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Length series on Teichmuller space

Greg McShane

Abstract

We prove that a certain series defines a constant function using Wolpert's formula for the variation of the length of a geodesic along a Fenchel Nielsen twist. Subsequently we determine the value viewing it as function on the the Deligne Mumford compactification $\overline{\mathcal{M}}_{1,1}$ and evaluating it at the stable curve at infinity.

Conventions:

1. For γ an essential closed curve on a surface $l_\gamma(x)$ is the length of the geodesic homotopic to γ where x is the point in the moduli space determined by the metric on the surface.
2. For a homeomorphism $h : M \to M$ and a geodesic γ, $h(\gamma)$ is the *geodesic* homotopic to the image of γ under h.
3. if γ is an oriented curve then $-\gamma$ is the curve with the opposite orientation.

1 Introduction

Let M be a *one holed torus*. The fundamental group of M is freely generated by two loops γ_1, γ_2 which meet in a single point and such that their commutator is a loop, δ, around the hole. Such a surface is uniformized by a representation

$$\rho : \pi_1 = \langle \gamma_1, \gamma_2 \rangle \to \mathrm{SL}_2(\mathbf{R})$$

such that the commutator of the generators is a hyperbolic element of translation length l_δ [8]. Denote by $\mathcal{T}_1(l_\delta)$ the *Teichmuller space of* M and $\mathcal{M}_1(l_\delta)$ the corresponding *moduli space*. The mapping class group, $\mathcal{MCG}$, is defined to be the group of orientation preserving diffeomorphisms up to isotopy i.e. $\pi_0(\mathrm{Diffeo}^+(M))$; it is critical that we take only the orientation preserving diffeomorphisms. By the work of Nielsen and Mangel:

$$\pi_0(\mathrm{Diffeo}(M)) \cong \mathrm{Aut}(\pi_1)/\mathrm{Inn}(\pi_1) \cong \mathrm{GL}_2(\mathbf{Z}).$$

The mapping class group is index 2 in $\pi_0(\text{homeo}(M))$ and so isomorphic to $\mathrm{SL}_2(\mathbf{Z})$.

Two questions

In [15] we showed that

$$\sum \frac{2}{1+\exp l_\gamma} = 1$$

where the sum extends over all closed simple curves γ on a hyperbolic punctured torus.

Question 1: Jorgensen asked if the above identity could be proved using the *Markoff cubic*

$$a^2+b^2+c^2-abc=0,\ \ a,b,c>2.$$

In [2] Bowditch answers this giving a proof using a summation argument over the edges of the tree, $\mathbf{T}$, of solutions to this equation.

Question 2: Jorgensen also asked can the identity be proved using Wolpert's formula for variation of length? It is this question that we address here. By clever "accounting" Bowditch avoids considering the following divergent series

$$\sum_{\{a,b,c\}\in\mathbf{T}} \left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right).$$

The series is divergent since T is infinite and, since a,b,c is a solution of the cubic, the value of each term is 1. Our approach is based on showing that the derivative of a divergent series like the one above is 0.

Another divergent series

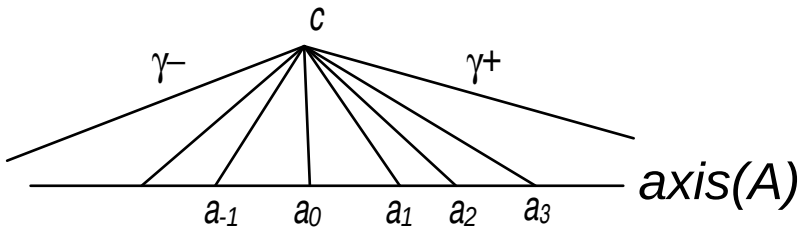
Let α,β a pair of oriented closed simple geodesics on M meeting in exactly one point and such that the *signed angle* $\alpha\vee\beta$ (see section 3) between them is positive. By viewing $[\alpha],[\beta]$ as a basis of $H_1(M,\mathbf{Z})\cong\mathbf{Z}\oplus\mathbf{Z}$ [17] one sees that the stabiliser of (α,β) in $\mathcal{MCG}$ is trivial. Moreover, since each $g\in\mathcal{MCG}$ is a homeomorphism, the pair of geodesics $g(\alpha),g(\beta)$ again meet in a single point so $g(\alpha)\vee g(\beta)$ is well defined and strictly positive (since g preserves the orientation.)

We begin with the formal series:

$$\mathcal{Q}=\sum_{g\in\mathcal{MCG}} g(\alpha)\vee g(\beta).$$

We determine the "sum" of this series using a coset decomposition and some hyperbolic geometry as follows.

