



**HAL**  
open science

## Homological properties of stratified spaces II

Martintxo Saralegi-Aranguren

► **To cite this version:**

| Martintxo Saralegi-Aranguren. Homological properties of stratified spaces II. 2004. hal-00001421v1

**HAL Id: hal-00001421**

**<https://hal.science/hal-00001421v1>**

Preprint submitted on 6 Apr 2004 (v1), last revised 21 Jul 2005 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Homological properties of stratified spaces II.

Martintxo Saralegi-Aranguren\*

Université d'Artois

## Abstract

For a stratified pseudomanifold, we have the de Rham Theorem

$$\mathbb{H}_{\bar{p}}^*(X) = \text{Hom} \left( \mathbb{H}_{*}^{\bar{t}-\bar{p}}(X), \mathbb{R} \right),$$

for a perversity  $\bar{0} \leq \bar{p} \leq \bar{t}$ , where  $\bar{t}$  denotes the top perversity. We extend this result to any perversity  $\bar{p}$ . In the sense cohomology  $\mapsto$  homology, we obtain the isomorphism

$$\mathbb{H}_{\bar{p}}^*(X) = \text{Hom} \left( \mathbb{H}_{*}^{\bar{t}-\bar{p}}(X, X_{\bar{p}}), \mathbb{R} \right),$$

where  $X_{\bar{p}} = \bigcup_{\bar{p}' < \bar{p}} \bar{S}$ . In the sense homology  $\mapsto$  cohomology, we obtain the isomorphism

$$\mathbb{H}_{*}^{\bar{p}}(X) = \text{Hom} \left( \mathbb{H}_{\max(\bar{0}, \bar{t}-\bar{p})}^*(X), \mathbb{R} \right).$$

In our context, the stratified pseudomanifolds with 1-codimensional strata are allowed.

The perverse de Rham Theorem  $\mathbb{H}_{\bar{p}}^*(X) = \text{Hom} \left( \mathbb{H}_{*}^{\bar{t}-\bar{p}}(X), \mathbb{R} \right)$  relates the (singular) intersection homology of [3] with the (de Rham) intersection cohomology of [1]. Here, the perversity  $\bar{p}$  lies between the zero perversity and the top perversity.

We find the first proof of this fact in [1]. The sheaf characterization of the intersection cohomology is used. The perversity  $\bar{p}$  is a filtration-depending perversity of Goresky-MacPherson [2], it verifies  $\bar{0} \leq \bar{p} \leq \bar{t}$ . The one codimensional strata are not allowed.

In [6], we give a geometrical proof by using the integration  $\int$  of differential forms on simplices, for a larger family of perversities. The perversity  $\bar{p}$  verifies  $\bar{0} \leq \bar{p} \leq \bar{t}$  but is a stratification-depending perversity. The one codimensional strata are not allowed.

In this work we extend this approach to any (stratification-depending) perversity  $\bar{p}$ . The one codimensional strata are allowed. We prove that, in the sense cohomology  $\mapsto$  homology, the integration  $\int$  induces the isomorphism

$$\mathbb{H}_{\bar{p}}^*(X) = \text{Hom} \left( \mathbb{H}_{*}^{\bar{t}-\bar{p}}(X, X_{\bar{p}}), \mathbb{R} \right),$$

where  $X_{\bar{p}} = \bigcup_{\bar{p}' < \bar{p}} \bar{S}$  (cf. Theorem 3.2.2). In the sense homology  $\mapsto$  cohomology, we have the isomorphism

$$\mathbb{H}_{*}^{\bar{p}}(X) = \text{Hom} \left( \mathbb{H}_{\max(\bar{0}, \bar{t}-\bar{p})}^*(X), \mathbb{R} \right)$$

(cf. Corollary 3.2.4).

We end the work by noticing that the Poincaré Duality of [1] and [6] is still true in our context.

---

\*UPRES-EA 2462 Géométrie-Algèbre. Faculté Jean Perrin. Université d'Artois. Rue Jean Souvraz SP 18. 62 307 Lens Cedex - France. [saralegi@euler.univ-artois.fr](mailto:saralegi@euler.univ-artois.fr).

*Key Words* : Intersection Homology, Stratifications.

*Classification AMS* : Primary : 55N33, Secondary : 57N80.

# 1 Stratified spaces and unfoldings.

We present the geometrical framework of this work, that is, the stratified pseudomanifolds and the unfoldings. For a more complete study of these notions, we refer the reader to, for example, [3] and [6].

In the sequel, any manifold will be considered connected, second countable, Hausdorff, without boundary and smooth (of class  $C^\infty$ ).

**1.1 Stratifications.** A *stratification* of a paracompact space  $X$  is a locally finite partition  $\mathcal{S}_X$  of  $X$  into disjoint smooth manifolds, called *strata*, such that

$$S \cap \overline{S'} \neq \emptyset \iff S \subset \overline{S'},$$

written  $S \preceq S'$ . Notice that  $(\mathcal{S}_X, \preceq)$  is a partially ordered set. A subset  $Z \subset X$  is *saturated* if it is the union of a family  $\mathcal{B} \subset \mathcal{S}_X$  of strata. This subset is closed when  $S \in \mathcal{B}$  and  $S' \preceq S$  imply  $S' \in \mathcal{B}$ .

We shall say that  $X$  is a *stratified space*. The *depth* of  $X$ , written  $\text{depth } X$ , is the length of the maximal chain contained in  $X$ . When  $X$  is compact, this length is finite. The minimal (resp. maximal) strata are the closed strata (resp. open) and viceversa. The open strata are the *regular strata* and the other ones are the *singular strata*. We shall write  $\mathcal{S}_X^{\text{sing}}$  the family of singular strata. The union  $\Sigma_X$  of singular strata is the *singular part*, which is a closed subset. The *regular part*  $X - \Sigma_X$  is an open dense subset. We ask the regular strata to have the same dimension, written  $\text{dim } X$ .

We shall consider, for each  $i \in \{-1, 0, \dots, \text{dim } X\}$ , the saturated subset:

$$X_i = \bigcup \{S \in \mathcal{S}_X \mid \text{dim } S \leq i\}.$$

This gives to *filtration*  $\mathcal{F}_X$ :

$$X_{\text{dim } X} \supset X_{\text{dim } X - 1} \supset X_1 \supset X_0 \supset X_{-1} = \emptyset.$$

The main example of stratified space is given by the following conical construction. Consider  $L$  a compact stratified space and write  $cL$  the *cone* of  $L$ , that is,  $cL = L \times [0, 1[ \big/ L \times \{0\}$ . The points of  $cL$  are denoted by  $[x, t]$ . The *vertex* of the cone is the point  $\vartheta = [x, 0]$ . This cone is naturally endowed with the following stratification:  $\mathcal{S}_{cL} = \{\{\vartheta\}\} \cup \{S \times ]0, 1[ \mid S \in \mathcal{S}_L\}$ . For the filtration  $\mathcal{F}_{cL}$  we have:  $(cL)_i = \begin{cases} \{\vartheta\} & \text{if } i = 0 \\ cL_{i-1} & \text{if } i > 0. \end{cases}$  Notice that  $\text{depth } cL = \text{depth } L + 1$ .

The *canonical stratification* of a manifold  $X$  is the family  $\mathcal{S}_X$  formed by the connected components of  $X$ . The filtration contains just one element:  $X_{\text{dim } X}$ .

A continuous map (resp. homeomorphism)  $f: Y \rightarrow X$  between two stratified spaces is a stratified morphism (resp. *isomorphism*) if it sends the strata of  $Y$  to the strata of  $X$  smoothly (resp. diffeomorphically).

