80, 100, 150, 200$ when $X_0 = 0$, $m = 50$ and $\rho = 5/6$

5 Conclusion

We developed a new transient solution of a fluid model with an input and output controlled by a homogeneous Markov chain. Our solution do not make use of any transform, as done in previous works. It is based on simple recurrence relations which are particularly interesting for their numerical properties. The algorithm implementing this solution is very accurate since it uses essentially non negative numbers bounded by one and it gives results with an error tolerance that can be specified in advance. We also develop an approximation method based on the detection of the stationary regime of the model. It has been shown though numerical examples that, as the exact method, this approximation method is highly accurate. Moreover its computational time can be very low with respect to the exact method.

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Appendix A. Proof of Theorem 2.1

For $t > 0$ and $x \in (r_{j-1}^+ t, r_j t)$, for $j = u, u+1, \ldots, m$, we write the solution of equation (1) for every $i \in S$, as

$$F_i(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} b_i^{(j)}(n,k),$$

and we determine the relations that must be satisfied by the coefficients $b_i^{(j)}(n,k)$. We have

$$\frac{\partial F_i(t,x)}{\partial t} = -\lambda F_i(t,x)$$
$$+ \frac{\lambda}{r_j - r_{j-1}^+} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} [r_j b_i^{(j)}(n+1,k) - r_{j-1}^+ b_i^{(j)}(n+1,k+1)],$$

and

$$\frac{\partial F_i(t,x)}{\partial x} = \frac{\lambda}{r_j - r_{j-1}^+} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} [b_i^{(j)}(n+1,k+1) - b_i^{(j)}(n+1,k)].$$

Using the uniformization technique, we have

$$\sum_{r \in S} F_r(t,x) A(r,i) = -\lambda F_i(t,x) + \lambda \sum_{r \in S} F_r(t,x) P(r,i),$$

that is,

$$\sum_{r \in S} F_r(t,x) A(r,i) = -\lambda F_i(t,x) + \lambda \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x_j^k (1-x_j)^{n-k} \sum_{r \in S} b_r^{(j)}(n,k) P(r,i).$$

It follows that if the $b_i^{(j)}(n,k)$ are such that

$$(d_i - r_{j-1}^+) b_i^{(j)}(n+1,k+1) + (r_j - d_i) b_i^{(j)}(n+1,k) = (r_j - r_{j-1}^+) \sum_{r \in S} b_r^{(j)}(n,k) P(r,i) \qquad (6)$$

then equation (1) is satisfied.
The recurrence relation (6) can also be written as follows, for $j = u, \ldots, m$.
For $i \in B_0 \cup \cdots \cup B_{j-1}$,

$$b_i^{(j)}(n,k) = \frac{r_{j-1}^+ - d_i}{r_j - d_i} b_i^{(j)}(n,k+1) + \frac{r_j - r_{j-1}^+}{r_j - d_i} \sum_{r \in S} b_r^{(j)}(n-1,k) P(r,i)$$

and for $i \in B_j \cup \cdots \cup B_m$,

$$b_i^{(j)}(n,k) = \frac{d_i - r_j}{d_i - r_{j-1}^+} b_i^{(j)}(n,k-1) + \frac{r_j - r_{j-1}^+}{d_i - r_{j-1}^+} \sum_{r \in S} b_r^{(j)}(n-1,k-1) P(r,i).$$

Using matrix and vector notation, we get for $j = u, \ldots, m$

$$b_{B_l}^{(j)}(n,k) = \frac{r_l - r_j}{r_l - r_{j-1}^+} b_{B_l}^{(j)}(n,k-1) + \frac{r_j - r_{j-1}^+}{r_l - r_{j-1}^+} \sum_{i=0}^{m} b_{B_i}^{(j)}(n-1,k-1) P_{B_i B_l} \text{ for } 0 \le l \le j-1$$

$$b_{B_l}^{(j)}(n,k) = \frac{r_{j-1}^+ - r_l}{r_j - r_l} b_{B_l}^{(j)}(n,k+1) + \frac{r_j - r_{j-1}^+}{r_j - r_l} \sum_{i=0}^{m} b_{B_i}^{(j)}(n-1,k) P_{B_i B_l} \text{ for } j \le l \le m.$$

