Real algebraic morphisms and Del Pezzo surfaces of degree 2

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REAL ALGEBRAIC MORPHISMS AND DEL PEZZO SURFACES OF DEGREE 2

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Abstract. Let X and Y be affine nonsingular real algebraic varieties. A general problem in Real Algebraic Geometry is to try to decide when a smooth map $f : X \to Y$ can be approximated by regular maps in the space of $\mathcal{C}^\infty$ mappings from X to Y, equipped with the $\mathcal{C}^\infty$ topology.

In this paper we give a complete solution to this problem when the target space is the standard 2-dimensional sphere and the source space is a geometrically rational real algebraic surface. The approximation result for real algebraic surfaces rational over $\mathbb{R}$ is due to J. Bochnak and W. Kucharz.

Here we give a detailed description of the more interesting case, namely real Del Pezzo surfaces of degree 2.

Introduction

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to a Zariski locally closed subset of $\mathbb{P}^n(\mathbb{R})$, for some n, endowed with the Zariski topology and the sheaf of real valued regular functions. Morphisms between real algebraic varieties are called *regular maps*. Every real algebraic variety is isomorphic to a Zariski closed subvariety of $\mathbb{R}^n$, see [BCR, Proposition 3.2.10 and Theorem 3.4.4]. All topological notions concerning real algebraic varieties will refer to the Euclidean topology inherited from the metric topology in $\mathbb{R}^n$.

An equivalent description of real algebraic varieties can be obtained using reduced quasi-projective schemes over $\mathbb{R}$. If $\mathcal{X}$ is a reduced quasi-projective scheme over $\mathbb{R}$, we denote by $\mathcal{X}(\mathbb{R})$ its set of $\mathbb{R}$-rational points. When $\mathcal{X}(\mathbb{R})$ is Zariski dense in $\mathcal{X}$, we regard it as a real algebraic variety whose structure sheaf is the restriction of the structure sheaf of $\mathcal{X}$. Up to isomorphism every real algebraic variety X is of this type.

Given two nonsingular real algebraic varieties X and Y, with X compact, we consider the set $\mathcal{R}(X, Y)$ of all regular maps from X into Y as a subset of the space $\mathcal{C}^\infty(X, Y)$ of all $\mathcal{C}^\infty$ maps from X into Y equipped with the $\mathcal{C}^\infty$ topology. We want to study which $\mathcal{C}^\infty$ maps from X into Y can be approximated by regular maps. The classical Stone-Weierstrass approximation theorem implies that all $\mathcal{C}^\infty$ maps from X into Y can be approximated by regular maps when $Y = \mathbb{R}^m$, for some m. In this paper we will always consider $Y = \mathbb{S}^2$, the standard unit two dimensional sphere.

Our main tool to attack this case of the general approximation problem is the following subgroup of the second cohomology group of our variety X. Given an n-dimensional smooth projective scheme $\mathcal{Y}$ over $\mathbb{C}$, we regard its set of $\mathbb{C}$-rational points $\mathcal{Y}(\mathbb{C})$ as a complex manifold, and denote by $H^2_{\mathrm{alg}}(\mathcal{Y}(\mathbb{C}), \mathbb{Z})$ the subgroup of $H^2(\mathcal{Y}(\mathbb{C}), \mathbb{Z})$ that consists of the cohomology classes Poincaré dual to the homology classes in $H_{2n-2}(\mathcal{Y}(\mathbb{C}), \mathbb{Z})$ represented by divisors in $\mathcal{Y}$. If $\mathcal{X}$ is a smooth projective scheme over $\mathbb{R}$, we let $\mathcal{X}_\mathbb{C} = \mathcal{X} \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ and we identify $\mathcal{X}_\mathbb{C}(\mathbb{C})$ with the set $\mathcal{X}(\mathbb{C})$ of $\mathbb{C}$-rational points of $\mathcal{X}$. Observe that $\mathcal{X}(\mathbb{R})$ can be viewed as the set of fixed points under the action of the Galois group $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ on $\mathcal{X}(\mathbb{C})$. Let us denote by σ

the antiholomorphic involution on $\mathcal{X}(\mathbb{C})$ ("complex conjugation") which represents the nontrivial element of $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$.

Given a compact nonsingular real algebraic variety X, by the resolution of singularities theorem [BiMi, Hi], there exists a smooth projective scheme $\mathcal{X}$ over $\mathbb{R}$, such that X and $\mathcal{X}(\mathbb{R})$ are isomorphic as real algebraic varieties. We identify X and $\mathcal{X}(\mathbb{R})$ and we set:

$$H^2_{\mathbb{C}-\mathrm{alg}}(X,\mathbb{Z}) = i^*(H^2_{\mathrm{alg}}(\mathcal{X}(\mathbb{C}),\mathbb{Z})) ,$$

where $i : X \hookrightarrow \mathcal{X}(\mathbb{C})$ is the inclusion map. It is easy to check that the group $H^2_{\mathbb{C}-\mathrm{alg}}(X,\mathbb{Z})$ does not depend on the scheme $\mathcal{X}$ associated to X. We will denote by $\Gamma(X)$ the quotient group $H^2(X,\mathbb{Z})/H^2_{\mathbb{C}-\mathrm{alg}}(X,\mathbb{Z})$. Throughout this paper, if $f : M \to N$ is a differentiable map between two manifolds, $f^* : H^l(N,\mathbb{Z}) \to H^l(M,\mathbb{Z})$ (respectively $f_2^* : H^l(N,\mathbb{Z}/2) \to H^l(M,\mathbb{Z}/2)$) denotes the pullback homomorphism induced by f in the corresponding cohomology groups with integer coefficients (respectively with $\mathbb{Z}/2$ coefficients).

There is a close connection between the subgroup $H^2_{\mathbb{C}-\mathrm{alg}}(X,\mathbb{Z})$ and the topological closure of the space $\mathcal{R}(X,Y)$ in $\mathcal{C}^\infty(X,Y)$. More precisely, the following result is well-known, cf. [BCR, Chapter 13] and [BBK].

Proposition 0.1. *Let X be a compact nonsingular real algebraic variety and let $f : X \to \mathbb{S}^2$ be a $\mathcal{C}^\infty$ differentiable mapping. Then, f can be approximated by regular maps in the $\mathcal{C}^\infty$ topology, if and only if $f^*(\kappa) \in H^2_{\mathbb{C}-\mathrm{alg}}(X,\mathbb{Z})$. Here κ is a fixed generator of the group $H^2(\mathbb{S}^2,\mathbb{Z})$.*

We say that a smooth projective surface $\mathcal{X}$ defined over $\mathbb{R}$ is *rational* (over $\mathbb{R}$) if it is birationally equivalent to $\mathbb{P}^2_{\mathbb{R}}$ as real varieties.