**1.2 Stratified pseudomanifolds.** A stratified space  $X$  is a *stratified pseudomanifold* when it possesses a conical local structure. More explicitly, when for each point  $x$  of a singular stratum  $S$  of  $X$  there exists a stratified isomorphism  $\varphi: U \rightarrow \mathbb{R}^n \times cL_S$ , where

- (a)  $U \subset X$  is an open neighborhood of  $x$  endowed with the induced stratification,
- (b)  $L_S$  is a compact stratified space, called *link* of  $S$ ,

- (c)  $\mathbb{R}^n \times cL_S$  is endowed with the stratification  $\{\mathbb{R}^n \times \{\vartheta\}\} \cup \{\mathbb{R}^n \times S' \times ]0, 1[ / S' \in \mathcal{S}_{L_S}\}$ , and
- (d)  $\varphi(x) = (0, \vartheta)$ .

The couple  $(U, \varphi)$  is a *chart* of  $x$ . Un *atlas*  $\mathcal{A}$  is a family of charts covering  $X$ . The stratified pseudomanifold is *normal* when all the links are connected. Notice that in this case each link is a connected normal stratified pseudomanifold.

**1.3 Unfoldings.** Consider  $X$  a stratified pseudomanifold. A continuous map  $\mathcal{L}: \tilde{X} \rightarrow X$ , where  $\tilde{X}$  is a (not necessarily connected) manifold, is an *unfolding* if the two following conditions hold:

1. The restriction  $\mathcal{L}_X: \mathcal{L}_X^{-1}(X - \Sigma_X) \rightarrow X - \Sigma_X$  is a local diffeomorphism.
2. There exist a family of unfoldings  $\{\mathcal{L}_{L_S}: \tilde{L}_S \rightarrow L_S\}_{S \in \mathcal{S}_X^{sing}}$  and an atlas  $\mathcal{A}$  of  $X$  such that for each chart  $(U, \varphi) \in \mathcal{A}$  there exists a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n \times \tilde{L}_S \times ]-1, 1[ & \xrightarrow{\tilde{\varphi}} & \mathcal{L}_X^{-1}(U) \\ Q \downarrow & & \mathcal{L}_X \downarrow \\ \mathbb{R}^n \times cL_S & \xrightarrow{\varphi} & U \end{array}$$

where

- (a)  $\tilde{\varphi}$  is a diffeomorphism and
- (b)  $Q(x, \tilde{\zeta}, t) = \left(x, \left[\mathcal{L}_{L_S}(\tilde{\zeta}), |t|\right]\right)$ .

We shall say that  $X$  is an *unfoldable pseudomanifold*. This definition makes sense because it is made by induction on depth  $X$ . When depth  $X = 0$  then  $\mathcal{L}_X$  is just a local diffeomorphism. The restriction  $\mathcal{L}_X: \mathcal{L}_X^{-1}(S) \rightarrow S$  is a fibration with  $\tilde{L}_S$  as a fiber, for any singular stratum  $S$ . The *canonical unfolding* of the cone  $L_S$  is the map  $\mathcal{L}_{cL_S}: \tilde{cL}_S = \tilde{L}_S \times ]-1, 1[ \rightarrow cL_S$  defined by  $\mathcal{L}_{cL_S}(\tilde{\zeta}, t) = \left[\mathcal{L}_{L_S}(\tilde{\zeta}), |t|\right]$ .

From now on,  $(X, \mathcal{S}_X)$  is a stratified pseudomanifold endowed with an unfolding  $\mathcal{L}_X: \tilde{X} \rightarrow X$ .

## 2 Intersection homology.

The intersection homology was introduced by Goresky-MacPherson in [2]. We use here the singular intersection homology of [4].

**2.1 Perversity.** A *perversity* is a map  $\bar{p}: \mathcal{S}_X^{sing} \rightarrow \mathbb{Z}$ . The *zero perversity*  $\bar{0}$  is defined by  $\bar{0}(S) = 0$ . The *top perversity*  $\bar{t}$  is defined by  $\bar{t}(S) = \text{codim}_X S - 2$ . Notice that the condition  $\bar{0} \leq \bar{t}$  implies  $\text{codim}_X S \geq 2$  for each singular stratum  $S$ .

The usual notion of perversity (see [2], [3], [1], ...) is not defined on the stratification  $\mathcal{S}_X$  but on the filtration  $\mathcal{F}_X$ . It also verifies the condition  $\bar{0} \leq \bar{p} \leq \bar{t}$ , which implies that  $X$  does not possess any one codimensional stratum.

We fix for the sequel  $\bar{p}$  a perversity.

**2.2 Intersection homology. First approach.** A singular simplex  $\sigma: \Delta \rightarrow X$  is a  $\bar{p}$ -allowable simplex if

(All)  $\sigma^{-1}(S) \subset (\dim \Delta - 2 - \bar{t}(S) + \bar{p}(S))$ -skeleton of  $\Delta$ , for each singular stratum  $S$ .

A singular chain  $\xi = \sum_{j=1}^m r_j \sigma_j \in S_*(X)$  is  $\bar{p}$ -allowable if each singular simplex  $\sigma_j$  is  $\bar{p}$ -allowable. The family of  $\bar{p}$ -allowable chains is a graded vector space denoted by  $AC_*^{\bar{p}}(X)$ . The associated differential complex is the complex of  $\bar{p}$ -intersection chains, that is,  $IC_*^{\bar{p}}(X) = AC_*^{\bar{p}}(X) \cap \partial^{-1} AC_{*-1}^{\bar{p}}(X)$ . Its cohomology  $\mathbb{H}_*^{\bar{p}}(X) = H_*(IC_*^{\bar{p}}(X))$  is the  $\bar{p}$ -intersection homology of  $X$ . This is the approach of [3]. The intersection homology verifies two important computational properties: Mayer-Vietoris and the product formula  $\mathbb{H}_*^{\bar{p}}(\mathbb{R} \times X) = \mathbb{H}_*^{\bar{p}}(X)$  (see [6]). The usual local calculation is (see [4]):

**Proposition 2.2.1** *Let  $L$  be a compact stratified pseudomanifold. Then*

$$\mathbb{H}_i^{\bar{p}}(cL) = \begin{cases} \mathbb{H}_i^{\bar{p}}(L) & \text{if } i \leq \bar{t}(\vartheta) - \bar{p}(\vartheta) \\ 0 & \text{if } 0 \neq i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta) \\ \mathbb{R} & \text{if } 0 = i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta). \end{cases}$$

*Proof.* For  $i \leq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$  we have  $SC_i^{\bar{p}}(cL) = SC_i^{\bar{p}}(L \times ]0, 1[)$  which gives  $\mathbb{H}_i^{\bar{p}}(cL) = \mathbb{H}_i^{\bar{p}}(L)$  if  $i \leq \bar{t}(\vartheta) - \bar{p}(\vartheta)$ .

