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▶ **To cite this version:**

Frank Loray. Versal deformation of the analytic saddle-node. Analyse complexe, Systèmes dynamiques, Sommabilité des séries divergentes et Théories Galoisiennes, Volume en l'honneur de Jean-Pierre Ramis, 2004, Toulouse, France. pp.167–187. hal-00001192

HAL Id: hal-00001192
https://hal.science/hal-00001192

Submitted on 3 Mar 2004

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ccsd-00001192 (version 1) : 3 Mar 2004

VERSAL DEFORMATION OF THE ANALYTIC SADDLE-NODE

par

Frank LORAY

To Jean-Pierre Ramis for his 60^{th} *birthday*

Résumé. — In the continuation of [**15**], we derive simple forms for saddle-node singular points of analytic foliations in the real or complex plane just by gluing foliated complex manifolds. We give the versal analytic deformation of the simplest model. We also derive a unique analytic form for those saddle-node having a central manifold. By this way, we recover and generalize results earlier proved by J. Écalle by using mould theory and partially answer to some questions asked by J. Martinet and J.-P. Ramis at the end of [**16**].

Introduction

Let X be a germ of analytic vector field at the origin of $\mathbb{C}^2$

$$X = f(x,y)\partial_x + g(x,y)\partial_y, \quad f,g \in \mathbb{R}\{x,y\} \text{ or } \mathbb{C}\{x,y\}$$

having a singularity at 0: $f(0) = g(0) = 0$. Consider $\mathcal{F}$ the germ of singular holomorphic foliation induced by the complex integral curves of X near 0. A question going back to H. Poincaré is the following:

Problem. — Find local coordinates in which the vector field defining the foliation has the simplest coefficients.

***Classification mathématique par sujets* (2000).** — 32S65.
Mots clefs. — Normal form, Singularity, Foliation.

For instance, if the vector field X has a linear part (in the matrix form)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+by)\partial_x + (cx+dy)\partial_y$$

having non zero eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ with non real ratio $\lambda_2/\lambda_1 \notin \mathbb{R}$, then H. Poincaré proved that the vector field X is actually linear in convenient analytic coordinates. In this situation, the eigenvalues $\{\lambda_1, \lambda_2\}$ (resp. the ratio λ_2/λ_1) provide a complete set of invariants for such vector fields (resp. foliations) modulo analytic change of coordinates.

In this paper, we consider *saddle-node* singularities of foliations, i.e. defined by a vector field with an isolated singularity of the form

$$X = y\partial_y + \text{higher order terms}$$

(one of the eigenvalues is zero). In this situation, H. Dulac proved that, for arbitrary large $N \in \mathbb{N}$, there exist analytic coordinates in which the foliation is defined by a vector field of the form

$$X = x^{k+1}\partial_x + y\partial_y + \mu x^k y\partial_y + x^{k+N} f(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\} \tag{1}$$

for unique $k \in \mathbb{N}^*$ and $\mu \in \mathbb{C}$. In this sense, the family of vector fields

$$x^{k+1}\partial_x + y\partial_y + \mu x^k y\partial_y \tag{2}$$

may be thought as formal normal forms for saddle-node foliations. The complete analytic classification of those singular points has been given by J. Martinet and J.-P. Ramis in 1982 (see [**16**]), giving rise to infinitely many invariants additional to the formal ones (k, μ) above. Let us recall their main result in the case $k = 1$ (for simplicity of notations). Given a saddle-node $\mathcal{F}$ of the form (1), one can associate the so-called *Martinet-Ramis' invariants* $(\varphi_0, \varphi_\infty)$ where

$$\begin{cases} \varphi_0(\zeta) &= e^{2i\pi\mu}\zeta + \sum_{n\geq 2} a_n \zeta^n \ \in \mathrm{Diff}(\mathbb{C},0) \\ \varphi_\infty(\zeta) &= \zeta + t \ \in \mathbb{C} \text{ (a translation)} \end{cases} \tag{3}$$

The map $\mathcal{F} \mapsto (\varphi_0, \varphi_\infty)$ is analytic and surjective onto the set of all pairs satisfying (3). Any two saddle-nodes in the form (1) with $k = 1$ are conjugated by a germ of analytic diffeomorphism tangent to the identity if and only if the corresponding invariants coincide. Finally, if we allow linear change of coordinates, then one has to consider the pairs (3) up to an action of $\mathbb{C}^*$ in order to derive the complete analytic classification.

Nevertheless, this final moduli space is singular at the formal models (2) due to an extra symmetry given by the linear vector field $y\partial_y$.

The classification above suggests that a generic saddle-node foliation cannot be defined by a polynomial vector field in any analytic coordinates. A direct application of our recent work [**15**] provides the following

Theorem 1. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) having formal invariants $k = 1$ and μ arbitrary. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad f \in \mathbb{C}\{y\} \tag{4}$$

where $f'(0) = \mu$.

This statement is a particular case of a general simple analytic form independantly announced by A. D. Brjuno and P. M. Elizarov for all resonant saddles ($\lambda_2/\lambda_1 \in \mathbb{Q}^-$) and saddle-nodes in 1983 (see [**2, 8**]). So far, only the case $k = 1$ and $\mu = 0$ has been proved in 1985 by J. Écalle at the end of [**7**], p.535, presented as an application of *resurgent functions and mould theory*. In 1994, P. M. Elizarov considered in [**9**] the general saddle-node $\mathcal{F}$ in Dulac's prenormal form (1) and computed the derivative of Martinet-Ramis' modular map $\mathcal{F} \mapsto (\varphi_0, \varphi_\infty)$ at all formal models (2). For instance, if we restrict to the family (4) given by Theorem 1, the modular map is tangent at $X_0 = x^2\partial_x + y\partial_y$ to the simpler following one

$$f(y) = \sum_{n\geq 0} a_n y^n \;\mapsto\; \begin{cases} \varphi_0(\zeta) &= e^{2i\pi a_1}\zeta + \sum_{n\geq 2} a_n\zeta^n \\ \varphi_\infty(\zeta) &= \zeta + a_0 \end{cases}$$

In particular, the derivative at X_0 is bijective. On the other hand, our proof of Theorem 1 can be achieved extra complex parameters. In other words, any finite dimensional deformation of X_0 inside the family (1) is analytically conjugated (by an analytic diffeomorphism depending analytically on the parameters) to a deformation inside (4). In this sense, Theorem 1 provides *the versal deformation of X_0*.

It is important to mention at this step that the form (4) is not unique! Of course, we can modify the functional coefficient f by conjugating the vector field with an homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$. But even if we restrict to tangent-to-the-identity conjugacies, the form (4) is perhaps locally unique at X_0 ($f \equiv 0$), but not globally for the following reason. By construction (see proof of Theorem 1), the form (4) is obtained with $f(0) \neq 0$, even if the saddle node has a *central manifold* (see below). For instance, the foliation from X_0 also has another form (4) with $f(0) \neq 0$.

From preliminary form (1), we see that $\{x = 0\}$ is an invariant curve for the vector field. Tangent to the zero eigendirection, there is also a "formal invariant curve" $\{y = \varphi(x)\}$, $\varphi \in \mathbb{R}[[x]]$ or $\mathbb{C}[[x]]$, which is generically divergent. Another remarquable result of [**16**] is that this formal invariant curve is actually convergent if, and only if, the translation part of Martinet-Ramis' invariant is trivial: in the case $k = 1$, this means $\varphi_\infty(\zeta) = \zeta$ (and $t = 0$). We then say that the saddle-node has a *central manifold*. For instance, saddle-nodes in the form (4) with $f(0) = 0$ have the central manifold $\{y = 0\}$. Conversely, a natural question is

Problem. — Given a saddle-node with a central manifold and formal invariant $k = 1$, is it possible to put it into the form (4) with $f(0) = 0$ (i.e. simultaneously straightening the central manifold onto $\{y = 0\}$) ?

