



**HAL**  
open science

# TIME-DELAY REGULARIZATION OF ANISOTROPIC DIFFUSION AND IMAGE PROCESSING

Abdelmounim Belahmidi, Antonin Chambolle

► **To cite this version:**

Abdelmounim Belahmidi, Antonin Chambolle. TIME-DELAY REGULARIZATION OF ANISOTROPIC DIFFUSION AND IMAGE PROCESSING. ESAIM: Mathematical Modelling and Numerical Analysis, 2005, Vol.39, No.2., pp. 231-251. 10.1051/m2an:2005010 . hal-00001401

**HAL Id: hal-00001401**

**<https://hal.science/hal-00001401v1>**

Submitted on 2 Apr 2004

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Time-delay regularization of anisotropic diffusion and image processing

A. Belahmidi, A. Chambolle

January 2004

## Abstract

We study a time-delay regularization of the anisotropic diffusion model for image denoising of Malik and Perona, which has been proposed by Nitzberg and Shiotu. In the two-dimensional case, we show the convergence of a numerical approximation and the existence of a weak solution. Finally, we show some experiments on images.

*Key words* : Image restoration, Variational methods, Numerical approximation, Time-delay regularization, Malik and Perona equation.

## 1 Introduction

In a well-known paper, Malik and Perona [15] have proposed a model for image restoration based on the following partial differential equation :

$$\frac{\partial u}{\partial t} = \operatorname{div} (g(|Du|^2)Du) \quad u(\cdot, 0) = u_0. \quad (1)$$

Here  $u_0$  is the grey level intensity of the original image,  $u(\cdot, t)$  is the restored version, that depends on the scale parameter  $t$ , and  $g$  is a smooth non-increasing positive function with  $g(0) = 1$  and  $sg(s^2) \rightarrow 0$  at infinity. The main idea is that the restoration process obtained by the equation is conditional: if  $x$  is an edge point, where the gradient is large, then the diffusion will be stopped and therefore the edge will be kept. If  $x$  is in homogeneous area, the gradient has to be small, and the diffusion will tend to smooth around  $x$ . By introducing an edge stopping function  $g(|Du|^2)$  in the diffusion process, the model has been considered as an important improvement of the theory of edge detection [17]. The experiments of Malik and Perona were very impressive, edges remained stable over a very long time. It was demonstrated in [16] that edge detection based on this process clearly outperforms the Canny edge detector [3].

Unfortunately, the Malik and Perona model is ill-posed. Indeed, among the functions which Malik and Perona advocate in their papers, we find  $g(s^2) = 1/(1+s^2)$  or  $g(s^2) = e^{-s^2}$  for which no correct theory of equation (1) is available. By writing the equation in dimension two:

$$\frac{\partial u}{\partial t} = g(|Du|^2)|Du|\operatorname{div}\left(\frac{Du}{|Du|}\right) + (g(|Du|^2) + 2|Du|^2g'(|Du|^2))D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right), \quad (2)$$

where  $D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right)$  is the second derivative of  $u$  in the gradient direction and  $|Du|\operatorname{div}\left(\frac{Du}{|Du|}\right)$  is the second derivative in the orthogonal direction, we observe that the diffusion runs backwards if  $sg(s^2)$  is non-increasing. Then, in the regions where the gradient of a solution is large, the process can be interpreted as a backward heat equation which is actually ill posed. In the continuous setting, it means that (1) may have no solution at all. One could also imagine very close pictures producing divergent solutions [11]. In practice, the equation is discretized into a (obviously well-posed) finite-dimensional version of (1), however, it does not seem correct to interpret such a discretization as an approximation of the ill-posed problem (1).

For these reasons, there have been many attempts to understand the Malik and Perona equation and find out whether (1) can be given a sound interpretation. There are essentially two approaches: The first, motivated by favorable numerical results, consists in studying the original equation and in establishing theoretical results that explain the observed behaviour. The second approach consists in modifying the equation by regularizing the term  $g(|Du|^2)$  in order to get a well-posed equation.

## 2 The Malik and Perona equation and the regularized versions

First, we expose the main mathematical results established on the Malik and Perona model. Most of these results are restricted to the dimension one; the unique result in dimension two, given by You et al. [21] confirms the ill-posedness of the equation. Kawohl and Kutev [13] establish, in 1D, nonexistence of global weak solution, and prove the existence and uniqueness of a classical solution only if the initial data has everywhere a small slope. In this case the equation remains parabolic for all time and there is no edge to preserve: the diffusion smoothes the data, like the heat equation would do. They also prove a comparison principle under special assumptions on the initial data.

Kichenassamy [14] shows that in general the Malik and Perona equation does not have a weak solution if the initial data is not analytic in a neighborhood of high gradient regions. His argument is based on interior regularity properties of parabolic equations. Only in dimension one, he proposes a notion of generalized solutions, which are piecewise linear with jumps, and shows existence.

Adopting a numerical viewpoint, Esedoglu [7] studies the one-dimensional Malik and Perona scheme. He establishes by a scaling argument the convergence to an evolution in the continuous setting. The resulting evolution solves a system of heat equations coupled to each other through nonlinear boundary conditions.

Working in dimension one clearly reduces the difficulty by eliminating the first term of (2) which is nothing but the mean curvature motion operator with the coefficient  $g(|Du|^2)$ . As it is known, the mean curvature motion evolves each level line  $\{u = C\}$  with a normal speed proportional to its curvature (see [8, 1] for more details).

In dimension two, You et al. [21] express the anisotropic diffusion of Malik and Perona as the steepest descent of an energy surface and analyze the behaviour of the model. They prove that the ill-posedness is caused by the fact that the energy functional has an infinite number of global minima that are dense in the image space. Each of these minima corresponds to a piecewise constant image. This means that slightly different initial images may end up in different minima for large  $t$ .

As mentioned, another approach relies on the idea that the ill-posedness may be alleviated through the introduction of a smooth version of  $g(|Du|^2)$ . There are essentially two propositions which we consider as a direct derivation from the Malik and Perona Model. The first consists in a spatial regularization, as in the following model:

$$\frac{\partial u}{\partial t} = \operatorname{div} (g(|D G_\sigma * u|^2) Du), \quad (3)$$

whereby  $g(|Du|^2)$  is replaced by  $g(|D G_\sigma * u|^2)$ , where  $G_\sigma$  is a Gaussian with variance  $\sigma$ . In [4], Catté et al. prove existence, uniqueness and regularity of a solution. It is known that  $G_\sigma * u(x, t)$  is nothing but the solution at scale  $\sigma$  of the heat equation with  $u(x, t)$  as initial data.

A first observation is that near a sharp corner, the diffusion coefficient  $g(|D G_\sigma * u|^2)$  may remain very large, hence this model will be unable to preserve corners.

Another problem is the choice of the regularization parameter  $\sigma$ . In fact, this choice is critical in the sense that the diffusion process would be ill-posed if  $\sigma = 0$ , while image features would be blurred for too large an  $\sigma$ . As proposed by Whitaker and Pizer [20], the regularization parameter  $\sigma$  should be a decreasing function in  $t$ , by using large  $\sigma$  initially to suppress noise and reducing  $\sigma$  so that image features are not further blurred. In spite of this, the choice of the initial and final values of  $\sigma$  remains an open question.

The second proposition is a time-delay regularization, where one replaces  $|Du|^2$  with an average of its

values from 0 to  $t$ . Then  $g(|Du|^2)$  is replaced with  $g(v)$  with:

$$v(x, t) = e^{-t}v_0(x) + \int_0^t e^{s-t}|Du(x, s)|^2 ds, \quad (4)$$

where  $v_0$  is an initial data, for example  $v_0 = 0$  or  $|Du_0|^2$ . Therefore the new diffusion process is described by the following system:

$$\frac{\partial u}{\partial t} = \operatorname{div}(g(v)Du) \quad u(\cdot, 0) = u_0, \quad (5)$$

$$\frac{\partial v}{\partial t} = |Du|^2 - v \quad v(\cdot, 0) = v_0. \quad (6)$$

Proposed by Nitzberg and Shiotani [19], this model is very close to the Malik and Perona equation since there is no spatial smoothing. In particular, it should mean that there is no previous movement of the features in the diffusion process. In [2] the authors of the present paper have shown that in any dimension, the system (5)-(6) admits a unique classical solution  $(u, v)$  which can blow up in finite time, and that as long as the solution exists, the equation satisfies the maximum principle and does not create spurious information (that is, strict local extrema). These properties of the system (5)-(6) have encouraged us to study it from a numerical viewpoint. Let us mention that time-delay regularization has been already used in image processing by Cottet and El Ayyadi [6] as anisotropic diffusion tensors.

This paper is organized as follows: In section 3 we propose a natural discretization in time of (5)-(6) with  $|Du|^2$  replaced by  $F(|Du|^2)$ ,  $F$  being a sort of truncation. Numerically this modification does not have any impact on the output images since the threshold implicitly exists in the numerical scheme. Indeed, if the discrete scheme satisfies the maximum principle, then the discrete gradient is always bounded (for example by  $(\max u_0 - \min u_0)/\Delta x$ ,  $\Delta x$  being the grid size). Theoretically, the introduction of  $F$  is a huge regularization of the system (we will see that it yields existence of a weak solution for all time). Section 4 proposes a numerical scheme for solving the system, and section 5 shows some experiments on synthetic and natural images. In section 6 we establish a priori estimates and regularity results on the proposed approximation and prove the main result of this paper. In the section 7 we give the proofs of two technical results on elliptic equations that are needed in section 6.

### 3 Numerical approximation

The goal of this paper is to study and approximate numerically the system:

$$\frac{\partial u}{\partial t} = \operatorname{div}(g(v)Du) \quad u(\cdot, 0) = u_0, \quad (7)$$

$$\frac{\partial v}{\partial t} = F(|Du|^2) - v \quad v(\cdot, 0) = v_0, \quad (8)$$

in  $\Omega \times (0, T)$  where  $\Omega = (0, 1)^2$ ,  $0 < T < \infty$ . We will show that the system admits a weak solution, under the following technical assumptions:

- $g \in C^1([0, +\infty))$  is a positive non-increasing function with  $g(0) = 1$  and  $g(+\infty) = 0$ .
- $F \in C^1([0, +\infty))$  is a smooth version of  $s \rightarrow \min(s, M)$ , where  $M > 0$  is a (large) real number (in particular, we assume  $0 \leq F' \leq 1$ ).