For a singular simplex  $\sigma: \Delta^i \rightarrow cL$  we define the cone  $c\sigma: \Delta^{i+1} \rightarrow cL$  by

$$c\sigma(x_0, \dots, x_{i+1}) = (1 - x_{i+1}) \cdot \sigma \left( \frac{x_0}{1 - x_{i+1}}, \dots, \frac{x_i}{1 - x_{i+1}} \right).$$

Here, we have written  $r \cdot [x, s] = [x, rs]$  for a point  $[x, s] \in cL$  and a number  $r \in [0, 1]$ . In the same way, we define the cone  $c\xi$  of a singular chain  $\xi$ . It defines the linear operator

$$c: AC_{\geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)}^{\bar{p}}(cL) \longrightarrow AC_{\geq 2 + \bar{t}(\vartheta) - \bar{p}(\vartheta)}^{\bar{p}}(cL).$$

Let us see that. Take  $\sigma \in AC_{\geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)}^{\bar{p}}(cL)$  and prove  $c\sigma \in AC_{\geq 2 + \bar{t}(\vartheta) - \bar{p}(\vartheta)}^{\bar{p}}(cL)$ . Notice first that, for  $x_{i+1} \neq 1$ , we have

$$\underbrace{\left( \frac{x_0}{1 - x_{i+1}}, \dots, \frac{x_i}{1 - x_{i+1}} \right)}_{\tau(x_0, \dots, x_{i+1})} \in j\text{-skeleton of } \Delta^i \implies (x_0, \dots, x_{i+1}) \in (j+1)\text{-skeleton of } \Delta^{i+1}.$$

So,

$$\begin{aligned} (c\sigma)^{-1}(\vartheta) &= \{(0, \dots, 0, 1)\} \cup \{(x_0, \dots, x_{i+1}) \in \Delta^{i+1} / \tau(x_0, \dots, x_{i+1}) \in \sigma^{-1}(\vartheta) \text{ and } x_{i+1} \neq 1\} \\ &\subset \{(0, \dots, 0, 1)\} \cup (i - 2 - \bar{t}(\vartheta) + \bar{p}(\vartheta) + 1)\text{-skeleton of } \Delta^{i+1} \\ &\subset (i + 1 - 2 - \bar{t}(\vartheta) + \bar{p}(\vartheta))\text{-skeleton of } \Delta^{i+1}, \end{aligned}$$

since  $i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$ . We have, for each singular stratum  $S$  of  $L$ :

$$\begin{aligned} (c\sigma)^{-1}(S \times ]0, 1[) &= \{(x_0, \dots, x_{i+1}) \in \Delta^{i+1} / \tau(x_0, \dots, x_{i+1}) \in \sigma^{-1}(S \times ]0, 1[) \text{ and } x_{i+1} \neq 1\} \\ &\subset (i + 1 - 2 - \bar{t}(S \times ]0, 1[) + \bar{p}(S \times ]0, 1[))\text{-skeleton of } \Delta^{i+1}. \end{aligned}$$

We conclude that  $c\sigma \in AC_{\geq 2+\bar{t}(\vartheta)-\bar{p}(\vartheta)}^{\bar{p}}(cL)$ . Notice that any singular simplex verifies the formula  $\partial c\sigma = c\partial\sigma + (-1)^{i+1}\sigma$ , if  $i > 0$  and  $\partial c\sigma = \vartheta - \sigma$ , if  $i = 0$ .

Consider now a cycle  $\xi \in IC_i^{\bar{p}}(cL)$  with  $i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$  and  $i \neq 0$ . Since  $\xi = (-1)^{i+1}\partial c\xi$  then  $c\xi \in IC_{i+1}^{\bar{p}}(cL)$  and therefore  $\mathbb{H}_i^{\bar{p}}(cL) = 0$ .

For  $i = 0 \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$  we get that for any point  $\sigma$  of  $cL - \{\vartheta\}$  the cone  $c\sigma$  is a  $\bar{p}$ -allowable chain with  $\partial c\sigma = \sigma - \vartheta$ . This gives  $\mathbb{H}_0^{\bar{p}}(cL) = \mathbb{R}$ .  $\clubsuit$

The intersection homology can be expressed in terms of the usual homology  $H_*(-)$  in some cases (see [2]).

**Proposition 2.2.2** *Let  $X$  be a stratified pseudomanifold. Then we have*

- $\mathbb{H}_*^{\bar{p}}(X) = H_*(X - \Sigma_X)$  if  $\bar{p} < \bar{0}$ , and
- $\mathbb{H}_*^{\bar{q}}(X) = H_*(X)$  if  $\bar{q} \geq \bar{t}$  and  $X$  is normal.

*Proof.* We prove, by induction on the depth, that the natural inclusions  $S_*(X - \Sigma_X) \hookrightarrow IC_*^{\bar{p}}(X)$  and  $IC_*^{\bar{q}}(X) \hookrightarrow S_*(X)$  are quasi-isomorphisms (i.e. isomorphisms in cohomology). When the depth of  $X$  is 0 then the above inclusions are in fact two identities. In the general case, we use the Mayer-Vietoris procedure and we reduce the problem to a chart  $X = \mathbb{R}^n \times cL$ . We apply the product formula and we arrive to  $X = cL$ . Here

- $\mathbb{H}_*^{\bar{p}}(cL) \stackrel{2.2.1, \bar{p}(\vartheta) < 0}{\cong} \mathbb{H}_{\leq \bar{t}(\vartheta) - \bar{p}(\vartheta)}^{\bar{p}}(L) \stackrel{ind}{=} H_{\leq \bar{t}(\vartheta) - \bar{p}(\vartheta)}(L - \Sigma_L) \stackrel{\bar{p}(\vartheta) < 0}{\cong} H_*(L - \Sigma_L) = H_*(cL - \Sigma_{cL})$ .
- For  $\bar{q}(\vartheta) = \bar{t}(\vartheta)$  we have:  $\mathbb{H}_*^{\bar{q}}(cL) \stackrel{2.2.1}{\cong} \mathbb{H}_0^{\bar{q}}(L) \stackrel{ind}{=} H_0(L) \stackrel{norm}{=} \mathbb{R} = H_*(cL)$ .
- For  $\bar{q}(\vartheta) > \bar{t}(\vartheta)$  we have:  $\mathbb{H}_*^{\bar{q}}(cL) \stackrel{2.2.1}{\cong} \mathbb{R} = H_*(cL)$ .

This ends the proof. Notice that we can replace the normality of  $X$  by the connectedness of the links  $\{L_S / \bar{q}(S) = \bar{t}(S)\}$ .  $\clubsuit$

The following result will be needed in the last section.

**Corollary 2.2.3** *Let  $X$  be a connected normal stratified pseudomanifold. For a perversity  $\bar{p}$  then  $\mathbb{H}_0^{\bar{p}}(X) = \mathbb{R}$ .*

*Proof.* We prove this result by induction on the depth. When the depth of  $X$  is 0 then  $\mathbb{H}_0^{\bar{p}}(X) \stackrel{\Sigma_X = \emptyset}{\cong} H_0(X) = \mathbb{R}$ . Consider now the general case. Notice that any point  $\sigma \in X - \Sigma_X$  is a  $\bar{p}$ -intersection cycle. So,  $\mathbb{H}_0^{\bar{p}}(X) \neq 0$ . We prove  $[\sigma_1] = [\sigma_2]$  in  $\mathbb{H}_0^{\bar{p}}(X)$  for two  $\bar{p}$ -allowable points. This is the case when  $\sigma_1, \sigma_2 \in X - \Sigma_X$  since we now from [5] that  $X - \Sigma_X$  is connected. Consider now a  $\bar{p}$ -intersection cycle  $\sigma_1 \in \Sigma_X$  and  $\varphi: U \rightarrow \mathbb{R}^n \times cL$  a chart of  $\sigma_1$ . Since

$$\mathbb{H}_0^{\bar{p}}(U) = \mathbb{H}_0^{\bar{p}}(\mathbb{R}^n \times cL) = \mathbb{H}_0^{\bar{p}}(cL) \stackrel{2.2.1, ind}{\cong} \mathbb{R},$$

we have  $[\sigma_1] = [\sigma_2]$  in  $\mathbb{H}_0^{\bar{p}}(X)$  for some  $\bar{p}$ -intersection point  $\sigma_2 \in X - \Sigma_X$ . Then  $\mathbb{H}_0^{\bar{p}}(X) = \mathbb{R}$ .  $\clubsuit$

**2.2.4 Relative case.** The conical formula given by the Proposition 2.2.1 for the intersection homology does not fit with that of Proposition 3.1.1 for the intersection cohomology. For this reason we introduce the following notion of relative intersection homology. If  $Z \subset X$  we put  $SC_*^{\bar{p}}(X, Z)$  the differential complex

$$\frac{\left( AC_*^{\bar{p}}(X) + S_*(Z) \right) \cap \partial^{-1} \left( AC_{*-1}^{\bar{p}}(X) + S_{*-1}(Z) \right)}{S_*(Z)}$$

and  $\mathbb{H}_p^*(X, Z)$  its cohomology. Of course,  $\mathbb{H}_p^*(X, Z) = \mathbb{H}_p^*(X)$  when  $Z = \emptyset$  and  $\mathbb{H}_p^*(X, Z) = H^*(X, Z)$  when  $\text{depth } X = 0$ .

