VaR and ES for linear portfolios with mixture of elliptic distributions Risk Factors

Jules Sadefo-Kamdem

VAR AND ES FOR LINEAR PORTFOLIOS WITH MIXTURE OF ELLIPTIC DISTRIBUTED RISK FACTORS

JULES SADEFO KAMDEM

UNIVERSITÉ DE REIMS
UNIVERSITÉ D'EVRY

Abstract. In this paper, following the generalization of *Delta Normal VaR* to *Delta Mixture Elliptic VaR* in Sadefo-Kamdem [3], we give and explicit formula to estimate linear VaR and ES when the risk factors changes with the mixture of t-Student distributions. In particular, we give rise to *Delta-Mixture-Student VaR* and the *Delta-Mixture-Elliptic ES*.

Key Words: Mixture of Elliptic distributions, Linear portfolio, Value-at-Risk, Expected Shortfall, Capital allocation.

1. Introduction

The original RiskMetrics methodology for estimating VaR was based on parametric methods, and used the multi-variate normal distribution. This approach works well for the so-called *linear portfolios*, that is, those portfolios whose aggregate return is, to a good approximation, a linear function of the returns of the individual assets which make up the portfolio, and in situations where the latter can be assumed to be jointly normally distributed. For other portfolios, like portfolios of derivatives depending non-linearly on the return of the underlying, or portfolios of non-normally distributed assets, one generally turns to Monte Carlo methods to estimate the VaR. This is an issue in situations demanding for real-time evaluation of financial risk. For non-linear portfolios, practitioners, as an alternative to Monte Carlo, use Δ-normal VaR methodology, in which the portfolio return is linearly approximated, and an assumption of normality is made. Such methods present us with a trade-off between accuracy and speed, in the sense that they are much faster than Monte Carlo, but are much less accurate unless the linear approximation is quite good and the normality hypothesis holds well. The assumption of normality simplifies the computation of VaR considerably. However it is inconsistent with the empirical evidence of assets returns, which finds that asset returns are fat tailed. This implies that extreme events are much more likely to occur in practice than would be predicted based on the assumption of normality.

Some alternative return distributions have been proposed in the world of elliptic distributions by Sadefo-Kamdem [3], that better reflect the empirical evidence. In this paper, following [3], I examine one such alternative that simultaneously allows for asset returns that are fat tailed and for tractable calculation of Value-at-Risk and Expected Shortfall, by giving attention to mixture of elliptic distributions,

This draft is a part of J.SADEFO-KAMDEM PhD Thesis at the Université de Reims (France). The author is a temporary lecturer at the Department of Mathematics (Université d'evry Val d'essonne France). Author's address: Université de Reims, Laboratoire de Mathématiques UMR 6056-CNRS , BP 1039 Moulin de la Housse , 51687 Reims cedex 2 FRANCE.
e-mail: sadefo@univ-reims.fr.

with an explicit formula of VaR and ES in the special case where assets returns changes with mixture of Student-t distributions. Note that, the particular case based on mixture of normal distributions, has been proposed by Zangari(1996)[7], Subu-Venkataraman [5] and some references therein.

An obvious first generalization is to keep the linearity assumption, but replace the normal distribution by some other family of multi-variate distributions. In Sadefo-Kamdem [3], we have such generalization concerning linear portfolios, in the case where the joint risk factors changes with mixture of elliptic distributions. In this paper, by using some generalized theorems concerning *Delta-Mixture-Elliptic* VaR and *Delta-Student* VaR in [3], we introduce the notion of *Delta-Mixture-Student VaR*, *Delta-Mixture-Elliptic ES* and the *Delta-Mixture-Student ES*.

So the particular subject of this paper, is to give an explicit formulas that will permit to obtain the linear VaR or linear ES, when the joint risk factors of the linear portfolios, changes with mixture of t-Student distributions. Note that, since one shortcoming of the multivariate t-distribution is that all the marginal distributions must have the same degrees of freedom, which implies that all risk factors have equally heavy tails, the mixture of t-Student will be view as a serious alternatives, to a simple t-Student-distribution. Therefore, the methodology proposes by this paper seem to be interesting to controlled thicker tails than the standard Student distribution.

The paper is organized, as follows: In section 2, we recall some theorems concerning the *Delta-Elliptic*, *Delta-Elliptic* and *Delta Mixture Elliptic VaR* given by Sadefo-Kamdem [3]. In section 3, following the theorem concerning *Delta Mixture Elliptic VaR*, we show how to reduce the computation of the *Delta-Mixture-Student VaR* to finding the zeros of a mixture of special function. In section 4, we introduces the notion of *Delta Mixture Elliptic ES* , by treat the expected shortfall for general mixture of elliptic distribution, with special attention to *Delta Mixture Elliptic ES* . Finally, in section 5 we discuss some potential application areas.

2. Some Notions on log-elliptic Linear VaR

In this section, following [3], we recall some notions on elliptic distributions and Linear VaR.

We will use the following notational conventions for the action of matrices on vectors: single letters $x, y, \cdots$ will denote *row vectors* $(x_1, \cdots, x_n)$, $(y_1, \cdots y_n)$. The corresponding column vectors will be denoted by x^t, y^t,the ^{t} standing more generally for taking the transpose of any matrix. Matrices $A = (A_{ij})_{i,j}$, B , etc. will be multiplied in the usual way. In particular, A will act on vectors by left-multiplication on column vectors, Ay^t, and by right multiplication on row vectors, xA; $x \cdot x = xx^t = x_1^2 + \cdots + x_n^2$ will stand for the Euclidean inner product.

A portfolio with time-t value $\Pi(t)$ is called linear if its profit and loss $\Delta\Pi(t) = \Pi(t) - \Pi(0)$ over a time window, [0 t] is a linear function of the returns $X_1(t), \ldots, X_n(t)$ of its constituents over the same time period:

$$\Delta\Pi(t) = \delta_1 X_1 + \delta_2 X_2 + ... + \delta_n X_n$$

This will for instance be the case for ordinary portfolios of common stock, if we use percentage returns, and will also hold to good approximation with log-returns, provided the time window [0,t] is small. We will drop the time t from our notations, since it will be kept fixed, and simply write X_j,$\Delta\Pi$, etc. We also put

$$X = (X_1, \cdots, X_n),$$

so that $\Delta\Pi = \delta \cdot X = \delta X^t$.

We now assume that the X_j are elliptically distributed with mean μ and correlation matrix $\Sigma = AA^t$:

$$(X_1, \ldots, X_n) \sim N(\mu, \Sigma, \phi).$$

This means that the pdf of X is of the form

$$f_X(x) = |\Sigma|^{-1/2} g((x-\mu)\Sigma^{-1}(x-\mu)^t),$$

where $|\Sigma|$ stands for the determinant of Σ, and where $g : \mathbb{R}_{\geq 0} \to 0$ is such that the Fourier transform of $g(|x|^2)$, as a generalized function on $\mathbb{R}^n$, is equal to $\phi(|\xi|^2)$ [1].