A smooth projective surface $\mathcal{X}$ defined over $\mathbb{R}$ is called *geometrically rational* if $\mathcal{X}_{\mathbb{C}}$ is irreducible and birationally equivalent to $\mathbb{P}^2_{\mathbb{C}}$ over $\mathbb{C}$. In this case $\mathcal{X}(\mathbb{C})$ is obviously birationally equivalent to $\mathbb{P}^2(\mathbb{C})$ as complex algebraic varieties. In the classical language, we say that $X = \mathcal{X}(\mathbb{R})$ is a geometrically rational real algebraic surface (respectively $X = \mathcal{X}(\mathbb{R})$ is a rational real algebraic surface) if $\mathcal{X}$ is geometrically rational (respectively $\mathcal{X}$ is rational over $\mathbb{R}$).

There is a nice characterization of real surfaces rational over $\mathbb{R}$ due to R. Silhol. Let $\mathcal{X}$ be a geometrically rational real algebraic surface with $\mathcal{X}(\mathbb{R}) \neq \emptyset$, then $\mathcal{X}$ is rational over $\mathbb{R}$ if and only if the real part of $\mathcal{X}$ is connected, cf. [Si, Chapter 6, VI.5].

The approximation problem of smooth maps from rational real algebraic surfaces into the 2-sphere has been solved by J. Bochnak and W. Kucharz, see [Ku] for references.

Theorem 0.2. *Let X be a rational real algebraic surface. Then*

$$\Gamma(X) = \begin{cases} \mathbb{Z} & \textit{if } X \textit{ is homeomorphic to } \mathbb{S}^1 \times \mathbb{S}^1 \\ 0 & \textit{in all other cases.} \end{cases}$$

On the other hand, we say that a smooth projective surface $\mathcal{X}$ over $\mathbb{R}$ is a *Del Pezzo surface* if $\mathcal{X}_{\mathbb{C}}$ is irreducible and the anticanonical divisor $-K_{\mathcal{X}}$ is ample. The degree of $\mathcal{X}$ is $d(\mathcal{X}) = K^2_{\mathcal{X}}$. It is known that for Del Pezzo surfaces, $1 \leq d(\mathcal{X}) \leq 9$.

The main results of this paper are the following.

Theorem 0.3. *Let $\mathcal{X}$ be a smooth projective surface over $\mathbb{R}$ whose $\mathbb{R}$-minimal model is a Del Pezzo surface of degree 2 and such that $\mathcal{X}(\mathbb{R})$ is homeomorphic to the disjoint union of 4 spheres, then $\Gamma(\mathcal{X}(\mathbb{R})) = \mathbb{Z}/2$.*

For the definition and remarks on $\mathbb{R}$-minimal models see Section 1.

In this paper, we also give the complete description of the quotient group $\Gamma(X)$ for a geometrically rational real surface X.

Theorem 0.4. *Let $X = \mathcal{X}(\mathbb{R})$ be a geometrically rational real algebraic surface. Then*

$$\Gamma(X) = \begin{cases} \mathbb{Z} & \text{if } X \sim \mathbb{S}^1 \times \mathbb{S}^1 \\ \mathbb{Z}/2 & \text{if } \mathcal{X} \text{ is as in Theorem 0.3} \\ 0 & \text{in all other cases.} \end{cases}$$

Remark 0.5. There exists a geometrically rational real algebraic surface $\mathcal{X}$ with $\mathcal{X}(\mathbb{R}) \sim \bigsqcup 4\mathbb{S}^2$ and $\Gamma(\mathcal{X}(\mathbb{R})) = 0$, namely $\mathcal{X}$ a real conic fibration over $\mathbb{P}^1$ with 8 singular fibers (see Section 4). Here $\bigsqcup 4\mathbb{S}^2$ denotes the disjoint union of four copies of the standard sphere. Observe that Γ is not a topological invariant.

In terms of regular maps, our results imply the following.

Corollary 0.6. *Let $X = \mathcal{X}(\mathbb{R})$ be a geometrically rational real algebraic surface. The space of regular maps $\mathcal{R}(X, \mathbb{S}^2)$ is dense in the space of $\mathcal{C}^\infty$ mappings $\mathcal{C}^\infty(X, \mathbb{S}^2)$ except when X is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ or when $\mathcal{X}$ is as in Theorem 0.3.*

When X is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$, W. Kucharz proved that only the nullhomotopic $\mathcal{C}^\infty$ mappings from X into $\mathbb{S}^2$ can be approximated by regular maps, cf. [Ku].

If all the connected components of X are nonorientable, N. Joglar-Prieto already proved that there was density, cf. [Jo].

In the last case, that is, when $\mathcal{X}$ is as in Theorem 0.3, only "half" of the $\mathcal{C}^\infty$ mappings from $\mathcal{X}(\mathbb{R})$ into $\mathbb{S}^2$ can be approximated by regular maps. See Corollary 5.5.

1. Minimal models of geometrically rational surfaces

In this section we will briefly recall the description of the minimal models of a given geometrically rational real algebraic surface, see e.g. [Si, Chap. III and VI].

Let $\mathcal{Y}$ be a smooth projective surface over $\mathbb{R}$. We say that a smooth complex curve E of $\mathcal{Y}_{\mathbb{C}}$ is a (-1)-*curve* if it is rational and $E^2 = -1$. If a (-1)-curve E is defined over $\mathbb{R}$, there exists a blowdown $\pi : \mathcal{Y} \to \mathcal{X}$ over $\mathbb{R}$ such that E contracts to a real point $P = \pi(E) \in \mathcal{X}(\mathbb{R})$ (in particular, $\mathcal{X}(\mathbb{R})$ and $\mathcal{Y}(\mathbb{R})$ are nonempty). If E is not defined over $\mathbb{R}$, then σE is another (-1)-curve. If the intersection number $E \cdot \sigma E = 0$, then we can blowdown over $\mathbb{R}$ the divisor $E + \sigma E$. We say that a smooth projective surface over $\mathbb{R}$ is $\mathbb{R}$-*minimal* if it does not contain either real (-1)-curves or pairs of disjoint complex conjugated (-1)-curves. If $E \cdot \sigma E \neq 0$, then we cannot blowdown $E + \sigma E$ over $\mathbb{R}$ and the surface can be $\mathbb{R}$-minimal but not $\mathbb{C}$-minimal.

If $\mathcal{X}$ is a smooth projective surface over $\mathbb{R}$, and $f : \mathcal{X} \to B$ is a morphism into a smooth curve whose generic fiber is isomorphic to a plane conic, then $(\mathcal{X}, f)$ is called a *conic fibration.*

Proposition 1.1. *Let $\mathcal{Y}$ be a geometrically rational real algebraic surface with $\mathcal{Y}(\mathbb{R}) \neq \emptyset$. If $\mathcal{Y}$ is not rational over $\mathbb{R}$, there exists a sequence*

$$\mathcal{Y} = \mathcal{Y}_r \xrightarrow{\pi_r} \mathcal{Y}_{r-1} \xrightarrow{\pi_{r-1}} \dots \xrightarrow{\pi_2} \mathcal{Y}_1 \xrightarrow{\pi_1} \mathcal{Y}_0$$

such that $r \geq 0$ and

(i) $\mathcal{Y}_0$ is either (i1) an $\mathbb{R}$-minimal conic fibration $f : \mathcal{Y}_0 \to B$, with $B \sim \mathbb{P}^1_{\mathbb{R}}$ or (i2) an $\mathbb{R}$-minimal Del Pezzo surface of degree 2 or 1;

(ii) each $\pi_i : \mathcal{Y}_i \to \mathcal{Y}_{i-1}$ is a blowup over $\mathbb{R}$ at a real point or at a pair of complex conjugate points of $\mathcal{Y}_{i-1}$.