The same arguments of 2.2 show that the relative intersection cohomology verifies the Mayer-Vietoris formula and the product formula  $\mathbb{H}_*^{\bar{p}}(\mathbb{R} \times X, \mathbb{R} \times Z) = \mathbb{H}_*^{\bar{p}}(X, Z)$ . For the typical local calculation we have:

**Corollary 2.2.5** *Let  $L$  be a compact stratified pseudomanifold and  $Z$  a subset of  $L$ . Then*

$$\mathbb{H}_i^{\bar{p}}(cL, cZ) = \begin{cases} \mathbb{H}_i^{\bar{p}}(L, Z) & \text{if } i \leq \bar{t}(\vartheta) - \bar{p}(\vartheta) \\ 0 & \text{if } i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta), \end{cases}$$

where we write  $c\emptyset = \vartheta$ .

*Proof.* For  $i \leq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$  we have  $SC_i^{\bar{p}}(cL, cZ) = SC_i^{\bar{p}}(L \times ]0, 1[, Z \times ]0, 1[)$  and therefore  $\mathbb{H}_i^{\bar{p}}(cL, cZ) = \mathbb{H}_i^{\bar{p}}(L, Z)$  if  $i \leq \bar{t}(\vartheta) - \bar{p}(\vartheta)$ .

Consider now a cycle  $\bar{\xi} \in SC_i^{\bar{p}}(cL, cZ)$  with  $i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$  and  $i \neq 0$ . We write  $\xi = \xi_1 + \xi_2 \in AC_i^{\bar{p}}(cL) + S_i(cZ)$  with  $\partial\xi \in S_{i-1}(cZ)$ . We have  $c\xi = c\xi_1 + c\xi_2 \in AC_{i+1}^{\bar{p}}(cL) + S_{i+1}(cZ)$ . Since  $\partial c\xi = (-1)^{i+1}\xi + c\partial\xi \in AC_i^{\bar{p}}(cL) + S_i(cZ)$  then  $\mathbb{H}_i^{\bar{p}}(cL, cZ) = 0$ .

For  $i = 0 \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$  we get that for any point  $\sigma$  of  $cL - \{\vartheta\}$  the cone  $c\sigma$  is a  $\bar{p}$ -allowable chain with  $\partial c\sigma = \sigma - \vartheta$ . This gives  $\mathbb{H}_0^{\bar{p}}(cL) = 0$ . ♣

**2.3 Intersection homology. Second approach.** (see [6]). In order to integrate differential forms on allowable simplices, we need to introduce some amount of smoothness on these simplices. Since  $X$  is not a manifold, we work in the manifold  $\tilde{X}$ . We consider in fact those allowable simplices which are liftable to a smooth simplex  $\tilde{X}$ .

**2.3.1 Linear unfolding.** The *unfolding* of the standard simplex  $\Delta$ , relative to the decomposition  $\Delta = \Delta_0 * \cdots * \Delta_j$ , is the map  $\mu_\Delta: \tilde{\Delta} = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{j-1} \times \Delta_j \longrightarrow \Delta$  defined by

$$\begin{aligned} \mu_\Delta([x_0, t_0], \dots, [x_{j-1}, t_{j-1}], x_j) &= t_0 x_0 + (1-t_0)t_1 x_1 + \cdots + (1-t_0) \cdots (1-t_{j-2})t_{j-1} x_{j-1} \\ &\quad + (1-t_0) \cdots (1-t_{j-1}) x_j. \end{aligned}$$

Here  $\bar{c}\Delta_i$  denotes the closed cone  $\Delta_i \times [0, 1] / \Delta_i \times \{0\}$  and  $[x_i, t_i]$  a point of it. This map is smooth and its restriction  $\mu_\Delta: \text{int}(\tilde{\Delta}) \longrightarrow \text{int}(\Delta)$  is a diffeomorphism (we write  $\text{int}(P) = P - \partial P$  the *interior* of the polyhedron  $P$ ). It sends a face  $U$  of  $\tilde{\Delta}$  on a face  $V$  of  $\Delta$  and the restriction  $\mu_\Delta: \text{int}(U) \rightarrow \text{int}(V)$  is a submersion.

On the boundary  $\partial\tilde{\Delta}$  we find not only the blow-up  $\tilde{\partial}\tilde{\Delta}$  of the boundary  $\partial\Delta$  of  $\Delta$  but also the faces  $F = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{i-1} \times (\Delta_i \times \{1\}) \times \bar{c}\Delta_{i+1} \times \cdots \times \bar{c}\Delta_{j-1} \times \Delta_j$  with  $i \in \{0, \dots, j-2\}$  or  $i = j-1$  and  $\dim \Delta_j > 0$ , which we shall call *bad faces*. This gives the decomposition

$$(1) \quad \partial\tilde{\Delta} = \tilde{\partial}\tilde{\Delta} + \delta\tilde{\Delta}$$

Notice that

$$(2) \quad \dim \mu_{\Delta}(F) = \dim(\Delta_0 * \cdots * \Delta_i) < \dim \Delta - 1 = \dim F.$$

**2.3.2 Lifiable simplices.** A *lifiable simplex* is a singular simplex  $\sigma: \Delta \rightarrow X$  verifying the two following conditions.

(Lif1) *Each pull back  $\sigma^{-1}(X_i)$  is a face of  $\Delta$ .*

(Lif2) *There exists a decomposition  $\Delta = \Delta_0 * \cdots * \Delta_j$  and a smooth map (called lifting)  $\tilde{\sigma}: \tilde{\Delta} \rightarrow \tilde{X}$  with  $\mathcal{L}_X \circ \tilde{\sigma} = \sigma \circ \mu_{\Delta}$ .*

A singular chain  $\xi = \sum_{j=1}^m r_j \sigma_j$  is a *lifiable* if each singular simplex  $\sigma_j$  is lifiable. Since a face of lifiable simplices is again a lifiable simplex then the family  $L_*(X)$  of lifiable chains is a differential complex. We write  $LC_*^{\bar{p}}(X) = AC_*^{\bar{p}}(X) \cap L_*(X)$ , the graded vector space of the  $\bar{p}$ -allowable lifiable chains. The associated differential complex is  $RC_*^{\bar{p}}(X) = LC_*^{\bar{p}}(X) \cap \partial^{-1}LC_{*-1}^{\bar{p}}(X)$ . The cohomology of this complex verifies the Mayer-Vietoris formula and the product formula  $H_*(RC_*^{\bar{p}}(\mathbb{R} \times X)) = H_*(RC_*^{\bar{p}}(X))$ . For the typical local calculation we have

**Proposition 2.3.3** *Let  $L$  be a compact unfoldable pseudomanifold. Consider on  $cL$  the canonical induced unfolding. Then*

$$H_i(RC_{\bar{p}}^*(cL)) = \begin{cases} H_i(RC_{\bar{p}}^*(L)) & \text{if } i \leq \bar{t}(\vartheta) - \bar{p}(\vartheta) \\ 0 & \text{if } 0 \neq i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta) \\ \mathbb{R} & \text{if } 0 = i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta). \end{cases}$$

*Proof.* We proceed as in Proposition 2.2.1. In fact, it suffices to prove that the cone  $c\sigma: \bar{c}\Delta \rightarrow cL$  of a  $\bar{p}$ -allowable lifiable simplex  $\sigma: \Delta \rightarrow cL$ , with  $\dim \Delta \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$ , is a  $\bar{p}$ -allowable lifiable simplex. Let us verify the properties  $(\text{All})_{c\sigma}$ ,  $(\text{Lif1})_{c\sigma}$  and  $(\text{Lif2})_{c\sigma}$ .