Assuming that g is continuous, and non-zero everywhere, the Value at Risk at a confidence level of $1-\alpha$ is given by solution of the following equation:

Here we follow the usual convention of recording portfolio losses by negative numbers, but stating the Value-at-Risk as a positive quantity of money.

2.1. **Linear VaR with mixtures of elliptic Distributions.** Mixture distributions can be used to model situations where the data can be viewed as arising from two or more distinct classes of populations; see also [1]. For example, in the context of Risk Management, if we divide trading days into two sets, quiet days and hectic days, a mixture model will be based on the fact that returns are moderate on quiet days, but can be unusually large or small on hectic days. Practical applications of mixture models to compute VaR can be found in Zangari [7] (1996), who uses a mixture normal to incorporate fat tails in VaR estimation. In Sadefo-Kamdem [3], we have generalized the preceding section to the situation where the joint log-returns follow a mixture of elliptic distributions, that is, a convex linear combination of elliptic distributions. In this section, a special attention will be give to mixture of Student-t distributions.

Definition 2.1. We say that $(X_1, ..., X_n)$ has a joint distribution that is the mixture of m elliptic distributions $N(\mu_j, \Sigma_j, \phi_j)$[2], with weights $\{\beta_j\}$ (j=1,..,m ; $\beta_j > 0$; $\sum_{j=1}^m \beta_j = 1$), if its cumulative distribution function can be written as

$$F_{X_1,\ldots,X_n}(x_1, ..., x_n) = \sum_{j=1}^{m} \beta_j F_j(x_1, ..., x_n)$$

with $F_j(x_1, ..., x_n)$ the cdf of $N(\mu_j, \Sigma_j, \phi_j)$.

Remark 2.2. In practice, one would usually limit oneself to $m = 2$, due to estimation and identification problems; see [1].

The following lemma is given by Sadefo-Kamdem [3].

Lemma 2.3. *Let $\Delta\Pi = \delta_1 X_1 + \ldots + \delta_n X_n$ with $(X_1, \ldots, X_n)$ is a mixture of elliptic distributions, with density*

$$f(x) = \sum_{j=1}^{m} \beta_j |\Sigma_j|^{-1/2} g_j((x-\mu_j)\Sigma_j^{-1}(x-\mu_j)^t)$$

where μ_j is the vector mean, and Σ_j the variance-covariance matrix of the j-th component of the mixture. We suppose that each g_j is integrable function over $\mathbb{R}$, and that the g_j never vanish jointly in a point of $\mathbb{R}^m$. Then the value-at-Risk,

[1] One uses ϕ as a parameter for the class of elliptic distributions, since it is always well-defined as a continuous function: $\phi(|\xi|^2)$ is simply the characteristic function of an $X \sim N(0, Id, \phi)$. Note, however, that in applications we'd rather know g

[2] or $N(\mu_j, \Sigma_j, g_j)$ if we parameterize elliptical distributions using g instead of ϕ

or Delta mixture-elliptic VaR*, at confidence* $1-\alpha$ *is given as the solution of the transcendental equation*

$$\alpha = \sum_{j=1}^{m} \beta_j G_j \left(\frac{\delta.\mu_j^t + VaR_\alpha}{(\delta \Sigma_j \delta)^{1/2}} \right), \tag{1}$$

where

$$G_j = \frac{|S_{n-2}|}{|\Sigma_j|^{1/2}} \int_0^{+\infty} r^{n-2} \Big[\int_{-\infty}^{\frac{-\delta \cdot \mu_j - VaR_\alpha}{|\delta A_j|}} g_j(z_1^2 + r^2) dz_1 \Big] dr.$$

Here $\delta = (\delta_1, \ldots, \delta_n)$.

Remark 2.4. In the case of a mixture of m elliptic distributions the VaR will not depend any more in a simple way on the total portfolio mean and variance-covariance. This is unfortunate, but already the case for a mixture of normal distributions.

Remark 2.5. One might, in certain situations, try to model with a mixture of elliptic distributions which all have the same variance-covariance and the same mean, and obtain for example a mixture of different tail behaviors by playing with the g_j's. In that case the VaR again simplifies to: $VaR_\alpha = -\delta \cdot \mu + q_\alpha \cdot \sqrt{\delta \Sigma \delta^t}$, with q_α now the unique positive solution to

$$\alpha = \sum_{j=1}^{m} \beta_j G_j(q_\alpha).$$

The preceding can immediately be specialized to a mixture of normal distributions. the details is left to the reader.

3. VaR with mixture Student-t distributions

We now consider in detail the case where our mixture of elliptic distributions is a mixture of multivariate Student-t. We will, unsurprisingly, call the corresponding VaR the *Delta mixture-Student VaR*.

In the case of our mixture of multi-variate t-Student distributions, the portfolio probability density function is given by:

$$h_X(x) = \sum_{j=1}^{m} \beta_j \frac{\Gamma(\frac{\nu_j+n}{2})}{\Gamma(\nu_j/2).\sqrt{|\Sigma_j|(\nu_j\pi)^n}} \Big(1 + \frac{(x-\mu_j)^t \Sigma_j^{-1}(x-\mu_j)}{\nu_j}\Big)^{(\frac{-\nu_j-n}{2})}, \tag{2}$$

$x \in \mathbb{R}^n$ and $\nu_j > 2$. Hence g_j is given by

$$g_j(s) = C(\nu_j, n)(1 + s/\nu_j)^{-\frac{(n+\nu_j)}{2}}, \quad s \geq 0,$$

where we have put

$$C(\nu_j, n) = \frac{\Gamma(\frac{\nu_j+n}{2})}{\Gamma(\nu_j/2)\sqrt{(\nu_j\pi)^n}}.$$

Using this g_j in (1), we find that

$$G_j(s) = \frac{\nu_j^{\frac{n+\nu_j}{2}}}{2} |S_{n-2}| C(\nu_j, n) \int_s^{\infty} I_j(z_1) dz_1, \tag{3}$$

where we have put

$$I_j(z_1) = \int_{z_1^2}^{+\infty} (u - z_1^2)^{\frac{n-3}{2}} (\nu_j + u)^{-\frac{(n+\nu_j)}{2}} du. \tag{4}$$

