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Dynamo threshold in a turbulent medium

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We study the magnetic self-generation in a turbulent conducting fluid by means of a stochastic equation. Both the effect of the mean flow and that of the turbulent fluctuations are considered. A quantitative criterion for the existence of a dynamo instability in the presence of turbulence is derived. This criterion is expressed in terms of an average over the possible orientations of the magnetic field. However, it can give insight about the influence of noise on the dynamo threshold.

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The process of magnetic field generation through the movement of an electrically conducting medium is called a dynamo. The dynamo process is akin to an instability problem. When the conducting medium is a fluid, the instability results from a competition between amplification of a seed magnetic field via stretching and folding, and magnetic field damping through diffusion. This is quantified by a dimensionless number, the magnetic Reynolds number (Rm), which must exceed some critical value Rm_c for the instability to operate. The back reaction of the magnetic field onto the velocity through the Lorentz force then implies existence of saturated equilibrium solutions.

Another dimensionless number of importance in dynamo theory is the kinetic Reynolds number (Re) of the conducting flow. As long as Re is small, the equations of motions are generally amenable either to analytical or numerical simulations. One interesting outcome of these studies is the computation of Rm_c , and its dependence as a function of the flow symmetries, or boundary conditions. Such studies for example evidenced the importance of helicity (of lack of parity invariance) in the magnetic field amplification and regeneration. When Re is sufficiently large, the flow becomes turbulent, forbidding both numerical and exact analytical studies. This situation is typically met in natural objects (the Sun, the Earth, Galaxies), or in liquid metal, where Re is typically 5 orders of magnitude larger than Rm .

Despite their obvious ubiquity in natural objects, turbulent dynamos are not so easy to study or model. Computer resources limit the numerical study of turbulent dynamos to a range of either small Re (laminar dynamo) or at best modest Re and Rm [1, 2]. Slightly more realistic values of the parameters can be obtained in the so-called “kinematic regime” where the magnetic field amplification is sufficiently weak for the back reaction of the magnetic field onto the flow to be negligible. This results in an uncoupling of the dynamics of the magnetic field and velocity field, allowing to consider the action of a *prescribed* velocity field, onto a magnetic field. The validity of this approximation in a strongly turbulent regime is however questionable, as we shall see in the sequel. These difficulties explain the recent development of ex-

periments involving liquid metals, as a way to study the dynamo problem at large Re . Among the various operating experiments, however, a clear distinction appears between set up with constrained or unconstrained geometry. In the former case [3, 4], the fluctuation level is very weak. The velocity field is then very close to its laminar (mean) value. In these experiments, dynamos have been observed, at critical magnetic Reynolds number comparable to the theoretical value. In contrast, unconstrained experiments [5, 6] are characterized by a large fluctuation level (as high as 50 per cent). A surrogate laminar Rm can then be computed, using the mean velocity field as an input [7] but it is not clear whether the actual Rm_c has anything to do with this value, owing to the possible influence of the correlations.

The purpose of this paper is to derive a description of the influence of the turbulence on the dynamo threshold. It is based on a stochastic description of small-scale turbulent motions. This subject has been pioneered by Kazantsev [8], Parker [9] and Kraichnan [10], and further developed by the Russian school [11]. It has recently been the subject of a renewed interest, in the framework of anomalous scaling and intermittency [12], or computation of turbulent transport coefficients and probability density functions (PDF) [13]. The present problem is an example of instability or bifurcation in the presence of a multiplicative noise. A classical associated difficulty is the ability to define a meaningful threshold. It is indeed well known that such systems are characterized by moment-dependent threshold, which forbids the direct use of, e.g., the mean or the energy to detect the bifurcation threshold [14]. In the present paper, we use the PDF of the magnetic field to identify a bifurcation on the shape of this function and show that it coincides with other estimates based on Lyapunov exponent.