To get the initial conditions for the $b_i^{(u)}(n,k)$, we consider the jumps of $F_i(t,x)$.
For $t > 0$ and $i \in B_u \cup \cdots \cup B_m$, we have

$$F_i(t,0) = \Pr\{X_t = i\} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} (\alpha P^n)(i).$$

It follows that

$$b_i^{(u)}(n,0) = (\alpha P^n)(i),$$

that is

$$b_{B_l}^{(u)}(n,0) = (\alpha P^n)_{B_l} \text{ for } u \le l \le m.$$

For $t > 0$ and $u \le j \le m-1$ and $i \notin B_j$, we have

$$F_i(t, r_j t) = \lim_{x \xrightarrow{<} r_j t} F_i(t,x),$$

since $i \notin B_j$ means that there is no jump at point $x = r_j t$. It follows that

$$b_i^{(j+1)}(n,0) = b_i^{(j)}(n,n) \text{ if } i \notin B_j,$$

That is,

$$b_{B_l}^{(j+1)}(n,0) = b_{B_l}^{(j)}(n,n) \text{ for } l \ne j.$$

This can also be written as

$$b_{B_l}^{(j)}(n,0) = b_{B_l}^{(j-1)}(n,n) \text{ for } u < j \leq l \leq m$$

$$b_{B_l}^{(j)}(n,n) = b_{B_l}^{(j+1)}(n,0) \text{ for } 0 \leq l \leq j-1 < m-1.$$

Finally, for $t > 0$ and $i \notin B_m$, we have

$$0 = F_i(t, r_m t) = \lim_{x \xrightarrow{<} r_m t} F_i(t,x).$$

It follows that $b_i^{(m)}(n,n) = 0$, that is

$$b_{B_l}^{(m)}(n,n) = 0 \text{ for } 0 \leq l \leq m-1.$$

The proof is now complete.

Appendix B. Proof of Theorem 3.1

The proof is made by successive inductions using the relations described in Theorem 2.1.

Step 0. For $n = 0$ and for every $j = u, \ldots, m$, we have

$$\begin{aligned} b_{B_l}^{(j)}(0,0) &= 0_{B_l} \text{ for } 0 \leq l \leq j-1 \\ b_{B_l}^{(j)}(0,0) &= \alpha_{B_l} \text{ for } j \leq l \leq m. \end{aligned}$$

So the relation (3) is satisfied for $n = 0$.

Step 1. Suppose the relation (3) is satisfied for integer $n-1$ and let us prove it is true for integer $n \geq 1$. Let $u \leq j \leq m$.

Step 1.1. We first consider the case where $0 \leq l \leq j-1$.

- For $j = m$ we have from Theorem 2.1,
  - ⋆ $b_{B_l}^{(m)}(n,n) = 0_{B_l} \leq (\alpha P^n)_{B_l}$.
  - ⋆ Suppose that $b_{B_l}^{(m)}(n,k+1) \leq (\alpha P^n)_{B_l}$ for integer $k \leq n-1$. Then,

$$\begin{aligned} b_{B_l}^{(m)}(n,k) &= \frac{r_{m-1}^+ - r_l}{r_m - r_l} b_{B_l}^{(m)}(n,k+1) + \frac{r_m - r_{m-1}^+}{r_m - r_l} \sum_{i=0}^{m} b_{B_i}^{(m)}(n-1,k) P_{B_i B_l} \\ &\leq \frac{r_{m-1}^+ - r_l}{r_m - r_l} (\alpha P^n)_{B_l} + \frac{r_m - r_{m-1}^+}{r_m - r_l} \sum_{i=0}^{m} (\alpha P^{n-1})_{B_i} P_{B_i B_l} \\ &= \frac{r_{m-1}^+ - r_l}{r_m - r_l} (\alpha P^n)_{B_l} + \frac{r_m - r_{m-1}^+}{r_m - r_l} (\alpha P^n)_{B_l} \\ &= (\alpha P^n)_{B_l}. \end{aligned}$$