Fixed  $\delta t > 0$ , we define the sequence  $(u_{\delta t}^n, v_{\delta t}^n)_n$  by the semi-implicit scheme:

$$(u_{\delta t}^0, v_{\delta t}^0) = (u_0, v_0) \in (H^1(\Omega) \cap L^\infty(\Omega)) \times (H^1(\Omega) \cap L^\infty(\Omega)), \quad v_0 \geq 0 \quad \text{and}$$

$$\frac{u_{\delta t}^{n+1} - u_{\delta t}^n}{\delta t} = \operatorname{div}(g(v_{\delta t}^n)Du_{\delta t}^{n+1}) \quad \frac{\partial u_{\delta t}^{n+1}}{\partial n} \Big|_{\partial\Omega} = 0 \quad (9)$$

$$\frac{v_{\delta t}^{n+1} - v_{\delta t}^n}{\delta t} = F(|Du_{\delta t}^{n+1}|^2) - v_{\delta t}^{n+1}. \quad (10)$$

We define the piecewise constant (in  $t > 0$ ), functions

$$u_{\delta t}(x, t) = u_{\delta t}^{\lfloor t/\delta t \rfloor + 1}(x),$$

where  $\lfloor \cdot \rfloor$  denotes the integer part. We also define  $(v_{\delta t})$  in the same way. Then we can write the discrete system (9)-(10) in the form ( $\tau^{-\delta t}$  is defined by  $\tau^{-\delta t}f(\cdot, t) = f(\cdot, t - \delta t)$ ):

$$\frac{u_{\delta t} - \tau^{-\delta t}u_{\delta t}}{\delta t} = \operatorname{div}(g(\tau^{-\delta t}v_{\delta t})Du_{\delta t}), \quad \frac{\partial u_{\delta t}}{\partial n} \Big|_{\partial\Omega} = 0, \quad (11)$$

$$\frac{v_{\delta t} - \tau^{-\delta t}v_{\delta t}}{\delta t} = F(|Du_{\delta t}|^2) - v_{\delta t}. \quad (12)$$

The main result of this paper is the following theorem:

**Theorem 1.** *Let  $T > 0$ . There exists a subsequence  $(u_{\delta t_j}, v_{\delta t_j})$  of  $(u_{\delta t}, v_{\delta t})$  and  $(u, v)$  a weak solution of the system (7)-(8) in  $(H^1(\Omega \times (0, T)) \cap L^\infty(\Omega \times (0, T))) \times (H^1(\Omega \times (0, T)) \cap L^\infty(\Omega \times (0, T)))$  such that, we have the convergences, as  $j \rightarrow +\infty$ :*

$$u_{\delta t_j} \longrightarrow u \quad \text{strongly in } L^2(0, T; H^1(\Omega)), \quad (13)$$

$$v_{\delta t_j} \longrightarrow v \quad \text{weakly in } L^2(0, T; H^1(\Omega)). \quad (14)$$

The proof of this theorem will be given in section 6.

## 4 Discretization

To discretize (9)-(10) we denote by  $u_{i,j}^n$  (resp.  $v_{i,j}^n$ ) the approximation of  $u$  (resp.  $v$ ) at point  $(ih, jh)$  ( $0 \leq i, j \leq N$ ) and time  $t = n\delta t$ , where the size of the initial image  $u_0$  is given by  $N \times N$  and  $h = 1/N$ . Using the following finite-differences formulas:

$$\begin{aligned} \Delta_+^x w &= w_{i+1,j} - w_{i,j}, & \Delta_-^x w &= w_{i,j} - w_{i-1,j}, \\ \Delta_+^y w &= w_{i,j+1} - w_{i,j} \quad \text{et} \quad \Delta_-^y w &= w_{i,j} - w_{i,j-1}, \end{aligned}$$

the approximation of  $\operatorname{div}(g(v)Du)$  at point  $(ih, jh)$  and at scale  $t = (n+1)\delta t$  is given by:

$$\frac{1}{h^2} \left( \Delta_-^x (g(v_{i,j}^n) \Delta_+^x u_{i,j}^{n+1}) + \Delta_-^y (g(v_{i,j}^n) \Delta_+^y u_{i,j}^{n+1}) \right).$$

Then the equation (9) becomes:

$$\begin{aligned} \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\delta t} &= \frac{1}{h^2} \left\{ g(v_{i,j}^n) (u_{i+1,j}^{n+1} - u_{i,j}^{n+1}) - g(v_{i-1,j}^n) (u_{i,j}^{n+1} - u_{i-1,j}^{n+1}) \right. \\ &\quad \left. + g(v_{i,j}^n) (u_{i,j+1}^{n+1} - u_{i,j}^{n+1}) + g(v_{i,j-1}^n) (u_{i,j}^{n+1} - u_{i,j-1}^{n+1}) \right\} \quad (15) \end{aligned}$$

with the Neumann boundary condition:

$$\begin{aligned} u_{i,0}^{n+1} - u_{i,1}^{n+1} &= 0, & u_{i,N-1}^{n+1} - u_{i,N}^{n+1} &= 0, \quad \text{for } 0 \leq i \leq N, \\ u_{0,j}^{n+1} - u_{1,j}^{n+1} &= 0, & u_{N-1,j}^{n+1} - u_{N,j}^{n+1} &= 0, \quad \text{for } 0 \leq j \leq N. \end{aligned}$$

Rearranging the right hand side of (15), we get

$$\frac{u^{n+1} - u^n}{\delta t} + h^{-2}A(v^n)u^{n+1} = 0,$$

where the matrix  $A(v^n)$  is tridiagonal by blocks, and positive defined. By classical arguments [5] we know that  $[I + \delta t h^{-2}A(v^n)]$  is invertible.

To avoid any additional anisotropy in the scheme, we try to build a discrete gradient of  $u$  in (10) as rotationally invariant as possible. We use the discretization proposed in [4] and [19] which writes:

$$\begin{aligned} \Delta^x w &= (1 + 2^{\frac{1}{2}})^{-1} \left\{ (w_{i+1,j} - w_{i-1,j}) + 2^{-\frac{1}{2}}(w_{i+1,j-1} - w_{i-1,j-1}) + 2^{-\frac{1}{2}}(w_{i+1,j+1} - w_{i-1,j+1}) \right\}, \\ \Delta^y w &= (1 + 2^{\frac{1}{2}})^{-1} \left\{ (w_{i,j+1} - w_{i,j-1}) + 2^{-\frac{1}{2}}(w_{i+1,j+1} - w_{i+1,j-1}) + 2^{-\frac{1}{2}}(w_{i-1,j+1} - w_{i-1,j-1}) \right\}. \end{aligned}$$

The discretization of (10) is then written (assuming that in the whole range  $[0, \max((\Delta^x u)^2 + (\Delta^y u)^2)]$ , we have  $F(s) = s$ )

$$v_{i,j}^{n+1} = \frac{1}{1 + \delta t} (\delta t h^{-2} ((\Delta^x u)^2 + (\Delta^y u)^2) + v_{i,j}^n). \quad (16)$$

We can now give a discrete version of the maximum principle and show that the proposed algorithm will not create new information (local extrema).

**Lemma 1.** *For all  $n > 0$  and  $(k, l)$ ,  $0 \leq k, l \leq N$ , we have:*

$$\min_{i,j} u_{i,j}^0 \leq \dots \leq \min_{i,j} u_{i,j}^n \leq u_{k,l}^{n+1} \leq \max_{i,j} u_{i,j}^n \leq \dots \leq \max_{i,j} u_{i,j}^0, \quad (17)$$

*In particular, if  $u_{k,l}^{n+1}$  is a strict local maximum (resp. strict local minimum) of  $(u_{i,j}^{n+1})$  then*

$$u_{k,l}^{n+1} < u_{k,l}^n \quad (\text{resp. } u_{k,l}^{n+1} > u_{k,l}^n). \quad (18)$$

PROOF : Let  $u_{k,l}^{n+1}$  a global maximum of  $(u_{i,j}^{n+1})$ , then in particular:

$$\begin{aligned} u_{k,l}^{n+1} - u_{k+1,l}^{n+1} &\geq 0, & u_{k,l}^{n+1} - u_{k-1,l}^{n+1} &\geq 0, \\ u_{k,l}^{n+1} - u_{k,l+1}^{n+1} &\geq 0, & u_{k,l}^{n+1} - u_{k,l-1}^{n+1} &\geq 0. \end{aligned} \quad (19)$$

Using (15), and the fact that  $g > 0$ , we obtain:

$$u_{k,l}^{n+1} \leq u_{k,l}^n,$$

and we deduce:

$$\max_{i,j} u_{i,j}^{n+1} \leq \max_{i,j} u_{i,j}^n \leq \dots \leq \max_{i,j} u_{i,j}^0.$$

In the same way we prove the ‘‘min’’ part of (17), by considering  $u_{k,l}^{n+1}$  a global minimum of  $(u_{i,j}^{n+1})$ . We prove (18) by using the same argument and the fact that we now have strict inequalities in (19).  $\square$

## 5 Experiments

In figure 1 we compare the performances of our scheme to the Catté and al. model [4] and in figure 2 we present an example of restoration on a natural image. The experiments have been done with the edge stopping function

$$g(s^2) = \frac{1}{1 + (s^2/\lambda^2)}.$$

We have chosen a value of  $\lambda = 6$ , for images in the range  $[0, 255]$  and with a spatial grid size  $h = 1$  (contrarily to the convention in the previous section). The temporal increment we have used is  $\delta t = 0.1$ .

Figure 1-(a) is a synthetic image ( $128 \times 128$ ) representing superimposed shapes having each one a constant grey level. Figure 1-b shows image 1-a where 20% of gaussian noise is added. We represent by figures 1-(c) the restoration of the noisy image with the Catté et al. model (3) at scales 4 and 8 (from left to right) and by figures 1-(d) the restoration with our scheme at the same scales. Notice that respectively, the scales 4 and 8 correspond to the stopping time  $t = 8$  and 32. As explained in [4] by the authors of the model (3), the scale  $\sigma$  used in the convolution term  $G_\sigma * u$  must be in taken in relation to the stopping time. Thus in figures 1-(c) we have used  $\sigma = 4$  and 8.