The dynamic of a magnetic field \mathbf{B} in an infinite conducting medium of diffusivity η and velocity \mathbf{V} , is governed by the induction equation:

$$\partial_t B_i = -V_k \partial_k B_i + B_k \partial_k V_i + \eta \partial_k \partial_k B_i \quad (1)$$

We decompose the velocity field into a mean part \bar{V}_i and a fluctuating part v_i . In most laboratory experiments, the mean part is provided by the forcing. As such, it is

generally composed of large scales, while the fluctuating part collects all short time scale, small-scale movements. In this regard, it is natural to consider the fluctuating part of the velocity as a noise, to be prescribed or computed in a physically plausible manner. The simplest, most widely used shape is the Gaussian, delta-correlated fluctuations, the so-called ‘‘Kraichnan’s ensemble’’:

$$\langle v_i(\mathbf{x}, t)v_j(\mathbf{x}', t') \rangle = 2G_{ij}(\mathbf{x}, \mathbf{x}')\delta(t - t') \quad (2)$$

Equation (1) then takes the shape of a stochastic partial differential equation for B . In that respect, we note that the induction equation is *linear* and does not include explicit back reaction term allowing saturation of any potential growth in the dynamo regime. This back reaction is provided through the velocity which is subject to the Lorentz-Force, a quadratic form of B . It is usually ignored in the so-called kinematic regime (see above). However, for reasons which will become clearer later, we prefer to work with a modified induction equation, so as to model this non-linear back reaction.

A practical way to include the effect of the Lorentz in the induction equation in the large Prandtl number regime has recently been proposed by Boldyrev [13]. It consists in a modification of the velocity gradient like:

$$\partial_k v_i \longrightarrow \partial_k v_i - \frac{1}{\nu} \frac{d-1}{d} B^2 B_i \quad (3)$$

where ν is the viscosity and d is the space dimension. The effect of the Lorentz force is then to lower the effect of the velocity gradients responsible for the growth of the magnetic field. In the small Prandtl number regime, we may generalize this prescription by considering that ν is an ad hoc turbulent viscosity. In some sense, this modification is akin to an amplitude equation, and the cubic shape for the non-linear term could be viewed as the only one allowed by the symmetries [15]. The magnitude of the prefactor is then unessential in the sequel.

Ignoring the magnetic diffusivity and using standard techniques [13, 16], one can derive the evolution equation for $P(\mathbf{B}, \mathbf{x}, t)$, the probability of having the field \mathbf{B} at point \mathbf{x} and time t (we assume an homogeneous turbulence for simplicity):

$$\begin{aligned} \partial_t P &= -\bar{V}_k \partial_k P - (\partial_k \bar{V}_i) \partial_{B_i} [B_k P] + \partial_k [\beta_{kl} \partial_l P] \quad (4) \\ &+ \frac{d-1}{\nu d} \partial_{B_i} [B^2 B_i P] + 2\partial_{B_i} [B_k \alpha_{lik} \partial_l P] \\ &+ \mu_{ijkl} \partial_{B_i} [B_j \partial_{B_k} (B_l P)] \end{aligned}$$

with the following turbulent tensors:

$$\begin{aligned} \beta_{kl} &= \langle v_k v_l \rangle \quad (5) \\ \alpha_{ijk} &= \langle v_i \partial_k v_j \rangle \\ \mu_{ijkl} &= \langle \partial_j v_i \partial_l v_k \rangle \end{aligned}$$

Due to incompressibility, the following relations hold: $\alpha^{kii} = \mu^{iikl} = \mu^{ijkk} = 0$.

The physical meaning of these tensors can be found by analogy with the ‘‘Mean-Field Dynamo theory’’ [17, 18]. Indeed, consider the equation for the evolution of the mean field, obtained by multiplication of equation (4) by B^i and integration:

$$\begin{aligned} \partial_t \langle B_i \rangle &= -\bar{V}_k \partial_k \langle B^i \rangle + (\partial_k \bar{V}_i) \langle B_k \rangle - 2\alpha_{kil} \partial_k \langle B_l \rangle \quad (6) \\ &+ \beta_{kl} \partial_k \partial_l \langle B_i \rangle - \frac{d-1}{\nu d} \langle B^2 B_i \rangle. \end{aligned}$$