Step one: we rewrite $\mathcal{Q}$ as a sum over the set of coset representatives $\mathcal{MCG}/\langle T_\gamma\rangle$ where T_γ is the Dehn twist along γ (see section 6):



$$\mathcal{Q} = \sum_{h \in \mathcal{MCG}/\langle T_\gamma \rangle} \sum_{p \in \langle T_\gamma \rangle} hp(\alpha) \vee hp(\beta) = \sum_{\gamma \in \mathcal{G}_0} \sum_{n \in \mathbf{Z}} T_\gamma^n(\gamma') \vee T_\gamma^{n+1}(\gamma'), \qquad (1)$$

where the outer sum is over all oriented simple closed geodesics $\mathcal{G}_0$ and γ' is any simple closed geodesic that meets γ exactly once. Note that, for a one holed torus, $\mathcal{MCG}$ acts transitively on $\mathcal{G}_0$ and the stabiliser of $\gamma \in \mathcal{G}_0$ is precisely $\langle T_\gamma \rangle$ so $\mathcal{MCG}/\langle T_\gamma \rangle$ is in 1-1 correspondence with $\mathcal{G}_0$.
Step two: we evaluate the inner sum over $\mathbf{Z}$ using (see section 7):

Lemma 1 *Let γ be a simple closed curve and γ' any simple closed geodesic meeting γ exactly once. Let $A \in \rho(\pi_1)$ be an element such that $axis(A)/\langle A \rangle = \gamma$ and let $\sqrt{A} \in SL_2(\mathbf{R})$ denote the square root of A . Then there exists $c \in H^2$, and $\hat{\alpha_n} \subset H^2$ such that $\forall n \in \mathbf{Z}$:*

1. *$\hat{\alpha_n}$ is a lift of $T_\gamma^n(\gamma')$.*
2. *if $a_n := \hat{\alpha_n} \cap axis(A)$ then $a_n = (\sqrt{A})^n(a_0)$.*
3. *$c \in \hat{\alpha_n}$.*

Morever, let γ^+ (resp. γ^-)be the geodesic passing through c and asymptotic to $axis(A)$ at the attracting (resp. repelling) fixed point of A. Then:

$$\hat{\alpha_n} \to \gamma^\pm,$$

as $n \to \pm\infty$ and where the convergence is uniform on compact sets.

The content of the lemma is that the above diagram is a true representation of lifts of the orbit of γ' under T_γ. Thus the angles in the inner sum are just the angles between consecutive $\hat{\alpha_n}$ at the point c. The sum "telescopes" over n and one obtains:

$$\sum_{n \in \mathbf{Z}} T_\gamma^n(\gamma') \vee T_\gamma^{n+1}(\gamma') = \gamma^- \vee \gamma^+ = \pi - 2\arctan\left(\frac{\cosh(l_\delta/4)}{\sinh(l_\gamma/2)}\right). \qquad (2)$$

We now determine the sum of $\mathcal{Q}$ using a different coset decomposition. There is an element of order 2, $q \in \mathcal{MCG}$, such that $(q(\alpha), q(\beta)) = (\beta, -\alpha)$. Rewriting $\mathcal{Q}$ as a sum over cosets of $\mathcal{MCG}/\langle q \rangle$, one obtains:

$$\mathcal{Q} = \sum_{g \in \mathcal{MCG}/\langle q \rangle} g(\alpha) \vee g(\beta) + gq(\alpha) \vee gq(\beta) = \sum_{\mathcal{MCG}/\langle q \rangle} \pi \tag{3}$$

since

$$g(\alpha) \vee g(\beta) + g(\beta) \vee g(-\alpha) = \pi.$$

Observation: although this last identity clearly implies that our series is divergent, it also suggests that the variation of the $\mathcal{Q}$ vanishes when viewed as a 1-form on Teichmuller space. Formally one sees that:

$$\mathcal{Q}' : x \mapsto \sum_{\gamma} 2 \arctan \left(\frac{\cosh(l_\delta(x)/4)}{\sinh(l_\gamma(x)/2)} \right), \ \mathcal{T}_1(l_\delta) \to \mathbf{R}$$

is constant since $d\mathcal{Q}' = d\mathcal{Q}$.

Statement of results

The above illustrates a *formal* method for finding constant functions defined by automorphic series over $\mathcal{MCG}$. To illustrate how this method is made rigorous we show:

Theorem 2 *For a one holed torus M :*

$$\sum_{\gamma} \arctan \left(\frac{\cosh(l_\delta/4)}{\sinh(l_\gamma/2)} \right) = \frac{3\pi}{2},$$

where the sum extends over all simple closed geodesics γ on M and l_δ is the length of the boundary geodesic δ.

First we show that $\mathcal{Q}'$ is constant. From their expansions as infinite series:

$$d\mathcal{Q}' = d\mathcal{Q} = \sum_{g} d(g(\alpha) \vee g(\beta)).$$

To conclude that the variation vanishes one must show that the series on the right converges absolutely justifying the rearrangements used above. Our point of view is similar to that of Kerckhoff [14] in that we do not explicitly work with a metric, although the Weil-Petersson metric is implicit, but with the "Fenchel-Nielsen geometry" of the cotangent bundle. We evaluate the pairing of $d\mathcal{Q}$ with the Fenchel-Nielsen vector field $t(\mu)$ associated to a simple closed geodesic μ. By a result of Wolpert [26][24] there are finitely many simple closed geodesics μ_i such that the associated Fenchel-Nielsen vector fields t_{μ_i} generate the tangent space at every point in the Teichmuller space of a surface of finite type. A 1-form vanishes iff its pairing with these fields vanishes. Using Wolpert's formula for variation of lengths [23] (section 5) and elementary estimates for the lengths of simple geodesics (section 4) we obtain as our main theorem:

Theorem 3 *Let μ be a simple closed geodesic $t(\mu)$ the associated Fenchel-Nielsen vector field then the series*

$$\sum_{g \in \mathcal{MCG}} d(g(\alpha) \vee g(\beta)).t(\mu)$$

converges absolutely and its sum vanishes.