We provide the complete following answer.

Theorem 2. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) having formal invariant $k = 1$. Assume that $\mathcal{F}$ has a central manifold. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad \text{with } f(0) = 0. \tag{5}$$

(and $\mu = f'(0)$) if, and only if, we are in one of the following cases

1. *$\mu \in \mathbb{C} - \mathbb{R}^-$,*
2. *$\mu < 0$ and φ_0 is linearizable up to conjugacy in Diff$(\mathbb{C}, 0)$,*
3. *$\mu = 0$ and Martinet-Ramis' invariants $(\tilde{\varphi}_0^i, \tilde{\varphi}_\infty^i)_i$ of φ_0 satisfy: all $\tilde{\varphi}_0^i$ are linear.*

When $\mu \notin \mathbb{Q}_*^-$, this form is unique up to homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$.

In case (2) of Theorem 2, the linearizability of φ_0 is automatic (see [**1**]) as soon as μ belongs to the set $\mathcal{B}$ of Brjuno numbers defined as follows

$$\forall \lambda \in \mathbb{R}, \quad \lambda \in \mathcal{B} \quad \Leftrightarrow \quad \sum_{n \geq 0} \frac{\log(q_{n+1})}{q_n} < \infty$$

(where $\frac{p_n}{q_n}$ stand for succesive truncatures of the continued fraction of $|\lambda|$). Recall that $\mathcal{B}$ has full Lebesgue measure in $\mathbb{R}$. But when $\mu \in \mathbb{R}_*^- \setminus \mathcal{B}$, the linearizability condition on φ_0 is very restrictive (see [**25**]). For instance, when $\mu \in \mathbb{Z} - \mathbb{N}$, this condition forces φ_0 to be the identity.

The case (3) of Theorem 2 has been proved by J. Écalle in [**7**], p.539. In this case, $\varphi_0(\zeta) = \zeta + \cdots$ is tangent to the identity. The classification of such map up to conjugacy in $\mathrm{Diff}(\mathbb{C}, 0)$ has been given independently by J. Écalle, B. Malgrange and S. M. Voronin in 1981. Following the presention of J. Martinet and J.-P. Ramis in [**17**], to any element

$$\varphi(\zeta) = \zeta + \zeta^{n+1} + \cdots \in \mathrm{Diff}(\mathbb{C}, 0)$$

tangent to the identity at the order $n \in \mathbb{N}$, one associates a formal invariant $\lambda \in \mathbb{C}$ (characterizing the class of φ up to formal change of coordinate) and a n-uple

$$(\tilde{\varphi}_0^i, \tilde{\varphi}_\infty^i)_{i=1,\ldots,n} \in (\mathrm{Diff}(\mathbb{C}, 0) \times \mathrm{Diff}(\mathbb{C}, \infty))^n$$

$$\text{with} \quad \begin{cases} \tilde{\varphi}_0^i(\zeta) &= e^{2i\pi\lambda}\zeta + \cdots \in \mathbb{C}\{\zeta\} \\ \tilde{\varphi}_\infty^i(\zeta) &= \zeta + \cdots \qquad\quad \in \mathbb{C}\{\frac{1}{\zeta}\} \end{cases} \quad i = 1, \ldots, n$$

The map $\varphi \mapsto (\tilde{\varphi}_0, \tilde{\varphi}_\infty)_i$ is well-defined, analytic and surjective onto the set of all pairs satisfying conditions above. Moreover, any two such maps are conjugated by a change of coordinate tangent to the identity if, and only if, the corresponding n-uples coincide. Finally, if we allow linear changes of coordinates, then the conjugacy class of φ within $\mathrm{Diff}(\mathbb{C}, 0)$ is characterized by the n-uple $(\tilde{\varphi}_0, \tilde{\varphi}_\infty)_i$ up to an explicit action of $\mathbb{C}^*$ and a cyclic permutation of indices $i = 1, \ldots, n$. Again, the condition of Theorem 2 that all $\tilde{\varphi}_0$ are linear is very restrictive (infinite codimension). In fact, the present work as well as [**15**] started while we were trying to understand this statement of J. Écalle in a geometric way. It is now important to say few words about the proofs, which are perhaps more interesting than the results above.

Both forms (4) and (5) are polynomial in the variable x, and therefore extend analytically along a tubular neighborhood of x-axis, including $x = \infty$: the corresponding singular foliation $\mathcal{F}$ is actually defined on some product $\overline{\mathbb{C}} \times \Delta$ where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere and Δ, a disc on which $f(y)$ is convergent. For instance, in the form (4), the foliation $\mathcal{F}$ is, apart from the singular point $(x, y) = 0$, transversal to the projective line $L = \overline{\mathbb{C}} \times \{0\}$ and his tangencies with the vertical projection $(x, y) \mapsto x$ concentrate along the invariant discs $\{x = 0\}$ and $\{x = \infty\}$. Conversely, it turns out that these geometrical properties essentially characterize the form (4).

The proof of Theorem 1 simply consists, given a germ of saddle-node $\mathcal{F}$, in gluing it with another foliated complex domain in order to obtain a germ of $\overline{\mathbb{C}}$-bundle $M \simeq \overline{\mathbb{C}} \times (\mathbb{C}, 0)$ together with a foliation $\mathcal{F}$ transversal to $\overline{\mathbb{C}} \times \{0\}$ outside the singular point. After a convenient uniformization, we obtain the expected form. Although Theorem 1 is already proved at the end of [**15**], we reprove it directly in order to be self contained.

By the same way, form (5) of Theorem 2 is characterized by the fact that the corresponding foliation $\mathcal{F}$ extends analytically on some product $\overline{\mathbb{C}} \times \Delta$, has central manifold $L = \overline{\mathbb{C}} \times \{0\}$ and has exactly one other singular point along this line, namely at $x = \infty$, with eigenratio $\lambda_2/\lambda_1 = -\mu$. Since φ_0 is also the *holonomy map* of the central manifold, necessary and sufficient conditions given in Theorem 2 exactly coincide with conditions imposed by the holonomy of the singular point at $x = \infty$: for instance, when $\mu < 0$ irrational, then φ_0 must be also the holonomy of a node with irrational eigenratio $-\mu$ and hence linearizable. The unicity therefore comes from the unicity of the gluing construction when the analytic type of the given saddle-node is fixed. In the case $\mu > 0$, we simultaneously obtain a simple form for the saddle at $x = \infty$ (negative eigenratio).

Corollary 3. — *Let $\mathcal{F}$ be a germ of saddle foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) with eigenratio $-\mu < 0$. Then there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = -x\partial_x + \mu(f(y) + x)y\partial_y, \quad \textit{with } f(0) = 1. \tag{6}$$

This latter form is not unique: when $\mu \in \mathcal{B}$, all X_f are conjugated.

For saddle-nodes with formal invariants $k = 1$ and $\mu < 0$ having a central manifold, it is possible to give unique forms by modifying our construction.

