



HAL
open science

On the rate of convergence in the central limit theorem for martingale difference sequences.

Lahcen Ouchti

► **To cite this version:**

Lahcen Ouchti. On the rate of convergence in the central limit theorem for martingale difference sequences.. 2004. hal-00001350v3

HAL Id: hal-00001350

<https://hal.science/hal-00001350v3>

Preprint submitted on 16 May 2004

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the rate of convergence in the central limit theorem for martingale difference sequences

Lahcen OUCHTI*

24 Mars 2004

ABSTRACT: We established the rate of convergence in the central limit theorem for stopped sums of a class of martingale difference sequences.

Sur la vitesse de convergence dans le théorème limite central pour les différences de martingale

RÉSUMÉ: On établit la vitesse de convergence dans le théorème limite central pour les sommes arrêtées issues d'une classe de suites de différences de martingale.

AMS Subject Classifications (2000): 60G42, 60F05

Keywords: central limit theorem, martingale difference sequence, rate of convergence.

1 Introduction

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall say that $(X_k)_{k \in \mathbb{N}}$ is a martingale difference sequence if, for any $k \geq 0$

1. $\mathbb{E}\{|X_k|\} < +\infty$.
2. $\mathbb{E}\{X_{k+1} | \mathcal{F}_k\} = 0$, where \mathcal{F}_k is the σ -algebra generated by $X_i, i \leq k$.

For each integer $n \geq 1$ and x real number, we denote

$$S_0 = 0, \quad S_n = \sum_{i=1}^n X_i, \quad \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-\frac{t^2}{2}) dt, \quad \sigma_{n-1}^2 = \mathbb{E}\{X_n^2 | \mathcal{F}_{n-1}\},$$

$$\nu(n) = \inf\{k \in \mathbb{N}^* / \sum_{i=0}^k \sigma_i^2 \geq n\}, \quad S_{\nu(n)}^2 = \sum_{k=1}^{+\infty} S_k^2 I_{\nu(n)=k}, \quad \sigma_{\nu(n)}^2 = \sum_{k=1}^{+\infty} \sigma_k^2 I_{\nu(n)=k},$$

*LMRS, UMR 6085, Université de Rouen, site Colbert 76821 Mont-Saint-Aignan cedex, France.
E-mail: lahcen.ouchti@univ-rouen.fr

$F_n(x) = \mathbb{P}(S_{\nu(n)} \leq x\sqrt{n})$, $S'_{\nu(n)} = S_{\nu(n)} + \sqrt{\gamma(n)}X_{\nu(n)+1}$, $H_n(x) = \mathbb{P}(S'_{\nu(n)} \leq x\sqrt{n})$, and $\gamma(n)$ is a random variable such that

$$\sum_{i=0}^{\nu(n)-1} \sigma_i^2 + \gamma(n)\sigma_{\nu(n)}^2 = n \quad p.s. \quad (1)$$

If the random variables X_i are independent and identically distributed with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$, we have by the central limit theorem (CLT)

$$\lim_{n \rightarrow +\infty} \sup_{x \in \mathbb{R}} |\mathbb{P}(S_n \leq x\sqrt{n}) - \phi(x)| = 0.$$

By the theorem of Berry ([1], 1941) and Esseen ([3], 1942), if moreover, $\mathbb{E}|X_i^3| < +\infty$, the rate of convergence in the limit is of order $n^{-\frac{1}{2}}$. If $(X_i)_{i \in \mathbb{N}}$ is an ergodic martingale difference sequence with $\mathbb{E}X_i^2 = 1$, by the theorem of Billingsley ([9], 1968) and Ibragimov ([6], 1963), see also ([10], 1980)) we have the CLT. The rate of convergence can, however, be arbitrarily slow even if X_i are bounded and α -mixing (cf [7]). There are several results showing that with certain assumption on the conditional variance $\mathbb{E}(X_i^2 | \mathcal{F}_{i-1})$, the rate of convergence becomes polynomial (Kato ([13], 1979), Grams ([12], 1972), Nakata ([11], 1976), Bolthausen ([4], 1982), Haeusler ([5], 1988), ...).