Observe that if we blowup over $\mathbb{R}$ a real closed point of $\mathcal{Y}$, the real part of the blowup changes by gluing a copy of $\mathbb{P}^2(\mathbb{R})$ to the connected component of $\mathcal{Y}(\mathbb{R})$ where the point we blow up lies in. If we blowup a pair of complex conjugate points of $\mathcal{Y}$, the real parts of $\mathcal{Y}$ and of the blowup are biregularly isomorphic.

We can also give the birational classification of geometrically rational real algebraic surfaces. We include the topological types of the real parts for completeness.

Proposition 1.2. *Let $\mathcal{Y}$ be a geometrically rational real algebraic surface with $\mathcal{Y}(\mathbb{R}) \neq \emptyset$. Then $\mathcal{Y}$ is birationally equivalent over $\mathbb{R}$ to an $\mathbb{R}$-minimal model $\mathcal{Y}_0$ in exactly one of the following cases:*

(a) $\mathcal{Y}_0 = \mathbb{P}^2_{\mathbb{R}}$. In this case $\mathcal{Y}(\mathbb{R})$ is connected and $\mathcal{Y}(\mathbb{R})$ can be homeomorphic to $\mathbb{S}^2$, $\mathbb{S}^1 \times \mathbb{S}^1$ or to any nonorientable compact connected smooth real surface.

(b) $\mathcal{Y}_0$ is an $\mathbb{R}$-minimal conic fibration over $\mathbb{P}^1$ with $2m$ singular fibers, for some $m \geq 2$. In this case, $\mathcal{Y}_0(\mathbb{R})$ has exactly m connected components. Each component is homeomorphic to $\mathbb{S}^2$.

(c) $\mathcal{Y}_0$ is a degree 2 $\mathbb{R}$-minimal Del Pezzo surface such that $\mathcal{Y}_0(\mathbb{R})$ has 4 components. In this case, each component is homeomorphic to $\mathbb{S}^2$.

(d) $\mathcal{Y}_0$ is a degree 1 $\mathbb{R}$-minimal Del Pezzo surface. In this case $\mathcal{Y}_0(\mathbb{R})$ has 5 connected components. One of them is homeomorphic to $\mathbb{P}^2(\mathbb{R})$ and the rest are homeomorphic to $\mathbb{S}^2$.

In all cases above, $\mathcal{Y}(\mathbb{R})$ and $\mathcal{Y}_0(\mathbb{R})$ have the same number of connected components. In the cases (b), (c) and (d), each component of $\mathcal{Y}(\mathbb{R})$ is homeomorphic either to $\mathbb{S}^2$ or to any nonorientable compact connected smooth real surface.

Remark 1.3. For a given geometrically rational surface $\mathcal{Y}$, there might be many different $\mathbb{R}$-minimal models $\mathcal{Y}_0$. For example, there exists a degree 2 $\mathbb{R}$-minimal Del Pezzo surface $\mathcal{X}$ such that $\mathcal{X}(\mathbb{R})$ has 3 connected components and $\mathcal{X}$ is also a conic fibration over $\mathbb{P}^1$ (so it falls into the case (b)). See [DIK, Theorem 6.11.11] for more details.

To end this section, let us mention here that the topological types of geometrically rational surfaces over $\mathbb{R}$ were already known to Comessatti in 1913. As we can see from Proposition 1.2, compact orientable real surfaces of genus bigger than 1 do not occur as connected components of the real part of a geometrically rational surface. Moreover, if the real part is not connected, the only orientable connected components that may appear are homeomorphic to a sphere.

If $\mathcal{X}$ is as in Theorem 0.3, all the blowups in the sequence given by Proposition 1.1 are at pairs of complex conjugate points of its minimal model $\mathcal{X}_0$. Therefore, $\mathcal{X}(\mathbb{R})$ and $\mathcal{X}_0(\mathbb{R})$ are biregularly isomorphic. In particular, $\Gamma(\mathcal{X}(\mathbb{R}))$ is isomorphic to $\Gamma(\mathcal{X}_0(\mathbb{R}))$.

2. Gysin map and transversality

Let $i : \mathcal{X}(\mathbb{R}) \hookrightarrow \mathcal{X}(\mathbb{C})$ be as above the canonical injection of a compact nonsingular real algebraic surface into its nonsingular projective complexification.

The manifolds $\mathcal{X}(\mathbb{C})$ and $\mathcal{X}(\mathbb{R})$ are compact and $\mathcal{X}(\mathbb{C})$ is orientable. If $\mathcal{X}(\mathbb{R})$ is orientable, we can define the Gysin morphism $i_!$ by the commutative diagram :

$$\begin{array}{ccc} H^2(\mathcal{X}(\mathbb{C}),\mathbb{Z}) & \xrightarrow{i^*} & H^2(\mathcal{X}(\mathbb{R}),\mathbb{Z}) \\ {\scriptstyle D_{\mathbb{C}}^{-1}}\uparrow\cong & & {\scriptstyle D_{\mathbb{R}}^{-1}}\uparrow\cong \\ H_2(\mathcal{X}(\mathbb{C}),\mathbb{Z}) & \xrightarrow{i_!} & H_0(\mathcal{X}(\mathbb{R}),\mathbb{Z}) \end{array}$$

where the isomorphisms $D_{\mathbb{C}}$ and $D_{\mathbb{R}}$ come from Poincaré duality.

Remark 2.1. If X is not orientable, then we will work similarly as above with $\mathbb{Z}/2$ coefficients. See also Section 3.