Absolute convergence in the usual sense for numerical series allows one to pair terms as in (3) above:

$$d(g(\alpha) \vee g(\beta) + g(\beta) \vee g(-\alpha)).t(\mu) = 0,$$

and so the sum for $d\mathcal{Q}.t(\mu)$ vanishes identically and $\mathcal{Q}'$ is constant.

Subsequently (section 8) we determine the value of the series by viewing it as function on the the Deligne-Mumford compactification $\overline{\mathcal{M}}_1(l_\delta)$ and evaluating at the stable curve added to obtain the compactification from $\mathcal{M}_1(l_\delta)$.On a neighborhood of infinity the systole $\mathrm{sys}(x)$, that is the shortest closed geodesic, is short. To evaluate the sum we prove:

Theorem 4 *Let $f : [0,1] \to \mathbf{R}$ continuous at 1 and satisfying*

$$f(x) = f'(0)x + O(x^{1+\delta}),$$

for some $\delta > 0$.

As $\mathrm{sys}(x) \to 0$,

$$\begin{aligned} \lim \textstyle\sum_\gamma f(\mathrm{sech}(l_\gamma/2) &= f(1) + f'(0) \lim(\textstyle\sum_{\gamma'} \mathrm{sech}(l_{\gamma'}/2)) \\ &= f(1) + (\pi \mathrm{sech}(l_\delta/4)) f'(0) \end{aligned}$$

where γ varies over over all simple geodesics and γ' over all simple closed geodesic that meets $\mathrm{sys}(x)$ *exactly once.*

Generalizations

There are two generalisations of Wolpert's formula. The first is due to Goldman [6][7] for the representation space of a surface group into semi simple Lie group and the second to Series [22] for quasi-Fuchsian deformation space. By replacing signed angle by *signed complex length* [22] one obtains a constant function for quasi-Fuchsian space (compare [10].)

Identities for higher genus surfaces with punctures/boundary components [16] can also be treated using this method. In addition interesting relations can be obtained by considering cyclic groups other than those generated by Dehn twists.

Note. Our approach is strikingly similar to that of Golse and Lochak [9] who give an infinitesimal version of the Selberg trace formula based on Wolpert's formula.

2 Markoff triples, $\mathcal{T}_1(l_\delta)$

A Markoff triple is a solution in positive integers of the *Markoff cubic*:

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 0,\ x_i > 2. \tag{4}$$

By work of Cohn [5], Haas [11] and others [21] there is a correspondence between Markoff triples and configurations $\gamma_1, \gamma_2, \gamma_3$ of simple closed geodesics on the punctured torus such that $\gamma_i \cap \gamma_j, i \neq j$ is a single point. Define a *Markoff triple of geodesics* to be such a configuration. Recall that a closed geodesic, since it has no base point, only determines a conjugacy class of the fundamental group. One can, however, view γ_1, γ_2 as elements of $\pi_1(M, \gamma_1 \cap \gamma_2)$. There is a choice of orientations for γ_1, γ_2 them such that:

1. the commutator $[\gamma_1, \gamma_2]$ represents a loop around the puncture on M.
2. $[\gamma_3] = [\gamma_1^{-1}\gamma_2]$ as conjugacy classes in π_1.

Observation: The second condition is equivalent to:

$$2'.\ \ \gamma_2 = T_{\gamma_3}(\gamma_1),$$

where T_{γ_3} is the Dehn twist round γ_3. This point of view is important in the proof of (1), see section 6.

Now consider a representation $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbf{R})$ uniformizing a hyperbolic structure. The condition on the commutator means that $\rho([\gamma_1, \gamma_2])$ is a parabolic and an elementary argument shows that its trace is negative hence -2. Using the trace relations in $\mathrm{SL}_2(R)$ one computes the trace of the commutator in terms of the traces of $\rho([\gamma_i])$:

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = \operatorname{tr}\rho([\gamma_1, \gamma_2]) + 2.$$

Thus the triple of traces $x_i = \operatorname{tr}\rho([\gamma_i])$ is always a solution of the Markoff cubic. To obtain integer solutions one specializes to a representation whose image is contained in $\mathrm{SL}_2(Z)$.