In 1963, Ibragimov [6] has shown that for X_i uniformly bounded, if instead of usual sums S_n , the stopped sums $S_{\nu(n)}$ or $S'_{\nu(n)}$ are considered, one gets the rate of convergence of order $n^{-\frac{1}{4}}$; the only assumption beside boundedness is that $\sum_{i=0}^{+\infty} \sigma_i^2$ diverge to infinity a.s.

In the present paper we give a rate of convergence for a larger class of martingale difference sequences, the Ibragimov's case will be a particular one.

2 Main result

We consider a sequence $(X_i)_{i \in \mathbb{N}}$ of square integrable martingale differences.

Theorem 1. *If the series $\sum_{i=0}^{+\infty} \sigma_i^2$ diverges a.s. and if there exists a nondecreasing sequence $(Y_i)_{i \in \mathbb{N}}$ adapted to the filtration $(\mathcal{F}_i, i \in \mathbb{N})$ such that, for all $i \in \mathbb{N}^*$*

$$\mathbb{E}(Y_i^4) < +\infty, \quad 1 \leq Y_i \quad \text{and} \quad \mathbb{E}(|X_i|^3 | \mathcal{F}_{i-1}) \leq Y_{i-1} \sigma_{i-1}^2 \quad a.s.$$

then for all n sufficiently large

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left(11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right), \quad (2)$$

$$\sup_{x \in \mathbb{R}} \left| H_n(x) - \phi(x) \right| \leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left(11 + \frac{9}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right) \quad (3)$$

where $a_n = (\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$.

If we put $Y_i = M$ a.s. where $M > 0$ is a constant, one obtains the following corollaries:

Corollary 1. *If the series $\sum_{i=0}^{+\infty} \sigma_i^2$ diverges a.s. and there exists $M > 0$ such that, for all $i \in \mathbb{N}^*$, $\mathbb{E}(|X_i|^3 | \mathcal{F}_{i-1}) \leq M \mathbb{E}(X_i^2 | \mathcal{F}_{i-1})$ a.s. then there is a constant $0 < c_M < +\infty$*

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{c_M}{n^{\frac{1}{4}}}, \quad (4)$$

$$\sup_{x \in \mathbb{R}} \left| H_n(x) - \phi(x) \right| \leq \frac{c_M}{n^{\frac{1}{4}}}. \quad (5)$$

Corollary 2. *If there exists $0 < \alpha \leq M < +\infty$ satisfying $\sigma_{i-1}^2 \geq \alpha$ and $\mathbb{E}(|X_i|^3 | \mathcal{F}_{i-1}) \leq M$ a.s. for all $i \in \mathbb{N}^*$, then there is a constant $0 < c_{(\alpha, M)} < +\infty$ such that (4) and (5) hold.*

Moreover, if we suppose that $(X_i)_{i \in \mathbb{N}}$ is uniformly bounded, we obtain the result of Ibragimov [6].

Corollary 3. *If the series $\sum_{i=0}^{+\infty} \sigma_i^2$ diverges a.s. and $|X_i| \leq M < +\infty$ a.s. for all $i \geq 0$, then (4) and (5) hold.*

Example. Let $A = (A_k)_{k \in \mathbb{N}}$ be a sequence of real valued random variables such that $\sup_{k \in \mathbb{N}} \mathbb{E}(A_k^4)^{1/4} = \beta < \infty$ and consider an arbitrary sequence of variables $\zeta = (\zeta_k)_{k \in \mathbb{N}^*}$ with zero means, unit variances, bounded third moments and which are also independent of A . We define $X = (A_{k-1} \zeta_k)_{k \in \mathbb{N}^*}$ and \mathcal{F}_k the σ -algebra generated by A_0, A_1, \dots, A_k .