Lemma 2.2. *Let S be an oriented 2-dimensional closed submanifold of $\mathcal{X}(\mathbb{C})$ transverse to $\mathcal{X}(\mathbb{R})$, then*

$$i_!([S]) = [S \cap \mathcal{X}(\mathbb{R})] \, .$$

Lemma 2.3. *Let $\mathcal{X}$ be a smooth projective surface over $\mathbb{R}$ with geometric genus $p_g(\mathcal{X}_{\mathbb{C}}) = 0$, then $H^2_{\mathbb{C}-\mathrm{alg}}(\mathcal{X}(\mathbb{R}), \mathbb{Z}) = \operatorname{Im} i^*$. Moreover, if $\mathcal{X}_{\mathbb{C}}$ has besides irregularity $q(\mathcal{X}_{\mathbb{C}}) = 0$, then $H^2(\mathcal{X}(\mathbb{C}), \mathbb{Z}) \cong \operatorname{Pic}(\mathcal{X}_{\mathbb{C}})$.*

Proof. By definition, we have $H^2_{\mathbb{C}-\mathrm{alg}}(\mathcal{X}(\mathbb{R}), \mathbb{Z}) \subseteq \operatorname{Im} i^*$ and $\operatorname{Im} i^* \cong \operatorname{Im} i_!$.
For a complex nonsingular variety V, we have the long exact sequence coming from the exponential exact sequence. Considering also the isomorphism $\operatorname{Pic}(V) \cong H^1(V, \mathcal{O}^*)$ we obtain the following exact sequence

$$\cdots \to H^1(V, \mathcal{O}) \to \operatorname{Pic}(V) \xrightarrow{c_1} H^2(V, \mathbb{Z}) \to H^2(V, \mathcal{O}) \to \cdots$$

The Lemma is now clear since $p_g(V) = \dim H^2(V, \mathcal{O})$ and $q = \dim H^1(V, \mathcal{O})$. $\square$

Corollary 2.4. *If X is a geometrically rational real algebraic surface, then $H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}) = \operatorname{Im} i^*$.*

In other words, for a geometrically rational real algebraic surface X, the quotient group $\Gamma(X)$ is trivial if and only if i^* is surjective.

Corollary 2.5. *Let X be a geometrically rational real algebraic surface. If X is orientable, then*

$$\Gamma(X) \cong H_0(X, \mathbb{Z}) / \operatorname{Im} i_! .$$

In the last three sections of the paper, we will use Lemma 2.2 and we will have to prove that a complex curve is transverse to the real part of a surface. In the Del Pezzo cases, we will use double coverings and we will obtain the transversality as "half" of a certain type of tangency. Here is an example of the type of computations we will use.

Example 2.6. Let Δ be a smooth real plane curve of even degree and let $\varphi : \mathcal{Y} \to \mathbb{P}^2$ be one of the two double coverings over $\mathbb{R}$ ramified along Δ. Let L be a real line and $p \in \Delta(\mathbb{R}) \cap L$ a point with simple tangency. In an affine neighborhood of p, the surface $\mathcal{Y}$ can be given by one of the two equations $z^2 = \pm(y - x^2)$ and in the plane $z = 0$, the curves Δ and L by equations $y = x^2$ and $y = 0$. Let C be the pullback of L in $\mathcal{Y}(\mathbb{C})$. Locally, C has equation $\{y = 0,\ (z - x)(z + x) = 0\}$ or $\{y = 0,\ (z - ix)(z + ix) = 0\}$. Here $i = \sqrt{-1}$.

Globally, the complex curve C can be reducible or irreducible. Suppose from now on that C is reducible. Then C decomposes as $C = E + \tau E$ where τ is the covering involution. For one real covering, E and τE are defined over $\mathbb{R}$. For the other real covering, E and τE are complex conjugated curves. In the latter case, since $\mathcal{Y}$ is smooth, the ring of local regular functions is factorial and both E and τE have local equations $\{y = 0,\ z - ix = 0\}$ and $\{y = 0,\ z + ix = 0\}$, respectively. The tangent plane to $\mathcal{Y}(\mathbb{R})$ at p is generated by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial z_1}$, where $x = x_1 + ix_2$, $y = y_1 + iy_2$ and $z = z_1 + iz_2$. It is easy to check that the tangent plane to E at p is generated by $\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$ and $i\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}$. Then E is transverse to $\mathcal{Y}(\mathbb{R})$ at p in $\mathcal{Y}(\mathbb{C})$.

Remark 2.7. The crucial point here is the reducibility of $\varphi^{-1}(L)$. This is a global datum with no information about the local computations.

3. Blowup and regular maps

Throughout this section $\mathcal{X}$ denotes a geometrically rational surface defined over $\mathbb{R}$ and we denote by X its real part. We want to study now the behavior of the quotient group $\Gamma(X)$ (see Introduction) under the blowup $\pi : \bar{\mathcal{X}} \to \mathcal{X}$ at a real rational point P of $\mathcal{X}$ (over $\mathbb{R}$). If we blowup a pair of complex conjugate points of $\mathcal{X}$, then the real part does not change biregularly, so we restrict our study to the case of the blowup at a real point. From now on, whenever we write blowup we will mean the blowup at a real point.

The main result of this section is the following.

Proposition 3.1. *If* $\Gamma(X) = 0$ *or* $\Gamma(X) = \mathbb{Z}/2$*, then* $\Gamma(\bar{X}) = 0$ *for every blowup* $\bar{X}$ *of* X*.*

Remark 3.2. As we will see in the last three sections of this paper, if X is also $\mathbb{R}$-minimal and nonconnected, then $\Gamma(X) = 0$ or $\Gamma(X) = \mathbb{Z}/2$. Therefore, in view of Section 1 and Proposition 3.1, to prove Theorem 0.4, it will be enough to compute $\Gamma(\mathcal{X}_0(\mathbb{R}))$ for the different minimal models $\mathcal{X}_0$ of a given geometrically rational surface $\mathcal{X}$ over $\mathbb{R}$. Theorem 0.4 will be proven in Proposition 4.2, Section 5 and Theorem 6.1. Theorem 0.3 is also included in Section 5.

Since $\bar{X}$ is the blowup of a geometrically rational real surface, it is also a geometrically rational real surface, so $\Gamma(\bar{X}) = 0$ if and only if $\bar{i}^*$ is surjective, see Corollary 2.4.

Before giving the proof of Proposition 3.1, we need several auxiliary results.

We denote by $VB^1_{\mathbb{C}}(X)$ the group of isomorphism classes of topological complex line bundles on X and by $\beta : H^1(X, \mathbb{Z}/2) \to H^2(X, \mathbb{Z})$ the *Bockstein* homomorphism induced in cohomology by the following exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0 \ .$$

Proposition 3.3. *Let* X *be a geometrically rational real surface. Then*

$$\beta(H^1(X, \mathbb{Z}/2)) \subseteq H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}).$$

Proof. We have a commutative diagram, cf. eg. [Jo]:

$$\begin{array}{ccccc}
\mathrm{Pic}(\mathcal{X}_{\mathbb{C}}) & \xrightarrow{|_{\mathcal{X}(\mathbb{R})}} & VB^1_{\mathbb{C}}(\mathcal{X}(\mathbb{R})) & \xleftarrow{\otimes\mathbb{C}} & \mathrm{Pic}(\mathcal{X}) \\
{\scriptstyle c_1}\downarrow & & {\scriptstyle c_1}\downarrow & & {\scriptstyle w_1}\downarrow \\
H^2(\mathcal{X}(\mathbb{C}), \mathbb{Z}) & \xrightarrow{i^*} & H^2(\mathcal{X}(\mathbb{R}), \mathbb{Z}) & \xleftarrow{\beta} & H^1(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2).
\end{array}$$

Here w_1 denotes the homomorphism induced by the first *Stiefel-Whitney class*, and c_1 denotes the homomorphism induced by the first *Chern class*.