Following [3], we have the following expression

$$G_j(s) = \frac{1}{\nu_j\sqrt{\pi}}\left(\frac{\nu_j}{s^2}\right)^{\nu_j/2}\frac{\Gamma\left(\frac{\nu_j+1}{2}\right)}{\Gamma\left(\frac{\nu_j}{2}\right)}{}_2F_1\Big(\frac{1+\nu_j}{2},\frac{\nu_j}{2};1+\frac{\nu_j}{2};-\frac{\nu_j}{s^2}\Big) \tag{5}$$

we obtain the following corollary

Corollary 3.1. *Let $\Delta\Pi = \delta_1 X_1 + \ldots + \delta_n X_n$ with $(X_1, \ldots, X_n)$ is a mixture of m t-Student distributions, with density h_X defined by (2),where μ_j is the vector mean, and Σ_j the variance-covariance matrix of the j-th component of the mixture. Then the value-at-Risk, or* Delta mixture-student VaR*, at confidence $1-\alpha$ is given as the solution of the transcendental equation*

$$\alpha = \sum_{j=1}^{m}\frac{\beta_j\Gamma\left(\frac{\nu_j+1}{2}\right)}{\nu_j\sqrt{\pi}\Gamma\left(\frac{\nu_j}{2}\right)}\nu_j^{\frac{\nu_j}{2}}\left(\frac{\delta.\mu_j^t + VaR_\alpha}{\delta\Sigma_j\delta}\right)^{\frac{-\nu_j}{2}}{}_2F_1\Big(\frac{1+\nu_j}{2},\frac{\nu_j}{2};1+\frac{\nu_j}{2};-\frac{\nu_j(\delta.\mu_j^t+VaR_\alpha)}{\delta\Sigma_j\delta}\Big) \tag{6}$$

where G_j is defined by (3) with $g = g_j$. Here $\delta = (\delta_1, \ldots, \delta_n)$.

Corollary 3.2. *One might, in certain situations, try to model with a mixture of t-Student distributions which all have the same variance-covariance $\Sigma = \Sigma_j$ and the same mean $\mu = \mu_j$, and obtain for example a mixture of different tail behaviors by playing with the ν_j's. In that case the VaR again simplifies to:*

$$VaR_\alpha = -\delta\cdot\mu + q_\alpha\cdot\sqrt{\delta\Sigma\delta^t},$$

with q_α now the unique positive solution to

$$\alpha = \sum_{j=1}^{m}\frac{\beta_j\Gamma\left(\frac{\nu_j+1}{2}\right)}{\nu_j\sqrt{\pi}\Gamma\left(\frac{\nu_j}{2}\right)}\nu_j^{\frac{\nu_j}{2}}\left(\frac{\delta.\mu^t + VaR_\alpha}{\delta\Sigma\delta}\right)^{\frac{-\nu_j}{2}}{}_2F_1\Big(\frac{1+\nu_j}{2},\frac{\nu_j}{2};1+\frac{\nu_j}{2};-\frac{\nu_j(\delta.\mu^t+VaR_\alpha)}{\delta\Sigma\delta}\Big).$$

Remark 3.3. One might, in certain situations, try to model with a mixture of t-Student distributions which all have the same $\nu_j = \nu$ and the same mean $\mu_j \approx 0$, and obtain for example a mixture of different tail behaviors by playing with the Σ_j's. In that case the VaR is the unique positive solution to

$$\alpha = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\nu\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\sum_{j=1}^{m}\beta_j\left(\frac{\nu(VaR_\alpha)}{\delta\Sigma_j\delta}\right)^{\frac{\nu}{2}}{}_2F_1\Big(\frac{1+\nu}{2},\frac{\nu}{2};1+\frac{\nu}{2};-\frac{\nu(VaR_\alpha)}{\delta\Sigma_j\delta}\Big).$$

3.1. **Some Numerical Result of Delta Mixture-Student VaR coefficient.** Here we give some numerical results when applying the corollary 3.2, in the situation where $m = 2$.

By introducing the function F such that

$$F(s,\beta,\nu_1,\nu_2) = \beta\cdot G_1(s) + (1-\beta)\cdot G_2(s), \tag{7}$$

where G_j is define in (5), for $j = 1, 2$, for given as inputs β, ν_1 and ν_2, we give a table that contains some solutions $s = q_{\beta,\nu_1,\nu_2} = q_\alpha^{MS-VaR}$ of the following transcendental equation:

$$F(s,\beta,\nu_1,\nu_2) = \alpha.$$

For given Σ, μ, and δ, these solutions will permit to calculate VaR_α, when the confidence is $1-\alpha$.

(1) In the case where $\alpha = 0.01$, we obtain some solutions of (3.1) in the following table:

(ν_1, ν_2)	(2,3)	(3,4)	(4,6)	(5,8)	(6,10)	(7, 15)	(8, 40)	(9, 16)
$q_{0.05,\nu_1,\nu_2}$	4.64839	3.78507	3.17184	3.91919	2.78228	2.62175	2.44602	2.59524
$q_{0.10,\nu_1,\nu_2}$	4.7586	3.82348	3.20124	2.94213	2.80092	2.64116	2.46906	2.60704
$q_{0.15,\nu_1,\nu_2}$	4.87115	3.86216	3.23086	2.9652	2.81965	2.6607	2.49235	2.61887
$q_{0.20,\nu_1,\nu_2}$	4.98587	3.9011	3.26066	2.98846	2.83846	2.68035	2.51586	2.63073
$q_{0.25,\nu_1,\nu_2}$	5.10258	3.94025	3.29063	3.01177	2.85734	2.70009	2.53957	2.64261
$q_{0.30,\nu_1,\nu_2}$	5.22106	3.97962	3.32075	3.03518	2.87629	2.71991	2.56344	2.65452
$q_{0.35,\nu_1,\nu_2}$	5.34113	4.01917	3.35100	3.05866	2.89528	2.7398	2.58744	2.66644
$q_{0.40,\nu_1,\nu_2}$	5.46259	4.05888	3.38136	3.08221	2.91432	2.75974	2.6115	2.67838
$q_{0.45,\nu_1,\nu_2}$	5.58523	4.09873	3.41180	3.10502	2.93339	2.77972	2.6357	2.69033
$q_{0.50,\nu_1,\nu_2}$	5.70886	4.13870	3.44231	3.12946	2.95248	2.79972	2.65989	2.70228

(ν_1, ν_2)	(10,20)	(20,30)	(200, 300)	(250,50)	(275,15)	(300,55)	(400,10)	(1000,5)
$q_{0.05,\nu_1,\nu_2}$	2.53963	2.46079	2.33916	2.40018	2.58957	2.39322	2.7432	3.3202
$q_{0.10,\nu_1,\nu_2}$	2.55132	2.46432	2.33947	2.39709	2.57661	2.39036	2.72242	3.27401
$q_{0.15,\nu_1,\nu_2}$	2.56304	2.46785	2.33978	2.39399	2.56359	2.38750	2.7014	3.22632
$q_{0.20,\nu_1,\nu_2}$	2.5748	2.47139	2.3401	2.3909	2.55051	2.38464	2.68019	3.17715
$q_{0.25,\nu_1,\nu_2}$	2.58658	2.47492	2.34041	2.3878	2.53738	2.38178	2.6588	3.12651
$q_{0.30,\nu_1,\nu_2}$	2.59838	2.47846	2.34073	2.38471	2.52422	2.37892	2.63726	3.07446
$q_{0.35,\nu_1,\nu_2}$	2.6102	2.482	2.34104	2.38161	2.51102	2.37605	2.61559	3.02112
$q_{0.40,\nu_1,\nu_2}$	2.62204	2.48553	2.34136	2.37851	2.49779	2.37319	2.59382	2.96663
$q_{0.45,\nu_1,\nu_2}$	2.63389	2.48907	2.34167	2.37541	2.48455	2.37033	2.57198	2.91121
$q_{0.50,\nu_1,\nu_2}$	2.64574	2.49261	2.34199	2.37232	2.4713	2.36746	2.55009	2.85513