This equation resembled the classical Mean Field Equation of dynamo theory, with generalized (anisotropic) ‘‘ α ’’ and ‘‘ β ’’. The first effect leads to a large scale instability for the mean-field, while the second one is akin to a turbulent diffusivity. A few remarks are in order at this point: i) our mean field equation has been derived without assumption of scale separation. ii) From the expression of β , it is easy to show that it is always positive: the magnetic diffusivity is enhanced in this case. This property is an artefact of our neglect of the molecular diffusivity, which may induce a ‘‘negative β effect’’ [19]. The account of molecular diffusivity requires functional derivative and integration, leading to a much more complex description than the present one. iii) The tensor μ does not appear at this level. In the sequel, it will be shown to govern the stochastic dynamo transition.

For this, we need to identify the threshold as a function of the noise properties. Here, we follow an idea by Mallick and Marcq [14], and focus on the properties of the stationary PDF of the system. Indeed, below the transition, the only stable state is $B = 0$ and the PDF should be a Dirac delta function. Above the transition, other equilibrium states are possible, with non zero magnetic field. However, in the general case, it is not possible to find analytical solution for the equation (4). We thus resort to the following mean field argument. Changing variable $B_i = B e_i$ where e is a unit vector (and can be characterized by $d-1$ angular variables), we can get an equation for $P(B, e_i, x)$. We now assume that there is an uncoupling for P as $P(B, e_i) = P(B)G(e_i, x)$, and perform an average over the angular variables, to find a closed equation for $P(B)$. In some sense, this can be regarded as a kind of angle/action variable separation, with average over the fast variables. The final equation for $P(B)$ becomes:

$$\begin{aligned} \frac{\partial P}{\partial t} &= a \frac{1}{B^{d-1}} \frac{\partial}{\partial B} [B \frac{\partial}{\partial B} (B^d P)] \quad (7) \\ &- b \frac{1}{B^{d-1}} \frac{\partial}{\partial B} (B^d P) + \frac{d-1}{d\nu} \frac{1}{B^{d-1}} \frac{\partial}{\partial B} (B^{d+2} P) \end{aligned}$$

where the coefficients a and b are given by averages over the position and the angular variables $\langle \bullet \rangle_\phi = \int \bullet G(\mathbf{e}, \mathbf{x}) d\mathbf{x} d\mathbf{e}$:

$$\begin{aligned} a &= \langle \mu_{ijkl} e_i e_j e_k e_l \rangle_\phi \quad (8) \\ b &= \langle \partial_k \bar{V}_i e_i e_k \rangle_\phi + \langle \mu_{ijkl} (\Delta_{ik} e_j e_l + \Delta_{kj} e_i e_l) \rangle_\phi \end{aligned}$$

where we used $\Delta_{ij} = \partial_{n_i}(n_j) = \delta_{ij} - e_i e_j$ an “angular Dirac tensor”. One can notice that these coefficients only explicitly involve the tensor μ . Nevertheless, one must keep in mind that the tensor α and β enter these expressions by mean of the angular distribution $G(e_i, x)$, whose expression involves this two tensors in the general case. In the derivation of equation (7), we use the following expression for the gradient with respect to the magnetic field:

$$\frac{\partial}{\partial B_i}(B_k G) = e_i e_k \frac{\partial}{\partial B}(B G) + \frac{\partial}{\partial e_i}(e_k G), \quad (9)$$

where the first part contains derivatives only with respect to the radial variable and the last one with respect to the angular ones. Using this decomposition in equation (4), integrating with respect to \mathbf{x} and \mathbf{e} and making use of the “integration by part on the angular variable” formula:

$$\begin{aligned} \langle F(\mathbf{e}) \partial_{e_i} [e_j G(\mathbf{e})] \rangle_\phi &= (d-1) \langle e_i e_k F(\mathbf{e}) G(\mathbf{e}) \rangle_\phi \\ &- \langle \partial_{e_i} [F(\mathbf{e})] e_j G(\mathbf{e}) \rangle_\phi, \end{aligned} \quad (10)$$

leads to equation (7).