Observe that the restriction homomorphism $|_{\mathcal{X}(\mathbb{R})}$ from the first row is surjective and the homomorphisms c_1 and w_1 are surjective in this case, see [Jo, Proof of Claim 2]. Recall also from Corollary 2.4 that $H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}) = i^*(H^2(\mathcal{X}_{\mathbb{C}}, \mathbb{Z}))$. □

Observe that in this proof we are using that $w_1 : \mathrm{Pic}(\mathcal{X}) \to H^1(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2)$ and $c_1 : \mathrm{Pic}(\mathcal{X}_{\mathbb{C}} \to H^2(\mathcal{X}(\mathbb{C}), \mathbb{Z}))$ are surjective. This fact is not true in general for a given smooth projective surface defined over $\mathbb{R}$.

Denote now by $X_1,\ X_2, \cdots, X_s$ the connected components of X. Assume that P (the point at which we are blowingup) belongs to X_1. Observe that the real part $\bar{X}$ of the blowup is homeomorphic to the disjoint union of $\bar{X}_1 = X_1 \# \mathbb{P}^2(\mathbb{R})$ and the other connected components of X.

Proposition 3.4. *If we identify* $H^2(\bar{X}, \mathbb{Z}) \cong H^2(\bar{X}_1, \mathbb{Z}) \oplus H^2(X_2, \mathbb{Z}) \oplus \cdots \oplus H^2(X_s, \mathbb{Z})$*, then we have that* $H^2(\bar{X}_1, \mathbb{Z}) \subseteq H^2_{\mathbb{C}-\mathrm{alg}}(\bar{X}, \mathbb{Z})$*.*

Proof. Consider the exact sequence in cohomology induced by the change of coefficients, see the proof of Proposition 3.3,

$$\cdots \to H^1(\bar{X}, \mathbb{Z}/2) \xrightarrow{\bar{\beta}} H^2(\bar{X}, \mathbb{Z}) \xrightarrow{\bar{\times}2} H^2(\bar{X}, \mathbb{Z}) \to \cdots$$

Observe that $H^2(\bar{X}_1, \mathbb{Z}) \cong \mathbb{Z}/2$. Then it is contained in the kernel of the homomorphism $\bar{\times}2$. Therefore, the subgroup $H^2(\bar{X}_1, \mathbb{Z})$ of $H^2(\bar{X}, \mathbb{Z})$ is in the image of the Bockstein homomorphism $\bar{\beta}$. By Proposition 3.3 the proof is complete. □

We continue with some remarks from cohomology theory.

Remark 3.5. Let $\mathcal{Y}$ be an affine nonsingular algebraic surface defined over $\mathbb{R}$. If the cohomology of $\mathcal{Y}(\mathbb{C})$ with integer coefficients is without 2-torsion, then $H^*(\mathcal{Y}(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}/2 \cong H^*(\mathcal{Y}(\mathbb{C}), \mathbb{Z}/2)$ and $H_*(\mathcal{Y}(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}/2 \cong H_*(\mathcal{Y}(\mathbb{C}), \mathbb{Z}/2)$. In this case we also have that $H^2(\mathcal{Y}(\mathbb{R}), \mathbb{Z}) \otimes \mathbb{Z}/2 \cong H^2(\mathcal{Y}(\mathbb{R}), \mathbb{Z}/2)$ because $\mathcal{Y}(\mathbb{R})$ is of dimension 2. Observe that this is the case for our geometrically rational surface $\mathcal{X}$.

In view of this remark and given that X does not have to be orientable in general, we will work with cohomology with $\mathbb{Z}/2$-coefficients. As above we call i (respectively $\bar{i}$) the inclusion mapping from X into its complexification $\mathcal{X}(\mathbb{C})$ (respectively $\bar{X}$ into $\bar{\mathcal{X}}(\mathbb{C})$). We can also define $H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}/2) = i_2^*(H^2_{\mathrm{alg}}(\mathcal{X}(\mathbb{C}), \mathbb{Z}/2))$.

Observe that $H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}/2) = i_2^*(H^2(\mathcal{X}(\mathbb{C}), \mathbb{Z}/2)) \cong H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}) \otimes \mathbb{Z}/2$.

After these remarks, Proposition 3.4 implies the following result.

Corollary 3.6. *Let Q be a point of $\bar{X}_1$. Then the Poincaré dual class $\bar{D}_2^{-1}([Q])$ belongs to $H^2_{\mathbb{C}-\mathrm{alg}}(\bar{X}, \mathbb{Z}/2)$.*

Here $\bar{D}_2 : H^2(\bar{X}, \mathbb{Z}/2) \to H_0(\bar{X}, \mathbb{Z}/2)$ denotes as usual the Poincaré duality isomorphism ($D_2 : H^2(X, \mathbb{Z}/2) \to H_0(X, \mathbb{Z}/2)$).

Remark 3.7. It is also easy to prove Corollary 3.6 directly for a given smooth projective surface $\mathcal{X}$ defined over $\mathbb{R}$ (not necessarily geometrically rational). If we call $E = \pi^{-1}(P)$ the exceptional divisor of our real blowup, and denote by $\mathcal{E}$ the associated complex line bundle, then $\bar{i}^*(c_1(\mathcal{E})) = c_1(\mathcal{L} \otimes \mathbb{C})$, where $\mathcal{L}$ is the real line bundle associated to $E(\mathbb{R})$. By the general theory of characteristic classes, see [MS, Exercise 15-D], $c_1(\mathcal{L} \otimes \mathbb{C}) = \bar{\beta}(w_1(\mathcal{L}))$. It is easy to see that the reduction modulo 2 of $\bar{\beta}(w_1(\mathcal{L}))$ is equal to $\bar{D}_2^{-1}([Q])$, the generator of $H^2(\bar{X}, \mathbb{Z}/2)$ generating the subspace $H^2(\bar{X}_1, \mathbb{Z}/2)$. Therefore $\bar{i}_2^*([c_1(\mathcal{E})]_2) = [\bar{\beta}(w_1(\mathcal{L}))]_2 = \bar{D}_2^{-1}([Q])$.

The following corollary is also valid, with the same proof, for any smooth projective surface defined over $\mathbb{R}$.