(2) In the case where $\alpha = 0.001$, we obtain some solutions of (3.1) in the following table:

(ν_1, ν_2)	(2,3)	(3,4)	(4,6)	(5,8)	(6,10)	(7, 15)	(8, 40)	(9, 16)
$q_{0.20,\nu_1,\nu_2}$	12.8878	7.84891	5.66393	4.82769	4.39245	3.98902	3.62286	3.82625
$q_{0.25,\nu_1,\nu_2}$	13.5577	8.01412	5.77451	4.90665	4.45334	4.05064	3.69896	3.86013
$q_{0.30,\nu_1,\nu_2}$	14.2205	8.17734	5.88317	4.98414	4.51241	4.11084	3.77242	3.89346
$q_{0.35,\nu_1,\nu_2}$	14.874	8.33840	5.98975	5.06004	4.57030	4.16948	3.84285	3.92621
$q_{0.40,\nu_1,\nu_2}$	15.5168	8.49717	6.09412	5.13427	4.62694	4.22648	3.91007	3.95838
$q_{0.45,\nu_1,\nu_2}$	16.1480	8.65357	6.19624	5.20677	4.68229	4.28179	3.97400	3.98993
$q_{0.50,\nu_1,\nu_2}$	16.7671	8.80753	6.29604	5.27752	4.73634	4.33537	3.03470	4.02087

(3) In the case where $\alpha = 0$, we obtain some solutions of (3.1) in the following table:

(ν_1, ν_2)	(2,3)	(3,4)	(4,6)	(5,8)	(6,10)	(7, 15)	(8, 40)	(9, 16)
$q_{0.20,\nu_1,\nu_2}$	322.785	82.6688	31.0894	20.7154	15.8813	11.4371	10.1089	9.25604
$q_{0.25,\nu_1,\nu_2}$	352.09	87.1881	32.5561	21.541	16.42471	11.7949	10.3957	9.47529
$q_{0.30,\nu_1,\nu_2}$	378.302	91.2285	33.8309	22.2487	16.88721	12.0958	10.6352	9.66243
$q_{0.35,\nu_1,\nu_2}$	402.155	94.8927	34.9619	22.8697	17.2907	12.3561	10.8414	9.82571
$q_{0.40,\nu_1,\nu_2}$	424.137	98.2529	35.981	23.4244	17.6493	12.5858	11.0227	9.97061
$q_{0.45,\nu_1,\nu_2}$	444.591	101.362	36.9102	23.9265	17.9726	12.7919	11.1848	10.1009
$q_{0.50,\nu_1,\nu_2}$	463.771	104.26	37.7655	24.3858	18.2673	12.9789	11.3316	10.2194

Remark 3.4. Note that, the precedent results are available when $\alpha = 0$. This means that with our model, one would calculate the linear VaR with mixture of elliptic distributions, for 100 percent confidence level.

4. Expected Shortfall with mixture of elliptic distributions

Expected shortfall is a sub-additive risk statistic that describes how large losses are on average when they exceed the VaR level. Expected shortfall will therefore give an indication of the size of extreme losses when the VaR threshold is breached. We will evaluate the expected shortfall for a linear portfolio under the hypothesis of mixture of elliptically distributed risk factors. Mathematically, the expected shortfall associated with a given VaR is defined as:

$$\text{Expected Shortfall } = \mathbb{E}(-\Delta\Pi | - \Delta\Pi > VaR),$$

see for example [1]. Assuming again a multivariate mixture of elliptic probability density $f_X(x) = \sum_{i=1}^n \beta_i |\Sigma_i|^{-1/2} g_i((x-\mu_i)\Sigma_i^{-1}(x-\mu_i)^t)$, the Expected Shortfall at confidence level $1-\alpha$ is given by

$$\begin{aligned} -ES_\alpha &= \mathbb{E}(\Delta\Pi \mid \Delta\Pi \leq -VaR_\alpha) \\ &= \frac{1}{\alpha}\mathbb{E}\left(\Delta\Pi \cdot 1_{\{\Delta\Pi \leq -VaR_\alpha\}}\right) \\ &= \frac{1}{\alpha}\int_{\{\delta x^t \leq -VaR_\alpha\}} \delta x^t \ f_X(x)\ dx \\ &= \sum_{i=1}^n \beta_i \frac{|\Sigma_i|^{-1/2}}{\alpha} \int_{\{\delta x^t \leq -VaR_\alpha\}} \delta x^t \ g_i((x-\mu_i)\Sigma_i^{-1}(x-\mu_i)^t) dx. \end{aligned}$$

Let $\Sigma = A_i^t \ A_i$, as before.Doing the same linear changes of variables as in section 2, we arrive at:

$$\begin{aligned} -ES_\alpha &= \frac{1}{\alpha}\sum_{i=1}^m \beta_i \int_{\{|\delta A_i| z_1 \leq -\delta\cdot\mu_i - VaR_\alpha\}} (|\delta A| z_1 + \delta\cdot\mu_i)\ g_i(\|z\|^2) dz \\ &= \frac{1}{\alpha}\sum_{i=1}^m \beta_i \Big[\int_{\{|\delta A| z_1 \leq -\delta\cdot\mu_i - VaR_\alpha\}} |\delta A_i| z_1\ g_i(\|z\|^2)\ dz \ + \ \delta\cdot\mu_i \Big]. \end{aligned}$$

The final integral on the right hand side can be treated as before, by writing $\|z\|^2 = z_1^2 + \|z^{'}\|^2$ and introducing spherical coordinates $z^{'} = r\xi$, $\xi \in S_{n-2}$, leading to:

$$-ES_\alpha = \sum_{i=1}^m \beta_i \delta\cdot\mu_i + \frac{|S_{n-2}|}{\alpha}\sum_{i=1}^m \beta_i \int_0^\infty r^{n-2}\Big[\int_{-\infty}^{\frac{-\delta\mu_i^t - VaR_\alpha}{|\delta A_i|}} |\delta A_i|\ z_1\ g_i(z_1^2+r^2) dz_1\Big] dr$$