Theorem 4. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) having formal invariant $k = 1$. Assume that $\mathcal{F}$ has a central manifold. If $\mu \leq 0$, then let $n \in \mathbb{N}$ be such that $\mu + n > 0$. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xyf(x^n y)\partial_y, \quad \text{where } f(0) = \mu. \tag{7}$$

Moreover, this form is unique up to homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$.

One could think that we obtain, without analysis, the complete analytic classification of saddle-nodes with formal invariant $k = 1$ having a central manifold. But the proof of Theorem 4 uses results of [**16**] obtained as corollaries of the classification itself.

Similarly, it is tempting to apply those methods to the *cuspidal* case. The story below has been strongly motivated by questions and encouragements of Jean-Pierre Ramis. Recall that, under a generic assumption on the quadratic part, the foliation $\mathcal{F}$ defined by a vector field with nilpotent linear part may be written in convenient analytic coordinates (see [**6**])

$$(2y\partial_x + 3x^2\partial_y) + f(x,y)(2x\partial_x + 3y\partial_y) \quad f(0,0) = 0.$$

We see that the cusp $\{y^2 - x^3 = 0\}$ is invariant. Following [**23**] (see also [**15**]), one can analytically reduce f to a function of the single variable x. Still, this form is far to be unique. In fact, one can formally reduce f to a unique form $f(x) = x + \tilde{f}(x^3)$, $\tilde{f} \in \mathbb{C}[[x]]$, $\tilde{f}(0) = 0$ (see [**14**]) or analytically reduce any finite jet of f to this form. But, after experimental tests done by M. Canalis-Durand, F. Michel and M. Teysseire showing experimental divergence of $\tilde{f}$ with Gevrey estimates (see [**4**]), M. Canalis-Durand and R. Schäfke finally established the generic divergence in [**5**]. In fact, the equivalent formal reduction to $f(x) = x + \tilde{f}(h)$ where $h = y^2 - x^3$ is the hamiltonian variable (see [**14**]) is Gevrey 1 and the first singularity in the Borel plane is computed in [**5**].

Clearly, there are at least infinitely many formal invariants but a complete analytic classification of cuspidal singularities is still missing. In [**6**], this problem is reduced to the analytic classification of a pair of an involution and a trivolution in $\mathrm{Diff}(\mathbb{C},0)$ up to simultaneous conjugacy, but this latter equivalence is not simpler that the initial one. One possibility could be to find a unique analytic form. For instance, an easy computation shows that an analytic form

$$(2y\partial_x + 3x^2\partial_y) + (x + \tilde{f}(y^2 - x^3))(2x\partial_x + 3y\partial_y) \quad \tilde{f}(0) = 0.$$

defines a foliation $\mathcal{F}$ that extends analytically along the cusp, and actually on a tubular neighborhood of the corresponding (singular) rational curve C after desingularization of the elliptic fibration $\{h = \text{constant}\}$. Along C, apart form the cuspidal singular point, there are exactly 2 other singular points for $\mathcal{F}$, namely a node and a saddle with respective eigenratios -6 and 6. Clearly the first one is non resonant since it has an invariant curve C in the good direction: it is linearizable and the corresponding holonomy is the identity.

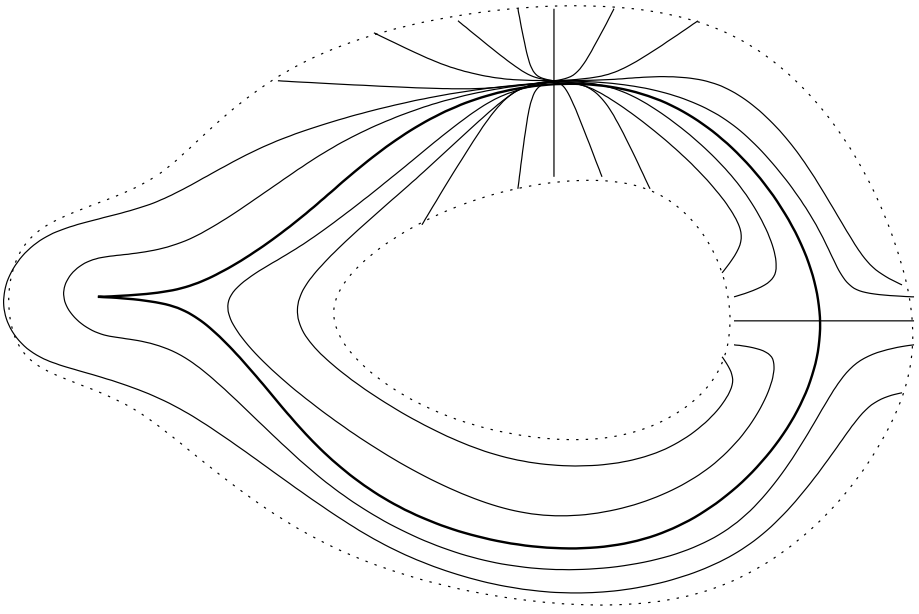


FIGURE 1. The geometry of normal forms from [**14**]

On the other hand, the saddle gives no obstruction to the holonomy of the cusp. Conversely, given a cuspidal foliation $\mathcal{F}$ (with a generic quadratic part), one can construct an elliptic fibration having the cusp as a singular fibre on which $\mathcal{F}$ extends with exactly 2 singular points as above. In order to do this, we have to construct the elliptic fibration at the same time. We give in Section 4 a Lemma which permits to do this (and avoid with the difficult Savelev's Theorem for the proof of Theorems 2 and 4). A first difficulty is that the constructed elliptic fibration is not necessarily isotrivial. But, following Kodaira's classification, the germ of fibration is classified by an integer, namely the order of modular $\mathcal{J}$-function at C. For instance, we have to allow alternate variables like $h = \frac{y^2-x^3}{1+x}$ in the non isotrivial case. Now, counting the parameters for instance in the isotrivial case, we can write the foliation into the form $X = \partial_t + f(t,h)h\partial_h$ where ∂_t is a translation along the elliptic fibres, $h\partial_h$ a multiple of $2x\partial_x + 3y\partial_y$, transversal to the regular fibres, and f an elliptic function on each fibre. By construction, f has 2 zeroes and thus 2 poles: it can be expressed by means of Weierstrass functions and 4 coefficients analytic in h. After a translation in the fibres, one can reduce to 3 analytic coefficients. But at the end, the analytic form that we could obtain by this way will be not unique because of the resonant node! Also, we strongly hope that the continuation of [**5**] will permit to provide the good approach for the complete analytic classification of cuspidal singularities.