As mentionned in section 2 the threshold introduced by  $F$  implicitly exists in the numerical scheme. Indeed, since the discrete scheme satisfies the maximum principle and the fact that spatial increment is assumed to be 1, then the discrete gradient is always bounded by  $\sqrt{2}(\max u_0 - \min u_0)$  and  $M$  can be chosen to be  $2(\max u_0 - \min u_0)^2$ .

In the left image of 1-(b), the noise is smoothed in the homogeneous areas but is kept near the edges. This drawback is caused by the fact that the diffusion is inhibited also in the neighborhood of edges. Whereas in the left image of 1-(c), the noise is only partially smoothed but in a uniform way. In the right image of 1-(b) the edges and corners are blurred. Indeed, we know that  $\|DG_\sigma * u\|_{L^\infty}$  decreases for large values of  $\sigma$  consequently for large values of  $\sigma$  we diffuse more near edges: in particular, if  $\|DG_\sigma * u\|_{L^\infty} < \lambda$ , the diffusion is never inhibited. Whereas in the right image of 1-(c), the noise has disappeared and the reconstructed image is very close to the original.

Figure 2-(Right) represents a natural image ( $256 \times 256$ ) without additive noise and figure 2-(Left) represents its restoration with our scheme at scale 5 that corresponds to the stopping time  $t = 12.5$ . We remark that salient edges and textures are preserved (see for example the top of the hat) whereas the noise in homogeneous areas is smoothed.

## 6 Numerical analysis

First we check that our schemes makes sense. Indeed, for all  $\delta t > 0$ , the sequence  $(u_{\delta t}^n, v_{\delta t}^n)$  exists and is unique. Equation (10) allows to write  $v_{\delta t}^{n+1}$  explicitly:

$$v_{\delta t}^{n+1} = \frac{1}{1 + \delta t} (\delta t F(|Du_{\delta t}^{n+1}|^2) + v_{\delta t}^n). \quad (20)$$

and by induction we find

$$0 \leq v_{\delta t}^{n+1} \leq (1 - (1 + \delta t)^{-(n+1)})M + (1 + \delta t)^{-(n+1)}\|v_0\|_{L^\infty(\Omega)}.$$

We deduce that  $(v_{\delta t}^n)$  is uniformly bounded in  $L^\infty(\Omega)$  and satisfies:

$$0 \leq v_{\delta t}^n \leq \max(M, \|v_0\|_{L^\infty(\Omega)}) := M', \text{ for all } n \text{ and } \delta t. \quad (21)$$

Using the fact that  $g$  is a positive non-increasing function, we have  $0 < g(M') \leq g(v_{\delta t}^n) \leq 1$ . Therefore equation (9) is strictly elliptic and we know that there exists a unique solution  $u_{\delta t}^{n+1}$  in  $H^1(\Omega)$ . In addition,  $u_{\delta t}^{n+1}$  is given by the problem

$$\min \left\{ E_{(\delta t, n)}(w) = \int_{\Omega} g(v_{\delta t}^n) |Dw|^2 dx + \frac{1}{2\delta t} \int_{\Omega} |w - u_{\delta t}^n|^2 dx : w \in H^1(\Omega) \right\}. \quad (22)$$

By the maximum principle, it is clear that for almost all  $x \in \Omega$  we have

$$\inf u_0 \leq \dots \leq \inf u_{\delta t}^n \leq u_{\delta t}^{n+1}(x) \leq \sup u_{\delta t}^n \leq \dots \leq \sup u_0. \quad (23)$$

Multiplying by  $u_{\delta t}^{n+1}$  the equation (9) and integrating on  $\Omega$  we get

$$0 \leq \delta t \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^{n+1}|^2 dx \leq \int_{\Omega} u_{\delta t}^n u_{\delta t}^{n+1} dx - \int_{\Omega} |u_{\delta t}^{n+1}|^2, \quad (24)$$

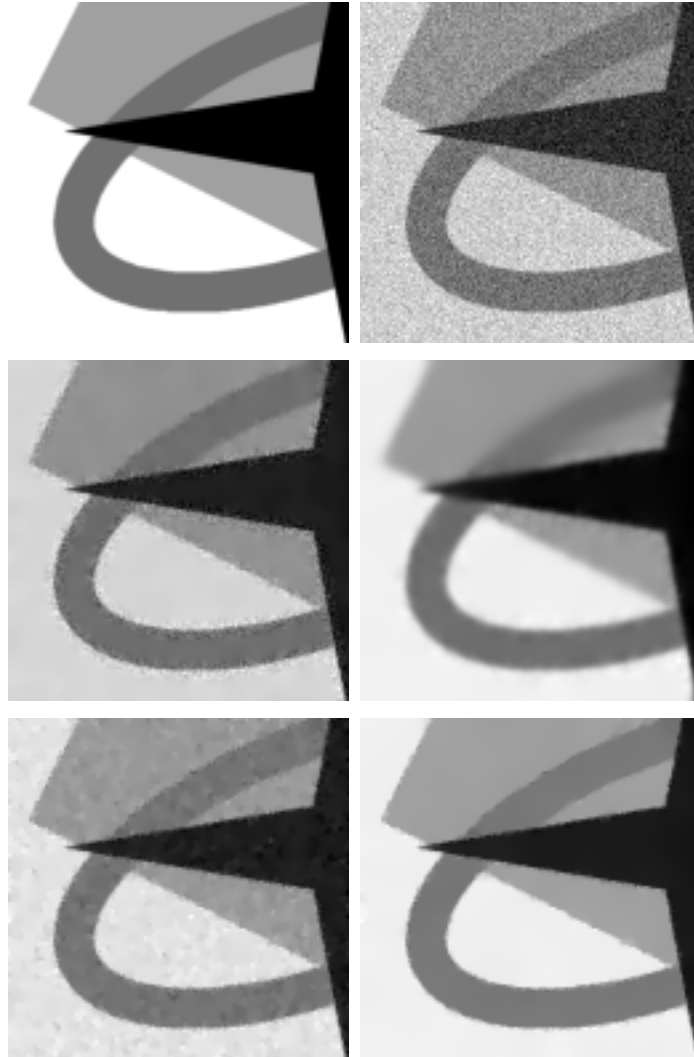


Figure 1: TOP LEFT: **(a)** Original image. TOP RIGHT: **(b)** Image (a) with 20% of gaussian noise. MIDDLE LINE: **(c)** Image (b) restored by the Catté et al. model [4] with scales 4, and 8. BOTTOM LINE: **(d)** Image (b) restored by our scheme with scales 4, and 8.





Figure 2: RIGHT : Original natural image. LEFT: The output of our scheme at scale 5.

from which we deduce

$$\|u_{\delta t}^{n+1}\|_{L^2(\Omega)} \leq \|u_{\delta t}^n\|_{L^2(\Omega)} \leq \dots \leq \|u_0\|_{L^2(\Omega)}. \quad (25)$$

We now define the piecewise affine (in  $t > 0$ )

$$\widehat{u}_{\delta t}(x, t) = (1 - \theta)u_{\delta t}^{\lfloor t/\delta t \rfloor}(x) + \theta u_{\delta t}^{\lfloor t/\delta t \rfloor + 1}(x)$$

with  $\theta = t/\delta t - \lfloor t/\delta t \rfloor \in [0, 1)$ . We also define  $(\widehat{v}_{\delta t})$  in the same way. Then we can write discrete system (11)-(12) in the form

$$\frac{\partial \widehat{u}_{\delta t}}{\partial t} = \operatorname{div}(g(\tau^{-\delta t} v_{\delta t}) D u_{\delta t}), \quad (26)$$

$$\frac{\partial \widehat{v}_{\delta t}}{\partial t} = F(|D u_{\delta t}|^2) - v_{\delta t}. \quad (27)$$

**Lemma 2.**  $(\partial \widehat{v}_{\delta t} / \partial t)$  is uniformly bounded in  $L^\infty((0, T) \times \Omega)$ , and in particular

$$\lim_{\delta t \rightarrow 0} \|\widehat{v}_{\delta t} - v_{\delta t}\|_{L^\infty((0, T); L^2(\Omega))} = 0. \quad (28)$$

PROOF : From (27) and the inequalities (21), we easily deduce the uniform bound of  $(\partial \widehat{v}_{\delta t} / \partial t)$ ,

$$\left\| \frac{\partial \widehat{v}_{\delta t}}{\partial t} \right\|_{L^\infty((0, T) \times \Omega)} \leq M'.$$

Let  $t \in (0, T)$ . We set  $\theta = t/\delta t - \lfloor t/\delta t \rfloor$ , and write

$$\int_{\Omega} |\widehat{v}_{\delta t}(x, t) - v_{\delta t}(x, t)|^2 dx = \int_{\Omega} |(1 - \theta)(v_{\delta t}^{\lfloor t/\delta t \rfloor}(x) - v_{\delta t}^{\lfloor t/\delta t \rfloor + 1}(x))|^2 dx. \quad (29)$$

Hence

$$\int_{\Omega} |\widehat{v}_{\delta t}(x, t) - v_{\delta t}(x, t)|^2 dx \leq \delta t^2 \int_{\Omega} \left| \frac{\partial \widehat{v}_{\delta t}}{\partial t}(x, t) \right|^2 dx \leq \delta t^2 M'^2,$$

(28) follows  $\square$

**Lemma 3.**  $(u_{\delta t})$  is uniformly bounded in  $L^\infty(0, T; H^1(\Omega))$ . More precisely we have:

$$\|Du_{\delta t}\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq \frac{1}{g(M')} \left( \|Du_0\|_{L^2(\Omega)}^2 + \frac{C}{g(M')} \|u_0\|_{L^2(\Omega)}^2 \right) \quad (30)$$

with  $C = (\sup |g'|) \left\| \frac{\partial \widehat{v}_{\delta t}}{\partial t} \right\|_{L^\infty((0, T) \times \Omega)}$ .