Corollary 3.8. *If $D_2^{-1}([P])$ belongs to $H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}/2)$, then $H^2_{\mathbb{C}-\mathrm{alg}}(\bar{X}, \mathbb{Z}/2) = \pi(\mathbb{R})^*(H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}/2))$. Otherwise, we have a canonical decomposition*

$$H^2_{\mathbb{C}-\mathrm{alg}}(\bar{X}, \mathbb{Z}/2) = \pi(\mathbb{R})^*(H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}/2)) \oplus \bar{D}_2^{-1}([Q]).$$

Proof. Since $\pi(\mathbb{R}) : \bar{X} \to X$ is the blowup at P, it is a regular map that induces a biregular isomorphism from $\bar{X} - E(\mathbb{R})$ into $X - \{P\}$. Given that X and $\bar{X}$ are surfaces, $\pi(\mathbb{R})$ induces an isomorphism $\pi(\mathbb{R})_2^* : H^2(X, \mathbb{Z}/2) \to H^2(\bar{X}, \mathbb{Z}/2)$. Since π is regular, this isomorphism satisfies

$$\pi(\mathbb{R})_2^*(H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}/2)) \subseteq H^2_{\mathbb{C}-\mathrm{alg}}(\bar{X}, \mathbb{Z}/2).$$

To finish the proof, recall that we can identify

$$\pi(\mathbb{R})_2^*(H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z}/2) \cap (H^2(X_2, \mathbb{Z}/2) \oplus \cdots \oplus H^2(X_s, \mathbb{Z}/2)))$$

with $H^2_{\mathbb{C}-\mathrm{alg}}(\bar{X},\mathbb{Z}/2)\cap(H^2(X_2,\mathbb{Z}/2)\oplus\cdots\oplus H^2(X_s,\mathbb{Z}/2))$ and $\pi(\mathbb{R})^*(D_2^{-1}([P]))=\bar{D}_2^{-1}([Q])$. □

Finally, to study the image of the homomorphism i^* induced by the inclusion mapping, we need the following lemma from linear algebra to use the information from cohomology with $\mathbb{Z}/2$ coefficients.

Lemma 3.9. *Let A and B be two Abelian groups and let $h : A \to B$ be a group homomorphism. We call A_2, B_2 and h_2 the corresponding objects modulo 2. Suppose that $2B \subseteq \mathrm{Im}(h)$. Then if h_2* is surjective *so is h.*

Proposition 3.10. *Let $\mathcal{X}$ be a geometrically rational algebraic surface defined over $\mathbb{R}$. Let $i : X = \mathcal{X}(\mathbb{R}) \hookrightarrow \mathcal{X}(\mathbb{C})$ be the inclusion map. Then i^* is surjective if and only if i_2^* is surjective.*

Proof. For a geometrically rational real algebraic surface X, we have that $2H^2(X,\mathbb{Z}) \subseteq \mathrm{Im}(i^*)$, see [Jo, Claim 1, proof of Theorem 1.1]. To conclude, it suffices to consider Remark 3.5 and Lemma 3.9. □

Proposition 3.11. *Let $\mathcal{X}$ be a geometrically rational algebraic surface defined over $\mathbb{R}$. If the restriction homomorphism*

$$i_2^* : H^2(\mathcal{X}(\mathbb{C}),\mathbb{Z}/2) \to H^2(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2)$$

is surjective, then so is the corresponding homomorphism for the blowup

$$\bar{i}_2^* : H^2(\bar{\mathcal{X}}(\mathbb{C}),\mathbb{Z}/2) \to H^2(\bar{\mathcal{X}}(\mathbb{R}),\mathbb{Z}/2)\ .$$

Proof. Let us fix points $P_j \in X_j,\ j=1,\ldots,s$, so that $D_2^{-1}([P_1]),\ D_2^{-1}([P_2]),\ldots,$ $D_2^{-1}([P_s])$ generate $H^2(X,\mathbb{Z}/2)$. Obviously, since P belongs to X_1 and $\bar{X}$ is the blowup of X centered at P, $\bar{D}_2^{-1}([P_2]),\ldots,\bar{D}_2^{-1}([P_s])$ are in $H^2(\bar{X},\mathbb{Z}/2)$. Moreover, since the restriction homomorphism i_2^* is surjective by hypothesis, the classes $\bar{D}_2^{-1}([P_2]),\ldots,\ \bar{D}_2^{-1}([P_s])$ are also in the image $\bar{i}_2^*(H^2(\bar{\mathcal{X}}(\mathbb{C}),\mathbb{Z}/2))$. We conclude by Corollary 3.8. □

Proof of Proposition 3.1. Case 1. If $\Gamma(X)=0$, then i^* is surjective. By Proposition 3.10, i_2^* is also surjective and so is $\bar{i}_2^*$, see Proposition 3.11. Then $\bar{i}^*$ is surjective again by Proposition 3.10. Therefore $\Gamma(\bar{X})=0$ as we wanted to prove.

Case 2. If $\Gamma(X)=\mathbb{Z}/2$, then one can easily see that $\Gamma_2(X)=\mathbb{Z}/2$. Here $\Gamma_2(X)=H^2(X,\mathbb{Z}/2)/H^2_{\mathbb{C}-\mathrm{alg}}(X,\mathbb{Z}/2)$. If we consider the two possible cases from Corollary 3.8, it is easy to see that $\Gamma_2(\bar{X})=0$, so $\bar{i}_2^*$ is surjective. We conclude now as in the previous case. □

To finish this section, we write the main result (Proposition 3.1) in terms of regular mappings.

Corollary 3.12. *Let X be a geometrically rational real algebraic surface. If X is nonconnected, then the space of regular mappings $\mathcal{R}(\bar{X},\mathbb{S}^2)$ is dense in the space $\mathcal{C}^\infty(\bar{X},\mathbb{S}^2)$ for every blowup $\bar{X}$ of X with the $\mathcal{C}^\infty$ topology.*

4. Real conic fibrations

Let us recall again the definition of a conic fibration.

Definition 4.1. *Let $\mathcal{Y}$ be a smooth projective surface over $\mathbb{R}$. A morphism $\rho : \mathcal{Y} \to B$ onto a smooth projective curve over $\mathbb{R}$ is called a real conic fibration if ρ is defined over $\mathbb{R}$ and the generic fiber of ρ is isomorphic to a plane conic over the function field of B.*

This is equivalent to say that the generic fiber of ρ is a smooth curve of genus 0 or that $\mathcal{Y}_{\mathbb{C}}$ is a birationally ruled surface.

Proposition 4.2. *Let $\mathcal{X}$ be a geometrically rational algebraic surface over $\mathbb{R}$ which admits a real conic fibration. If $X = \mathcal{X}(\mathbb{R})$ is nonconnected, then $\Gamma(X) = 0$.*

Proof. Since $\mathcal{X}$ is geometrically rational, it admits a real conic fibration ρ over $\mathbb{P}^1$. Using the results from Section 3, it is enough to see that $\Gamma(X') = 0$ for any $\mathbb{R}$-minimal model $\mathcal{X}'$ of $\mathcal{X}$ (here $X' = \mathcal{X}'(\mathbb{R})$). In particular, it was already known to Comessatti that ρ is birational over $\mathbb{R}$ to a conic fibration $\rho' : \mathcal{X}' \to \mathbb{P}^1$ with affine equation

$$x^2 + y^2 = \pm \prod_{c=1}^{2m} (z - a_c)$$

where the a_c are distinct real numbers and $m \geq 2$, see [Si, VI.3.1] or [Ko, 4.6]. From this equation, the topology of X' is easy to understand. The roots of the polynomial $\prod_{c=1}^{2m}(z - a_c)$ determine m intervals in $\mathbb{P}^1(\mathbb{R})$ over which the real fibers $\mathcal{X}_z(\mathbb{R})$ are non empty. Over each of these intervals, there is a connected component of X' which is homeomorphic to a sphere. In particular X' is orientable.