A *generalised Markoff triple* is a solution x_1, x_2, x_3 of:

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = -2\cosh(l_\delta/2) + 2,\ x_i > 2. \tag{5}$$

For $l_\delta = 0$, $\mathcal{T}_1(l_\delta)$ is more usually denoted $\mathcal{T}_{1,1}$; it is a result of Keen [13] that the solution set of (4) can be identified with the Teichmuller space $\mathcal{T}_{1,1}$ of M. Similarly for $\mathcal{T}_1(l_\delta), l_\delta > 0$ is identified with the set of generalised Markoff triples. More precisely, to each $x \in \mathcal{T}_1(l_\delta)$ one associates a point $[\rho_x] \in \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbf{R}))//\mathrm{SL}_2(\mathbf{R})$, the $\mathrm{SL}_2(\mathbf{R})$ character variety of the free group on 2 generators. The map:

$$[\rho_x] \to (\operatorname{tr}\rho_x([\gamma_1]), \operatorname{tr}\rho_x([\gamma_2]), \operatorname{tr}\rho_x([\gamma_3]))$$

gives an embedding of Teichmuller space into $\mathbf{R}^3$ and the image satisfies (5) ; see Goldman [8] for more details.

3 Signed angles

We now define the signed angle between two geodesics at an intersection (compare Kerckhoff [14], Series [22] or Jorgensen [12] for a discussion of signed complex lengths in general.) Subsequently we find an explicit expression in terms of lengths of geodesics and study its behaviour on Teichmuller space.

Definition: Let $\alpha \neq \beta$ be a pair of oriented geodesics in H^2 meeting in at a point x. There is a well defined *signed angle* between α, β at x; this is a function from ordered pairs of oriented geodesics to $]-\pi, \pi[$:

$$(\alpha, \beta) \mapsto \alpha \vee_x \beta.$$

One way to define $\alpha \vee \beta$ is to work in the disc model for H^2. After conjugating we may assume:

$$\alpha = (-1, 1),\ \beta = (-e^{i\theta}, e^{i\theta}),$$

for some $\theta \in]-\pi, \pi[$ so that $\alpha \cap \beta = \{0\}$. Now $z \mapsto e^{i\theta} z$ is the unique rotation fixing 0 and taking 1 to $e^{i\theta}$ and hence α (oriented in the direction from -1 to 1) to β (oriented in the direction from $-e^{i\theta}$ to $e^{i\theta}$.) Set $\alpha \vee_x \beta = \theta$.

For a pair of geodesics $\alpha \neq \beta$ meeting at a point x in a surface M one defines the signed angle at x by lifting to H^2. When $\alpha \neq \beta$ meet at a single point x in the surface we shall omit x and use simply $\alpha \vee \beta$ to denote this angle.

Computation: Let α, β be a pair of simple closed geodesics meeting in a single point on M. We now calculate the angle $\alpha \vee \beta$ in terms of l_α, l_β and the length of the boundary l_δ. Let γ be the unique simple geodesic which satisfies $[\gamma] + [\alpha] = [\beta]$ in the homology. Using the intersection form one verifies that α, β, γ meet pairwise in a point and so form a Markoff triple. Now quotient M by the elliptic involution J to obtain an orbifold M/J with three cone points one for each of the fixed points of J . Simple geodesics are invariant for this involution and the three intersection points $\alpha \cap \beta, \beta \cap \gamma, \gamma \cap \alpha$ coincide with the fixed points of J. So the quotient of $\alpha \cup \beta \cup \gamma$ is an embedded geodesic triangle on M/J with vertices at the 3 cone points. The sides of this triangle have lengths $l_\alpha/2, l_\beta/2, l_\gamma/2$, and the cosine rule for this triangle is:

$$\cosh(l_\gamma/2) = \cosh(l_\alpha/2)\cosh(l_\beta/2) - \sinh(l_\alpha/2)\sinh(l_\beta/2)\cos(\alpha \vee \beta),$$

so that

$$\alpha \vee \beta = \arccos\left(\frac{\cosh(l_\alpha/2)\cosh(l_\beta/2) - \cosh(l_\gamma/2)}{\sinh(l_\alpha/2)\sinh(l_\beta/2)}\right). \tag{6}$$

One sees immediately that $\alpha \vee \beta$ is continuous on Teichmuller space.

Finally we derive another expression for $\alpha \vee \beta$ which will be useful in the proof of theorem 3. Replacing in the trace relation/Markoff cubic one obtains:

$$\sinh^2(l_\alpha/2)\sinh^2(l_\beta/2)\sin^2(\alpha \vee \beta) = \cosh^2(l_\delta/4), \tag{7}$$

where δ is the boundary geodesic. If, $\alpha \vee \beta \in]0, \pi/2[$ then ,

$$\alpha \vee \beta = \arcsin\left(\frac{\cosh(l_\delta/4)}{\sinh(l_\alpha/2)\sinh(l_\beta/2)}\right). \tag{8}$$

Remark: Another way of thinking of the relation (7) is as a hyperbolic version of the usual formula for the area of a euclidean torus:

$$2l_\alpha l_\beta \sin(\alpha \vee \beta) = \text{area of torus},$$

where α, β are closed Euclidean geodesics meeting in a single point at angle $\alpha \vee \beta$.

The reader is left to check that, unfortunately, the analogous series for the variation of the Euclidean angles does not converge absolutely and so we obtain no new identity on the moduli space of Euclidean structures.