Clearly $(X_k, \mathcal{F}_k, k \in \mathbb{N}^*)$ is a martingale difference sequence, and for all $k \in \mathbb{N}^*$,

$$\mathbb{E}(A_{k-1}^2 \zeta_k^2 | \mathcal{F}_{k-1}) = A_{k-1}^2 \quad \text{a.s.},$$

$$\mathbb{E}(|A_{k-1} \zeta_k|^3 | \mathcal{F}_{k-1}) \leq |A_{k-1}| \sup_{i \in \mathbb{N}^*} \mathbb{E}(|\zeta_i|^3) A_{k-1}^2 \quad \text{a.s.}$$

If $(|A_k|)_{k \in \mathbb{N}}$ is nondecreasing, then using Theorem 1, one obtains

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq c \beta \frac{\sup_{k \in \mathbb{N}^*} \mathbb{E}(|\zeta_k|^3)^{\frac{1}{4}}}{n^{\frac{1}{4}}}$$

where c is a positive constant.

3 Proof of Theorem

According to Esseen's theorem (see, e.g., ([2], 1954) p. 210 and ([8], 1955) p. 285), for all $y > 0$,

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{1}{\pi} \int_{-y}^y \left| \mathbb{E} \left\{ \exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}\right) \right\} - \exp\left(-\frac{t^2}{2}\right) \right| \frac{dt}{|t|} + \frac{24}{\pi \sqrt{2\pi y}}. \quad (6)$$

Below we shall prove the following inequalities

$$\left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \leq a_n e^{\frac{t^2}{2}} \left(\frac{|t|}{3\sqrt{n}} + \frac{t^2}{4n} + \frac{a_n |t|^3}{3n^{\frac{3}{2}}} + \frac{a_n t^4}{4n^2} \right), \quad (7)$$

$$\left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2} \right) \right\} \right| \leq \frac{a_n t^2}{2n} \exp\left(\frac{t^2}{2}\right), \quad (8)$$

$$\left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} \right) \right\} - \mathbb{E} \left\{ \exp \left(\frac{itS'_{\nu(n)}}{\sqrt{n}} \right) \right\} \right| \leq \frac{3a_n t^2}{2n} \quad (9)$$

where $a_n = (\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$.

3.1 Proof of the Inequality (7)

We have

$$\begin{aligned} & \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \\ &= \sum_{k=1}^{+\infty} \mathbb{E} \left\{ \left(\exp \left(\frac{itS_k}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{k-1} \sigma_p^2 \right) - 1 \right) I_{\nu(n)=k} \right\} \\ &= \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ \exp \left(\frac{itS_{j-1}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \left(e^{\frac{itX_j}{\sqrt{n}}} - e^{-\frac{t^2\sigma_{j-1}^2}{2n}} \right) I_{\nu(n)=k} \right\}. \end{aligned}$$

For real x , put

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + u(x), \quad e^{-x} = 1 - x + \beta(x) \frac{x^2}{2} \quad (*)$$

It is easily seen that, for all $x \in \mathbb{R}$

$$|u(x)| \leq \frac{|x|^3}{6}, \quad |u(x)| \leq \frac{x^2}{2}, \quad \text{and} \quad |\beta(|x|)| \leq 1.$$

Observing that the random variable $W_{j-1}^n = \exp \left(\frac{itS_{j-1}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right)$ is measurable with respect to the σ -algebra \mathcal{F}_{j-1} and using the identities (*), we obtain

$$\begin{aligned} & \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \\ &= \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ W_{j-1}^n \mathbb{E} \left\{ \left(\frac{itX_j}{\sqrt{n}} - \frac{t^2 X_j^2}{2n} + u\left(\frac{tX_j}{\sqrt{n}}\right) + \frac{t^2 \sigma_{j-1}^2}{2n} + \beta\left(\frac{t^2 \sigma_{j-1}^2}{2n}\right) \frac{t^4 \sigma_{j-1}^4}{8n^2} \right) I_{\nu(n)=k} \middle| \mathcal{F}_{j-1} \right\} \right\} \end{aligned} \quad (10)$$

Since $\{\nu(n) = k\}$ is measurable with respect to the σ -algebra \mathcal{F}_k , for all $j \geq 2$, we have

$$\sum_{k=1}^{j-1} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k=1}^{j-1} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

On the other hand, for all $j \geq 1$ we have

$$\sum_{k=1}^{+\infty} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k=1}^{+\infty} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