PROOF : First we establish a uniform bound on  $(u_{\delta t})$  in  $L^2(0, T; H^1(\Omega))$ , and prove the lemma by showing the following inequality:

$$\|Du_{\delta t}\|_{L^2(0, T; L^2(\Omega))}^2 \leq C_1 \|Du_0\|_{L^2(\Omega)}^2 + C_2 \|Du_{\delta t}\|_{L^2(0, T; L^2(\Omega))}^2 \quad (31)$$

where  $C_1, C_2 > 0$  are constants that will be made precise.

Multiplying (9) by  $u_{\delta t}^{n+1}$  and integrating by part in  $\Omega$  as in (24), we obtain

$$\begin{aligned} g(M') \int_{n\delta t}^{(n+1)\delta t} \int_{\Omega} |Du_{\delta t}|^2 dx dt &= g(M') \delta t \int_{\Omega} |Du_{\delta t}^{n+1}|^2 dx = \\ &\leq \delta t \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^{n+1}|^2 dx \\ &\leq \|u_{\delta t}^n\|_{L^2(\Omega)}^2 - \|u_{\delta t}^{n+1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Then we deduce (for simplicity we use the notation  $k := [T/\delta t]$ )

$$\begin{aligned} g(M') \int_0^T \int_{\Omega} |Du_{\delta t}|^2 dx dt &= g(M') \int_0^{\delta tk} \int_{\Omega} |Du_{\delta t}|^2 dx dt + g(M') \int_{\delta tk}^T \int_{\Omega} |Du_{\delta t}|^2 dx dt \\ &\leq (\|u_0\|_{L^2(\Omega)}^2 - \|u_{\delta t}^k\|_{L^2(\Omega)}^2) + \frac{T - \delta tk}{\delta t} (\|u_{\delta t}^k\|_{L^2(\Omega)}^2 - \|u_{\delta t}^{k+1}\|_{L^2(\Omega)}^2) \\ &\leq \|u_0\|_{L^2(\Omega)}^2 + \frac{T - \delta t(k+1)}{\delta t} \|u_{\delta t}^k\|_{L^2(\Omega)}^2 - \frac{T - \delta tk}{\delta t} \|u_{\delta t}^{k+1}\|_{L^2(\Omega)}^2 \end{aligned}$$

and since  $\delta tk \leq T \leq \delta t(k+1)$ , we obtain

$$\int_0^T \int_{\Omega} |Du_{\delta t}|^2 dx dt \leq \frac{1}{g(M')} \|u_0\|_{L^2(\Omega)}^2. \quad (32)$$

This prove that  $(u_{\delta t})$  is uniformly bounded in  $L^2(0, T; H^1(\Omega))$ .

Now we show (31). For all  $n \geq 0$ , we have

$$\begin{aligned} \int_{\Omega} g(v_{\delta t}^{n+1}) |Du_{\delta t}^{n+1}|^2 dx - \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^n|^2 dx &= \\ \int_{\Omega} (g(v_{\delta t}^{n+1}) - g(v_{\delta t}^n)) |Du_{\delta t}^{n+1}|^2 dx + \int_{\Omega} g(v_{\delta t}^n) (|Du_{\delta t}^{n+1}|^2 - |Du_{\delta t}^n|^2) dx. \end{aligned} \quad (33)$$

Using the minimum problem (22) we have  $E_{(\delta t, n)}(u_{\delta t}^{n+1}) \leq E_{(\delta t, n)}(u_{\delta t}^n)$ , that is,

$$\int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^{n+1}|^2 dx + \frac{1}{2\delta t} \int_{\Omega} |u_{\delta t}^{n+1} - u_{\delta t}^n|^2 dx \leq \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^n|^2 dx, \quad (34)$$

Then the second integral of (33) satisfies

$$\int_{\Omega} g(v_{\delta t}^n) (|Du_{\delta t}^{n+1}|^2 - |Du_{\delta t}^n|^2) dx \leq 0.$$

The first integral of (33) we can be written

$$\begin{aligned} \int_{\Omega} (g(v_{\delta t}^{n+1}) - g(v_{\delta t}^n)) |Du_{\delta t}^{n+1}|^2 dx &\leq \delta t \int_{\Omega} \left| \frac{g(v_{\delta t}^{n+1}) - g(v_{\delta t}^n)}{\delta t} \right| |Du_{\delta t}^{n+1}|^2 dx \\ &\leq C \delta t \int_{\Omega} |Du_{\delta t}^{n+1}|^2 dx, \end{aligned} \quad (35)$$

with  $C := (\sup |g'|) \left\| \frac{\partial \widehat{u}_{\delta t}}{\partial t} \right\|_{L^\infty((0,T) \times \Omega)}$ . Using this estimate in (33), we get

$$\int_{\Omega} g(v_{\delta t}^{n+1}) |Du_{\delta t}^{n+1}|^2 dx - \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^n|^2 dx \leq C \delta t \int_{\Omega} |Du_{\delta t}^{n+1}|^2 dx.$$

Taking the sum as  $n$  varies from 0 to  $[t/\delta t]$  and using the fact that  $u_{\delta t}(\cdot, t) = u_{\delta t}^{[t/\delta t]+1}$ , we obtain

$$\begin{aligned} \int_{\Omega} g(v_{\delta t}^{[t/\delta t]+1}) |Du_{\delta t}^{[t/\delta t]+1}|^2 dx - \int_{\Omega} g(v_0) |Du_0|^2 dx &\leq C \sum_{n=0}^{[t/\delta t]} \delta t \int_{\Omega} |Du_{\delta t}^{n+1}|^2 dx \\ &\leq C \int_0^{\delta t([t/\delta t]+1)} \int_{\Omega} |Du_{\delta t}|^2 dx dt. \end{aligned}$$

We conclude that for all  $t \in (0, T)$

$$\int_{\Omega} g(v_{\delta t}(\cdot, t)) |Du_{\delta t}(\cdot, t)|^2 dx \leq \int_{\Omega} g(v_0) |Du_0|^2 dx + C \int_0^T \int_{\Omega} |Du_{\delta t}|^2 dx dt.$$

Since  $g(M') \leq g(v_{\delta t}) \leq 1$ , we get

$$\|Du_{\delta t}(\cdot, t)\|_{L^2(\Omega)}^2 \leq \frac{1}{g(M')} \left( \int_{\Omega} |Du_0|^2 dx + C \int_0^T \int_{\Omega} |Du_{\delta t}|^2 dx dt \right),$$

this proves (31). To conclude the proof of the lemma we combine the last result with (32) to obtain (30).  $\square$

**Lemma 4.**  $(\widehat{u}_{\delta t})$  is uniformly bounded in  $H^1((0, T) \times \Omega)$  and satisfies the inequality

$$\frac{1}{2} \left\| \frac{\partial \widehat{u}_{\delta t}}{\partial t} \right\|_{L^2((0,T) \times \Omega)}^2 \leq \|Du_0\|_{L^2(\Omega)}^2 + \frac{C}{g(M')} \|u_0\|_{L^2(\Omega)}^2, \quad (36)$$

where  $C$  is the constant of lemma 3. In particular we have

$$\lim_{\delta t \rightarrow 0} \|\widehat{u}_{\delta t} - u_{\delta t}\|_{L^2((0,T); L^2(\Omega))} = 0. \quad (37)$$

PROOF : First we rewrite the inequality (34) in the form

$$\begin{aligned} \int_{n\delta t}^{(n+1)\delta t} \left\| \frac{\partial \widehat{u}_{\delta t}}{\partial t} \right\|_{L^2(\Omega)}^2 dt &= \delta t \int_{\Omega} \left| \frac{u_{\delta t}^{n+1} - u_{\delta t}^n}{\delta t} \right|^2 dx \\ &\leq 2 \left( \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^n|^2 dx - \int_{\Omega} g(v_{\delta t}^{n+1}) |Du_{\delta t}^{n+1}|^2 dx \right). \end{aligned}$$

Let  $k := [T/\delta t]$ ; we obtain using the last inequality

$$\begin{aligned}
& \frac{1}{2} \int_0^T \left\| \frac{\partial \widehat{u}_{\delta t}}{\partial t} \right\|_{L^2(\Omega)}^2 dt = \\
& = \frac{1}{2} \left( \sum_{n=0}^{k-1} \delta t \left\| \frac{u_{\delta t}^{n+1} - u_{\delta t}^n}{\delta t} \right\|_{L^2(\Omega)}^2 + (T - \delta t k) \left\| \frac{u_{\delta t}^{k+1} - u_{\delta t}^k}{\delta t} \right\|_{L^2(\Omega)}^2 \right), \\
& \leq \sum_{n=0}^{k-1} \left( \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^n|^2 dx - \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^{n+1}|^2 dx \right) \\
& \quad + \frac{T - \delta t k}{\delta t} \left( \int_{\Omega} g(v_{\delta t}^k) |Du_{\delta t}^k|^2 dx - g(v_{\delta t}^k) |Du_{\delta t}^{k+1}|^2 dx \right), \\
& \leq \int_{\Omega} g(v_0) |Du_0|^2 dx \tag{38}
\end{aligned}$$

$$+ \sum_{n=1}^{k-1} \left( \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^n|^2 dx - \int_{\Omega} g(v_{\delta t}^{n-1}) |Du_{\delta t}^n|^2 dx \right) \tag{39}$$

$$+ \left( \frac{T - \delta t k}{\delta t} \int_{\Omega} g(v_{\delta t}^k) |Du_{\delta t}^k|^2 dx - \int_{\Omega} g(v_{\delta t}^{k-1}) |Du_{\delta t}^k|^2 dx \right) \tag{40}$$

$$- \frac{T - \delta t k}{\delta t} \int_{\Omega} g(v_{\delta t}^{k+1}) |Du_{\delta t}^{k+1}|^2 dx. \tag{41}$$

Let us estimate the last four terms. Since  $g \leq 1$ , (38) satisfies

$$\int_{\Omega} g(v_0) |Du_0|^2 dx \leq \|Du_0\|_{L^2(\Omega)}^2.$$

For the term (39), we proceed as in (35)

$$\sum_{n=1}^{k-1} \left( \int_{\Omega} g(v_{\delta t}^n) |Du_{\delta t}^n|^2 dx - \int_{\Omega} g(v_{\delta t}^{n-1}) |Du_{\delta t}^n|^2 dx \right) \leq C \int_{\delta t}^{\delta t(k-1)} \int_{\Omega} |Du_{\delta t}|^2 dx dt.$$

We use  $T - \delta t k \leq \delta t$  to estimate the term (40),

$$\left( \frac{T - \delta t k}{\delta t} \int_{\Omega} g(v_{\delta t}^k) |Du_{\delta t}^k|^2 dx - \int_{\Omega} g(v_{\delta t}^{k-1}) |Du_{\delta t}^k|^2 dx \right) \leq C \int_{\delta t(k-1)}^{\delta t k} \int_{\Omega} |Du_{\delta t}|^2 dx dt,$$

and since  $T - \delta t k \geq 0$ , (41) is non-positive. Thus

$$\frac{1}{2} \int_0^T \left\| \frac{\partial \widehat{u}_{\delta t}}{\partial t} \right\|_{L^2(\Omega)}^2 dt \leq \|Du_0\|_{L^2(\Omega)}^2 + C \int_0^T \int_{\Omega} |Du_{\delta t}|^2 dx dt.$$

To find (36), in the last inequality we put the right hand side of (32) in place of the integral term.