So the relation is satisfied for $n \geq 0$, for $j = m$, for $0 \leq k \leq n$, and for $0 \leq l \leq m-1$.

- Suppose now that the relation is satisfied for integer $j+1$, $j \leq m-1$.

  Using Theorem 2.1, we have

  ⋆ $b_{B_l}^{(j)}(n,n) = b_{B_l}^{(j+1)}(n,0) \leq (\alpha P^n)_{B_l}$.

  ⋆ Suppose that $b_{B_l}^{(j)}(n,k+1) \leq (\alpha P^n)_{B_l}$ for integer $k \leq n-1$. Then,

$$\begin{aligned} b_{B_l}^{(j)}(n,k) &= \frac{r_{j-1}^+ - r_l}{r_j - r_l} b_{B_l}^{(j)}(n,k+1) + \frac{r_j - r_{j-1}^+}{r_j - r_l} \sum_{i=0}^{m} b_{B_i}^{(j)}(n-1,k) P_{B_i B_l} \\ &\leq \frac{r_{j-1}^+ - r_l}{r_j - r_l} (\alpha P^n)_{B_l} + \frac{r_j - r_{j-1}^+}{r_j - r_l} \sum_{i=0}^{m} (\alpha P^{n-1})_{B_i} P_{B_i B_l} \\ &= \frac{r_{j-1}^+ - r_l}{r_j - r_l} (\alpha P^n)_{B_l} + \frac{r_j - r_{j-1}^+}{r_j - r_l} (\alpha P^n)_{B_l} \\ &= (\alpha P^n)_{B_l}. \end{aligned}$$

  So the relation is satisfied for $n \geq 0$, for $j = u, \ldots, m$, for $0 \leq k \leq n$, and for $0 \leq l \leq j-1$.

Step 1.2. In the same way, we now consider the case where $j \leq l \leq m$.

- For $j = u$ we have from Theorem 2.1,

  ⋆ $b_{B_l}^{(u)}(n,0) = (\alpha P^n)_{B_l}$.

  ⋆ Suppose that $b_{B_l}^{(m)}(n,k-1) \leq (\alpha P^n)_{B_l}$ for integer $k \geq 1$. Then,

$$\begin{aligned} b_{B_l}^{(u)}(n,k) &= \frac{r_l - r_u}{r_l} b_{B_l}^{(u)}(n,k-1) + \frac{r_u}{r_l} \sum_{i=0}^{m} b_{B_i}^{(u)}(n-1,k-1) P_{B_i B_l} \\ &\leq \frac{r_l - r_u}{r_l} (\alpha P^n)_{B_l} + \frac{r_u}{r_l} \sum_{i=0}^{m} (\alpha P^{n-1})_{B_i} P_{B_i B_l} \\ &= \frac{r_l - r_u}{r_l} (\alpha P^n)_{B_l} + \frac{r_u}{r_l} (\alpha P^n)_{B_l} \\ &= (\alpha P^n)_{B_l}. \end{aligned}$$

  So the relation is satisfied for $n \geq 0$, for $j = u$, for $0 \leq k \leq n$, and for $u \leq l \leq m$.

- Suppose now that the relation is satisfied for integer $j-1$, $j \geq u+1$.