1. Proof of Theorem 1

We repeat the geometric construction of [**15**]. Consider the germ of foliation $\mathcal{F}_0$ defined by a vector field X_0 of the form (1)

$$X_0 = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

(saddle-node with formal invariant $k=1$ in the form (1) with $N=0$). Maybe replacing $y := x+y$, the linear part of X_0 is given by

$$\begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix} = (cx+y)\partial_y \quad \text{with} \quad c = f(0) \neq 0.$$

Therefore, the vector field X_0 is well-defined on the neighborhood of any small horizontal disc $\Delta_0 = \{|x| < \varepsilon\} \times \{0\}$, $\varepsilon > 0$, and transversal to Δ_0 outside the singular point. Consider inside the horizontal line $L = \overline{\mathbb{C}} \times \{0\}$ the covering given by Δ_0 and $\Delta_\infty = \{|x| > \varepsilon/2\} \times \{0\}$, and denote by $C = \Delta_0 \cap \Delta_\infty$ the intersection corona. By the flow-box Theorem, there exists a unique germ of diffeomorphism of the form

$$\Phi : (\mathbb{C}^2, C) \to (\mathbb{C}^2, C) \ ; \ (x,y) \mapsto (\phi(x,y), y), \quad \phi(x,0) = x$$

straightening $\mathcal{F}_0$ onto the vertical foliation $\mathcal{F}_\infty$ (defined by ∂_y) at the neighborhood of the corona C. Therefore, after gluing the germs of complex surfaces $(\overline{\mathbb{C}} \times \mathbb{C}, \Delta_0)$ and $(\overline{\mathbb{C}} \times \mathbb{C}, \Delta_\infty)$ along the corona by means of Φ, we obtain a germ of smooth complex surface S along a rational curve L equipped with a singular holomorphic foliation $\mathcal{F}$ and a (germ of) rational fibration $y : (S, L) \to (\mathbb{C}, 0)$. Following [**10**], there exists a germ of submersion $x : (S, L) \to \overline{\mathbb{C}}$ completing y into a system of trivializing coordinates: $(x, y) : (S, L) \to \overline{\mathbb{C}} \times (\mathbb{C}, 0)$.

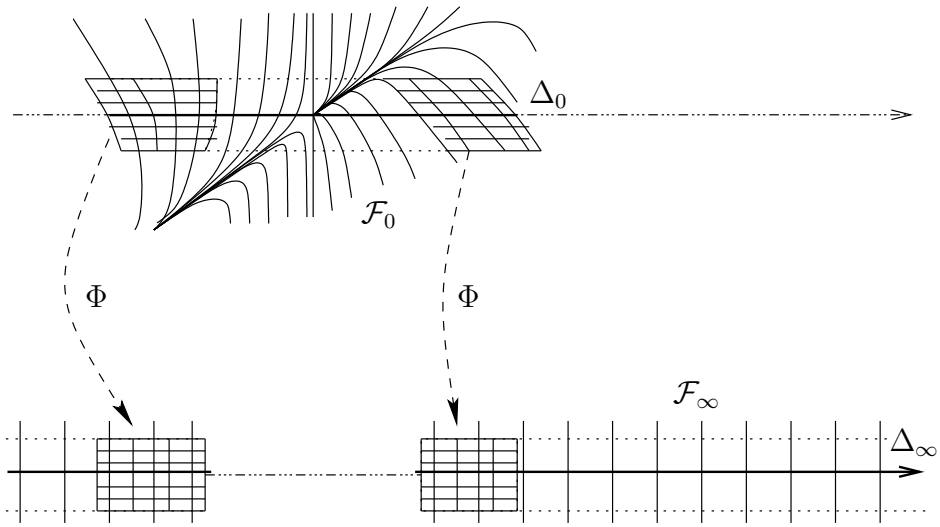


FIGURE 2. Gluing (bi)foliated surfaces

At the neighborhood of any point $p \in L$, the foliation $\mathcal{F}$ is defined by a (non unique) germ of holomorphic vector field, or equivalently by a unique germ of meromorphic vector field of the form

$$X = f(x, y)\partial_x + \partial_y$$

with f meromorphic at p. By unicity, this meromorphic vector field is actually globally defined on the neighborhood of L and is therefore rational in x, i.e. f is the quotient of two Weierstrass polynomials. For y fixed (close to 0), the horizontal component $f(x, y)\partial_x$ defines a meromorphic vector field on the corresponding horizontal line $\overline{\mathbb{C}} \times \{y\}$ whose zeroes and poles coincide with the tangencies between $\mathcal{F}$ and the respective vertical and horizontal fibrations. By construction, we control the number of poles: in the second chart, $\mathcal{F} = \mathcal{F}_\infty$ is transversal to y, although in the first chart, $\mathcal{F} = \mathcal{F}_0$ has exactly one simple tangency with any horizontal line. It follows that, for y fixed, the meromorphic vector field $f(x, y)\partial_x$ has exactly 1 simple pole and thus 3 zeroes (counted with multiplicity).

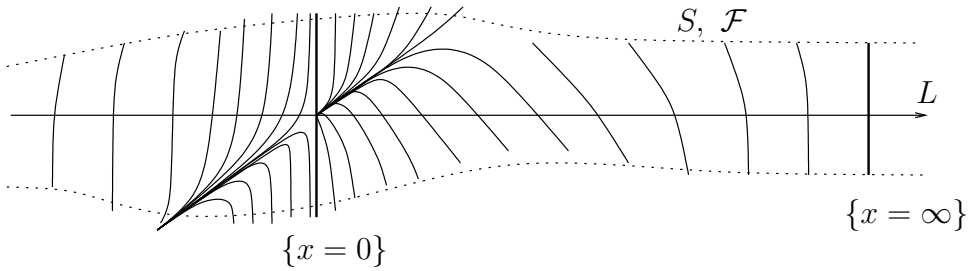


FIGURE 3. Uniformization

Of course, in restriction to L, the pole vanishes together with one zero at the singular point of $\mathcal{F}$. We conclude that the vector field X defining the foliation $\mathcal{F}$ takes the form

$$X = \frac{f_0(y) + f_1(y)x + f_2(y)x^2 + f_3(y)x^3}{g_0(y) + g_1(y)x}\partial_x + \partial_y \tag{8}$$

with $f_i, g_j \in \mathbb{C}\{y\}$. Up to a change of projective horizontal coordinate $x := \{\frac{a(y)x+b(y)}{c(y)x+d(y)}\}$ on S, one can assume that $\{x=\infty\}$ is a vertical leaf of $\mathcal{F}$, that $\{x=0\}$ is the invariant curve of the saddle-node tangent to the non zero eigendirection and that $\mathcal{F}$ has a contact of order 2 with the vertical fibration along $\{x=0\}$ (likely as in the local form (1) with $k=1$). Therefore, $f_0, f_1, f_3 \equiv 0$ and, reminding that $\mathcal{F}_0$ is a saddle-node singular point with invariant $k=1$ and 0-eigendirection transversal to L, we also have $f_2(0) \neq 0$, $g_1(0) \neq 0$, $g_0(0) = 0$ and $g_0'(0) \neq 0$. After division, $\mathcal{F}$ is actually defined by a vector field of the form

$$\tilde{X} = x^2\partial_x + (f(y)x + yg(y))\partial_y, \quad f(0), g(0) \neq 0.$$

After change of y-coordinate, one may normalize the holomorphic vector field $yg(y)\partial_y$ to $g(0)y\partial_y$; after division by $g(0)$ and linear change of the x-coordinate, we finally obtain the form (4).

2. Proof of Theorem 2

Given a saddle-node foliation $\mathcal{F}$ of the form (5), it is easy to verify that its analytic continuation at the neighborhood of the horizontal line $L = \overline{\mathbb{C}} \times \{0\}$ satisfies

1. the line L is a global invariant curve for $\mathcal{F}$, the union of a smooth leaf together with 2 singular points;
2. the point $x = 0$ is a saddle-node singular point with formal invariants $k = 1$ and $\mu = f'(0)$ and invariant curve $\{xy = 0\}$; in particular, the sadle-node has an analytic central manifold which is contained in L;
3. the point $x = \infty$ is a singular point with eigenratio $-\mu$ and invariant curve $\{x = \infty\} \cup \{y = 0\}$;
4. the contact between $\mathcal{F}$ and the vertical fibration is double along the invariant curve $\{x = 0\}$.