To show (37) we proceed as in (29), then we obtain after integrating on  $(0, T)$

$$\int_0^T \int_{\Omega} |\widehat{u}_{\delta t}(x, t) - u_{\delta t}(x, t)|^2 dx dt \leq \delta t^2 \int_0^T \int_{\Omega} \left| \frac{\partial \widehat{u}_{\delta t}}{\partial t} \right|^2 dx dt,$$

which goes to 0 as  $\delta t$  goes to 0.  $\square$

Now we will focus on the regularity and the convergence of the sequence  $(v_{\delta t})$ . The idea is the following: the fact that  $v_{\delta t}^0 \in H^1(\Omega)$ , allows us to establish a regularity on the second derivative of  $u_{\delta t}^1$  which in his turn used to show that  $v_{\delta t}^1 \in H^1(\Omega)$ , and so on. For this, we will use the classical topological degree's theory and a regularity results of solutions of elliptic equations given by D. Gilbarg and N-S. Trudinger [9] and N-G. Meyers [18], to establish the following lemma.

**Lemma 5.** Let  $w \in H^1(\Omega) \cap L^\infty(\Omega; \mathbb{R}_+)$  such that  $0 < \lambda \leq w(x)$  a.e. in  $\Omega$ ,  $f \in L^2(\Omega)$  and  $u \in H^1(\Omega)$  the solution of the elliptic problem

$$\operatorname{div}(wDu) = f, \quad \frac{\partial u}{\partial n} \Big|_{\partial\Omega} = 0, \quad \int_{\Omega} u(x) dx = 0. \quad (42)$$

Then for all bounded continuous function  $\psi \in C(]0, \infty[, \mathbb{R})$  satisfying  $|\psi(s)| \leq \lambda_0$  and  $|s\psi(s)| \leq \lambda_1$  for all  $s > 0$ , we have

$$\|\psi(|Du|)D^2u\|_{L^2(\Omega)} \leq \lambda^{-1} \left( \lambda_0 \|f\|_{L^2(\Omega)} + \lambda_1 \|Dw\|_{L^2(\Omega)} \right). \quad (43)$$

The proof of the lemma 5 is given in section 7.

**Remark 1.** Since the divergence term of (9) has zero average, we have for all  $\delta t > 0$  and all  $n$   $\int_{\Omega} u_0(x) dx = \int_{\Omega} u_{\delta t}^n(x) dx$ . Thus, using the fact that our model is grey level shift invariant, we can assume that  $u_0$  has zero average in  $\Omega$ : it is not restrictive as we may always replace  $u_0$  with  $u_0 - \int_{\Omega} u_0(x) dx$ . This allows to have  $\int_{\Omega} u_{\delta t}^n(x) dx = 0$  for all  $\delta t$  and all  $n$ .

**Lemma 6.** For all  $n \geq 0$ , we have  $v_{\delta t}^n \in H^1(\Omega)$ .

PROOF : We begin by proving that  $v_{\delta t}^1 \in H^1(\Omega)$  Since  $v_{\delta t}^1$  is a linear combination of  $F(|Du_{\delta t}^1|^2)$  and  $v_0 \in H^1(\Omega)$ , its amounts to show that  $F(|Du_{\delta t}^1|^2) \in H^1(\Omega)$ . The first step is to determine the distributional derivative of  $F(|Du_{\delta t}^1|^2)$ . For simplicity we use the notation  $u := u_{\delta t}^1$ .

By applying lemma 5 to the equation (9) with  $n = 0$ , we know (from the proof of the same lemma) that there exists  $\epsilon \in (0, 1)$  such that  $D^2u \in L^{1+\epsilon}(\Omega)$ , and since  $u \in H^1(\Omega)$  in particular we have  $u \in W^{2,1}(\Omega)$ . Then, there exists a sequence of  $C^2$  functions  $(u_n)_n$  that strongly converges to  $u$  in  $W^{2,1}(\Omega)$  and satisfy  $u_n \rightarrow u$ ,  $Du_n \rightarrow Du$  a.e. in  $\Omega$ .

Let  $\phi \in C_0^\infty(\Omega)$ . Since  $F$  is bounded and continuous we have  $|F(|Du_n|^2) \partial_i \phi| \leq M |\partial_i \phi|$  and  $F(|Du_n|^2) \rightarrow F(|Du|^2)$  a.e. in  $\Omega$ . Then by applying the Lebesgue Theorem we obtain the convergence

$$\int_{\Omega} F(|Du_n|^2) \partial_i \phi dx \rightarrow \int_{\Omega} F(|Du|^2) \partial_i \phi dx, \quad \text{as } n \rightarrow \infty. \quad (44)$$

By the fact that  $F'(s) = 0$  for large values of  $s$  its clear that  $(F'(|Du_n|^2) \partial_j u_n)_n$  is bounded in  $L^\infty(\Omega)$ . Then there exists a function  $\xi \in L^\infty(\Omega)$  and a subsequence still denoted by  $(F'(|Du_n|^2) \partial_j u_n)_n$  such that  $F'(|Du_n|^2) \partial_j u_n \xrightarrow{*} \xi$  in  $L^\infty(\Omega)$ . By using the continuity of  $F'$  we have  $F'(|Du_n|^2) \partial_j u_n \rightarrow F'(|Du|^2) \partial_j u$  a.e. in  $\Omega$ . Then  $\xi = F'(|Du|^2) \partial_j u$ . Combining the last weak convergence with the strong convergence  $\partial_{ij} u_n \phi \rightarrow \partial_{ij} u \phi$  in  $L^1(\Omega)$  (here we use  $\|\partial_{ij} u_n \phi - \partial_{ij} u \phi\|_{L^1} \leq \|\phi\|_{L^\infty} \|\partial_{ij} u_n - \partial_{ij} u\|_{L^1} \rightarrow 0$ ), we obtain

$$\int_{\Omega} -2F'(|Du_n|^2) \partial_{ij} u_n \partial_j u_n \phi dx \rightarrow \int_{\Omega} -2F'(|Du|^2) \partial_{ij} u \partial_j u \phi dx, \quad \text{as } n \rightarrow \infty. \quad (45)$$

The fact that the two sequences in the left hand side of (44) and (45) are identical, proves that the distributional derivative of  $F(|Du|^2)$  is given by  $-2F'(|Du|^2) D^2u Du$ .

The second step is to show that  $D(F(|Du|^2)) \in L^2(\Omega)$ . Indeed, we have  $|F'(|Du|^2) \partial_i u (1 + |Du|)| \leq C(M)$  a.e. in  $\Omega$  with  $C(M) = (M^{\frac{1}{2}} + M)$ . Then we can write

$$\|D(F(|Du|^2))\|_{L^2(\Omega)} \leq C(M) \left\| \frac{D^2u}{1 + |Du|} \right\|_{L^2(\Omega)}. \quad (46)$$

Applying once more the lemma 5 to the equation (9), with  $n = 0$  and  $\psi(s) = 1/(1 + s)$  to conclude that the right hand side of (46) is bounded in  $L^2(\Omega)$  by writing

$$\left\| \frac{D^2u}{1 + |Du|} \right\|_{L^2(\Omega)} \leq (g(M'))^{-1} \left( \left\| \frac{u - u_0}{\delta t} \right\|_{L^2(\Omega)} + \|Dv_0\|_{L^2(\Omega)} \right). \quad (47)$$

We return to equation (20). Since  $v_0 \in H^1(\Omega)$ , we deduce that  $v_{\delta t}^1 \in H^1(\Omega)$ . By induction we conclude that  $v_{\delta t}^n \in H^1(\Omega)$  for all  $n > 0$ . This proves the lemma.  $\square$

**Lemma 7.** *The sequence  $(v_{\delta t})$  is uniformly bounded in  $L^\infty(0, T; H^1(\Omega))$ . In addition we have*

$$\|Dv_{\delta t}(\cdot, t)\|_{L^2(\Omega)} \leq e^{Kt} \|Dv_0\|_{L^2(\Omega)} + K \int_0^t e^{K(t-s)} \left\| \frac{\partial \widehat{u}_{\delta t}}{\partial t}(\cdot, s) \right\|_{L^2(\Omega)} ds.$$

with  $K = (g(M'))^{-1}(M^{\frac{1}{2}} + M)$ .