An obvious stationary solution of (7) is a Dirac function, representing a solution with vanishing magnetic field. Another stationary solution can be found by setting $\partial_t P = 0$ in (7), with solution:

$$P(B) = \frac{1}{Z} B^{b/a-d} \exp\left[-\frac{d-1}{2\nu da} B^2\right] \quad (11)$$

where Z is a normalization constant. This solution can represent a meaningful probability density function only if it can be normalized, i.e., it is integrable over the volume element B^{d-1} . This remark provides us with a bifurcation threshold: there is dynamo whenever (11) is integrable, i.e., when solution other than vanishing magnetic field are possible.

Condition of integrability at infinity of (11) requires a be positive. This illustrates the importance of the non-linear term which is essential to ensure vanishing of the probability density at infinity. Condition of integrability near zero requires b/a be positive. This leads us to identify a necessary and sufficient condition for existence of a stationary dynamo as

$$a > 0 \quad \text{and} \quad \frac{b}{a} > 0 \quad \text{DYNAMO} \quad (12)$$

In some sense, this bifurcation (I) is obtained using the mean field as control parameter. Another bifurcation threshold can be defined using the most probable field as control parameter. Indeed, an elementary calculation shows that the condition for a maximum in the PDF is $b > ad$. Therefore, the bifurcation threshold (II) with the most probable field as control parameter is defined by $b = ad$. This difference may have some relevance when analyzing real data from experiment. To illustrate this,

we performed one dimensional simulations of a surrogate equation mimicking our non-linear stochastic system. Of course, the simulated equation is just a paradigm for the type of bifurcation we encounters: strictly speaking, the induction equation has no meaning in one dimension. In that sense, one dimensional equation may be misleading, so higher dimensional models are currently under study. The time series and PDF for three different values of the control parameter are shown on figure 1.

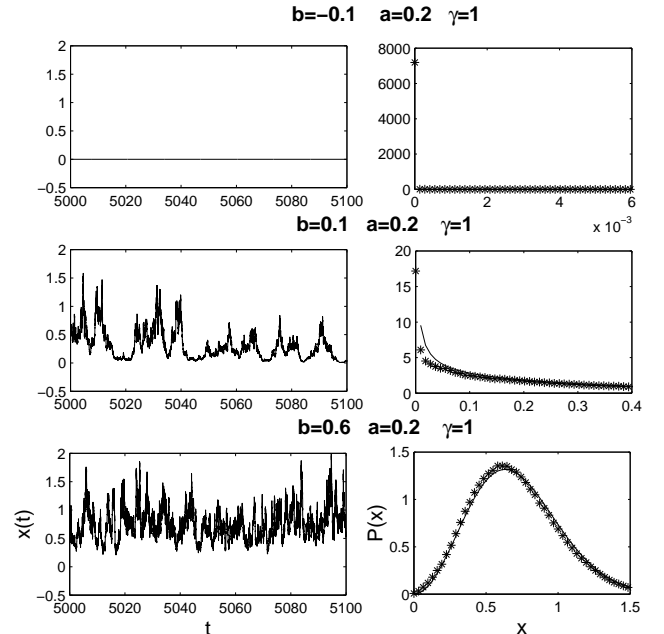


FIG. 1: Result of the surrogate 1D model: $\partial_t x = (b + \xi(t))x - \gamma x^3$ with $\langle \xi(t)\xi(t') \rangle = 2a\delta(t-t')$. On the left side we show time series for 3 different values of the parameter b and on the right side, the corresponding PDF and the theoretical curve corresponding to equation (11).