The singular fiber over a point a_c is the union of two complex conjugated lines E and $\sigma(E)$ whose intersection point p is the only real point of the fiber. The tangent plane to X' at p is generated by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial y_1}$, where $x = x_1 + ix_2$ and $y = y_1 + iy_2$. It is easy to check that the tangent plane to E at p is generated by $i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}$, see Example 2.6. Then E is transverse to X' at p in $\mathcal{X}'(\mathbb{C})$. By Lemma 2.2, the Gysin morphism $i'_! : H_2(\mathcal{X}'(\mathbb{C}), \mathbb{Z}) \to H_0(X', \mathbb{Z})$ is surjective. Therefore, using the commutative diagram from Section 2, we obtain that $i'^* : H^2(\mathcal{X}'(\mathbb{C}), \mathbb{Z}) \to H^2(X', \mathbb{Z})$ is surjective as well. Finally by Corollary 2.4 we get $H^2_{\mathbb{C}-\mathrm{alg}}(X', \mathbb{Z}) = H^2(X', \mathbb{Z})$. □

5. Del Pezzo surfaces of degree 2

Recall that a smooth projective algebraic surface $\mathcal{X}$ defined over $\mathbb{R}$ is a *Del Pezzo surface* over $\mathbb{R}$ if $\mathcal{X}_{\mathbb{C}}$ is irreducible and the anticanonical divisor $-K_{\mathcal{X}}$ is ample. The degree of $\mathcal{X}$ is $d = K^2_{\mathcal{X}}$. In addition to the surface $\mathbb{P}^1 \times \mathbb{P}^1$, degree d Del Pezzo surfaces are classically described as the blowup of the plane $\mathbb{P}^2$ at $9 - d$ closed points (so Del Pezzo surfaces are geometrically rational). Depending on whether the centers of blowups are real or not, we obtain different real surfaces. For the degree 2 and degree 1 cases, there is an alternative model as ramified double coverings. For a degree 2 surface, this model is obtained with the anti-canonical morphism. Because of the deck involution, there are two real structures on these Del Pezzo surfaces.

Let $\mathcal{X}$ be a degree 2 Del Pezzo surface defined over $\mathbb{R}$ and let $\varphi : \mathcal{X} \to \mathbb{P}^2$ be the anti-canonical morphism. For simplicity, suppose that $\mathcal{X}(\mathbb{R}) \neq \emptyset$ (see the crucial remark in [DIK, 6.12.5]). It is known that φ is a double covering of the projective plane ramified along a curve $\Delta \subset \mathbb{P}^2$ of degree 4 defined over $\mathbb{R}$, see [Ko, Section 6]. If the real curve Δ is maximal, i.e. if it has 4 connected components, then the real part of $\mathcal{X}$ is homeomorphic to 4 spheres or to the connected nonorientable surface of Euler characteristic $\chi = -6$.

This section is devoted to the proof of Theorem 0.3. Before giving this proof we recall some properties of Del Pezzo surfaces of degree 2 that will be used.

Lemma 5.1. *Let $\mathcal{X}$ be a Del Pezzo surface of degree 2 (resp. degree 1) then there are exactly 56 (-1)-curves in $\mathrm{Pic}(\mathcal{X}_{\mathbb{C}})$ (resp. 240). Each (-1)-curve projects by φ to a bitangent line to Δ. There are 28 such bitangent lines.*

Proof. Cf. e.g. [De, II, Table 3]. □

Lemma 5.2. *Let Δ be a degree 4 smooth real plane curve and $\varphi : \mathcal{X} \to \mathbb{P}^2$ the double covering ramified along Δ. The following are equivalent.*

(1) *All the 28 bitangents are real;*
(2) *the real part $\mathcal{X}(\mathbb{R})$ is homeomorphic to 4 spheres or to the nonorientable surface $\#8\mathbb{P}^2(\mathbb{R})$;*

Moreover, if $\mathcal{X}(\mathbb{R}) \cong \bigsqcup 4\mathbb{S}^2$, the 56 (-1)-curves are arranged by pairs of complex conjugated curves.

Proof. Cf. e.g. [Ko, Theorem 6.3]. □

Proof of Theorem 0.3. Let $X = \mathcal{X}(\mathbb{R})$ as usual. Since the connected components of X are spheres, if $\mathcal{X}$ is not $\mathbb{R}$-minimal, then its $\mathbb{R}$-minimal model $\mathcal{X}_0$ is obtained by blowingdown pairs of complex conjugated curves and X is biregularly isomorphic to $\mathcal{X}_0(\mathbb{R})$. Therefore we can suppose that $\mathcal{X}$ itself is $\mathbb{R}$-minimal, see Section 3.
By Corollary 2.5, we have the following

$$\Gamma(X) \cong H_0(X,\mathbb{Z})/\mathrm{Im}(i_!)\ .$$

Therefore, it is enough to see that

$$H_0(X,\mathbb{Z})/\mathrm{Im}(i_!) \cong \mathbb{Z}/2\ .$$

By construction, the group $\mathrm{Pic}(\mathcal{X}_{\mathbb{C}})$ is generated by the hyperplane section and the exceptional curves coming from the 7 blowups. Therefore, $\mathrm{Pic}(\mathcal{X}_{\mathbb{C}})$ is generated by $\varphi^{-1}(H)$, where H is a generic line in $\mathbb{P}^2$ defined over $\mathbb{R}$, and by the (-1)-curves. It is easy to see that $i_!(\varphi^{-1}(H)) = 0$ in $H_0(\mathcal{X}(\mathbb{R}),\mathbb{Z})$. Using again Corollary 2.5, it is clear that

$$H^2_{\mathbb{C}-\mathrm{alg}}(X,\mathbb{Z}) \cong < i_!([E_k]) >_{1\le k\le 56} \subseteq H_0(X,\mathbb{Z})\ .$$

After Lemma 5.2, let us look at the bitangent lines to Δ. The real bitangents have been studied by Zeuthen [Ze]. Zeuthen called *bitangent of the second kind* a bitangent meeting two connected components of $\Delta(\mathbb{R})$ and *bitangent of the first kind* a bitangent meeting only one component. Let $\Delta_1, \cdots, \Delta_4$ be the four connected components of $\Delta(\mathbb{R})$.

Remark 5.3. For each pair of distinct connected components (Δ_j, Δ_l) of $\Delta(\mathbb{R})$, there is an integer $k(j,l)$, $1 \le k(j,l) \le 56$ and a line $L_{k(j,l)}$ bitangent of the second kind to Δ such that $L_{k(j,l)} \cap \Delta(\mathbb{R}) = \{p_j, p_l\}$, for some $p_j \in \Delta_j$ and $p_l \in \Delta_l$. To see this fact, one just have to look at the classification given by Zeuthen or at Theorem 6.4 given below.