Differentiability: We now study the regularity of $\alpha \vee \beta$ as we vary the surface over Teichmuller space. It is well known [1] that for any closed geodesic γ the function:

$$x \mapsto l_\gamma(x), \mathcal{T}_1(l_\delta) \to \mathbf{R}^+$$

is differentiable and even real analytic. It is not difficult to see from (6) that $\alpha \vee \beta$ is also real analytic.

From the expression (8) for the angle obtained above:

$$d(\alpha \vee \beta) = \cosh(l_\delta)/4) \frac{\coth(l_\alpha/2)dl_\alpha + \coth(l_\beta/2)dl_\beta}{4(\sinh^2(l_\alpha/2)\sinh^2(l_\beta/2) - \cosh^2(l_\delta/4))^{1/2}},$$

provided $\alpha \vee \beta \neq \pi/2$ (by equation (7)) , that is off the subset where $\alpha \vee \beta$ attains its maximum. It is left to the reader that the right hand side defines a form which extends to a continuously to the whole of Teichmuller space by 0 on this exceptional set.

4 The length spectrum of simple geodesics

We prove two lemmata used in the proof of Theorem 3 in the next section. For a discussion of length spectra in general see Schmutz [20].

Notation: Sections 4 and 5 deal with lengths of geodesics and the mapping class group will not figure explicitly. To make this clear we set

$$B^+ := \mathcal{MCG}.(\alpha, \beta).$$

Let $M_{g,n}$ be a hyperbolic surface of genus g with n punctures and x the corresponding point in the moduli space. Let $\mathcal{G}_0$ be the set of all closed simple geodesics $\gamma \neq \delta$. Define the *simple length spectrum*, denoted $\sigma_0(x) \subset \mathbf{R}^+$, to be the set $\{l_\gamma, \gamma \in \mathcal{G}_0\}$ counted *with* multiplicities. It is more useful to think in terms of the associated *counting function*:

$$N(\mathcal{G}_0, t) := \sharp\{\gamma \in \mathcal{G}_0, l_\gamma(x) < t\}.$$

There are two important features of the simple length spectrum:

1. The infimum over all lengths $l_\gamma(x)$ of all geodesics of is strictly positive and attained for a simple closed geodesic *the systole*, $\mathrm{sys}(x)$. We shall also denote by $\mathrm{sys}(x)$ the length of this geodesic.

2. $\sigma_0(M)$ is *discrete* that is $N(\mathcal{G}_0, t)$ is finite for all $t \geq 0$ and moreover has *polynomial growth* [19],[18] that is

$$N(\mathcal{G}_0, t) \leq At^{6g-6+2n},$$

for some $A = A(x) > 0$.

Remark: If M is a holed torus then it has quadratic growth [17], $N(\mathcal{G}_0, t) \sim At^2$.

Lemma 4.1 *Let $x \in \mathcal{M}_1(l_\delta)$ then $\forall t > 0$ there exists $N = N(t, x) > 0$ such that the inequality:*

$$\sinh(l_\alpha(x)/2)\sinh(l_\beta(x)) \geq t,$$

for all but N pairs $(\alpha, \beta) \in B^+$.

Proof:

$$\begin{aligned}\sinh(l_\alpha)\sinh(l_\beta) &\geq 1/2(\sinh(l_\alpha/2)\sinh(\mathrm{sys}(x)/2) + \sinh(\mathrm{sys}(x)/2)\sinh(l_\beta/2)) \\ &\geq 1/2\sinh(\mathrm{sys}(x)/2)(l_\alpha + l_\beta),\end{aligned}$$

and the lemma follows by the discreteness of the length spectrum.

Lemma 4.2 *$\sharp\{(\alpha, \beta) \in B^+ : l_\alpha(x) + l_\beta(x) < t\}$ grows polynomially in t.*

Proof: View B^+ as a subset of $\mathcal{G}_0 \times \mathcal{G}_0$. By an elementary counting argument the growth of $N(\mathcal{G}_0 \times \mathcal{G}_0, t) \leq CN(\mathcal{G}_0, t)^2$ for some $C > 0$

The Collar lemma: Useful information about the length spectrum can be obtained from the *collar lemma* see Buser [4] chapter 4. Given a closed simple geodesic μ there is an embedded collar (= regular tubular neighbourhood of μ) such that

$$(\text{width of collar round } \mu) \geq w(l_\mu),$$

for $w(s) := 2\mathrm{arcsinh}(1/\sinh(s/2))$. One bounds the length of any closed geodesic γ such that $\gamma \cap \mu \neq \emptyset$ by theintersection number times the width the collar round μ, that is

$$i(\gamma, \mu) \leq \frac{l_\gamma}{w(l_\mu)}, \tag{9}$$

where $i(\gamma, \mu) := \sharp\gamma \cap \mu$ is the *geometric intersection number.*

5 Wolperts formula and the variation of $\mathcal{Q}$

Let μ_1, μ_2 be closed simple geodesics on a hyperbolic surface. *Wolpert's formula* [23] gives an expression for the variation of l_{μ_1} along the Fenchel-Nielsen vector field $t(\mu_2)$ associated to μ_2:

$$dl_{\mu_1}.t(\mu_2) = \sum_{z \in \mu_1 \cap \mu_2} \cos(\theta_z),$$

where the sum is over all the intersections between the geodesics. An immediate corollary, also due to Wolpert [25], is a bound on the amplitude of the variation in terms of intersection numbers:

$$|dl_{\mu_1}.t(\mu_2)| \leq \sum_{x \in \mu_1 \cap \mu_2} 1 = \sharp(\mu_1 \cap \mu_2) := i(\mu_1, \mu_2).$$

Together with the estimates obtained in the previous section this is all that is required to prove theorem 3.