Put  $\bar{c}\Delta = \Delta * \{Q\}$  with  $c\sigma(tP + (1-t)Q) = t \cdot \sigma(P)$ . We have:

$$(c\sigma)^{-1}(cL)_i = \begin{cases} \{Q\} & \text{if } \sigma^{-1}(cL)_i = \emptyset \\ \bar{c}(\sigma^{-1}(cL)_i) & \text{if } \sigma^{-1}(cL)_i \neq \emptyset. \end{cases}$$



We obtain  $(\text{Lif1})_{c\sigma}$  from  $(\text{Lif1})_\sigma$ . For the property  $(\text{All})_{c\sigma}$  we consider a stratum  $S \in \mathcal{S}_{cL}$ . We have

$$\begin{aligned}
 (c\sigma)^{-1}(S) &= \begin{cases} \{Q\} & \text{if } S = \{\vartheta\}; \sigma^{-1}(\vartheta) = \emptyset \\ \bar{c}(\sigma^{-1}(\vartheta)) & \text{if } S = \{\vartheta\}; \sigma^{-1}(\vartheta) \neq \emptyset \\ \emptyset & \text{if } S \neq \{\vartheta\}; \sigma^{-1}(S) = \emptyset \\ \bar{c}(\sigma^{-1}(S)) - \{Q\} & \text{if } S \neq \{\vartheta\}; \sigma^{-1}(S) \neq \emptyset \end{cases} \\
 (\text{All})_\sigma &\subset \begin{cases} 0\text{-skeleton of } \bar{c}\Delta \\ (1 + (\dim \Delta - 2 - \bar{t}(\vartheta) + \bar{p}(\vartheta)))\text{-skeleton of } \bar{c}\Delta \\ \emptyset \\ (1 + (\dim \Delta - 2 - \bar{t}(S) + \bar{p}(S)))\text{-skeleton of } \bar{c}\Delta \end{cases} \\
 &\subset (\dim \bar{c}\Delta - 2 - \bar{t}(S) + \bar{p}(S))\text{-skeleton of } \bar{c}\Delta,
 \end{aligned}$$

since  $\dim \Delta \geq 1 + \bar{t}(\vartheta) - \bar{p}(\vartheta)$ . We prove  $(\text{Lif2})_{c\sigma}$ . Consider the decomposition  $\Delta = \Delta_0 * \cdots * \Delta_j$  and the smooth map  $\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2): \tilde{\Delta} \rightarrow \tilde{L} \times ]-1, 1[$  given by  $(\text{Lif2})_\sigma$ . We have the decomposition  $\bar{c}\Delta = \{Q\} * \Delta_0 * \cdots * \Delta_j$  whose lifting  $\mu_{\bar{c}\Delta}: \bar{c}\tilde{\Delta} = \bar{c}\{Q\} \times \tilde{\Delta} \rightarrow \bar{c}\Delta$  is defined by

$$\mu_{\bar{c}\Delta}([Q, t], x) = tQ + (1-t)\mu_\Delta(x).$$

We put  $\tilde{c}\sigma: \bar{c}\{Q\} \times \tilde{\Delta} \rightarrow \tilde{L} \times ]-1, 1[$  the smooth map defined by

$$\tilde{c}\sigma([Q, t], x) = (\tilde{\sigma}_1(x), (1-t) \cdot \tilde{\sigma}_2(x)).$$

Finally, for each  $([Q, t], x) \in \bar{c}\{Q\} \times \tilde{\Delta}$  we have:

$$\begin{aligned}
 c\sigma\mu_{\bar{c}\Delta}([Q, t], x) &= c\sigma(tQ + (1-t)\mu_\Delta(x)) = (1-t) \cdot \sigma\mu_\Delta(x) = (1-t) \cdot \mathcal{L}_{cL}\tilde{\sigma}(x) \\
 &= (1-t) \cdot [\mathcal{L}_L\tilde{\sigma}_1(x), |\tilde{\sigma}_2(x)|] = \mathcal{L}_{cL}\tilde{c}\sigma([Q, t], x).
 \end{aligned}$$

This gives  $(\text{Lif2})_{c\sigma}$ . ♣

**2.3.4 Relative case.** If  $Z \subset X$  we put  $RC_*^{\bar{p}}(X, Z)$  the differential complex

$$\frac{\left( LC_*^{\bar{p}}(X) + S_*(Z) \right) \cap \partial^{-1} \left( LC_{* - 1}^{\bar{p}}(X) + S_{* - 1}(Z) \right)}{S_*(Z)}.$$

Of course, when  $Z = \emptyset$  then  $RC_*^{\bar{p}}(X, Z) = RC_*^{\bar{p}}(X)$  and  $RC_*^{\bar{p}}(X, Z) = S_*(X, Z)$  when  $\text{depth } X = 0$ . Proceeding as in [4], [6] one sees that this cohomology verifies the Mayer-Vietoris formula and the product formula  $H_* \left( RC_*^{\bar{p}}(\mathbb{R} \times X, \mathbb{R} \times Z) \right) = H_* \left( RC_*^{\bar{p}}(X, Z) \right)$ . For the typical local calculation we have (see [6]):

**Corollary 2.3.5** *Let  $L$  be a compact unfoldable pseudomanifold and  $Z$  a closed subset of  $L$ . Consider on  $cL$  the canonical induced unfolding. Then*

$$H_i \left( RC_*^{\bar{p}}(cL, cZ) \right) = \begin{cases} H_i \left( RC_*^{\bar{p}}(L, Z) \right) & \text{if } i \leq \bar{t}(\vartheta) - \bar{p}(\{\vartheta\}) \\ 0 & \text{if } i \geq 1 + \bar{t}(\vartheta) - \bar{p}(\{\vartheta\}), \end{cases}$$

where we write  $c\emptyset = \vartheta$ .

*Proof.* Same proof of the Corollary 2.2.5. ♣

**2.4 Comparing the two approaches.** When the perversity  $\bar{p}$  lies between  $\bar{0}$  and  $\bar{1}$  then we have the isomorphism  $\mathbb{H}_*^{\bar{p}}(X) = H_*\left(RC_*^{\bar{p}}(X)\right)$  (cf. [6]). We are going to see that this property extends to any perversity for the absolute and the relative case.