The simulations show that the bifurcation (I) leads to an intermittent behavior for the magnetic field reminiscent of the characteristic behavior of instability in presence of multiplicative noise, a phenomenon sometimes called “on-off” intermittency [20]: typically, the magnetic energy exhibits bursts separated by long quiescent (zero magnetic energy) period. Equation (11) cannot rigorously capture these intermittent states. However, two facts are very suggestive of such a type of bifurcation in our solution: (a) the distribution looks like a pure fluctuation distribution, with ill-defined mean value (b) the scaling for $\|B\| \ll 1$ (magnetic energy) is the same as that of [20]: $P(\|B\|) = \|B\|^\gamma$ with $\gamma = b/D - 1$, where D is a “diffusion coefficient” for the finite-time Lyapunov exponent. On the contrary, the bifurcation (II) is quite different in nature because of a well defined mean value for the magnetic field and fluctuations around this mean. Note also that if we consider the dynamo instability in absence of noise, $a = 0$, the two bifurcation threshold col-

lapse. As stated by [20], the bifurcation corresponding to $b > 0$ may be difficult to observe in real experiments because, under the threshold, the presence of the Earth external magnetic field always gives rise to magnetic fluctuations qualitatively similar to that above the threshold (magnetic bursts separated by quiescent period). This effect can be taken into account by adding an additive noise to equation (1).

In classical theory of dynamical systems stability, the instability criterion is usually associated with the existence of a positive Lyapunov exponent for the growth of the system energy: $\lim \ln B^2/2t = \lim \ln B/t$. It is possible to find this exponent in our case, by multiplying equation (7) by $B^{d-1} \ln B$ and integrating with respect to B . This yields:

$$\partial_t \langle \ln B \rangle = b, \quad (13)$$

meaning that the Lyapunov exponent in our system is equal to b . The two instability criteria (existence of a normalizable solution or a positive Lyapunov exponent) are therefore identical provided $a > 0$, a necessary condition for integrability of the PDF at infinity.

It is now interesting to discuss qualitatively the meaning of our main result (12). It is possible to show that for isotropic or axisymmetric velocity fluctuations, the coefficient a is positive. So we suspect that the main condition for existence of a dynamo is positivity of b . Therefore, in the limit of zero noise, the term proportional to μ is negligible and the dynamo threshold is only determined by the condition $\langle \partial_k \bar{V}_i e_i e_k \rangle_\phi > 0$. Since the magnetic field will mainly grow in the direction given by the largest eigenvalue λ_{max} of $S_{ij} = \partial_j \bar{V}_i$, this average will be equal to λ_{max} , and the dynamo threshold will be the same as in the deterministic case. Consider now a situation where you increase the noise level. Two different influences on the sign of b then result: one through the factor proportional by μ . According to the sign of this factor, it can therefore favor or hinder the dynamo. Another less obvious influence is through vector orientation. Indeed, noise changes the distribution of magnetic field orientation. For example, if noise induces a flat distribution for e_i , then $\langle \partial_k \bar{V}_i e_i e_k \rangle_\phi = S_{ii} = 0$. Even more dramatic results can be obtained if the noise tends to align the vector along a direction of negative eigenvalue for S_{ij} , since in that case the factor $\langle \partial_k \bar{V}_i e_i e_k \rangle_\phi$ becomes negative, decreasing the dynamo threshold. From these two examples, one sees that in the presence of noise, the dynamo threshold can be completely disconnected from the dynamo threshold in the deterministic system ! It is therefore dangerous to dimension laboratory experiments from arguments based only on the deterministic system.

Our approach gives a quantitative criterion on the dynamo threshold (namely $b > 0$). However, its practical implementation requires the measure on the angular and position variables $G(\mathbf{x}, \mathbf{n}, t)$ and the average of the tensor μ with this measure. One can obtain the measure by

multiplying equation (4) by B^{d-1} and integrating with respect to B . Unfortunately the equation for G can not be solved in the general case and particular types of turbulence statistics have to be considered (isotropic, axisymmetric, etc...). Work is under progress to determine the angular measure in these simple cases. It is however interesting to note that these measure explicitly depend on the tensors α and β defined in (5) and appearing in the Mean Field Equations (6). In that sense, the dynamo threshold does depend on these tensors, albeit in a less explicit way than in the Mean Field Equations (MFE). It would therefore be interesting to confront threshold derived from (MFE), which are μ independent, and from our theory, to see what kind of error in the threshold determination one can expect by using MFE instead of the true, non-perturbative theory.

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