Proof of theorem 3: Fix a metric on M and let $x \in \mathcal{M}_1(l_\delta)$ be the corresponding point in the moduli space.

Using the formula obtained for $\alpha \vee \beta$ in section 3:

$$d(\alpha \vee \beta) = \cosh(r) \frac{\coth(a)da + \coth(b)db}{(\sinh^2(a)\sinh^2(b) - \cosh(r))^{1/2}}$$

where to simplify notation:

$$a = l_\alpha/2, \ b = l_\beta/2, \ r = l_\delta/4.$$

Fix a geodesic μ, by Wolpert's formula:

$$|d(\alpha \vee \beta).t(\mu)| \leq \left| \frac{\cosh(r)(\coth(a)i(\alpha,\mu) + \coth(b)i(\beta,\mu))}{\sinh(a)\sinh(b)(\sinh^2(a)\sinh^2(b) - \cosh^2(r))^{1/2}} \right|.$$

Firstly, note that $\coth(a), \coth(b) \leq \coth(\mathrm{sys}(x)/2)$ since $a, b \geq \mathrm{sys}(x)/2$. Secondly, replacing for $i(\alpha,\mu), i(\beta,\mu)$ using (9) above we obtain the following majoration for the variation:

$$\left(\frac{\cosh(r)\coth(\mathrm{sys}(x)/2)}{w(l_\mu)} \right) \cdot \left(\frac{l_\alpha + l_\beta}{(\sinh^2(a)\sinh^2(b) - \cosh^2(r))^{1/2}} \right),$$

Note that the leading factor does not depend on l_α, l_β.

Thirdly, by the corollary to lemma 4.2 for all but finitely many pairs (α, β) in B^+ one has:

$$\sinh^2(a)\sinh^2(b) - \cosh^2(r) \geq \frac{1}{2}\sinh^2(a)\sinh^2(b) \geq \frac{1}{8}\exp(a+b).$$

Finally, the sum over all the configurations.

$$\sum_{B^+} (l_\alpha + l_\beta)\exp(-1/2(l_\alpha + l_\beta)),$$

converges since $\sharp\{(\alpha,\beta) \in B^+ : l_\alpha + l_\beta < t\}$ grows polynomially in t by lemma 4.2.

6 Action of the mapping class group and summation of series

We now explain Bowditch's summation argument [3] which decomposes a sum over $\mathcal{MCG}$ into a sum over orbits of all Dehn twists.

Action on length functions: A geodesic γ on M determines a conjugacy class $[\gamma]$ in π_1. A point x in Teichmuller space determines a point in the so-called character variety, that is an equivalence class of $\mathrm{SL}_2(\mathbf{R})$ representations of $\pi_1(M)$, $[\rho_x]$. For any representative $\rho_x \in [\rho_x]$ the length of γ geodesic at x satisfies:

$$2\cosh(\frac{1}{2}l_\gamma(x)) = \operatorname{tr}\rho_x([\gamma])$$

Now viewing $g \in \mathcal{MCG}$ as a diffeomorphism of M, g acts (on the left) by automorphism g_* on $\pi_1(M)$ and so (on the left) on the character variety by $g : [\rho] \mapsto [\rho \circ g_*^{-1}]$. This induces an action on the set of geodesic length functions as follows:

$$2\cosh(1/2 l_{g(\gamma)}(x)) = \operatorname{tr}(\rho_x \circ g_*([\gamma])) = \operatorname{tr}(\rho_{g^-1x}([\gamma])) = 2\cosh(1/2 l_\gamma(g^{-1}x)),$$

for $\gamma \in \pi_1(M)$.

Summation: Let $f : \mathbf{R}^3 \to C$, and $\gamma_1, \gamma_2, \gamma_3$ be a Markoff triple of geodesics. One associates a function $\Psi : \mathcal{T}_1(l_\delta) \to C$:

$$\Psi(x) = f(l_{\gamma_1}(x), l_{\gamma_2}(x), l_{\gamma_3}(x)).$$

Examples of such functions are our angle $\alpha \vee \beta$,

$$(u, v, t) \mapsto \arccos\left(\frac{\cosh(u/2)\cosh(v/2) - \cosh(t/2)}{\sinh(u/2)\sinh(v/2)}\right),$$

and Bowditch's function,

$$B : (u, v, t) \mapsto \frac{\cosh(u/2)}{2\cosh(v/2)\cosh(t/2)}.$$

We impose a growth condition on Ψ to guarantee the absolute convergence of the sum of Ψ over orbits $\mathcal{MCG}.x$. From the proof of Theorem 3 a suitable condition is:

$$|f(l_{\gamma_1}(g(x)), l_{\gamma_2}(g(x)), l_{\gamma_3}(g(x)))| < K \exp -s(l_{\gamma_1}(g(x)) + l_{\gamma_2}(g(x))),\ \forall g \in \mathcal{MCG}$$

for some $s > 0, K > 0$.