PROOF : Deriving the equation (20) with  $n = 0$ , and using the  $L^2$  norm, we get

$$\|Dv_{\delta t}^1\|_{L^2(\Omega)} \leq \frac{\delta t}{1 + \delta t} \|D(F(|Du_{\delta t}^1|^2))\|_{L^2(\Omega)} + \frac{1}{1 + \delta t} \|Dv_0\|_{L^2(\Omega)}.$$

Then using (46) and (47), we obtain

$$\|Dv_{\delta t}^1\|_{L^2(\Omega)} \leq \frac{K\delta t}{1 + \delta t} \left\| \frac{u_{\delta t}^1 - u_0}{\delta t} \right\|_{L^2(\Omega)} + \frac{1 + K\delta t}{1 + \delta t} \|Dv_0\|_{L^2(\Omega)},$$

with  $K := (g(M'))^{-1}C(M)$ . We may prove in the same way that

$$\|Dv_{\delta t}^{n+1}\|_{L^2(\Omega)} \leq \frac{K\delta t}{1 + \delta t} \left\| \frac{u_{\delta t}^{n+1} - u_{\delta t}^n}{\delta t} \right\|_{L^2(\Omega)} + \frac{1 + K\delta t}{1 + \delta t} \|Dv_{\delta t}^n\|_{L^2(\Omega)}.$$

Then by induction we get for all  $n$

$$\|Dv_{\delta t}^n\|_{L^2(\Omega)} \leq \frac{K}{1 + \delta t} \sum_{j=1}^n \left\{ \left( \frac{1 + K\delta t}{1 + \delta t} \right)^{n-j} \delta t \left\| \frac{u_{\delta t}^j - u_{\delta t}^{j-1}}{\delta t} \right\|_{L^2(\Omega)} \right\} + \left( \frac{1 + K\delta t}{1 + \delta t} \right)^n \|Dv_0\|_{L^2(\Omega)}.$$

Finally by using the inequality

$$\left( \frac{1 + K\delta t}{1 + \delta t} \right)^{n-j} \leq e^{K\delta t(n-j)},$$

we obtain for all  $t \in (0, T)$  ( $n = [t/\delta t]$ ),

$$\|Dv_{\delta t}(\cdot, t)\|_{L^2(\Omega)} \leq K \int_0^t e^{K(t-s)} \left\| \frac{\partial \widehat{u}_{\delta t}}{\partial t}(\cdot, s) \right\|_{L^2(\Omega)} ds + e^{Kt} \|Dv_0\|_{L^2(\Omega)}.$$

In particular we have

$$\|Dv_{\delta t}(\cdot, t)\|_{L^2(\Omega)} \leq KT e^{KT} \left\| \frac{\partial \widehat{u}_{\delta t}}{\partial t} \right\|_{L^2((0, T); L^2(\Omega))} + e^{KT} \|Dv_0\|_{L^2(\Omega)},$$

which is bounded according to lemma 4.  $\square$

PROOF OF THEOREM 1. According to (23) and the lemma 3 and 4 there exists two subsequences,  $(u_{\delta t_j})$  and  $(\widehat{u}_{\delta t_j})$ , and a function  $u \in H^1(\Omega \times (0, T)) \cap L^\infty(\Omega \times (0, T))$  such that

$$\begin{aligned} \widehat{u}_{\delta t_j}, u_{\delta t_j} &\xrightarrow{j \rightarrow +\infty} u \quad \text{strongly in } L^2(\Omega \times (0, T)), \\ \widehat{u}_{\delta t_j}, u_{\delta t_j} &\xrightarrow{j \rightarrow +\infty} u \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \\ \widehat{u}_{\delta t_j} &\xrightarrow{j \rightarrow +\infty} u \quad \text{weakly in } H^1(\Omega \times (0, T)). \end{aligned}$$

We draw the same conclusion from (21) and the lemma 2 and 7 if we replace  $(u_{\delta t_j}, \widehat{u}_{\delta t_j})$  with  $(v_{\delta t_j}, \widehat{v}_{\delta t_j})$  and  $u$  with  $v$ . It remains to prove the strong convergence (13) and the fact that  $(u, v)$  is a solution of a system (7)-(8).

Up to a subsequence, we can assume that  $v_{\delta t_j} \rightarrow v$  a.e. in  $\Omega \times (0, T)$ . In addition, by using the fact that  $(\partial \widehat{v}_{\delta t} / \partial t)$  is bounded in  $L^\infty(\Omega \times (0, T))$ , we deduce that  $\tau^{-\delta t_j} v_{\delta t_j} \rightarrow v$  a.e. in  $\Omega \times (0, T)$  and by continuity of  $g$ , we also obtain that  $g(\tau^{-\delta t_j} v_{\delta t_j}) \rightarrow g(v)$  a.e. in  $\Omega \times (0, T)$ .

Let  $\phi \in C^\infty(\overline{\Omega} \times (0, T))$ . Multiplying the equation (26) by  $\phi$  with  $\delta t_j$  in place of  $\delta t$ , and integrating the result in  $\Omega \times (0, T)$ , we get

$$\int_0^T \int_\Omega \frac{\partial \widehat{u}_{\delta t_j}}{\partial t} \phi \, dx \, dt = - \int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) Du_{\delta t_j} D\phi \, dx \, dt.$$

We study the limit of the two terms of the equality as  $j$  goes to  $+\infty$ . By using the weak convergence  $\partial \widehat{u}_{\delta t_j} / \partial t \rightharpoonup \partial u / \partial t$  in  $L^2(\Omega \times (0, T))$ ,

$$\int_0^T \int_\Omega \frac{\partial \widehat{u}_{\delta t_j}}{\partial t} \phi \, dx \, dt \xrightarrow{j \rightarrow +\infty} \int_0^T \int_\Omega \frac{\partial u}{\partial t} \phi \, dx \, dt.$$

Combining the weak convergence of  $Du_{\delta t_j}$  to  $Du$  in  $L^2((\Omega \times (0, T)); \mathbb{R}^2)$  with the strong convergence of  $g(\tau^{-\delta t_j} v_{\delta t_j}) D\phi$  to  $g(v) D\phi$  in  $L^2((\Omega \times (0, T)); \mathbb{R}^2)$  (using Lebesgue's dominated convergence theorem), we deduce that

$$\int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) Du_{\delta t_j} D\phi \, dx \, dt \xrightarrow{j \rightarrow +\infty} \int_0^T \int_\Omega g(v) Du D\phi \, dx \, dt.$$

Then we obtain

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t} \phi \, dx \, dt = - \int_0^T \int_\Omega g(v) Du D\phi \, dx \, dt, \quad (48)$$

which means that  $u$  is a weak solution of (7).

Now, to prove the convergence (13), it remains to show that  $Du_{\delta t_j}$  strongly converges to  $Du$  in  $L^2((\Omega \times (0, T)); \mathbb{R}^2)$ . For this we write

$$\begin{aligned} & \int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) |Du_{\delta t_j} - Du|^2 \, dx \, dt = \\ &= \int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) |Du_{\delta t_j}|^2 \, dx \, dt - 2 \int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) (Du_{\delta t_j} \cdot Du) \, dx \, dt \\ & \quad + \int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) |Du|^2 \, dx \, dt \\ &= - \int_0^T \int_\Omega \frac{\partial \widehat{u}_{\delta t_j}}{\partial t} u_{\delta t_j} \, dx \, dt + 2 \int_0^T \int_\Omega \frac{\partial \widehat{u}_{\delta t_j}}{\partial t} u \, dx \, dt + \int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) |Du|^2 \, dx \, dt. \end{aligned} \quad (49)$$

Let us study the convergence of each term of (49) as  $j$  goes to  $+\infty$ . Taking into account that  $\partial \widehat{u}_{\delta t_j} / \partial t$  converges weakly to  $\partial u / \partial t$  and that  $u_{\delta t_j}$  converges strongly to  $u$ , both in  $L^2(\Omega \times (0, T))$ , we obtain:

$$\begin{aligned} & \int_0^T \int_\Omega \frac{\partial \widehat{u}_{\delta t_j}}{\partial t} u_{\delta t_j} \, dx \, dt \xrightarrow{j \rightarrow +\infty} \int_0^T \int_\Omega \frac{\partial u}{\partial t} u \, dx \, dt, \\ & \int_0^T \int_\Omega \frac{\partial \widehat{u}_{\delta t_j}}{\partial t} u \, dx \, dt \xrightarrow{j \rightarrow +\infty} \int_0^T \int_\Omega \frac{\partial u}{\partial t} u \, dx \, dt. \end{aligned}$$

Using the fact that  $g(\tau^{-\delta t_j} v_{\delta t_j}) \rightarrow g(v)$  a.e. and  $g(\tau^{-\delta t_j} v_{\delta t_j}) |Du|^2 \leq |Du|^2$ , we deduce (by Lebesgue's theorem)

$$\int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) |Du|^2 \, dx \, dt \xrightarrow{j \rightarrow +\infty} \int_0^T \int_\Omega g(v) |Du|^2 \, dx \, dt.$$

thus

$$\lim_{j \rightarrow +\infty} \int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) |Du_{\delta t_j} - Du|^2 \, dx \, dt = \int_0^T \int_\Omega \frac{\partial u}{\partial t} u \, dx \, dt + \int_0^T \int_\Omega g(v) |Du|^2 \, dx \, dt.$$

By density of  $C^\infty(\overline{\Omega} \times (0, T))$  in  $H^1(\Omega \times (0, T))$  we substitute  $\phi$  by  $u$  in the equation (48), and get

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t} u \, dx \, dt = - \int_0^T \int_\Omega g(v) |Du|^2 \, dx \, dt,$$

then

$$g(M') \int_0^T \int_\Omega |Du_{\delta t_j} - Du|^2 \, dx \, dt \leq \lim_{j \rightarrow +\infty} \int_0^T \int_\Omega g(\tau^{-\delta t_j} v_{\delta t_j}) |Du_{\delta t_j} - Du|^2 \, dx \, dt = 0$$

this shows the strong convergence (13).

Finally, up to a subsequence we have that  $Du_{\delta t_j} \rightarrow Du$  a.e. in  $\Omega \times (0, T)$  and the continuity of  $F$  allows to obtain that  $F(|Du_{\delta t_j}|^2) \rightarrow F(|Du|^2)$  a.e. in  $\Omega \times (0, T)$ . This prove that  $v$  is a solution of (8).  $\square$

## 7 Appendix

PROOF OF LEMMA 5. By a density argument there exists a sequence  $(w_n)$  in  $C^\infty(\overline{\Omega})$  that converges to  $w$  in  $H^1(\Omega)$ . Using classical truncation and convolution arguments we can chose  $(w_n)$  in  $L^\infty(\Omega; \mathbb{R}_+)$  and which satisfy  $\lambda \leq w_n$  for all  $n \geq 0$ . We deduce that the solution  $u_n$  of the elliptic problem:

$$\operatorname{div}(w_n Du_n) = f, \quad \frac{\partial u_n}{\partial n} \Big|_{\partial\Omega} = 0, \quad \int_\Omega u_n(x) \, dx = 0. \quad (50)$$

exists and unique in  $H^1(\Omega)$ . Moreover, from  $\|Dw_n\|_{L^\infty(\Omega)} < \infty$ ,  $f \in L^2(\Omega)$  and using classical regularity results it appears that  $u_n \in H^2(\Omega)$ .