It follows that, for all $j \geq 1$

$$\sum_{k \geq j} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k \geq j} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

So, from (10) we derive

$$\begin{aligned} & \left| \mathbb{E}\left\{ \exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \\ &= \left| \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E}\left\{ W_{j-1}^n \mathbb{E}\left\{ \left(u\left(\frac{tX_j}{\sqrt{n}}\right) + \beta\left(\frac{t^2\sigma_{j-1}^2}{2n}\right) \frac{t^4\sigma_{j-1}^4}{8n^2} \right) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \right| \\ &\leq \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \left(\frac{|t|^3 |X_j|^3}{6n^{\frac{3}{2}}} + \frac{t^4\sigma_{j-1}^4}{8n^2} \right) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\}. \end{aligned} \quad (11)$$

For any $j \geq 2$ and any real function ψ such that $\mathbb{E}(\psi(X_k)) < \infty$ for any positive k , we have

$$\begin{aligned} & \sum_{k=1}^{j-1} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \\ &= \sum_{k=1}^{j-1} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} \right\}. \end{aligned} \quad (12)$$

On the other hand, for all $j \geq 1$, we have

$$\begin{aligned} & \sum_{k=1}^{+\infty} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \\ &= \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \psi(X_j) \right\} \\ &= \sum_{k=1}^{+\infty} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} \right\}. \end{aligned} \quad (13)$$

It follows from (12) and (13) that

$$\begin{aligned} & \sum_{j=1}^{+\infty} \sum_{k \geq j} \mathbb{E} \left\{ \exp \left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E} \left\{ \psi(X_j) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \\ &= \sum_{j=1}^{+\infty} \sum_{k \geq j} \mathbb{E} \left\{ \exp \left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E} \left\{ \psi(X_j) | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} \right\}. \end{aligned} \quad (14)$$

Applying (11) and (14) for $\psi(x) = |x|^3$ we deduce that

$$\begin{aligned} & \left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \\ & \leq \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ \exp \left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \left(\mathbb{E} \left\{ \frac{|t|^3 |X_j|^3}{6n^{\frac{3}{2}}} | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} + \mathbb{E} \left\{ \frac{t^4 \sigma_{j-1}^4}{8n^2} I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right) \right\} \\ & \leq \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ \exp \left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \left(\frac{|t|^3 Y_{j-1} \sigma_{j-1}^2}{6n^{\frac{3}{2}}} I_{\nu(n)=k} + \frac{t^4 \sigma_{j-1}^4}{8n^2} I_{\nu(n)=k} \right) \right\} \end{aligned} \quad (15)$$

By the Hölder inequality, for all $j \in \mathbb{N}^*$

$$\sigma_{j-1}^2 = \mathbb{E}(X_j^2 | \mathcal{F}_{j-1}) \leq \mathbb{E}(|X_j|^3 | \mathcal{F}_{j-1})^{\frac{2}{3}} \leq Y_{j-1}^{\frac{2}{3}} \sigma_{j-1}^{\frac{4}{3}} \quad a.s.,$$

whence

$$\sigma_{j-1}^2 \leq Y_{j-1}^2 \quad a.s. \quad (16)$$

From (15), (16) and using the fact that $Y_k \geq Y_{j-1} \geq 1$ for all $j \leq k$, we deduce that

$$\begin{aligned} & \left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \\ & \leq \left(\frac{|t|^3}{6n^{\frac{3}{2}}} + \frac{t^4}{8n^2} \right) \sum_{k=1}^{+\infty} \sum_{j=1}^k \mathbb{E} \left\{ Y_{j-1}^2 \sigma_{j-1}^2 \exp \left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) I_{\nu(n)=k} \right\} \\ & \leq \left(\frac{|t|^3}{6n^{\frac{3}{2}}} + \frac{t^4}{8n^2} \right) \sum_{k=1}^{+\infty} \mathbb{E} \left\{ Y_k^2 \sum_{j=1}^k \sigma_{j-1}^2 \exp \left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) I_{\nu(n)=k} \right\}. \end{aligned} \quad (17)$$

To bound up the terms appearing in (17), we will use the following elementary lemma.