  Using Theorem 2.1, we have

- $\star$ $b^{(j)}_{B_l}(n,0) = b^{(j-1)}_{B_l}(n,n) \leq (\alpha P^n)_{B_l}$.
- $\star$ Suppose that $b^{(j)}_{B_l}(n,k-1) \leq (\alpha P^n)_{B_l}$ for integer $k \geq 1$. Then,

$$\begin{aligned} b^{(j)}_{B_l}(n,k) &= \frac{r_l - r_j}{r_l - r_{j-1}} b^{(j)}_{B_l}(n,k-1) + \frac{r_j - r_{j-1}}{r_l - r_{j-1}} \sum_{i=0}^{m} b^{(j)}_{B_i}(n-1,k-1) P_{B_i B_l} \\ &\leq \frac{r_l - r_j}{r_l - r_{j-1}} (\alpha P^n)_{B_l} + \frac{r_j - r_{j-1}}{r_l - r_{j-1}} \sum_{i=0}^{m} (\alpha P^{n-1})_{B_i} P_{B_i B_l} \\ &= \frac{r_l - r_j}{r_l - r_{j-1}} (\alpha P^n)_{B_l} + \frac{r_j - r_{j-1}}{r_l - r_{j-1}} (\alpha P^n)_{B_l} \\ &= (\alpha P^n)_{B_l}. \end{aligned}$$

So the relation is satisfied for $n \geq 0$, for $j = u, \ldots, m$, for $0 \leq k \leq n$, and for $j \leq l \leq m$, which completes the proof.

Appendix C. Proof of Theorem 3.2

The proof of relation (4) is immediate since, for $j > u$, we have

$$b^{(j)}_{B_l}(n,0) = b^{(j-1)}_{B_l}(n,n) - \alpha_{B_{j-1}} P^n_{B_{j-1}B_{j-1}} \mathbf{1}_{B_{j-1}} 1_{\{l=j-1\}},$$

where $1_{\{c\}} = 1$ if condition c is satisfied and 0 otherwise. The proof of relation (5) is made by successive inductions using the relations described in Theorem 2.1. Note that from Theorem 2.1, $b^{(j)}_{B_l}(n,k)$ is a convex combination of two terms. It follows that

$$b^{(j)}_{B_l}(n,k) \leq b^{(j)}_{B_l}(n,k-1) \Longleftrightarrow \sum_{i=0}^{m} b^{(j)}_{B_i}(n-1,k-1) P_{B_i B_l} \leq b^{(j)}_{B_l}(n,k) \leq b^{(j)}_{B_l}(n,k-1) \text{ for } j \leq l \leq m$$

and

$$b^{(j)}_{B_l}(n,k) \geq b^{(j)}_{B_l}(n,k+1) \Longleftrightarrow b^{(j)}_{B_l}(n,k+1) \leq b^{(j)}_{B_l}(n,k) \leq \sum_{i=0}^{m} b^{(j)}_{B_i}(n-1,k) P_{B_i B_l} \text{ for } 0 \leq l \leq j-1.$$

Step 0. We prove the relation for $n = 1$. For $n = 1$, $j = u$ and $u \leq l \leq m$ we have from Theorem 3.1

$$b^{(u)}_{B_l}(1,1) \leq (\alpha P)_{B_l} = b^{(u)}_{B_l}(1,0).$$

Suppose that the relation is satisfied at level $j-1$, $j<m$. Then for $j \le l \le m$, we have from Theorem 2.1 and using the equivalence above

$$\begin{aligned} b_{B_l}^{(j)}(1,0) - b_{B_l}^{(j)}(1,1) &= \frac{r_j - r_{j-1}^+}{r_l - r_{j-1}^+}\left[b_{B_l}^{(j)}(1,0) - \sum_{i=0}^{m} b_{B_i}^{(j)}(0,0)P_{B_iB_l}\right] \\ &\ge \frac{r_j - r_{j-1}^+}{r_l - r_{j-1}^+}\left[b_{B_l}^{(j-1)}(1,1) - \sum_{i=0}^{m} b_{B_i}^{(j-1)}(0,0)P_{B_iB_l}\right] \ge 0 \end{aligned}$$

So the relation is satisfied for $n=1$ and $j \le l \le m$.