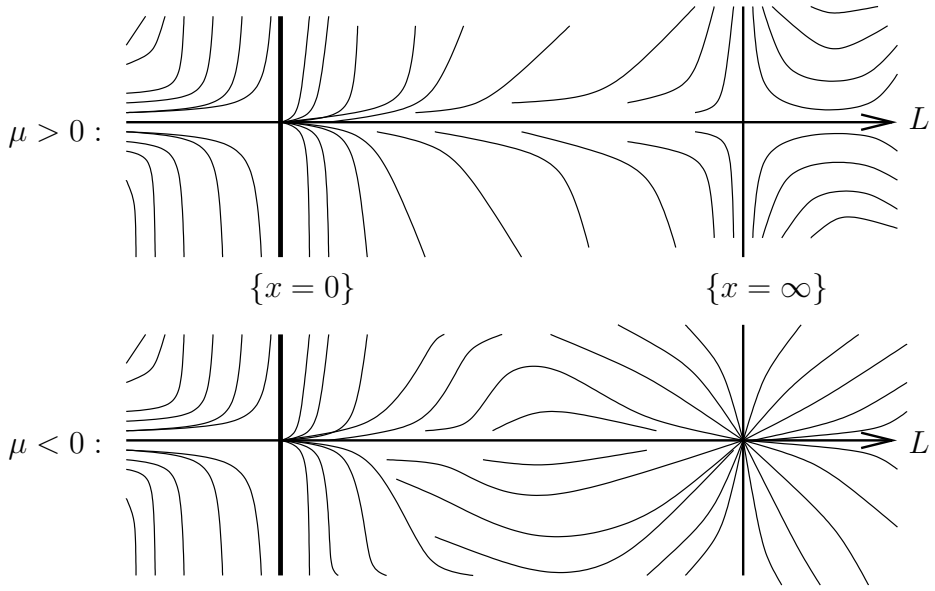


FIGURE 4. Geometry of the second normal form

Conversely, a germ of foliation $\mathcal{F}$ on $\overline{\mathbb{C}} \times (\mathbb{C}, 0)$ satisfying conditions above is actually defined by a vector field of the form

$$\tilde{X} = x^2\partial_x + (f(y)x + g(y))y\partial_y, \quad f(0), g(0) \neq 0.$$

After change of y-coordinate, one may normalize the holomorphic vector field $yg(y)\partial_y$ to $g(0)y\partial_y$; after division by $g(0)$ and linear change of the x-coordinate, we finally obtain the form (5).

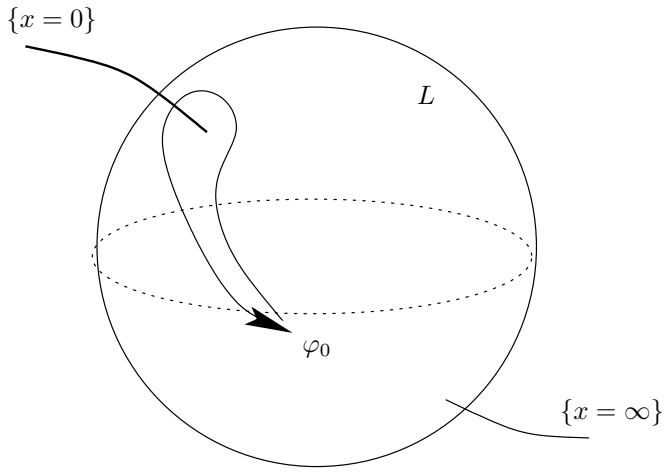


FIGURE 5. Holonomy compatibility

A necessary condition for a saddle-node to admit a form (5) is that the holonomy of the central manifold, which actually coincide with Martinet-Ramis' invariant φ_0, is also the anti-holonomy of the invariant curve L around the singular point $x = \infty$. In the case $\mu < 0$, the other singular point is linearizable by Poincaré's Theorem implying the linearizability of the holonomy map φ_0. Here, we use property 3 above and the fact that, in the resonant (non linearizable) case, the node has only one irreducible germ of invariant curve. In the case $\mu = 0$, the holonomy φ_0 is tangent to the identity and its inverse φ_0^{-1} must be the holonomy of the *strong manifold* (the invariant curve tangent to the non zero eigendirection) of a saddle-node having a central manifold. Following [**16**], this is equivalent to condition (3) of Theorem 2.

We now prove that conditions (1), (2) and (3) of Theorem 2 are sufficient. Like in Section 1, we start with a germ of saddle-node $\mathcal{F}_0$ defined on the neighborhood of some disc Δ_0 and glue it with a germ of foliation $\mathcal{F}_\infty$ along a complementary disc Δ_∞ in other to obtain a germ of 2-dimensional neighborhood S along a rational curve L equipped with a singular foliation $\mathcal{F}$. The difference with Section 1 is that we now glue $\mathcal{F}_0$ and $\mathcal{F}_\infty$ along a common invariant curve, in such a way that L becomes a global invariant curve for the foliation $\mathcal{F}$. In this case, it becomes very technical to do such gluing while preserving the horizontal foliation. We prefer to glue the foliations with respect to the vertical fibration and recover latter the triviality of the neighborhood by Savelev's Theorem.

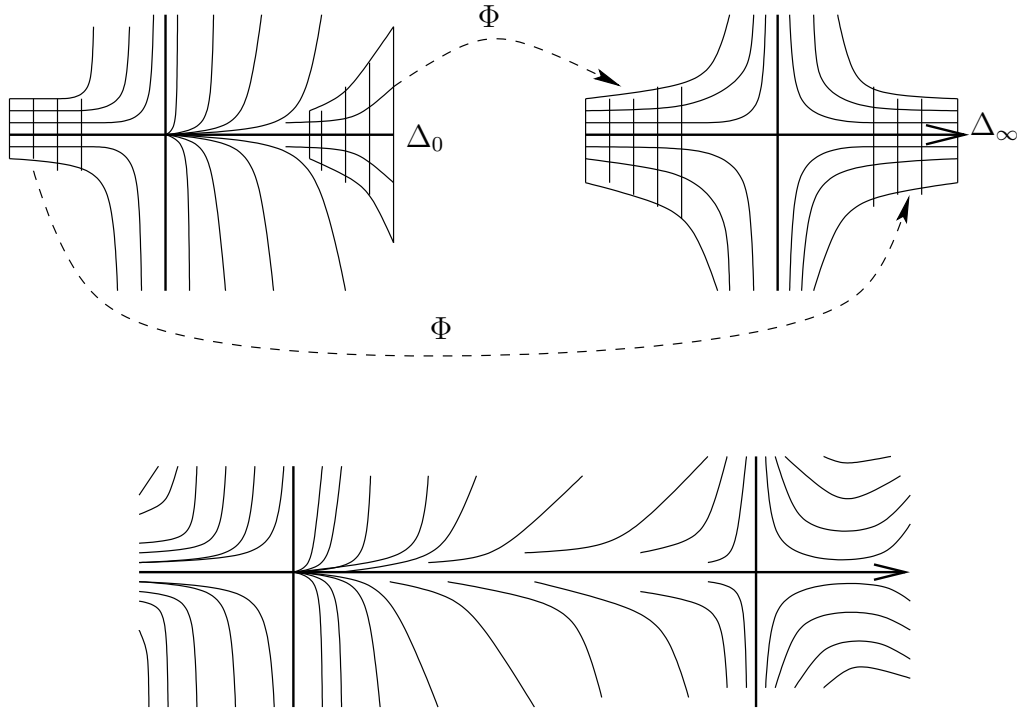


FIGURE 6. Gluing picture

We start with $\mathcal{F}_0$ into the form

$$X_0 = x^2\partial_x + y\partial_y + xyf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