Examples of such functions are $d(\alpha \vee \beta).t(\mu)$ and the variation of Bowditch's function $d(B).t(\mu)$

By substituting in the $d(\alpha \vee \beta).t(\mu)$ in the following theorem one obtains the summation formula (1).

Lemma 6.1 *Let Ψ, f be as above. Let T be the Dehn twist a along γ_3 and let $\mathcal{MCG}/\langle T\rangle$ denote a choice of coset representatives for $\langle T\rangle \subset \mathcal{MCG}$, then*

$$\sum_{g\in\mathcal{MCG}} \Psi(g(x)) = \sum_h \sum_{n\in\mathbf{Z}} f(l_{hT^n(\gamma_1)}, l_{hT^{n+1}(\gamma_1)}, l_{h(\gamma_3)}).$$

Proof: Consider the sum:

$$\begin{aligned}
\textstyle\sum_{g\in\mathcal{MCG}} \Psi(g^{-1}(x)) &= \textstyle\sum_{h\in\mathcal{MCG}/\langle T\rangle} \sum_{p\in\langle T\rangle} \Psi((ph)^{-1}(x)) \\
&= \textstyle\sum_h \sum_{n\in\mathbf{Z}} \Psi(hT^n(x)) \\
&= \textstyle\sum_h \sum_{n\in\mathbf{Z}} f(l_{hT^n(\gamma_1)}, l_{hT^n(\gamma_2)}, l_{hT^n(\gamma_3)})
\end{aligned}$$

Since T is the Dehn twist along γ_3, $T^n(\gamma_3) = \gamma_3$ and $T(\gamma_1) = \gamma_2$ (recall the definition of a Markoff triple) the lemma follows. □

7 Dehn twist orbits

We prove the lifting lemma of the introduction. We subsequently carry out two calculations. The first is to determine $\gamma^- \vee \gamma^+$, thus proving equation (2), in terms of length functions and the second to determine a formula for lengths of simple geodesics under iterated Dehn twists needed in the proof of theorem 4.
Proof of lemma 1: Let γ be a simple closed curve and γ' any simple closed geodesic meeting γ exactly once. Let $A \in \rho(\pi_1)$ be an element such that $axis(A)/\langle A\rangle = \gamma$.

For the first part of the lemma notice that since T_γ^n is a homeomorphism of M, $T_\gamma^n(\gamma')$ and γ meet in exactly one point. This point is necessarily one of the two Weierstrass points on γ. Since every simple closed geodesic passes through exactly two Weierstrass points, every curve $T_\gamma^n(\gamma')$ passes through the unique Weierstrass point c not on γ. The point $\hat{c} \in H^2$ is a lift of this Weierstrass point.

Choose a lift $\hat{\alpha_0}$ of γ' that meets $axis(A)$ and choose $\hat{c}$ to be a lift of this Weierstrass point on $\hat{\alpha_0}$ minimising the distance to $axis(A)$. One constructs the geodesics $\hat{\alpha_n}$ passing through $\hat{c}$, $a_n = (\sqrt{A})^n(a_0)$ so that these geodesics automatically satisfy (2), (3) in the statement of the lemma. It remains to check *(1)* in the statement of the lemma: that the geodesic arc joining $\hat{c}$ to a_n projects to a simple arc in the surface in the same homotopy class as (half of) $T_\gamma^n(\gamma')$ rel the two Weierstrass points on this latter geodesic. This is a simple exercise left to the reader.

The second part is a simple consequence of the fact that a_n converges to the attracting (resp. repelling) fixed point of A as $n \to \infty$ (resp. $n \to -\infty$). □

Calculation 1: angles

One computes the angle $\gamma^- \vee \gamma^+$ as follows again using the diagram **??**. The three geodesics $\gamma^-, \gamma^+, axis(A)$ form a triangle with angles $0, 0, \gamma^- \vee \gamma^+$. Break

this triangle up into two right angled triangles formed by the perpendicular dropped from $\hat{c}$ to a$xis(A)$ and one half of a$xis(A)$. One calculates l the length of the perpendicular obtaining:

$$\sinh(l)\sinh(l_\gamma/2) = \cosh(l_\delta/4). \tag{10}$$

Now standard hyperbolic trigonometry for the right angled triangle gives:

$$\tan(\gamma^- \vee \gamma^+/2) = 1/\sinh(l).$$

One easily obtains the quantity that appears on the right hand side of 2 from this.

Calculation 2: lengths

Note that Lemma 1 allows one to determine the lengths $l_{T^n(\gamma')}$ explicitly. Let θ be the signed distance between a_0 and the foot of the perpendicular dropped from $\hat{c}$ to a$xis(A)$. Then using Pythagorus' theorem:

$$\cosh(l_{T^n(\gamma')}/2) = \cosh(n.l_\gamma/2 + \theta)\cosh(l). \tag{11}$$

One can replace for $\cosh(l)$ from (10) above.