Lemma 1. *Let $k \geq 1$, then on the event $\{\nu(n) = k\}$ we have*

$$\sum_{j=1}^k \exp \left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \frac{t^2}{2n} \sigma_{j-1}^2 \leq \exp \left(\frac{t^2}{2} \right) \left(1 + \frac{Y_k^2 t^2}{n} \right).$$

Proof of Lemma. On the event $\{\nu(n) = k\}$, we have

$$\begin{aligned} \exp\left(\frac{t^2}{2}\right) &\geq \exp\left(\frac{t^2}{2n} \sum_{p=0}^{k-1} \sigma_p^2\right) - \exp\left(\frac{t^2}{2n} \sigma_0^2\right) \\ &\geq \sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \left(\exp\left(\frac{t^2 \sigma_j^2}{2n}\right) - 1\right) \end{aligned}$$

Using the inequality, $\exp(x) - 1 \geq x$ for all $x \geq 0$, one obtains

$$\exp\left(\frac{t^2}{2}\right) \geq \sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \frac{t^2}{2n} \sigma_j^2.$$

Therefore

$$\begin{aligned} &\sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \frac{t^2}{2n} \sigma_{j-1}^2 \\ &\leq \sum_{j=1}^{k-1} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \frac{t^2}{2n} (\sigma_{j-1}^2 - \sigma_j^2) + \exp\left(\frac{t^2}{2}\right) \\ &= \sum_{j=1}^{k-2} \left(\exp\left(\frac{t^2}{2n} \sum_{p=0}^j \sigma_p^2\right) - \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \right) \frac{t^2}{2n} \sigma_j^2 - \frac{t^2}{2n} \exp\left(\frac{t^2}{2n} \sum_{p=0}^{k-2} \sigma_p^2\right) \sigma_{k-1}^2 \\ &\quad + \frac{t^2}{2n} \exp\left(\frac{t^2}{2n} \sigma_0^2\right) \sigma_0^2 + \exp\left(\frac{t^2}{2}\right) \\ &\leq \frac{t^2}{2n} Y_k^2 \sum_{j=1}^{k-2} \left(\exp\left(\frac{t^2}{2n} \sum_{p=0}^j \sigma_p^2\right) - \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2\right) \right) + \frac{t^2}{2n} Y_k^2 \exp\left(\frac{t^2}{2n} \sigma_0^2\right) + \exp\left(\frac{t^2}{2}\right) \\ &\leq \left(1 + \frac{t^2}{2n} Y_k^2\right) \exp\left(\frac{t^2}{2}\right). \end{aligned}$$

We conclude the proof of the lemma by noting that $\sigma_{k-1}^2 \leq Y_k^2$ and $\sum_{p=0}^{k-1} \sigma_p^2 \leq n$ a.s..

Finally, according to Lemma 1 and the (17) we get

$$\left| \mathbb{E} \left\{ \exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2\right) \right\} - 1 \right| \leq a_n \exp\left(\frac{t^2}{2}\right) \left(\frac{|t|}{3\sqrt{n}} + \frac{t^2}{4n} + \frac{a_n |t|^3}{3n^{\frac{3}{2}}} + \frac{a_n t^4}{4n^2} \right),$$

where $a_n = (\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$.

3.2 Proof of the Inequality (8)

Using (1) and the inequality $|1 - \exp(-x)| \leq x$, for all $x \geq 0$ we see that

$$\begin{aligned}
& \left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2} \right) \right\} \right| \\
&= \left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2} \right) \left(\exp \left(-\frac{t^2}{2n} \gamma(n) \sigma_{\nu(n)}^2 \right) - 1 \right) \right\} \right| \\
&\leq \mathbb{E} \left\{ \left| 1 - \exp \left(-\frac{t^2}{2n} \gamma(n) \sigma_{\nu(n)}^2 \right) \right| \right\} \exp \left(\frac{t^2}{2} \right) \\
&\leq \mathbb{E} \left\{ \frac{t^2}{2n} |\gamma(n)| \sigma_{\nu(n)}^2 \right\} \exp \left(\frac{t^2}{2} \right) \\
&\leq (\mathbb{E} Y_{\nu(n)}^4)^{\frac{1}{2}} \frac{t^2}{2n} \exp \left(\frac{t^2}{2} \right).
\end{aligned}$$

Therefore (8) holds true.