Consider, in local coordinates ($\tilde{x} = 1/x, y$) at infinity, a germ of singular foliation $\mathcal{F}_\infty$ defined by

$$X_\infty = \tilde{x}\partial_{\tilde{x}} - \mu(1+g(\tilde{x},y))y\partial_y, \quad g(0) = 0.$$

Up to a linear change of $\tilde{x}$-coordinate, one may assume that $\mathcal{F}_\infty$ is actually defined on the neighborhood of Δ_∞. Obviously, there exists a germ of diffeomorphism of the form

$$\Phi : (\mathbb{C}^2, C) \to (\mathbb{C}^2, C) \ ; \ (x,y) \mapsto (x, \phi(x,y)), \quad \phi(x,0) = 0$$

straightening $\mathcal{F}_0$ onto $\mathcal{F}_\infty$ if, and only if, the respective holonomy maps around the corona $C = \Delta_0 \cap \Delta_\infty$ are conjugated in $\mathrm{Diff}(\mathbb{C},0)$. When $\mu \notin \mathbb{R}$, the holonomy map φ_0 of $\mathcal{F}_0$ around C (or $x=0$) is hyperbolic and hence linearizable by Kœnigs' Theorem. It is therefore enough to choose X_∞ linear. When $\mu > 0$, then the holonomy map φ_0 can be realized as the holonomy of a saddle $\mathcal{F}_\infty$ like above following [**17, 21**]. When $\mu < 0$ and φ_0 is linearizable, we obviously realize it with X_∞ linear. Finally, when $\mu = 0$, condition (3) of Theorem 2 is exactly the one to realize φ_0^{-1} as the holonomy of a saddle node $\mathcal{F}_\infty$. After gluing $\mathcal{F}_0$ and $\mathcal{F}_\infty$ along C, we obtain a germ of surface S containing a rational curve L which, by Camacho-Sad's Formula (see [**3**]), has 0 self-intersection in S. Following Savelev's Theorem [**22**], there exists a system of trivializing coordinates: $(x,y) : (S,L) \to \overline{\mathbb{C}} \times (\mathbb{C},0)$. Up to a change of trivializing coordinates $x := \{\frac{a(y)x+b(y)}{c(y)x+d(y)}\}$ and $y = \varphi(y)$ on S, one may assume properties (1), (2), (3) and (4) of the begining of the section all satisfied. Therefore, $\mathcal{F}$ is defined by a vector field of the form (5).

Finally, we prove the unicity of form (5) in case μ is not rational negative. Assume that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are of the form (5) and are analytically conjugated on a neighborhood of $(x, y) = 0$. Following [**16**], they are also conjugated by a germ of diffeomorphism of the form

$$\Phi_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \; ; \; (x, y) \mapsto (x, \phi_0(x, y))$$

which must preserve the central manifold: $\phi_0(x, 0) = 0$. One can extend analytically Φ_0 on a neighborhood of $L - \{x = \infty\}$ in the obvious way, by lifting-path-property. The problem is whether Φ_0 extends at the other singular point $x = \infty$ or not. If it is the case, then we obtain a global diffeomorphism Φ along L conjugating $\mathcal{F}$ and $\tilde{\mathcal{F}}$ and, by Blanchard's Lemma, permutting the horizontal lines: $\Phi(x, y) = (x, \phi(y))$. Due to the form (5), ϕ has to commute with $y\partial_y$ and must be linear. Finally, the fact that Φ_0 extends at the singular point at infinity is due to J.-F. Mattei and R. Moussu [**18**] in the case $\mu > 0$, to M. Berthier, R. Meziani and P. Sad in the case $\mu = 0$ and we now detail the simpler case $\mathcal{F}_\infty$ is in the Poincaré domain. Recall that property (3) implies that $\mathcal{F}_\infty$ is non resonant and hence linearizable by a local change of coordinates of the form $(\tilde{x}, y) \mapsto (\tilde{x}, \phi_\infty(\tilde{x}, y))$. Therefore, we can assume that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are defined by

$$X_\infty = \tilde{x}\partial_{\tilde{x}} - \mu y \partial_y, \quad \mu \not\geq 0.$$

and that $\Phi_0(\tilde{x}, y) = (\tilde{x}, \phi(\tilde{x}, y))$ is a self-conjugacy of $\mathcal{F}_\infty$ at the neighborhood of the punctured disc $\Delta^* := \Delta_\infty - \{\tilde{x} = 0\}$. The question is, when does Φ_0 coincide with a symetry of $\mathcal{F}_\infty$

$$\Phi_\infty(\tilde{x}, y) = (\tilde{x}, c \cdot y), \quad c \in \mathbb{C}^*.$$

Of course, this is the case if, and only if, $\phi(\tilde{x}, y)$ is linear in y. In fact, for x fixed, $\phi(\tilde{x}, y)$ commutes with the holonomy $y \mapsto e^{-2i\pi\mu} y$ of $\mathcal{F}_\infty$ and is therefore linear as soon as μ is not rational. Conversely, in the case μ is rational, it is easy to construct examples of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ like above that are not globally conjugated and giving rise to non unique form (5).

3. Proof of Theorem 4

Let us start by blowing-up a saddle-node of the form (5)

$$X_f = x^2\partial_x + y\partial_y + xyf(y)\partial_y, \quad f(0) = \mu_0.$$

Along the exceptional diviseur, we have one saddle with eigenratio -1 and a saddle-node, given in the chart (x,t), $y = tx$, by

$$\tilde{X}_f = x^2\partial_x + t\partial_t + xt(f(xt) - 1)\partial_t.$$

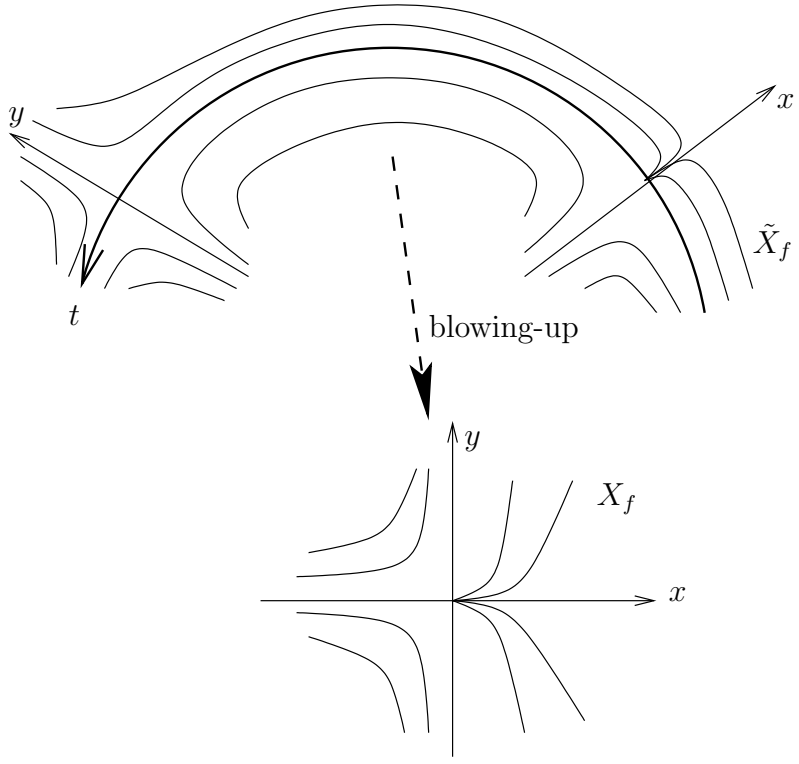


FIGURE 7. Blowing-up a saddle-node

After n successive blowing-up of the saddle-nodes, we obtain an exceptional divisor like in the picture below where the new saddle-node takes the form of Theorem 4 with formal invariant $\mu = \mu_0 - n$. All other singular points are saddles with -1 eigenratio.