**Proposition 2.4.1** *For any perversity  $\bar{p}$ , the inclusion  $IC_*^{\bar{p}}(X) \hookrightarrow RC_*^{\bar{p}}(X)$  induces the isomorphism  $\mathbb{H}_*^{\bar{p}}(X) = H_*\left(RC_*^{\bar{p}}(X)\right)$ .*

*Proof.* We proceed by induction on the depth. When  $\text{depth } X = 0$  then  $IC_*^{\bar{p}}(X) = RC_*^{\bar{p}}(X) = S_*(X)$ . In the general case, we use the Mayer-Vietoris procedure and we reduce the problem to a chart  $X = \mathbb{R}^n \times cL$ . Here, we apply the product formula and we arrive to  $X = cL$ . We end the proof by applying the Propositions 2.2.1, 2.3.3 and the induction hypothesis. ♣

**Proposition 2.4.2** *Consider  $Z$  a saturated closed subset of  $X$ . For any perversity  $\bar{p}$ , the inclusion  $IC_*^{\bar{p}}(X, Z) \subset RC_*^{\bar{p}}(X, Z)$  induces the isomorphism  $\mathbb{H}_*^{\bar{p}}(X, Z) = H_*\left(RC_*^{\bar{p}}(X, Z)\right)$ .*

*Proof.* We proceed by induction on the depth. If  $\text{depth } X = 0$  then  $IC_*^{\bar{p}}(X, Z) = RC_*^{\bar{p}}(X, Z) = S_*(X, Z)$ . In the general case, we use the Mayer-Vietoris procedure and we reduce the problem to a chart. Since  $Z$  is a closed saturated subset then  $(X, Z) = (\mathbb{R}^n \times cL, \emptyset)$  or  $(X, Z) = (\mathbb{R}^n \times cL, \mathbb{R}^n \times \{\vartheta\})$  or  $(X, Z) = (\mathbb{R}^n \times cL, \mathbb{R}^n \times cN)$ , where  $N$  is a closed saturated subset of  $L$  (cf. 1.1). Here, we apply the product formula and we arrive to  $(X, Z) = (cL, \emptyset)$  or  $(X, Z) = (cL, \{\vartheta\})$  or  $(X, Z) = (cL, cN)$ . We end the proof by applying the Propositions 2.2.1, 2.3.3, the Corollaries 2.2.5, 2.3.5 and the induction hypothesis. ♣

### 3 Intersection cohomology

The deRham intersection cohomology was introduced by Brylinski in [1]. We use here the presentation of [6].

**3.1 Perverse forms.** A *liftable form* is a differential form  $\omega \in \Omega^*(X - \Sigma_X)$  possessing a *lifting*, that is, a differential form  $\tilde{\omega} \in \Omega^*\left(\tilde{X}\right)$  verifying  $\tilde{\omega} = \mathcal{L}_X^* \omega$  on  $\mathcal{L}_X^{-1}(X - \Sigma_X)$ .

Given two liftable differential forms  $\omega, \eta$  we have the equalities:

$$\widetilde{\omega + \eta} = \tilde{\omega} + \tilde{\eta} \quad ; \quad \widetilde{\omega \wedge \eta} = \tilde{\omega} + \wedge \tilde{\eta} \quad ; \quad \widetilde{d\omega} = d\tilde{\omega}.$$

We write  $\Pi^*(X)$  the differential complex of liftable forms.

Recall that, for each singular stratum  $S$ , the restriction  $\mathcal{L}_S: \mathcal{L}_S^{-1}(S) \rightarrow S$  is a fiber bundle. The *perverse degree*  $\|\omega\|_S$  of  $\omega$  relatively to  $S$  is the vertical degree of the restriction  $\tilde{\omega}$  relatively to  $\mathcal{L}_S: \mathcal{L}_S^{-1}(S) \rightarrow S$ . The differential complex of  $\bar{p}$ -intersection differential forms is

$$\Omega_{\bar{p}}^*(X) = \{\omega \in \Pi^*(X) / \max(\|\omega\|_S, \|d\omega\|_S) \leq \bar{p}(S) \forall \text{ singular stratum } S\}.$$

The cohomology  $\mathbb{H}_{\bar{p}}^*(X)$  of this complex is the  $\bar{p}$ -intersection cohomology of  $X$ . The intersection cohomology verifies two important computational properties: Mayer-Vietoris and the product formula  $\mathbb{H}_{\bar{p}}^*(\mathbb{R} \times X) = \mathbb{H}_{\bar{p}}^*(X)$ . The usual local calculation is (see [1], [6])

**Proposition 3.1.1** *Let  $L$  be a compact stratified pseudomanifold. Then*

$$\mathbb{H}_{\bar{p}}^i(cL) = \begin{cases} \mathbb{H}_{\bar{p}}^i(L) & \text{if } i \leq \bar{p}(\vartheta) \\ 0 & \text{if } i > \bar{p}(\vartheta). \end{cases}$$

**3.2 Integration.** The relationship between the intersection homology and cohomology is established by using the integration of differential forms on simplices. Since  $X$  is not a manifold, we work on the blow up  $\tilde{X}$ .

Consider a  $\varphi: \Delta \rightarrow X$  a liftable simplex. We know that there exists a stratum  $S$  containing  $\sigma(\text{int}(\Delta))$ . Since  $\mu_\Delta: \text{int}(\tilde{\Delta}) \rightarrow \text{int}(\Delta)$  is a diffeomorphism then  $\sigma = \mathcal{L}_X \circ \tilde{\sigma} \circ \mu_\Delta^{-1}: \text{int}(\Delta) \rightarrow S$  is a smooth map.

Consider now a liftable differential form  $\omega \in \Pi^*(X)$  we define the *integration* as

$$\int_\sigma \omega = \begin{cases} \int_{\text{int}(\Delta)} \sigma^* \omega & \text{if } S \text{ a regular stratum (i.e. } \sigma(\Delta) \not\subset \Sigma_X) \\ 0 & \text{if } S \text{ a singular stratum (i.e. } \sigma(\Delta) \subset \Sigma_X) \end{cases}$$

This definition makes sense since  $\int_{\text{int}(\Delta)} \sigma^* \omega = \int_{\text{int}(\tilde{\Delta})} \tilde{\sigma}^* \tilde{\omega} = \int_{\tilde{\Delta}} \tilde{\sigma}^* \tilde{\omega}$ . By linearity, we have the linear pairing  $\int: \Pi^*(X) \rightarrow \text{Hom}(L_*(X), \mathbb{R})$ . This operator commutes with the differential  $d$  in some cases.

**Lemma 3.2.1** *If  $\bar{p}$  is a perversity then  $\int: \Omega_{\bar{p}}^*(X) \rightarrow \text{Hom}(RC_*^{\bar{t}-\bar{p}}(X), \mathbb{R})$  is differential pairing.*

*Proof.* Consider  $\sigma: \Delta^i \rightarrow X$  a liftable  $(\bar{t}-\bar{p})$ -allowable simplex with  $\sigma(\Delta) \not\subset \Sigma_X$  and  $\omega \in \Omega_{\bar{q}}^{i-1}(X)$ . It suffices to prove

$$(3) \quad \int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

The boundary of  $\Delta$  can be written  $\partial\Delta = \partial_1\Delta + \partial_2\Delta$  where  $\partial_1\Delta$  (resp.  $\partial_2\Delta$ ) is composed by the faces  $F$  with  $\sigma(F) \not\subset \Sigma_X$  (resp.  $\sigma(F) \subset \Sigma_X$ ). This gives the decomposition (see (1))

$$\partial\tilde{\Delta} = \partial_1\tilde{\Delta} + \partial_2\tilde{\Delta} + \delta\tilde{\Delta}.$$

By the Stokes formula the equation (3) becomes

$$\int_{\partial\tilde{\Delta}} \tilde{\sigma}^* \tilde{\omega} = \int_{\tilde{\Delta}} d\tilde{\sigma}^* \tilde{\omega} = \int_{\tilde{\Delta}} \tilde{\sigma}^* d\omega = \int_\sigma d\omega = \int_{\partial\sigma} \omega = \int_{\partial_1\tilde{\Delta}} \tilde{\sigma}^* \tilde{\omega}.$$