Let $X_1, \cdots, X_4$ be the connected components of X. For each $1 \le j \le 4$, let p_j be the point in $\Delta_j \subset X_j$ given by Remark 5.3. We fix from now on the following generators $[p_1],[p_2],[p_3],[p_4]$ of $H_0(X,\mathbb{Z})$. For each pair of distinct connected components (X_j, X_l), $1 \le j < l \le 4$ of X, there is a (-1)-curve $E_{k(j,l)}$ of $\mathcal{X}_{\mathbb{C}}$ such that $\varphi(E_{k(j,l)}) = L_{k(j,l)}$ and $E_{k(j,l)} \cap X = \{p_j, p_l\}$. In other words, given that $E_{k(j,l)}$ is transverse to X (see Example 2.6) we have $i_!([E_{k(j,l)}]) = [p_j] + [p_l]$ by Lemma 2.2. Furthermore, there is at least one component Δ_t of $\Delta(\mathbb{R})$ with a bitangent L_t of the first kind. For the same reasons as before, there is a (-1)-curve E_t of $\mathcal{X}_{\mathbb{C}}$ such that $\varphi(E_t) = L_t$ and $i_!([E_t]) = 2[p_t]$.

Let $\eta : H_0(X,\mathbb{Z}) \to \mathbb{Z}/2$ be the degree map defined by

$$\eta(\sum_{j=1}^{4} m_j[p_j]) = \sum_{j=1}^{4} m_j \pmod 2\ .$$

We have proved that $\mathrm{Ker}(\eta) \subset \mathrm{Im}(i_!)$. To prove the other inclusion, it suffices to recall that $\mathrm{Im}(i_!) =< i_!([E_k]) >_{1\le k\le 56}$. Now for every k, the curve E_k projects by φ to a line defined over $\mathbb{R}$ and bitangent to $\Delta(\mathbb{R})$, then $\deg(E_k \cap X) = 2$.
Therefore, $\mathrm{Im}(i_!) = \mathrm{Ker}(\eta)$ and finally, by the remarks above

$$\Gamma(X) \cong H_0(X,\mathbb{Z})/\mathrm{Ker}(\eta) \cong \mathrm{Im}(\eta) \cong \mathbb{Z}/2\ .$$

□

Remark 5.4. For a bitangent L of the first kind, we have $L \cap X = \{p, q\}$. For a generic curve, we have $p \neq q$ and the transversality argument of Example 2.6 works. Since the invariant $H^2_{\mathbb{C}-\mathrm{alg}}(X, \mathbb{Z})$ is stable by small deformation on geometrically rational surfaces (see Corollary 2.4), this argument is sufficient to conclude. It is also a nice exercise to make a direct calculation for a tangency point of order 4 in the same way as Example 2.6. This can happen, C. Raffalli and the second author construct a (necessarily noncomplete) familly of real quartics such that every bitangent of the first kind has only one meeting point of order 4 on the quartic.

In terms of regular maps we have the following result.

Corollary 5.5. *Let X be a real algebraic surface that admits a degree 2 Del Pezzo surface as $\mathbb{R}$-minimal model. Suppose that X is homeomorphic to the disjoint union of 4 spheres and let $X_1, \cdots, X_4$ be the connected components of X. For a smooth map $f : X \to \mathbb{S}^2$, let $d(f) = \sum_k \deg(f_{|X_k})$, where $\deg : \mathcal{C}^\infty(\mathbb{S}^2, \mathbb{S}^2) \to \mathbb{Z}/2$ is the reduction modulo 2 of the degree map. Then f can be approximated by regular maps if and only if $d(f) = 0$.*

6. Del Pezzo surfaces of degree 1

Let $\mathcal{X}$ be a degree 1 Del Pezzo surface defined over $\mathbb{R}$ and let $\varphi : \mathcal{X} \to Q$ be the anti-bicanonical morphism. Then φ is a double covering of a quadric cone $Q \subset \mathbb{P}^3$ ramified along a real curve Δ which is a complete intersection of a cubic surface with Q, see [Ko, Section 6]. By the comments given in Section 3, to finish the proof of Theorem 0.4, we just need to study the quotient group Γ for a degree 1 real Del Pezzo surface $\mathcal{X}$ with $X = \mathcal{X}(\mathbb{R})$ homeomorphic to the disjoint union of 4 spheres and a projective plane. We will keep these notations for the rest of the paper.

Theorem 6.1. *For the given X, the quotient group $\Gamma(X)$ is equal to 0. In particular, every smooth map $f : X \to \mathbb{S}^2$ can be approximated by regular maps.*

Let Δ be the branch curve on the quadric cone Q. By hypothesis, the real part of the branch curve $\Delta(\mathbb{R})$ has 5 connected components. Any hyperplane of $\mathbb{P}^3$ which intersects one of these components at exactly one point p is tangent to Δ at p because $\Delta(\mathbb{R})$ is contained in an affine chart. Then for any hyperplane H of $\mathbb{P}^3$ intersecting $\Delta(\mathbb{R})$ at exactly 3 distinct points, the conic $C = H \cap Q$ is a tritangent conic to Δ. In fact for p a point of $C \cap \Delta$, the tangent line T_pC is the intersection $T_pQ \cap H$.

Lemma 6.2. *For each point in the intersection of C with $\Delta(\mathbb{R})$, there exists one irreducible component of the pullback by φ of C in $\mathcal{X}(\mathbb{C})$ that is transverse to $\mathcal{X}(\mathbb{R})$.*

Locally, we have the same situation as in Example 2.6.

Proposition 6.3. *Let Δ_1, Δ_2, Δ_3 be any three connected components of $\Delta(\mathbb{R})$. Then there exists a hyperplane of $\mathbb{P}^3(\mathbb{R})$ which is tangent to Δ_1, Δ_2 and Δ_3.*

To prove the last proposition, we use the following result [MR, Theorem 9].

Theorem 6.4. *Let $n > 1$ be a natural integer. Consider n closed connected subsets of $\mathbb{P}^n(\mathbb{R})$. Suppose there exists a point $p \in \mathbb{P}^n$ such that no hyperplane passing by p meets all the n subsets. There exists a hyperplane which is a support hyperplane for each of the n subsets.*

Proof of Proposition 6.3. Let p be a point of $\Delta(\mathbb{R}) \setminus \cup\Delta_k$. By definition of the degree, a hyperplane passing through p cannot meet three other components of $\Delta(\mathbb{R})$ because Δ has degree 6 in $\mathbb{P}^3$. □

Proof of Theorem 6.1. After Corollary 2.4 and Proposition 3.10, it suffices to prove that $i_!^2 : H_2(\mathcal{X}(\mathbb{C}), \mathbb{Z}/2) \to H_0(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2)$ is surjective. Choose a basis $\{[p_1], \dots, [p_5]\}$ of $H_0(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2)$. For each triple $1 \le j, k, l \le 5$, we have $[p_j] + [p_k] + [p_l] \in \mathrm{Im}(i_!^2)$ by Proposition 6.3 and Lemma 6.2. Then for each $1 \le j \le 5$, we have $[p_j] \in \mathrm{Im}(i_!^2)$ and $i_!^2$ is surjective. $\square$

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