8 The value of a series at infinity

We now deteremine the value $\mathcal{Q}'$ by studying it in a neighborhood of infinity in the moduli space.

Mumford and Deligne compactified the moduli space, the resulting space is called the *augmented moduli space* $\overline{\mathcal{M}}_1(l_\delta)$, by adding certain *singular surfaces* [1]; these surfaces have double points as singularities – each double point is the result of pinching an essential simple curve to a point. Since the modular group of M acts transitively on simple curves $\neq \delta$ one adds a single point to $\mathcal{M}_1(l_\delta)$ to obtain the Deligne-Mumford compactification. The Mumford-Mahler compactness criterion says that a subset X of the moduli space is precompact iff $\mathrm{sys}(x) \geq \epsilon > 0$ on X. Thus a sequence of surfaces tends to infinity in the moduli space iff the length of the systole tends to 0.

Our main tool is:

Theorem 8.1 *Let M be a punctured torus. For $t \leq 0$, as $\mathrm{sys}(x)$ tends to 0*

$$\sum_\gamma \exp(-tl_\gamma) = o(\mathrm{sys}(x)^{-Nt-2}),$$

where the sum extends over all simple geodesics which meet α at least $N \geq 1$ times.

Proof: We need two estimates for lengths of curves on the torus.

Our first estimate of lengths comes via a version of the collar lemma. Let α be a simple closed geodesic representing the systole and choose α' so that $l_{\alpha'}$ minimizes the lengths of geodesics $\gamma \neq \alpha$. Since $\alpha' \neq \alpha$, α, α' meet at least once. The collar lemma yields:

$$\sinh(l_\alpha/2)\sinh(l_{\alpha'}/2) \geq 1$$

so for α short one has:

$$\exp(l_{\alpha'}) > \frac{32}{l_\alpha^2}.$$

For our second estimate of lengths we use the stable norm. By [17], for any simple closed curve γ, one has:

$$l_\gamma = \|[\gamma]\|_s, [l_\gamma] \in H_1(M, \mathbf{Z}),$$

where $\|.\|_s$ denotes the *stable norm* on $H_1(M, \mathbf{R})$. The simple geodesics are in 1-1 correspondence with the primitive elements of $H_1(M, \mathbf{Z})$. Viewing $[\alpha], [\alpha']$ as a basis of the homology one writes $[\gamma] = m[\alpha]+n[\alpha']$ for m, n coprime integers and the intersection is equivalent to $n > N$. Thus, by dropping the condition that the integers are coprime, one bounds the sum by:

$$2 \sum_{m,n \geq N} \exp(t\|m[\alpha] + n[\alpha']\|_s.)$$

To find upper bounds for sums of this form we need the following inequality:
claim: For any norm $\|.\|$ on $\mathbf{R}^2$ such that e_1, e_2 are the two shortest vectors with integer coefficients there exists $K \geq 1/2$ such that:

$$\|(x, y)\| \geq K(|x|\|e_1\| + |y|\|e_2\|).$$

By hypothesis $[\alpha], [\alpha']$ are the shortest vectors in $H_1(M, \mathbf{Z})$ for the stable norm so for $K > 1/2$ as above:

$$\begin{aligned}
\textstyle\sum_{m\geq 0, n\geq N} \exp(t\|m[\alpha] + n[\alpha']\|_s\,) &< \textstyle\sum_{m,n\geq N} \exp(tK(m\|[\alpha]|_s + n\|[\alpha']\|_s) \\
&= (\textstyle\sum_m \exp(tKml_\alpha))\ (\textstyle\sum_{n\geq N} \exp(tKnl_{\alpha'})) \\
&= \tfrac{1}{1-\exp(tKl_\alpha)}\ \tfrac{\exp(sNKl_\alpha)}{1-\exp(tKl_{\alpha'})} \\
&< \tfrac{1}{1-\exp(tKl_\alpha)}\ \tfrac{(R+1)l_\alpha^{-2tNK}}{1-l_\alpha^{-2tK}}.
\end{aligned}$$

For some $R > 0$. An elementary estimate shows that this latter function is $O(l_\alpha^{-2tNK-1})$ as $l_\alpha \to 0$. $\square$

Proof of theorem 4: Let γ be the closed simple geodesic that realises $\mathrm{sys}(x)$. The first equation is a consequence of the preceding theorem and the fact that

the Dehn twist round γ, T, acts transitivley on the set of geodesics that meet the systole exactly once. It remains to justify:

$$\lim_{\mathrm{sys}(x)\to 0} \sum_{n\in\mathbf{Z}} \cosh(l_\delta/4)\mathrm{sech}(l_{T^n(\gamma')}) = \pi.$$

By a straightforward calculation using equation (11) one has:

$$\text{L.H.S.} = \left(\sum_{n\in Z} \mathrm{sech}(1/2.n.\mathrm{sys}(x)+\theta)(1/2\mathrm{sys}(x))\right) + o(1) = \int_{-\infty}^{\infty} \mathrm{sech}(u)du + o(1),$$

as $\mathrm{sys}(x) \to 0$.

One evaluates the integral as usual and the theorem is proven. □

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