We now first change z_1 into $-z_1$, and then introduce $u = z_1^2 + r^2$, as before. If we recall that, by theorem 2.1,

$$q_{\alpha,i}^g = \frac{\delta\cdot\mu_i + VaR_\alpha}{|\delta A_i|}$$

then, simply writing $q_{\alpha,i}$ for $q_{\alpha,n}^{f_X}$, we arrive at:

$$\begin{aligned} ES_\alpha &= -\sum_{i=1}^m \beta_i \Big[\delta\cdot\mu_i + |\delta A|\ \frac{|S_{n-2}|}{\alpha}\cdot\int_{q_{\alpha,i}}^\infty \int_{z_1^2}^\infty z_1 (u-z_1^2)^{\frac{n-3}{2}} g(u)\ du\ dz_1\Big] \\ &= -\sum_{i=1}^m \beta_i(\delta\cdot\mu) + \sum_{i=1}^n \beta_i |\delta A_i|\ \frac{|S_{n-2}|}{\alpha}\cdot\int_{q_{\alpha,i}^2}^\infty \frac{1}{n-1}\left(u - q_{\alpha,i}^2\right)^{\frac{n-1}{2}}\ g_i(u)\ du, \end{aligned}$$

since

$$\int_{q_{\alpha,i}}^{\sqrt{u}} z_1 \left(u - z_1^2\right)^{\frac{n-3}{2}} dz_1 = \frac{1}{n-1}\left(u - q_{\alpha,i}^2\right)^{\frac{n-1}{2}}.$$

After substituting the formula for $|S_{n-2}|$ and using the functional equation for the Γ-function, $\Gamma(x+1) = x\Gamma(x)$, we arrive at the following result:

Theorem 4.1. *Suppose that the portfolio is linear in the risk-factors $X = (X_1, \cdots, X_n)$: $\Delta\Pi = \delta \cdot X$ and that $X \sim N(\mu, \Sigma, \phi)$, with pdf $f_X(x) = \sum_{i=1}^{n} \beta_i |\Sigma_i|^{-1/2} g((x - \mu_i)\Sigma_i^{-1}(x - \mu_i)^t)$. If we replace q_α by his value, then the expected Shortfall at level α is given by :*

(8)
$$ES_\alpha = -\sum_{i=1}^{m} \beta_i(\delta \cdot \mu_i) + \sum_{i=1}^{m} \beta_i |\delta\Sigma_i\delta^t|^{1/2} \cdot \frac{\pi^{\frac{n-1}{2}}}{\alpha \cdot \Gamma(\frac{n+1}{2})} \cdot \int_{(q^g_{\alpha,i})^2}^{\infty} \left(u - (q^g_{\alpha,i})^2\right)^{\frac{n-1}{2}} \; g_i(u)\, du.$$

Remark 4.2. If we are in situations where $\mu = \mu_i$ and $\Sigma_i = \Sigma$ for all $i = 1, \ldots, n$, therefore $q_{\alpha,i}$ does not depend to i. It will depend only to the q_α given by the mixture of elliptic VaR. In effect, $q_{\alpha,i} = q_\alpha = q_\alpha^{ME-VaR_\alpha}$ such that

$$VaR_\alpha = -\delta \cdot \mu + q_\alpha^{ME-VaR_\alpha} \cdot \sqrt{\delta\Sigma\delta^t}.$$

We therefore obtain the following corollary:

Corollary 4.3. *Suppose that the portfolio is linear in the risk-factors $X = (X_1, \cdots, X_n)$: $\Delta\Pi = \delta \cdot X$ and that $X \sim N(\mu, \Sigma, \phi)$, with pdf $f_X(x) = \sum_{i=1}^{m} \beta_i |\Sigma|^{-1/2} g_i((x - \mu)\Sigma^{-1}(x - \mu)^t)$. If we replace q_α by his value, then the expected Shortfall at level α is given by :*

$$ES_\alpha = -\delta \cdot \mu \; + \; q_\alpha^{ME-ES} \cdot \sqrt{\delta\Sigma\delta^t} \tag{9}$$

where

$$q_\alpha^{ME-ES} = \frac{\pi^{\frac{n-1}{2}}}{\alpha \cdot \Gamma(\frac{n+1}{2})} \sum_{i=1}^{m} \beta_i \cdot \int_{(q_\alpha^{ME-VaR})^2}^{\infty} \left(u - (q_\alpha^{ME-VaR})^2\right)^{\frac{n-1}{2}} \; g_i(u)\, du. \tag{10}$$

4.1. **Application: Mixture of Student-t Expected Shortfall.** In the case of multi-variate t-student distributions we have that $g_i(u) = C(\nu_i, n)(1 + u/\nu_i)^{-\frac{(n+\nu_i)}{2}}$, with $C(\nu_i, n)$ given in section 2. Let us momentarily write q for $q^{\mathrm{t}}_{\alpha,\nu_i}$. Following [3], we can evaluate the integral as follows:

$$\begin{aligned} & \int_{q^2}^{\infty} (u - q)^{\frac{n-1}{2}} \left(1 + \frac{u}{\nu_i}\right)^{-\frac{n+\nu_i}{2}} du \\ = \;& \nu_i^{\frac{n+\nu_i}{2}} (q^2 + \nu_i)^{-(\frac{\nu_i - 1}{2})} B\left(\frac{\nu_i - 1}{2}, \frac{n+1}{2}\right). \end{aligned}$$

If we pose that :

$$q_\alpha^{MST-ES} = \frac{1}{\alpha \cdot \sqrt{\pi}} \sum_{i=1}^{m} \beta_i \frac{\Gamma\left(\frac{\nu_i - 1}{2}\right)}{\Gamma\left(\frac{\nu_i}{2}\right)} \nu_i^{\nu_i/2} \left((q^{\mathrm{t}}_{\alpha,\nu_i})^2 + \nu_i\right)^{\frac{1-\nu_i}{2}}$$

After substitution in (8), we find, after some computations, the following result:

Theorem 4.4. *The Expected Shortfall at confidence level $1 - \alpha$ for a multi-variate Student-distributed linear portfolio $\delta \cdot X$, with*

$$f_X(x) = \sum_{i=1}^{m} \beta_i \frac{\Gamma(\frac{\nu_i + n}{2})}{\Gamma(\nu_i/2) \cdot \sqrt{|\Sigma|(\nu_i \pi)^n}} \Big(1 + \frac{(x - \mu)^t \Sigma^{-1} (x - \mu)}{\nu_i}\Big)^{-(\frac{\nu_i + n}{2})},$$

is given by:

$$\begin{aligned} ES^{\mathrm{t}}_{\alpha,\nu} &= -\delta\cdot\mu\sum_{i=1}^{m}\beta_i + |\delta\Sigma\delta^t|^{1/2}\sum_{i=1}^{m}\beta_i\frac{1}{\alpha\cdot\sqrt{\pi}}\frac{\Gamma\left(\frac{\nu_i-1}{2}\right)}{\Gamma\left(\frac{\nu_i}{2}\right)}\nu_i^{\nu_i/2}\left((q_\alpha^{MST-VaR})^2+\nu_i\right)^{\left(\frac{1-\nu_i}{2}\right)} \\ &= -\delta\cdot\mu + |\delta\Sigma\delta^t|^{1/2}\sum_{i=1}^{m}\frac{\beta_i}{\alpha\cdot\sqrt{\pi}}\frac{\Gamma\left(\frac{\nu_i-1}{2}\right)\nu_i^{\nu_i/2}}{\Gamma\left(\frac{\nu_i}{2}\right)}\left(\left(\frac{\delta\cdot\mu+VaR_\alpha}{|\delta\Sigma\delta|^{1/2}}\right)^2+\nu_i\right)^{\left(\frac{1-\nu_i}{2}\right)} \\ &= -\delta\cdot\mu \;+\; q_\alpha^{MST-ES}\cdot\sqrt{\delta\Sigma\delta^t} \end{aligned}$$

The Expected Shortfall for a linear Student portfolio is therefore given by a completely explicit formula, once the VaR is known. Observe that, as for the VaR, the only dependence on the portfolio dimension is through the portfolio mean $\delta\cdot\mu$ and the portfolio variance $\delta\Sigma\delta^t$.

4.2. **Some Numerical Result of Delta Mixture-Student Expected Shortfall.** Here we give some numerical results when applying the corollary 3.2, in the situation where $m=2$.

For given $s = q_{\beta,\nu_1,\nu_2} = q_\alpha^{MS-VaR}$, which is the solution of

$$F(s,\beta,\nu_1,\nu_2)=\alpha,$$

by introducing the fonction H such that

$$H(s,\beta,\nu_1,\nu_2)=\beta\cdot H_1(s)+(1-\beta)\cdot H_2(s), \tag{11}$$

where

$$H_j(s)=\frac{\beta_i}{\alpha\cdot\sqrt{\pi}}\frac{\Gamma\left(\frac{\nu_i-1}{2}\right)}{\Gamma\left(\frac{\nu_i}{2}\right)}\nu_i^{\nu_i/2}\left(s^2+\nu_i\right)^{\frac{1-\nu_i}{2}}$$

for $j=1,2$. For given as inputs β, ν_1 and ν_2, we give a table that contains some values of $q_\alpha^{MST-ES} = H(q_\alpha^{MST-VaR},\beta,\nu_1,\nu_2) = q_{\beta,\nu_1,\nu_2}^{MST-ES}$.

(1) In the case where $\alpha=0.01$, we obtain some solutions of (4.2) in the following table:

(ν_1,ν_2)	(2,3)	(3,4)	(4,6)	(7,15)	(8,40)
$q_{0.25,\nu_1,\nu_2}^{MST-ES}$	6.36587	1.29375	0.243125	0.00290856	0.000681262
$q_{0.30,\nu_1,\nu_2}^{MST-ES}$	7.01881	1.41000	0.279435	0.00341273	0.000793844
$q_{0.35,\nu_1,\nu_2}^{MST-ES}$	7.64714	1.52252	0.31424	0.00389277	0.0008997532
$q_{0.40,\nu_1,\nu_2}^{MST-ES}$	8.25196	1.63141	0.34759	0.0043495	0.000997532
$q_{0.45,\nu_1,\nu_2}^{MST-ES}$	8.83444	1.73679	0.379538	0.00478369	0.00108926
$q_{0.50,\nu_1,\nu_2}^{MST-ES}$	9.3957	1.83877	0.410131	0.00519619	0.00117468

(2) In the case where $\alpha=0.001$, we obtain some solutions of (4.2) in the following table:

(ν_1,ν_2)	(2,3)	(3,4)	(4,6)	(7,15)	(8,40)
$q_{0.25,\nu_1,\nu_2}^{MST-ES}$	20.8961	3.03289	0.576689	0.00661826	0.00164597
$q_{0.30,\nu_1,\nu_2}^{MST-ES}$	23.1642	3.32289	0.666054	0.0074621	0.00180969
$q_{0.35,\nu_1,\nu_2}^{MST-ES}$	25.2707	3.58757	0.716427	0.008196	0.00194229
$q_{0.40,\nu_1,\nu_2}^{MST-ES}$	27.239	3.83719	0.776394	0.00883632	0.00205071
$q_{0.45,\nu_1,\nu_2}^{MST-ES}$	29.0885	4.07077	0.830853	0.00939711	0.00214048
$q_{0.50,\nu_1,\nu_2}^{MST-ES}$	30.8351	4.28993	0.880508	0.00989055	0.00221577

4.3. **Delta-Theta Approximation of a Portfolio.** In the case where we dealt with portfolio that contains derivatives, we will consider the Greek Theta of the portfolio by replace the Delta approximation known in financial literature by the *Delta-Theta approximation*.

In clear, suppose that we are holding a portfolio of derivatives depending on n underlying assets $X(t) = (X_1(t), \ldots, X_n(t))$, with elliptically distributed log-returns $r_j = log(X_j(t)/X_j(0))$, over some fixed small time-window [0,t]. The portfolio's present value V will in general be some complicated non-linear function of the X_i's. To obtain a first approximation of its VaR, we simply approximate the present Value V of the position using a first order Taylor expansion:

$$\begin{aligned} V(X(t),t) &\approx V(X(0),0) + \sum_{i=1}^{n} \frac{\partial V}{\partial X_i}(X(0),0)(X_i(t) - X_i(0)) + t \cdot \frac{\partial V}{\partial t}(X(0),0) \\ &= V(X(0),0) + \sum_{i=1}^{n} \frac{\partial V}{\partial X_i}(X(0),0) X_i(0) \left(\exp(r_i) - 1\right) + \Theta \cdot t \\ &\approx V(X(0),0) + \sum_{i=1}^{n} \delta_i r_i + \Theta \cdot t \end{aligned} \tag{12}$$

From this, we can then approximate the Profit & Loss function as

$$\Delta V \approx \delta \cdot r^t + \Theta \cdot t,$$

where we put $r = (r_1, \ldots, r_n)$ and $\delta = (\delta_1, ..., \delta_n)$ with $\delta_i = X_i(0) \cdot \frac{\partial V}{\partial X_i}(X(0),0)$. The entries of the δ vector are called the "delta equivalents " for the position, and they can be interpreted as the sensitivities of the position with respect to changes in each of the risk factors. In this particular case, we have substitute the *Delta normal VaR* as known in the financial literature, by the *Delta-Theta Elliptic VaR* given by the following corollary of the theorem (2.1) :