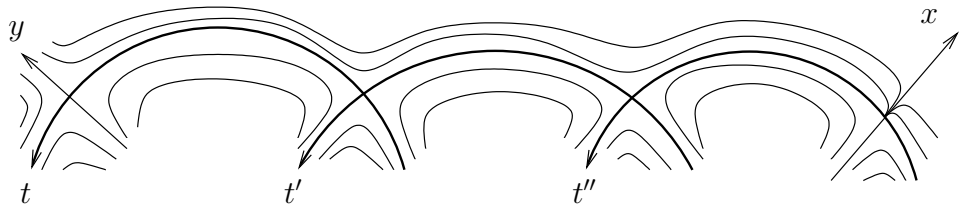


FIGURE 8. After 3 blowing-up

If we are able to prove that any saddle-node $\mathcal{F}$ with invariants $k = 1$ and $\mu > -n$ (or $\mu \notin \mathbb{R}$) having a central manifold is actually the n-time blowing-up of a saddle-node like above, then Theorem 2 implies Theorem 4. To show this, we construct the configuration above by gluing $\mathcal{F}$ with successive saddles. Of course, this is possible because any tangent-to-the-identity element of $\mathrm{Diff}(\mathbb{C}, 0)$ can be realized as the holonomy map of such saddle (see [**17**]). Then, Grauert's Theorem permits to blow down successively the components of the divisor having -1 self-intersection.

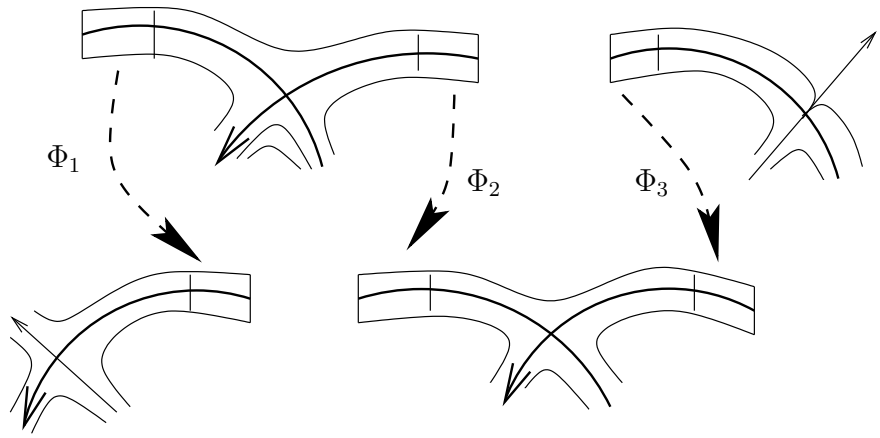


FIGURE 9. Gluing foliations along an exceptional divisor

The unicity of form (7) also follows from that of form (5) proved in Section 2. Indeed, if two such foliations are locally conjugated, then the corresponding holonomies of all -1 saddle along the exceptional divisor are also conjugated by [**18**] and, after blowing down, the holonomies of strong manifold of the sadddle-nodes are so. By [**16**], the saddle-nodes are also conjugated after blowing down and they are in the form (5).

4. Gluing Lemmae

The *order of contact* between two germs of regular holomorphic vector fields X_1 and X_2 at $0 \in \mathbb{C}^2$, or between the corresponding foliations, is by definition the order at 0 of the determinant $\det(X_1, X_2)$. For instance, X_1 and X_2 are transversal if and only if they have a contact of order $k = 0$. Now, if those two foliations share a common leaf, and if moreover there is no contact between them outside this leaf, then the contact order $k \in \mathbb{N}^*$ is constant along this common leaf and classifies locally the pair of foliations:

Lemma 5. — *Let $\mathcal{F}$ be a germ of regular analytic foliation at the origin of $\mathbb{C}^2$ (or $\mathbb{R}^2$) having the horizontal axis $L_0 : \{y = 0\}$ as a particular leaf and having no other contact with the vertical fibration $\{y = constant\}$: $\mathcal{F}$ is defined by a unique function (resp. vector field) of the form*

$$F(x, y) = y + y^k x f(x, y) \quad (resp.\ X = f(x, y)\partial x + y^k \partial_y), \quad f(0, 0) \neq 0$$

where $k \in \mathbb{N}^*$ denotes the contact order between $\mathcal{F}$ and the fibration. Then, up to a change of coordinates of the form $\Phi(x, y) = (\phi(x, y), y)$, the foliation $\mathcal{F}$ is defined by the function (resp. vector field)

$$F_0(x, y) = y + xy^k \quad (resp.\ X_0 = \partial x + y^k \partial_y).$$

The restriction of Φ to L_0 is the identity if, and only if, $f(0, w) \equiv 1$. Moreover, the normalizing coordinate Φ is unique once we have decided that it fixes the horizontal axis, i.e. $\phi(x, y) = x\tilde{\phi}(x, y)$.

Proof. — Given $\mathcal{F}$ as in the statement, choose $F(x,y)$ to be the unique function which is constant on the leaves and has restriction $F(0,y) = y$ on the horizontal axis: $F(x,y) = y(1 + x\tilde{F}(x,y))$. The assumption $dF \wedge dy = y^k u(x,y)$, $u(0,0) \neq 0$, yields $\tilde{F}(x,y) = y^{k-1} f(x,y)$ with $f(0,0) \neq 0$, whence the form $F(x,y) = y + xy^k f(x,y)$. Now, we have

$$F = F_0 \circ \Phi_0 \quad \text{with} \quad \Phi_0(x,y) = (xf(x,y), y).$$

Thus, Φ_0 is the unique change of x-coordinate which conjugates the functions F and F_0; in particular, it conjugates the induced foliations.

Conversely, assume that $\Phi(x,y) = (\phi(x,y), y)$ is conjugating the foliations respectively induced by F and F_0: we have

$$F_0 \circ \Phi(x,y) = \varphi \circ F(x,y) \quad \text{with} \quad \varphi(y) = y + y^k \phi(0,y)$$

(the germ φ is determined by the equality restricted to $\{w = 0\}$). If we decompose $f(x,y) = u(x) + yv(x,y)$, we notice that $\varphi \circ F(x,y) = y + xy^k(u(x) + y\tilde{v}(x,y))$, so that $\phi(x,0) = xu(x) = xf(x,0)$. Finally, if $\phi(x,y) = x\tilde{\phi}(x,y)$, then $\varphi(y) = y$ and Φ actually conjugates the functions: we must have $\Phi = \Phi_0$ whence the unicity.