For $n=1$, $j=m$ and $0 \le l \le m-1$ we have

$$b_{B_l}^{(m)}(1,1) = 0 \le b_{B_l}^{(m)}(1,0).$$

Suppose that the relation is satisfied at level $j+1$, $j>u$. Then for $0 \le l \le j-1$, we have from Theorem 2.1 and using the equivalence above

$$\begin{aligned} b_{B_l}^{(j)}(1,0) - b_{B_l}^{(j)}(1,1) &= \frac{r_j - r_{j-1}}{r_j - r_l^+}\left[\sum_{i=0}^{m} b_{B_i}^{(j)}(0,0)P_{B_iB_l} - b_{B_l}^{(j)}(1,1)\right] \\ &\ge \frac{r_j - r_{j-1}}{r_j - r_l^+}\left[\sum_{i=0}^{m} b_{B_i}^{(j+1)}(0,0)P_{B_iB_l} - b_{B_l}^{(j+1)}(1,0)\right] \ge 0 \end{aligned}$$

So the relation is satisfied for $n=1$ and $0 \le l \le j-1$.

Step 1. Suppose the relation (5) is satisfied for integer $n-1$ and let us prove it is true for integer n, $n \ge 2$. Let $u \le j \le m$.

Step 1.1. We first consider the case where $0 \le l \le j-1$.

- For $j=m$ we have from Theorem 2.1,
  - ⋆ $b_{B_l}^{(m)}(n,n) = 0_{B_l} \le b_{B_l}^{(m)}(n,n-1)$.
  - ⋆ Suppose that $b_{B_l}^{(m)}(n,k+1) \ge b_{B_l}^{(m)}(n,k+2)$ for integer $k \le n-2$. Then,

$$\begin{aligned} b_{B_l}^{(m)}(n,k) - b_{B_l}^{(m)}(n,k+1) &= \frac{r_{m-1}^+ - r_l}{r_m - r_l}\left[b_{B_l}^{(m)}(n,k+1) - b_{B_l}^{(m)}(n,k+2)\right] \\ &+ \frac{r_m - r_{m-1}^+}{r_m - r_l}\sum_{i=0}^{m}[b_{B_i}^{(m)}(n-1,k) - b_{B_i}^{(m)}(n-1,k+1)]P_{B_iB_l}, \end{aligned}$$

  which shows that $b_{B_l}^{(m)}(n,k) - b_{B_l}^{(m)}(n,k+1) \ge 0$.

So the relation is satisfied for $n \ge 1$, for $j=m$, for $1 \le k \le n$, and for $0 \le l \le m-1$.

- Suppose now that the relation is satisfied at level $j+1$, $j \leq m-1$.

  ⋆ Using Theorem 2.1, we have

$$\begin{aligned} b_{B_l}^{(j)}(n,n-1) - b_{B_l}^{(j)}(n,n) &= \frac{r_j - r_{j-1}}{r_j - r_l^+}\left[\sum_{i=0}^{m} b_{B_i}^{(j)}(n-1,n-1) - b_{B_l}^{(j)}(n,n)\right] \\ &\geq \frac{r_j - r_{j-1}^+}{r_j - r_l}\left[\sum_{i=0}^{m} b_{B_i}^{(j+1)}(n-1,0) - b_{B_l}^{(j+1)}(n,0)\right] \geq 0 \end{aligned}$$

  ⋆ Suppose that $b_{B_l}^{(j)}(n,k+1) \geq b_{B_l}^{(j)}(n,k+2)$ for integer $k \leq n-2$. Then,

$$\begin{aligned} b_{B_l}^{(j)}(n,k) - b_{B_l}^{(j)}(n,k+1) &= \frac{r_{j-1}^+ - r_l}{r_j - r_l}[b_{B_l}^{(j)}(n,k+1) - b_{B_l}^{(j)}(n,k+2)] \\ &+ \frac{r_j - r_{j-1}^+}{r_j - r_l}\sum_{i=0}^{m}[b_{B_i}^{(j)}(n-1,k) - b_{B_i}^{(j)}(n-1,k+1)]P_{B_iB_l}, \end{aligned}$$

  which shows that $b_{B_l}^{(j)}(n,k) - b_{B_l}^{(j)}(n,k+1) \geq 0$.