Corollary 4.5. *Suppose that the portfolio's Profit & Loss function over the time window of interest is, to good approximation, given by* $\Delta\Pi = \delta \cdot r^t + \Theta \cdot t$ *, with constant portfolio weights* $\delta = (\delta_1, \ldots, \delta_n)$*. Suppose moreover that the random vector* $r = (r_1, \cdots, r_n)$ *of underlying log-returns follows a continuous elliptic distribution, with probability density given by* $f_r(x) = |\Sigma|^{-1/2} g((x-\mu)\Sigma^{-1}(x-\mu)^t)$ *where* μ *is the vector mean and* Σ *is the variance-covariance matrix, and where we suppose that* $g(s^2)$ *is integrable over* $\mathbb{R}$*, continuous and nowhere 0. Then the portfolio's* Delta-Theta-elliptic VaR VaR_α *at confidence* $1-\alpha$ *is given by*

$$VaR_\alpha = -\delta \cdot \mu^t + \Theta \cdot t + q^g_{\alpha,n} \cdot \sqrt{\delta \Sigma \delta^t},$$

where $s = q^g_{\alpha,n}$ *is the unique positive solution of the transcendental equation*

$$\alpha = G(q^g_{\alpha,n}).$$

The Expected Shortfall of such portfolios is given by the following corollary

Corollary 4.6. *Suppose that the portfolio's Profit & Loss function over the time window of interest is, to good approximation, given by* $\Delta\Pi = \delta \cdot r^t + \Theta \cdot t$ *,and that* $r \sim N(\mu, \Sigma, \phi)$*, with pdf* $f(x) = |\Sigma|^{-1/2} g((x-\mu)\Sigma^{-1}(x-\mu)^t)$*, then the* Delta-Theta Elliptic Expected Shortfall *or* Delta-Theta ES *at confidence level* $1-\alpha$ *is :*

$$ES_\alpha = -\delta \cdot \mu^t + \Theta \cdot t + |\delta \Sigma \delta^t|^{1/2} \cdot \frac{\pi^{\frac{n-1}{2}}}{\alpha \cdot \Gamma(\frac{n+1}{2})} \cdot \int_{(q^g_{\alpha,n})^2}^{\infty} \left(u - (q^g_{\alpha,n})^2\right)^{\frac{n-1}{2}} g(u)\, du. \tag{13}$$

Remark 4.7. In short-term Risk Management, one can usually assume that $\mu \simeq 0$. In that case, for $t = 1$ we have

$$VaR_\alpha = \Theta + \sqrt{\delta \Sigma \delta^t} \cdot q^g_{\alpha,n},$$

$$ES_\alpha = \Theta + |\delta\Sigma\delta^t|^{1/2} \cdot \frac{\pi^{\frac{n-1}{2}}}{\alpha \cdot \Gamma(\frac{n+1}{2})} \cdot \int_{(q^g_{\alpha,n})^2}^{\infty} \left(u - (q^g_{\alpha,n})^2\right)^{\frac{n-1}{2}} \; g(u)\; du.$$

As before, The preceding can immediately be specialized to a Student t-distributions to estimate the *Delta-Theta Student VaR* and the *Delta-Theta Student ES*. The details will be left to the reader.

4.4. **Portfolios of Equities.** A special case of the preceding is that of an equity portfolio, build of stock $S_1, \ldots, S_n$ with joint log-returns $r = (r_1(t), \ldots, r_n(t))$. In this case, the portfolio's Profit & Loss function over the time window [0,t] of interest is, to good approximation, given by

$$\begin{aligned} \Pi(t) - \Pi(0) &= \sum_{i=1}^{n} w_i S_i(0)(S_i(t)/S_i(0) - 1) \\ &\approx \sum_{i=1}^{n} w_i S_i(0) r_i(t) = \delta \cdot r^t, \end{aligned}$$

where this approximation will be good if the $r_i(t)$ are small. In this case the preceded theorems are applicable where $\delta = (w_1 S_1(0), \ldots, w_n S_n(0))$ and $r_j(t) = log(X_j(t)/X_j(0))$ for j=1,...,n.

4.5. **Businesses as Linear Portfolios of Business Units.** An interesting way of looking upon an big enterprize, e.g. a multi-national or a big financial institution, is by considering it as a sum of its individual business units, cf. Dowd [2]. If X_j, is the variation of price or of profitability of business unit j in one period, then the variation of price of the agglomerate in the same period will be

$$\Delta\Pi = X_1 + \cdots + X_n.$$

The entire institution is therefore modelled by a linear portfolio, with $\delta = (1, 1, \ldots, 1)$, to which the results of this paper can be applied, if we model the vector of individual price variations by a multi-variate elliptic distribution. VaR, incremental VaR (see below) and Expected Shortfall will be relevant here. For more details see Dowd [2], chapter XI .

4.6. **Incremental VaR.** Incremental VaR is defined in [1] as the statistic that provides information regarding the sensitivity of VaR to changes in the portfolio holdings. It therefore gives an estimation of the change in VaR resulting from a risk management decision. Results from [1] for incremental VaR with normally distributed risk-factors generalize straightforwardly to elliptically distributed ones: if we denote by $IVaR_i$ the incremental VaR for each position in the portfolio, with θ_i the percentage change in size of each position, then the change in VaR will be given by

$$\Delta VaR = \sum \theta_i IVaR_i$$

By using the definition of $IVaR_i$ as in [1] (2001), we have that

$$IVaR_i = \omega_i \frac{\partial VaR}{\partial \omega_i} \tag{14}$$

with ω_i is the amount of money invested in instrument i. In the case of an equity portfolio in the elliptically distributed assets, we have seen that, assuming $\mu = 0$,

$$VaR_\alpha = -q^g_{\alpha,n}\sqrt{\delta\Sigma\delta^t},$$

We can then calculate $IVaR_i$ for the i-th constituent of portfolio as

$$IVaR_i = \omega_i \frac{\partial VaR}{\partial \omega_i} = \omega_i \gamma_i$$

with

$$\gamma = -q^g_{\alpha,n}\frac{\Sigma\omega}{\sqrt{\delta\Sigma\delta^t}}.$$

The vector γ can be interpreted as a gradient of sensitivities of VaR with respect to the risk factors. This is the same as in [1], except of course that the quantile has changed from the normal one to the one associated to g.

4.7. **Problem of the aggregation of risks.** Suppose that we have a constituted portfolio with several under portfolios of assets from different markets. Given the Value-at-Risk of the portfolios constituting the global portfolio, under the hypothesis that the joined risks factors follow an *elliptic distribution* , the question is how to get the VaR of the global portfolio.