We end the proof if we show that  $\tilde{\omega} = 0$  on  $F$  for a bad face  $F$  or a face  $F$  verifying  $\sigma(F) \subset \Sigma_X$ . Put  $C$  the face  $\mu(F)$  of  $\Delta$  and  $S$  the stratum of  $X$  containing  $\sigma(\text{int}(C))$ . Notice that the condition (All) implies

$$(4) \quad \dim C \leq \dim F + 1 - 2 - \bar{t}(S) + (\bar{t}(S) - \bar{p}(S)) = \dim F - 1 - \bar{p}(S).$$

We have the following commutative diagram

$$\begin{array}{ccc} \text{int}(F) & \xrightarrow{\tilde{\sigma}} & \mathcal{L}_X^{-1}(S) \\ \mu_\Delta \downarrow & & \mathcal{L}_X \downarrow \\ \text{int}(C) & \xrightarrow{\sigma} & S \end{array}$$

It suffices to prove that the vertical degree of  $\tilde{\sigma}^* \tilde{\omega}$  relatively to  $\mu_\Delta$  is strictly lower than the dimension of the fibers of  $\mu_\Delta$ , that is

$$\|\tilde{\sigma}^* \tilde{\omega}\| < \dim F - \dim C.$$

When  $S$  is a regular stratum the differential form  $\omega$  is defined on  $S$ . We have

$$\tilde{\sigma}^* \tilde{\omega} = \tilde{\sigma}^* \mathcal{L}_X^* \omega = \mu_\Delta^* \sigma^* \omega,$$

which is basic form relatively to  $\mu_\Delta$ . So,

$$\|\tilde{\sigma}^* \tilde{\omega}\| \leq 0 \stackrel{(2)}{<} \dim F - \dim C.$$

When  $S$  is a singular stratum, we have

$$\|\tilde{\sigma}^* \tilde{\omega}\| \leq \|\omega\|_S \leq \bar{p}(S) \stackrel{(4)}{\leq} \dim F - \dim C - 1 < \dim F - \dim C.$$

This ends the proof. ♣

The above pairing induces the pairing

$$\int : \mathbb{H}_{\bar{p}}^*(X) \longrightarrow \text{Hom} \left( \mathbb{H}_*^{\bar{i}-\bar{p}}(X), \mathbb{R} \right),$$

(cf. Proposition 2.4.1) which is not an isomorphism: for the cone we have Proposition 2.2.1 and Proposition 3.1.1. For this reason we consider the relative intersection homology. Define

$$X_{\bar{p}} = \bigcup_{\bar{p} < 0} \bar{S},$$

which is a saturated closed subset of  $\Sigma_X$ . Since the integration  $\int$  vanishes on  $\Sigma_X$  then

$$\int : \Omega_{\bar{p}}^*(X) \longrightarrow \text{Hom} \left( RC_*^{\bar{i}-\bar{p}}(X, X_{\bar{p}}), \mathbb{R} \right).$$

is a well defined differential operator. We arrive to the deRham duality (in the sense cohomology  $\mapsto$  homology):

**Theorem 3.2.2** *Let  $X$  be an unfoldable pseudomanifold. If  $\bar{p}$  is a perversity then the integration induces the isomorphism*

$$\mathbb{H}_{\bar{p}}^*(X) = \text{Hom} \left( \mathbb{H}_*^{\bar{i}-\bar{p}}(X, X_{\bar{p}}); \mathbb{R} \right)$$

*Proof.* Following the Proposition 2.4.2 it suffices to prove that the pairing

$$\int : \Omega_{\bar{p}}^*(X) \longrightarrow \text{Hom} \left( RC_*^{\bar{i}-\bar{p}}(X, X_{\bar{p}}), \mathbb{R} \right).$$

induces an isomorphism in cohomology. We proceed by induction on the depth. If depth  $X = 0$  then  $X_{\bar{p}} = \emptyset$  and we have the usual deRham theorem. In the general case, we use the Mayer-Vietoris procedure and we reduce the problem to a chart. By definition of  $X_{\bar{p}}$  we have  $(X, X_{\bar{p}}) = (\mathbb{R}^n \times cL, \mathbb{R}^n \times (cL)_{\bar{p}})$ . Here, we apply the product formula and we arrive to  $(X, X_{\bar{p}}) = (cL, (cL)_{\bar{p}})$ . Now, we distinguish three cases

- $\bar{p} \geq \bar{0}$ . Here  $(cL)_{\bar{p}} = \emptyset$  and we have

$$\mathbb{H}_{\bar{p}}^*(cL) \stackrel{3.1.1}{=} \mathbb{H}_{\bar{p}}^{\leq \bar{p}(\emptyset)}(L) \stackrel{ind}{=} \mathbb{H}_{\leq \bar{p}(\emptyset)}^{\bar{i}-\bar{p}}(L) \stackrel{2.2.1}{=} \mathbb{H}_*^{\bar{i}-\bar{p}}(cL).$$

- $\bar{p}(\vartheta) < 0$  and  $\bar{p} \geq \bar{0}$  on  $L$ . Here  $L_{\bar{p}} = \emptyset$ ,  $(cL)_{\bar{p}} = \{\vartheta\}$  and we have

$$\mathbb{H}_{\bar{p}}^*(cL) \stackrel{3.1.1}{=} \mathbb{H}_{\bar{p}}^{\leq \bar{p}(\vartheta)}(L) \stackrel{ind}{=} \mathbb{H}_{\leq \bar{p}(\vartheta)}^{\bar{t}-\bar{p}}(L) \stackrel{2.2.5}{=} \mathbb{H}_*^{\bar{t}-\bar{p}}(cL, \{\vartheta\}) = \mathbb{H}_*^{\bar{t}-\bar{p}}(cL, (cL)_{\bar{p}}).$$

- $\bar{p} \not\geq \bar{0}$  on  $L$ . Here  $(cL)_{\bar{p}} = cL_{\bar{p}}$  with  $L_{\bar{p}} \neq \emptyset$  and we have

$$\mathbb{H}_{\bar{p}}^*(cL) \stackrel{3.1.1}{=} \mathbb{H}_{\bar{p}}^{\leq \bar{p}(\vartheta)}(L) \stackrel{ind}{=} \mathbb{H}_{\leq \bar{p}(\vartheta)}^{\bar{t}-\bar{p}}(L, L_{\bar{p}}) \stackrel{2.2.5}{=} \mathbb{H}_*^{\bar{t}-\bar{p}}(cL, (cL)_{\bar{p}}).$$

This ends the proof. ♣

The intersection cohomology can be expressed in terms of the usual cohomology  $H^*(-)$  in some cases (see [1]).

**Proposition 3.2.3** *Let  $X$  be an unfoldable pseudomanifold. Then we have*

- $\mathbb{H}_{\bar{p}}^*(X) = H^*(X - \Sigma_X)$  if  $\bar{p} > \bar{t}$ , and
- $\mathbb{H}_{\bar{q}}^*(X) = H^*(X, X_{\bar{q}})$  if  $\bar{q} \leq \bar{0}$  and  $X$  is normal.