  So the relation is satisfied for $n \geq 1$, for $j = u, \ldots, m$, for $1 \leq k \leq n$, and for $0 \leq l \leq j-1$.

Step 1.2. In the same way, we now consider the case where $j \leq l \leq m$.

- For $j = u$ we have from Theorem 3.2,

  ⋆ $b_{B_l}^{(u)}(n,0) = (\alpha P^n)_{B_l} \geq b_{B_l}^{(u)}(n,1)$.

  ⋆ Suppose that $b_{B_l}^{(u)}(n,k-2) - b_{B_l}^{(u)}(n,k-1) \geq 0$, for integer $k \geq 2$. Then,

$$\begin{aligned} b_{B_l}^{(u)}(n,k-1) - b_{B_l}^{(u)}(n,k) &= \frac{r_l - r_u}{r_l}[b_{B_l}^{(u)}(n,k-2) - b_{B_l}^{(u)}(n,k-1)] \\ &+ \frac{r_u}{r_l}\sum_{i=0}^{m}[b_{B_i}^{(u)}(n-1,k-2) - b_{B_i}^{(u)}(n-1,k-1)]P_{B_iB_l}, \end{aligned}$$

  which shows that $b_{B_l}^{(u)}(n,k-1) - b_{B_l}^{(u)}(n,k) \geq 0$.

  So the relation is satisfied for $n \geq 1$, for $j = u$, for $1 \leq k \leq n$, and for $u \leq l \leq m$.

- Suppose now that the relation is satisfied for integer $j-1$, $j \geq u+1$.

⋆ Using Theorem 2.1, we have

$$\begin{aligned} b_{B_l}^{(j)}(n,0) - b_{B_l}^{(j)}(n,1) &= \frac{r_j - r_{j-1}}{r_l - r_{j-1}}[b_{B_l}^{(j)}(n,0) - \sum_{i=0}^{m} b_{B_i}^{(j)}(n-1,0)P_{B_iB_l}] \\ &\geq \frac{r_j - r_{j-1}}{r_l - r_{j-1}}[b_{B_l}^{(j-1)}(n,n) - \sum_{i=0}^{m} b_{B_i}^{(j-1)}(n-1,n-1)P_{B_iB_l}] \geq 0 \end{aligned}$$

⋆ Suppose that $b_{B_l}^{(j)}(n,k-2) - b_{B_l}^{(j)}(n,k-1) \geq 0$, for integer $k \geq 2$. Then,

$$\begin{aligned} b_{B_l}^{(j)}(n,k-1) - b_{B_l}^{(j)}(n,k) &= \frac{r_l - r_j}{r_l - r_{j-1}^{+}}[b_{B_l}^{(j)}(n,k-2) - b_{B_l}^{(u)}(n,k-1)] \\ &+ \frac{r_j - r_{j-1}^{+}}{r_l - r_{j-1}^{+}} \sum_{i=0}^{m}[b_{B_i}^{(u)}(n-1,k-2) - b_{B_i}^{(u)}(n-1,k-1)]P_{B_iB_l}, \end{aligned}$$

which shows that $b_{B_l}^{(j)}(n,k-1) - b_{B_l}^{(j)}(n,k) \geq 0$.

So the relation is satisfied for $n \geq 1$, for $j = u, \ldots, m$, for $1 \leq k \leq n$, and for $j \leq l \leq m$, which completes the proof.