First we prove that the inequality (43) is true for the regularized problem (50). For this writing (50) in the form

$$w_n \Delta u_n = -f + Dw_n Du_n, \quad (51)$$

and multiplying the equation by  $\psi(|Du_n|)$ , then we get using the  $L^2$ -norm

$$\|\psi(|Du_n|) w_n \Delta u_n\|_{L^2(\Omega)} \leq \|\psi(|Du_n|) f\|_{L^2(\Omega)} + \|\psi(|Du_n|) Dw_n Du_n\|_{L^2(\Omega)}.$$

As  $0 < \lambda \leq w_n$ ,  $|\psi| \leq \lambda_0$  and  $|s\psi(s)| \leq \lambda_1$  together with the last inequality, we obtain

$$\|\psi(|Du_n|) \Delta u_n\|_{L^2(\Omega)} \leq \lambda^{-1} \left( \lambda_0 \|f\|_{L^2(\Omega)} + \lambda_1 \|Dw_n\|_{L^2(\Omega)} \right). \quad (52)$$

To continue we need the following lemma that we prove in the end of this section (let us recall  $\Omega$  is the plane square  $(0, 1)^2$ ):

**Lemma 8.** *Let  $u \in H^2(\Omega)$  an  $\Omega$ -periodic function, then for all bounded continuous function  $\psi \in C([0, \infty[, \mathbb{R})$ , we have the equality*

$$\|\psi(|Du|) \Delta u\|_{L^2(\Omega)} = \|\psi(|Du|) D^2 u\|_{L^2(\Omega)}.$$

Now we extend  $u_n$  on all  $\mathbb{R}^2$  by symmetry and periodicity to a function  $\tilde{u}_n$   $2\Omega$ -periodic. Then invoking lemma 8, we get

$$\|\psi(|D\tilde{u}_n|) D^2 \tilde{u}_n\|_{L^2(2\Omega)} = \|\psi(|D\tilde{u}_n|) \Delta \tilde{u}_n\|_{L^2(2\Omega)},$$

and by symmetry of  $\tilde{u}_n$  in  $2\Omega$ , we easily deduce that

$$\|\psi(|Du_n|) D^2 u_n\|_{L^2(\Omega)} = \|\psi(|Du_n|) \Delta u_n\|_{L^2(\Omega)},$$

that we use in (52) to obtain

$$\|\psi(|Du_n|) D^2 u_n\|_{L^2(\Omega)} \leq \lambda^{-1} \left( \lambda_0 \|f\|_{L^2(\Omega)} + \lambda_1 \|Dw_n\|_{L^2(\Omega)} \right). \quad (53)$$



This proves that the inequality (43) is true for the regularized problem (50).

Inequality (53) implies that the sequence  $(\psi(|Du_n|)D^2u_n)$  is bounded. Then there exists a subsequence (still denoted by  $(\psi(|Du_n|)D^2u_n)$ ) weakly convergent in  $L^2(\Omega)$ . Let us assume that its weak limit is  $\psi(|Du|)D^2u$ . Then by applying Fatou's lemma we obtain

$$\begin{aligned} \|\psi(|Du|)D^2u\|_{L^2(\Omega)} &\leq \liminf_{n \rightarrow +\infty} \|\psi(|Du_n|)D^2u_n\|_{L^2(\Omega)} \\ &\leq \lambda^{-1} \liminf_{n \rightarrow +\infty} \left( \lambda_0 \|f\|_{L^2(\Omega)} + \lambda_1 \|Dw_n\|_{L^2(\Omega)} \right), \\ &\leq \lambda^{-1} \left( \lambda_0 \|f\|_{L^2(\Omega)} + \lambda_1 \|Dw\|_{L^2(\Omega)} \right), \end{aligned}$$

showing in particular (43).

Before proving that the weak limit of  $(\psi(|Du_n|)D^2u_n)$  is  $\psi(|Du|)D^2u$ , let us show that up to a subsequence,  $(u_n)$  is strongly convergent in  $H^1(\Omega)$ . Indeed, multiplying  $\operatorname{div}(w_n Du_n) = f$  by  $u_n$  and integrating the result on  $\Omega$ ,

$$\int_{\Omega} w_n |Du_n|^2 dx = - \int_{\Omega} f u_n dx.$$

The left hand side satisfies

$$\lambda \|Du_n\|_{L^2(\Omega)}^2 \leq \int_{\Omega} w_n |Du_n|^2 dx,$$

and using the Hölder and Poincare-Wirtinger inequalities, we get for the right hand side

$$\int_{\Omega} f u_n dx \leq \|f\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \|Du_n\|_{L^2(\Omega)}. \quad (54)$$

We deduce that  $(u_n)$  is uniformly bounded in  $H^1(\Omega)$  and we have

$$\|Du_n\|_{L^2(\Omega)} \leq \frac{C}{\lambda} \|f\|_{L^2(\Omega)}. \quad (55)$$

Then there exists  $u \in H^1(\Omega)$  and a subsequence still denoted by  $(u_n)$  that strongly converges to  $u$  in  $L^2(\Omega)$ , weakly in  $H^1(\Omega)$  and a.e. in  $\Omega$ . To prove the strong convergence  $Du_n \rightarrow Du$  in  $L^2(\Omega)$  we can write

$$\int_{\Omega} w_n |Du_n - Du|^2 dx = \int_{\Omega} w_n |Du_n|^2 dx - 2 \int_{\Omega} w_n Du_n Du dx + \int_{\Omega} w_n |Du|^2 dx,$$

and

$$\begin{aligned} \int_{\Omega} w_n |Du_n|^2 dx &= - \int_{\Omega} f u_n dx \xrightarrow{n \rightarrow +\infty} - \int_{\Omega} f u dx, \\ \int_{\Omega} w_n |Du|^2 dx &\xrightarrow{n \rightarrow +\infty} \int_{\Omega} w |Du|^2 dx = - \int_{\Omega} f u dx, \\ -2 \int_{\Omega} w_n Du_n Du dx &= 2 \int_{\Omega} f u dx, \end{aligned}$$

thus

$$\lambda \int_{\Omega} |Du_n - Du|^2 dx \leq \int_{\Omega} w_n |Du_n - Du|^2 dx \xrightarrow{n \rightarrow +\infty} 0.$$

Now we are going to study the convergence of the sequences  $(D^2u_n)$  and  $(\psi(|Du_n|))$  to prove that  $(\psi(|Du_n|)D^2u_n) \rightharpoonup \psi(|Du|)D^2u$  weakly in  $L^2(\Omega)$ .

Denoting  $\Omega_1 = (-1, 2) \times (-1, 2)$  and choosing a function  $\xi \in C_c^\infty(\mathbb{R}^2)$ , such that  $\Omega \subset\subset \text{supp}(\xi) \subset\subset \Omega_1$ , and satisfies

$$\xi(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^2/\Omega_1. \end{cases}$$

Remark that  $\tilde{u}_n \xi \in W_0^{2,1+\epsilon}(\Omega_1)$  for all  $\epsilon \in (0, 1)$  ( $\epsilon$  fixed later). Then, according to Gilbarg-Trudinger [9] Chap. 9, Corollary 10, there exists a constant  $C(\epsilon) > 0$  such that

$$\|D^2(\tilde{u}_n \xi)\|_{L^{1+\epsilon}(\Omega_1)} \leq C(\epsilon) \|\Delta(\tilde{u}_n \xi)\|_{L^{1+\epsilon}(\Omega_1)}.$$

It follows that

$$\begin{aligned} \|D^2 u_n\|_{L^{1+\epsilon}(\Omega)} &\leq \|D^2(\tilde{u}_n \xi)\|_{L^{1+\epsilon}(\Omega_1)}, \\ &\leq C(\epsilon) \|\Delta(\tilde{u}_n \xi)\|_{L^{1+\epsilon}(\Omega_1)}, \\ &\leq C(\epsilon) \left\{ \|\xi \Delta \tilde{u}_n\|_{L^{1+\epsilon}(\Omega_1)} + \|\tilde{u}_n \Delta \xi\|_{L^{1+\epsilon}(\Omega_1)} + 2\|D\tilde{u}_n D\xi\|_{L^{1+\epsilon}(\Omega_1)} \right\}, \\ &\leq C(\epsilon) C \left\{ \|\Delta \tilde{u}_n\|_{L^{1+\epsilon}(\Omega_1)} + \|\tilde{u}_n\|_{L^{1+\epsilon}(\Omega_1)} + \|D\tilde{u}_n\|_{L^{1+\epsilon}(\Omega_1)} \right\}, \end{aligned}$$

where  $C := \max(\|\xi\|_\infty, 2\|D\xi\|_\infty, \|\Delta\xi\|_\infty)$ . Thus

$$\|D^2 u_n\|_{L^{1+\epsilon}(\Omega)} \leq 9C(\epsilon)C \left\{ \|\Delta u_n\|_{L^{1+\epsilon}(\Omega)} + \|u_n\|_{L^{1+\epsilon}(\Omega)} + 2\|Du_n\|_{L^{1+\epsilon}(\Omega)} \right\}. \quad (56)$$

The two sequences  $(\|u_n\|_{L^{1+\epsilon}(\Omega)})$  and  $(\|Du_n\|_{L^{1+\epsilon}(\Omega)})$  are bounded according to (54) and (55). For the sequence  $(\|\Delta u_n\|_{L^{1+\epsilon}(\Omega)})$  we use (51), and write

$$\|\Delta u_n\|_{L^{1+\epsilon}(\Omega)} \leq \lambda^{-1} (\|f\|_{L^{1+\epsilon}(\Omega)} + \|Dv_n Du_n\|_{L^{1+\epsilon}(\Omega)}). \quad (57)$$