From (7) and (8) we conclude that

$$\left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} \right) \right\} - \exp \left(-\frac{t^2}{2} \right) \right| \leq a_n \left(\frac{|t|}{3\sqrt{n}} + \frac{3t^2}{4n} + \frac{|t|^3}{3n^{\frac{3}{2}}} a_n + \frac{t^4}{4n^2} a_n \right).$$

Using Esseen's theorem, we derive

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{a_n}{\pi} \int_{-y}^y \left(\frac{1}{3\sqrt{n}} + \frac{3|t|}{4n} + \frac{t^2}{3n^{\frac{3}{2}}} a_n + \frac{|t|^3}{4n^2} a_n \right) dt + \frac{24}{\pi \sqrt{2\pi} y}.$$

Hence

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{a_n}{\pi} \left(\frac{2y}{3\sqrt{n}} + \frac{3y^2}{4n} + \frac{2y^3}{9n^{\frac{3}{2}}} a_n + \frac{y^4}{8n^2} a_n \right) + \frac{24}{\pi \sqrt{2\pi} y}.$$

Choosing y in such a way that $y/\sqrt{n} = 1/(ya_n)$, i.e. $y = (n/a_n^2)^{\frac{1}{4}}$, we infer that

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - \phi(x) \right| \leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left(11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right).$$

The proof of the inequality (2) in theorem is complete.

3.3 Proof of the Inequality (9)

Observing that the random events $\{\gamma(n) \leq x\} \cap \{\nu(n) = k\}$ and consequently the random variables $\sqrt{\gamma(n)} I_{\nu(n)=k}$ are measurable with respect to \mathcal{F}_k , we find that

$$\begin{aligned}
& \left| \mathbb{E} \left\{ \exp \left(\frac{itS_{\nu(n)}}{\sqrt{n}} \right) \right\} - \mathbb{E} \left\{ \exp \left(\frac{itS'_{\nu(n)}}{\sqrt{n}} \right) \right\} \right| \\
&= \left| \sum_{k=0}^{+\infty} \mathbb{E} \left\{ \left(\exp \left(\frac{itS_k}{\sqrt{n}} \right) - \exp \left(\frac{itS_k}{\sqrt{n}} + \frac{it\sqrt{\gamma(n)}}{\sqrt{n}} x_{\nu(n)+1} \right) \right) I_{\nu(n)=k} \right\} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{+\infty} \left| \mathbb{E} \left\{ \exp \left(\frac{itS_k}{\sqrt{n}} \right) \left(1 - \exp \left(\frac{it\sqrt{\gamma(n)}}{\sqrt{n}} X_{\nu(n)+1} \right) \right) I_{\nu(n)=k} \right\} \right| \\
&= \sum_{k=0}^{+\infty} \left| \mathbb{E} \left\{ \exp \left(\frac{itS_k}{\sqrt{n}} \right) \left(-\frac{it\sqrt{\gamma(n)}}{\sqrt{n}} X_{k+1} + \frac{t^2}{2n} \gamma(n) X_{k+1}^2 - u \left(\frac{t\sqrt{\gamma(n)}}{\sqrt{n}} X_{k+1} \right) \right) I_{\nu(n)=k} \right\} \right| \\
&= \sum_{k=0}^{+\infty} \left| \mathbb{E} \left\{ \exp \left(\frac{itS_k}{\sqrt{n}} \right) \mathbb{E} \left\{ -\frac{it\sqrt{\gamma(n)}}{\sqrt{n}} X_{k+1} + \frac{t^2}{2n} \gamma(n) X_{k+1}^2 - u \left(\frac{t\sqrt{\gamma(n)}}{\sqrt{n}} X_{k+1} \right) \middle| \mathcal{F}_k \right\} I_{\nu(n)=k} \right\} \right| \\
&= \sum_{k=0}^{+\infty} \left| \mathbb{E} \left\{ \exp \left(\frac{itS_k}{\sqrt{n}} \right) \left(\frac{t^2}{2n} \gamma(n) X_{k+1}^2 - u \left(\frac{t}{\sqrt{n}} \sqrt{\gamma(n)} X_{k+1} \right) \right) I_{\nu(n)=k} \right\} \right| \\
&\leq \sum_{k=0}^{+\infty} \mathbb{E} \left\{ I_{\nu(n)=k} \frac{3t^2}{2n} \gamma(n) X_{k+1}^2 \right\} \\
&\leq \frac{3t^2}{2n} \sum_{k=0}^{+\infty} \mathbb{E} \left\{ I_{\nu(n)=k} \mathbb{E} \{ X_{k+1}^2 | \mathcal{F}_k \} \right\} \\
&\leq \frac{3t^2}{2n} \mathbb{E} (Y_{\nu(n)}^4)^{\frac{1}{2}}.
\end{aligned}$$