In order to be clearer and simpler, let us consider a global constituted portfolio of 2 under portfolios from different markets with respective weights $\delta_{\mathbf{1}}$ and $\delta_{\mathbf{2}}$. $\mathbf{\Sigma_1}$ represents the matrix of interrelationship in the under portfolio of market 1; $\mathbf{\Sigma_2}$ represents the matrix of interrelationship in the under portfolio of market 2. One will be able to write the matrix of interrelationship of a global portfolio like this:

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma_1} & \mathbf{\Sigma_{12}} \\ \mathbf{\Sigma_{12}}^t & \mathbf{\Sigma_2} \end{pmatrix},$$

where $\mathbf{\Sigma_{12}}$ is the correlation matrix that takes into consideration the interaction between the market $\mathbf{M_1}$ and the market $\mathbf{M_2}$. If $\delta^t = (\delta_{\mathbf{1}}, \delta_{\mathbf{2}})$, we have

$$\delta^t\mathbf{\Sigma}\delta = {\delta_{\mathbf{1}}}^t\mathbf{\Sigma_1}\delta_{\mathbf{1}} + {\delta_{\mathbf{2}}}^t\mathbf{\Sigma_2}\delta_{\mathbf{2}} + 2\cdot{\delta_{\mathbf{1}}}^t\mathbf{\Sigma_{12}}\delta_{\mathbf{2}}. \tag{15}$$

Therefore, since we know that when $\mu \approx 0$, we have

$$VaR_\alpha = q^g_{\alpha,n}\cdot\sqrt{\delta\Sigma\delta^t},$$

the Value-at-Risk of the global portfolio will be given by

$$VaR_\alpha(\mathbf{M}) = \sqrt{\mathbf{VaR}_\alpha(\mathbf{M_1})^{\mathbf{2}} + \mathbf{VaR}_\alpha(\mathbf{M_2})^{\mathbf{2}} + \mathbf{2}[\mathbf{q}^{\mathbf{g}}_{\alpha,\mathbf{n}}]^{\mathbf{2}}\cdot{\delta_{\mathbf{1}}}^{\mathbf{t}}\mathbf{\Sigma_{12}}\delta_{\mathbf{2}}}. \tag{16}$$

An implicit interrelationship with the hypothesis of *elliptic distribution* is obtained in an analogous way, like in the case where one works with the hypothesis of the normal distribution. Note that, one will distinguish several situations from the behavior of $\mathbf{\Sigma_{12}}$. With some simple operations, the implicit interrelationship is

$$\phi = \frac{{\delta_{\mathbf{1}}}^t\mathbf{\Sigma_{12}}\delta_{\mathbf{2}}}{\sqrt{({\delta_{\mathbf{1}}}^t\mathbf{\Sigma_1}\delta_{\mathbf{1}})({\delta_{\mathbf{2}}}^t\mathbf{\Sigma_2}\delta_{\mathbf{2}})}} \tag{17}$$

with the Value-at-Risk $VaR_\alpha(\mathbf{M})$ of the global portfolio being given as follows:

$$VaR_\alpha(\mathbf{M}) = \sqrt{[\mathbf{VaR}_\alpha(\mathbf{M_1})]^{\mathbf{2}} + [\mathbf{VaR}_\alpha(\mathbf{M_2})]^{\mathbf{2}} + \mathbf{2}\phi\cdot\mathbf{VaR}_\alpha(\mathbf{M_1})\mathbf{VaR}_\alpha(\mathbf{M_2}))}. \tag{18}$$

Also, for $\mu \approx 0$,

$$ES_\alpha = K^g_{ES,\alpha}\cdot\sqrt{\delta\Sigma\delta^t},$$

therefore by using the same technics that proves (16), we have that the expected shortfall of the global portfolio is given by:

$$ES_\alpha(\mathbf{M}) = \sqrt{\mathbf{ES}_\alpha(\mathbf{M_1})^{\mathbf{2}} + \mathbf{ES}_\alpha(\mathbf{M_2})^{\mathbf{2}} + \mathbf{2}[\mathbf{K}^{\mathbf{g}}_{\mathbf{ES},\alpha}]^{\mathbf{2}}\cdot{\delta_{\mathbf{1}}}^{\mathbf{t}}\mathbf{\Sigma_{12}}\delta_{\mathbf{2}}}. \tag{19}$$

This imply that

$$ES_\alpha(\mathbf{M}) = \sqrt{[\mathbf{ES}_\alpha(\mathbf{M_1})]^{\mathbf{2}} + [\mathbf{ES}_\alpha(\mathbf{M_2})]^{\mathbf{2}} + \mathbf{2}\phi_{\mathbf{ES}}\cdot\mathbf{ES}_\alpha(\mathbf{M_1})\mathbf{ES}_\alpha(\mathbf{M_2}))}, \tag{20}$$

where

$$\phi_{ES} = \frac{{\delta_1}^t \mathbf{\Sigma_{12}} \delta_2}{\sqrt{({\delta_1}^t \mathbf{\Sigma_1} \delta_1)({\delta_2}^t \mathbf{\Sigma_2} \delta_2)}} \tag{21}$$

Remark 4.8. The result about the agregation of risks work so well in the situation where, the joint risk factors of our portfolio changes with mixture of ellitic distributions as define in (2.3), and where all $\Sigma_i = \Sigma$, for $i = 1, \ldots, m$. In particular, when $\mu_i = \mu$, we have the results (20) and (16).

5. CONCLUSION

In this paper, we have shown how to reduce the estimation of Value-at-Risk for linear elliptic portfolios to the evaluation of one dimensional integrals which, for the special case of a mixture of t-Student distributions, can be explicitly evaluated in terms of a hypergeometric function. We have also given a similar, but simpler, integral formula for the expected shortfall of such portfolios which, again, can be completely evaluated in the Student case. Following the calculations in the case of Delta mixture-Student VaR, we indicated how to extend it to the case of mixture of t-distributions expected shortfall . We finally surveyed some potential application areas.

References

[1] J.Mina and J.Y.Xiao (2001), Return to Riskmetrics: The Evolution of the standard (www.riskmetrics.com). 2001 RiskMetrics Group Inc.
[2] K.Dowd,(1998) Beyond Value-at-Risk: the new science of risk Management (Wiley series in Frontiers in Finance).
[3] J.Sadefo-Kamdem, (2003) Value-at-Risk and Expected Shortfall for Linear Portfolio with Elliptically Distributed Risk Factors,submitted
[4] J.Sadefo-Kamdem, (2003): Value-at-Risk and Expected Shortfall for quadratic portfolio of securities with mixture of elliptic distributed risk factors. Submitted.
[5] Subu Venkataraman, (1998): Value at risk for a mixture of normal distributions: The use of quasi-bayesian estimation techniques. Economic Perspectives
[6] Yasuhiro and Toshinao Yoshiba, (2002): On the validity of Value-at-Risk: Comparative analysis with Expected shortfall
[7] Zangari, Peter, " An Improved methodology for measuring VaR," RiskMetrics Monitor, 2nd quarter pp.7-25. http://www.riskmetrics.com/research/journals