*Proof.* From the above Theorem it suffices to prove  $\mathbb{H}_*^{\bar{t}-\bar{p}}(X) = H_*(X - \Sigma_X)$  and  $\mathbb{H}_*^{\bar{t}-\bar{q}}(X, X_{\bar{q}}) = H_*(X, X_{\bar{q}})$ . The first assertion comes directly from the Proposition 2.2.2. We consider for the second one the differential morphism

$$A: \frac{\left( AC_*^{\bar{t}-\bar{q}}(X) + S_*(X_{\bar{q}}) \right) \cap \partial^{-1} \left( AC_{*-1}^{\bar{t}-\bar{q}}(X) + S_{*-1}(X_{\bar{q}}) \right)}{S_*(X_{\bar{q}})} \longrightarrow \frac{S_*(X)}{S_*(X_{\bar{q}})}$$

defined by  $A\{\xi\} = \{\xi\}$ . We prove, by induction on the depth, that the morphism  $A$  is a quasi-isomorphism. When the depth of  $X$  is 0 then  $A$  is the identity. In the general case, we use the Mayer-Vietoris procedure and we reduce the problem to a chart  $X = \mathbb{R}^n \times cL$ . We apply the product formula and we arrive to  $X = cL$ . Now, we have three cases

- $\bar{q} = \bar{0}$ . Here  $(cL)_{\bar{q}} = \emptyset$  and we have

$$\mathbb{H}_*^{\bar{t}-\bar{q}}(cL, (cL)_{\bar{q}}) = \mathbb{H}_*^{\bar{t}-\bar{q}}(cL) \stackrel{2.2.2}{=} H_*(cL) = H_*(cL, (cL)_{\bar{q}})$$

- $\bar{q}(\vartheta) = 0$  and  $\bar{q} \neq \bar{0}$  on  $L$ . Here  $(cL)_{\bar{q}} = cL_{\bar{q}}$  and  $L_{\bar{q}} \neq \emptyset$  and we have

$$\mathbb{H}_*^{\bar{t}-\bar{q}}(cL, (cL)_{\bar{q}}) = \mathbb{H}_*^{\bar{t}-\bar{q}}(cL, cL_{\bar{q}}) \stackrel{2.2.5}{=} \mathbb{H}_0^{\bar{t}-\bar{q}}(L, L_{\bar{q}}) \stackrel{ind}{=} H_0(L, L_{\bar{q}}) \stackrel{norm}{=} 0 = H_*(cL, (cL)_{\bar{q}})$$

- $\bar{q}(\vartheta) < 0$ . Here  $(cL)_{\bar{q}} = cL_{\bar{q}} \neq \emptyset$  and we have

$$\mathbb{H}_*^{\bar{t}-\bar{q}}(cL, (cL)_{\bar{q}}) = \mathbb{H}_*^{\bar{t}-\bar{q}}(cL, cL_{\bar{q}}) \stackrel{2.2.5}{=} 0 = H_*(cL, (cL)_{\bar{q}})$$

This ends the proof. Notice that we can replace the normality of  $X$  by the connectedness of the links  $\{L_S / \bar{q}(S) = 0\}$ . ♣

In the sense homology  $\mapsto$  cohomology we have the following de Rham Theorem

**Corollary 3.2.4** *Let  $X$  be a normal unfoldable pseudomanifold. If  $\bar{p}$  is a perversity then we have the isomorphism*

$$\mathbb{H}_*^{\bar{p}}(X) = \text{Hom} \left( \mathbb{H}_{\max(\bar{0}, \bar{t} - \bar{p})}^*(X), \mathbb{R} \right)$$

*Proof.* Since  $X_{\max(\bar{0}, \bar{t} - \bar{p})} = \emptyset$  then we have that  $\mathbb{H}_{\max(\bar{0}, \bar{t} - \bar{p})}^*(X)$  is isomorphic to  $\mathbb{H}_*^{\bar{t} - \max(\bar{0}, \bar{t} - \bar{p})}(X) = \mathbb{H}_*^{\min(\bar{p}, \bar{t})}(X)$  (cf. Theorem 3.2.2). It suffices to prove that the inclusion  $IC_*^{\min(\bar{p}, \bar{t})}(X) \hookrightarrow IC_*^{\bar{p}}(X)$  induces an isomorphism in cohomology. We proceed by induction on the depth. When the depth of  $X$  is 0 then  $IC_*^{\min(\bar{p}, \bar{t})}(X) = IC_*^{\bar{p}}(X) = S_*(X)$ . In the general case, we use the Mayer-Vietoris procedure and we reduce the problem to a chart  $X = \mathbb{R}^n \times cL$ . We apply the product formula and we arrive to  $X = cL$ . Now, we have two cases

- $\bar{t}(\vartheta) < \bar{p}(\vartheta)$ . We have

$$\mathbb{H}_*^{\min(\bar{p}, \bar{t})}(cL) \stackrel{2.2.1}{=} \mathbb{H}_0^{\min(\bar{p}, \bar{t})}(L) \stackrel{ind}{=} \mathbb{H}_0^{\bar{p}}(L) \stackrel{2.2.3}{=} \mathbb{R} \stackrel{2.2.1}{=} \mathbb{H}_*^{\bar{p}}(cL).$$

- $\bar{t}(\vartheta) \geq \bar{p}(\vartheta)$ . We have

$$\mathbb{H}_*^{\min(\bar{p}, \bar{t})}(cL) \stackrel{2.2.1}{=} \mathbb{H}_{\leq \bar{t}(\vartheta) - \bar{p}(\vartheta)}^{\min(\bar{p}, \bar{t})}(L) \stackrel{ind}{=} \mathbb{H}_{\leq \bar{t}(\vartheta) - \bar{p}(\vartheta)}^{\bar{p}}(L) \stackrel{2.2.1}{=} \mathbb{H}_*^{\bar{p}}(cL).$$

This ends the proof. ♣

**3.3 Poincaré Duality.** The intersection homology was introduced with the purpose of extending the Poincaré Duality to singular manifolds (see [2]). The pairing is given there by the intersection of cycles. In the context of manifolds the Poincaré Duality also derives from the integration of the wedge product of differential forms. This is also the case for stratified pseudomanifolds.

Consider that the stratified pseudomanifold  $X$  is compact and *orientable*, that is, the manifold  $X - \Sigma_X$  is an orientable manifold. We put  $m$  the dimension of  $X$ . It has been proved in [1] (see also [6]) that, for a perversity  $\bar{p}$ , with  $\bar{0} \leq \bar{p} \leq \bar{t}$ , the pairing  $P: \Omega_{\bar{p}}^i(X) \times \Omega_{\bar{t} - \bar{p}}^{m-i}(X) \longrightarrow \mathbb{R}$ , defined by  $P(\alpha, \beta) = \int_{X - \Sigma_X} \alpha \wedge \beta$ , induces the isomorphism:  $\mathbb{H}_{\bar{p}}^*(X) = \mathbb{H}_{\bar{t} - \bar{p}}^{m-*}(X)$ . The same proof works for any perversity. For example, if  $\bar{p} < \bar{0}$  or  $\bar{p} > \bar{t}$ , we obtain the Lefschetz Duality  $H^*(X, \Sigma_X) = H^{m-*}(X - \Sigma_X)$ .

## References

- [1] J.L. Brylinsky *Equivariant intersection cohomology*. - Contemporary Math. **132**(1992), 5-32.
- [2] M. Goresky and R. MacPherson: *Intersection homology theory*. - Topology **19**(1980), 135-162.
- [3] M. Goresky and R. MacPherson: *Intersection homology II*. - Invent. Math. **71**(1983), 77-129.
- [4] H. King: *Topology invariance of intersection homology without sheaves*. - Topology Appl. **20**(1985), 149-160.
- [5] G. Padilla: *On normal stratified pseudomanifolds*. - To appear in Extracta Mathematica. <http://arxiv.org/abs/math.AT/0210022>.
- [6] M. Saralegi: *Homological properties of stratified spaces with boundary*. - Ill. J. Math. **38**(1994), 47-70.