Then it remains to show that the sequence  $(\|Dw_n Du_n\|_{L^{1+\epsilon}(\Omega)})$  is bounded. Remark that by the symmetry and the fact that  $u_n$  satisfies the Neumann type boundary, the equation

$$\tilde{f} = \text{div}(\tilde{w}_n D\tilde{u}_n)$$

still holds in  $\Omega_2 := (-2, 3) \times (-2, 3)$  where the functions  $\tilde{f}$  and  $\tilde{w}_n$  are defined in the same way that  $\tilde{u}_n$ . Then according to Meyers' Theorem 2 in [18], there exists  $p > 2$ , depending only on  $\max \tilde{w} / \min \tilde{w} \leq \|w\|_\infty / \lambda$ , such that for all  $x \in \Omega_2$  and for all  $R > 0$  satisfying  $B(x, R) \subset B(x, 2R) \subset \Omega_2$ , we have:

$$\|D\tilde{u}_n\|_{L^p(B(x, R))} \leq C \left\{ R^{2(\frac{1}{p} - \frac{1}{2}) - 1} \|\tilde{u}_n\|_{L^2(B(x, 2R))} + R^{2(\frac{1}{p} - \frac{1}{2}) + 1} \|\tilde{f}\|_{L^2(B(x, 2R))} \right\}.$$

Choosing  $x \in \Omega$  and  $R$  such that  $\Omega \subset B(x, R)$ , for example  $x = (\frac{1}{2}, \frac{1}{2})$  and  $R = \sqrt{2}$ , we deduce that  $(|Du_n|)$  is bounded in  $L^p(\Omega)$ .

Coming back to (57) and choosing  $\epsilon = (p - 2)/(p + 2)$ , then the sequence  $(Dw_n Du_n)$  is bounded in  $L^{1+\epsilon}(\Omega)$  and consequently  $(D^2 u_n)$  is bounded in  $L^{1+\epsilon}(\Omega)$ . We conclude that up to extracting a subsequence

$$D^2 u_n \xrightarrow{n \rightarrow +\infty} D^2 u \quad \text{weakly in } L^{1+\epsilon}(\Omega). \quad (58)$$

Consider  $\phi \in C^\infty(\overline{\Omega})$ . Since  $Du_n \rightarrow Du$  a.e. in  $\Omega$ , the continuity and boundness of  $\psi$ , we obtain using the Lebesgue theorem

$$\psi(|Du_n|)\phi \xrightarrow{n \rightarrow +\infty} \psi(|Du|)\phi \quad \text{in } L^q(\Omega) \quad \text{for all } q \in [1, \infty).$$

Finally, choosing  $q$  such that  $1/(1 + \epsilon) + 1/q = 1$  and combining the last strong convergence with the weak convergence (58), we obtain

$$\int_{\Omega} \psi(|Du_n|) D^2 u_n \phi \, dx \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \psi(|Du|) D^2 u \phi \, dx.$$

This concludes the proof of the lemma.  $\square$

To prove the lemma 8 we need the following result which comes from the topological degree's theory, and is a particular case of the lemma 2.10 of [12] (see also the theorem 6 of [10]):

**Lemma 9.** *Let  $f \in C^1(\mathbb{R}^2; \mathbb{R}^2)$  an  $\Omega$ -periodic function. Then for all  $\psi \in C_0(]0, +\infty[)$  we have*

$$\int_{\Omega} \psi(|f(x)|) J_f(x) dx = 0, \quad (59)$$

where  $J_f(x)$  is the Jacobian of  $f$  at point  $x$ , i.e.  $J_f(x) = \det [\partial_i f^j(x)]$ .

PROOF OF LEMMA 8. Let  $u \in C^2(\mathbb{R}^2)$  an  $\Omega$ -periodic function,  $\psi \in C_0(]0, \infty[; \mathbb{R})$ . Using (59):

$$\int_{\Omega} \psi(|Du|) J_{Du} dx dy = 0,$$

that is,

$$\int_{\Omega} \psi(|Du|) \partial_{xx} u \partial_{yy} u dx dy = \int_{\Omega} \psi(|Du|) (\partial_{xy} u)^2 dx dy.$$

This implies

$$\int_{\Omega} \psi(|Du|) \left( (\partial_{xx} u)^2 + (\partial_{yy} u)^2 + 2\partial_{xx} u \partial_{yy} u \right) dx dy = \int_{\Omega} \psi(|Du|) \left( (\partial_{xx} u)^2 + (\partial_{yy} u)^2 + 2(\partial_{xy} u)^2 \right) dx dy,$$

that is,

$$\int_{\Omega} \psi(|Du|) |\Delta u|^2 dx dy = \int_{\Omega} \psi(|Du|) |D^2 u|^2 dx dy. \quad (60)$$

Since  $|Du|$  is bounded ( $u \in C^2(\mathbb{R}^2)$  and  $\Omega$ -periodic) it's clear that (60) remains true if  $\psi$  is only bounded and continuous function from  $\mathbb{R}^+$  to  $\mathbb{R}$ .

Now we show that (60) is true for  $u$  in  $H^2(\Omega)$ . Indeed, by density argument there exists a sequence  $(w_n) \in C^2(\mathbb{R}^2) \cap H^2(\Omega)$  that converges to  $u$  in  $H^2(\Omega)$ . In particular we have  $\Delta w_n \rightarrow \Delta u$ ,  $D^2 w_n \rightarrow D^2 u$  et  $Dw_n \rightarrow Du$  in  $L^2(\Omega)$ . In addition, there exists a subsequence still denoted by  $(w_n)$  such that  $Dw_n \rightarrow Du$  a.e. in  $\Omega$ .

Using the fact that  $\psi$  is bounded, there exists a function  $h \in L^\infty(\Omega)$  and a subsequence still denoted by  $(w_n)$  such that  $\psi^2(Dw_n) \xrightarrow{*} h$  in  $L^\infty(\Omega)$ . Combining this with the strong convergences  $|\Delta w_n|^2 \rightarrow |\Delta u|^2$  and  $|D^2 w_n|^2 \rightarrow |D^2 u|^2$  in  $L^1(\Omega)$ , then we obtain:

$$\psi^2(Dw_n) |\Delta w_n|^2 \rightarrow h |\Delta u|^2 \quad \text{and} \quad \psi^2(Dw_n) |D^2 w_n|^2 \rightarrow h |D^2 u|^2 \quad \text{in} \quad L^1(\Omega).$$

Now using the fact that  $Dw_n \rightarrow Du$  strongly and a.e. in  $\Omega$  and the continuity of  $\psi$  we deduce that  $h = \psi^2(Du)$ . Finally, the sequence  $(w_n)$  can be chosen  $\Omega$ -periodic (for example as a convolution of  $u$  with a smooth kernel), then we obtain the desired result by applying the equality (60) to  $w_n$  and passing to the limit.  $\square$

**Acknowledgement :** The first author would like to thank Prof. G. H. Cottet for valuable conversations.

## References

- [1] L. Alvares, F. Guichard, P. L. Lions, and J. M. Morel. Axioms and fundamental equations of image processing. *Arch. Rational Mech. Anal.*, 123(3):199–257, 1993.
- [2] A. Belahmidi and A. Chambolle. Classical solution of time-delay regularisation of the anisotropic diffusion. *Preprint*, 2003.

- [3] J. Canny. A computational approach to edge detection. *IEEE Trans. Pattern Anal. Mach. Intell.*, 8:679–698, 1986.
- [4] F. Catté, P. L. Lions, J. M. Morel, and T. Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM J. Numer. Anal.*, 29(1):182–193, 1992.
- [5] P. Ciarlet. *Introduction à l'Analyse Numérique Matricielle et à l'Optimisation*. Masson, Paris, 1982.
- [6] G. H. Cottet and M. El-Ayyadi. A volterra type model for image processing. *IEEE Transactions on image processing*, 7(3), March 1998.
- [7] S. Esedoglu. An analysis of the Perona-Malik scheme. *Comm. Pure Appl. Math.*, 54(12):1442–1487, 2001.
- [8] Y. Giga and S. goto. Motion of hypersurfaces and geometric equations. *J. Math. Soc. Japan*, 44(1):99–111, 1992.
- [9] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, 1977.
- [10] E. Heinz. An elementary analytic theory of the degree of mapping in  $n$ -dimensional space. *J. Math. Mech.*, 8:231–247, 1959.
- [11] K. Höllig and J. A. Nohel. A diffusion equation with a nonmonotone constitutive function. In *Systems of nonlinear partial differential equations*, volume 111 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pages 409–422, Dordrecht, 1983. Reidel.
- [12] O. Kavian. *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*. Springer-Verlag, Paris, 1993.
- [13] B. Kawohl and N. Kutev. Maximum and comparison principle for one-dimensional anisotropic diffusion. *Math. Ann.*, 311(1):107–123, 1998.
- [14] S. Kichenassamy. The Perona-Malik paradox. *SIAM J. Appl. Math.*, 57(5):1328–1342, 1997.
- [15] J. Malik and P. Perona. A scale space and edge detection using anitropic diffusion. *IEEE Comp. Soc. Workshop on Comp. Vision, Miami*, pages 16–22, 1987.
- [16] J. Malik and P. Perona. A scale space and edge detection using anitropic diffusion. *IEEE Trans. Pattern Anal. Mach. Intell.*, 12:629–639, 1990.
- [17] D. Marr and E. Hildreth. Theory of edge detection. In *Roy. Soc.*, volume 207 of *B*, pages 187–217, London, 1980.
- [18] N. G. Meyers. An  $L^p$ -estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:189–206, 1963.
- [19] M. Nitzberg and T. Shiota. Nonlinear image feltring whith edge and corner enhancement. *IEEE Trans. Pattern Anal. Mach. Intell.*, 14:826–833, 1992.
- [20] R. T. Whitaker and S. M. Pizer. A multi-scale approach to nonuniform diffusion. *CVGIP: Image Understanding*, 57:99–110, 1993.
- [21] Y. You, W. Xu, A. Tannenbaum, and M. Kaveh. Behavioral analysis of anisotropic diffusion in image processing. *IEEE Transactions on Image processing*, 5(11):1539–1553, 1996.