The proof of the inequality (9) is complete.

3.4 Proof of the inequality (3).

According to Esseen's theorem where $y = (n/a_n^2)^{\frac{1}{4}}$ and the inequality (9), one obtains

$$\begin{aligned}
\sup_{x \in \mathbb{R}} |H_n(x) - \phi(x)| &\leq \frac{1}{\pi} \int_{-y}^y \left| \mathbb{E} \left\{ \exp \left(\frac{itS'_{\nu(n)}}{\sqrt{n}} \right) \right\} - \exp \left(-\frac{t^2}{2} \right) \right| \frac{dt}{|t|} + \frac{24}{\pi \sqrt{2\pi} y} \\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left(11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right) + \frac{1}{\pi} \int_{-y}^y \frac{3|t|}{2n} E(Y_{\nu(n)}^4)^{\frac{1}{2}} dt \\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left(11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right) + \frac{3}{2\pi \sqrt{n}} \\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left(11 + \frac{9}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right).
\end{aligned}$$

The proof of theorem is complete. \square

Proofs of corollaries 1, 2 and 3 are easy so, it is left to the reader.

Acknowledgements: The author thanks the referee for careful reading of the manuscript and for valuable suggestions which improved the presentation of this paper.

References

- [1] A.C. Berry, The accuracy of the Gaussian approximation to the sum of independent variates, *Trans. Amer. Math. Soc.* 49 (1941), 122-136.
- [2] B. V. Gnedenko and A. N. Kolomogorov, Limit Distributions for Sums of Independent Random Variables, *translated by K. L. Chung, Addison-Wesley, Reading, Mass,* (1954).
- [3] C. G. Esseen, On the Liapounoff limit of error in the theory of probability, *Ark. Math. Astr. och Fysik 28A* (1942), 1-19.
- [4] E. Bolthausen, Exact convergence rates in some martingale central limit theorems, *The Annals of Probability, vol. 10, No 3* (1982), 672-688.
- [5] E. Haeusler, On the rate of convergence in the central limit theorem for martingales with discrete and continuous time, *Annals of Probability, vol 16, No 1* (1988), 275-299,
- [6] I. A. Ibragimov, A central limit theorem for a class of dependent random variables, *Theory Probab. Appl.* 8 (1963), 83-89.
- [7] M. El Machkouri and D. Volný, On the central and local limit theorems for martingale difference sequences, To appear in *Stochastics and Dynamics*.
- [8] M. Loève, Probability Theory, *D. Van Nostrand. Princeton, N. J.,* (1955).
- [9] P. Billingsley, Convergence of probability measures, wiley. (1968).
- [10] P. Hall and C.C. Heyde, Martingale Limit Theory and Its Application. Academic, New York. (1980).
- [11] T. Nakata, On the rate of convergence in mean central limit theorem for martingale differences . *Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.* 23 (1976), 10-15.
- [12] W. F. Grams, Rate of convergence in the central limit theorem for dependent variables. Dissertation, Florida State Univ. (1972).
- [13] Y. Kato, Rates of convergence in central limit theorem for martingale differences, *Bull. Math. Statist.* 18 (1979), 1-8.