Now, if $\mathcal{F}$ is defined by $X = f(x,y)\partial_x + g(x,y)\partial_y$, assumption gives $dy(X) = g(x,y) = y^k \tilde{g}(x,y)$ with $f(0,0), \tilde{g}(0,0) \neq 0$. After dividing X by g, we can write $X = f(x,y)\partial x + y^k \partial_y$. We have already proved that any two such foliations (in particular those induced by X and X_0) are conjugate by a unique diffeomorphism of the form $\Phi(x,y) = (x\tilde{\phi}(x,y), y)$. Now, if $\Phi(x,y) = (\phi(x,y), y)$ conjugates the foliations respectively induced by X and X_0, it actually conjugates these vector fields. In restriction to the trajectory L_0, we see that $\phi(x,0)$ conjugates $\tilde{X}|_{L_0} = f(x,0)\partial_x$ to the constant vector field ∂_x. Therefore, $\phi(x,0) = \int_0^x \frac{1}{f(\zeta,0)} d\zeta$ and $\phi(x,0) \equiv x$ if, and only if, $f(x,0) \equiv 1$. □

For the next statement, denote by $\Omega \subset (\mathbb{C} \times \{0\})$ a connected open domain inside the horizontal axis.

Lemma 6. — *Let $\mathcal{F}$ and $\mathcal{F}'$ be regular holomorphic foliations defined at the neighborhood of Ω in $\mathbb{C}^2$ both having Ω as a particular leaf. Assume that the contact between each foliation with the horizontal fibration $\{y = constant\}$ reduces to Ω, with same order $k \in \mathbb{N}^*$. In other words, $\mathcal{F}$ and $\mathcal{F}'$ are respectively defined by vector fields*

$$X = f(x,y)\partial_x + y^k\partial_y \quad \textit{and} \quad X' = f'(x,y)\partial_x + y^k\partial_y$$

where f and f' are non vanishing functions in the neighborhood of Ω. Then, $\mathcal{F}$ and $\mathcal{F}'$ are conjugated in a neighborhood of Ω by a diffeomorphism of the form $\Phi(x,y) = (x + y\phi(x,y), \psi(y))$ (fixing Ω) if, and only if, the two following conditions hold

1. *$f(x,0) \equiv f'(x,0)$;*
2. *the respective holonomies φ and φ' of $\mathcal{F}$ and $\mathcal{F}'$ along Ω are analytically conjugated: $\psi \circ \varphi = \varphi' \circ \psi$.*

Proof. — Following Lemma 5, condition (1) is the necessary and sufficient condition for the existence of local conjugacies $\Phi = (y\phi(x,y), y)$ between $\mathcal{F}$ and $\mathcal{F}'$ at the neighborhood of any point $w_0 \in \Omega$. Fix one of these points and consider the respective holonomy maps φ and φ' computed on the transversal $T : \{x = x_0\}$ in the variable y. By condition (2), up to conjugate, say $\mathcal{F}'$, by a diffeomorphism of the form $(x, \psi(y))$, we may assume without loss of generality $\varphi(y) = \varphi'(y)$. We start with the local diffeomorphism $\Phi(x,y) = (y\phi(x,y), y)$ given by Lemma 5 conjugating the foliations and fixing T. Since Φ conjugates the corresponding vector fields X and X', it extends analytically along the whole of Ω by the formula $\Phi(p) := \Phi_{X'}^{-t} \circ \Phi \circ \Phi_X^t(p)$. The condition $\varphi(y) = \varphi'(y)$ implies that Φ is uniform. □

Here is another gluing Lemma for pairs of regular foliations $\mathcal{F}$ and $\mathcal{G}$ at the neighborhood of the common leaf Ω (before, $\mathcal{G}$ was the horizontal fibration).

Lemma 7. — *Let $\mathcal{F}$ and $\mathcal{G}$ (resp. $\mathcal{F}'$ and $\mathcal{G}'$) be regular holomorphic foliations defined at the neighborhood of the corona Ω in $\mathbb{C}^2$ both having Ω as a regular leaf. Assume that the contact between $\mathcal{F}$ and $\mathcal{G}$ (resp. $\mathcal{F}'$ and $\mathcal{G}'$) reduces to Ω, with same order $k \in \mathbb{N}^*$. In other words, the foliations above are respectively defined by vector fields*

$$X = \partial_x + yf(x,y)\partial_y \quad \textit{and} \quad Y = X + y^k g(x,y)\partial_y,$$

$$(\textit{resp.} \quad X' = \partial_x + yf'(x,y)\partial_y \quad \textit{and} \quad Y' = X' + y^k g'(x,y)\partial_y)$$

where g and g' are non vanishing functions in the neighborhood of Ω. Then, $\mathcal{F}$ and $\mathcal{G}$ are simultaneously conjugated to $\mathcal{F}'$ and $\mathcal{G}'$ in a neighborhood of Ω by a diffeomorphism of the form $\Phi(x,y) = (x + y\phi(x,y), y\psi(x,y))$ (fixing point-wise Ω) if, and only if, the two conditions hold

1. *for any (and for all) $x_0 \in \Omega$, we have*
$$\frac{g(x,0)}{\exp(-\int_{x_0}^{x} f(\zeta,0)d\zeta)} \equiv \frac{g'(x,0)}{\exp(-\int_{x_0}^{x} f'(\zeta,0)d\zeta)};$$
2. *the respective pairs of holonomies $(\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}})$ and $(\varphi_{\mathcal{F}'}, \varphi_{\mathcal{G}'})$ along Ω are simultaneously analytically conjugated: $\psi \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{F}'} \circ \psi$ and $\psi \circ \varphi_{\mathcal{G}} = \varphi_{\mathcal{G}'} \circ \psi$.*

Proof. — It is similar to that of the previous Lemma. Up to a change of coordinate $y := \psi(y)$ (which does not affect neither $f(x,0)$, nor $g(x,0)$ and hence preserves equality (i)), we may assume that holonomies $(\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}) \equiv (\varphi_{\mathcal{F}'}, \varphi_{\mathcal{G}'})$ actually coincide on a transversal $T : \{x = x_0\}$. We just detail that condition (i) exactly provides the existence of local conjugacies between the given pairs of foliations fixing point-wise Ω; the unique conjugacy fixing T will extend uniformly along Ω by (ii).

At the neighborhood of any point $(x_0, 0)$ of the corona Ω, say $x_0 = 0$ for simplicity, we preliminary conjugate X to $X_0 = \partial_x$ by respective local changes of y-coordinate $\Psi(x,y) = (x, y\psi(x,y))$, $\psi(0,0) \neq 0$

$$\Psi_* X = X_0 = \partial_x \quad \text{and} \quad \Psi_* Y = Y_0 = z^k g_0(x,y)\partial_z + \partial_w$$

Doing the same with the pair X' and Y', we see by Lemma 5 that the corresponding pairs of foliations are conjugated by a diffeomorphism fixing point-wise $\{y=0\}$ if, and only if, the differential form $\omega = g_0(x,0)dx$ along Ω coincide with the corresponding one $\omega' = g_0'(x,0)dx$ for $X_0' = X_0$ and $Y_0' = \partial_x + y^k g_0(x,y)\partial_y$. This 1-form ω can be redefined in the following intrinsic way: the holonomy of $\mathcal{G}$ between two transversal cross-sections T_0 and T_1 computed in any coordinate y which is $\mathcal{F}$-invariant (here $\mathcal{F}$ is defined by ∂_x) takes the form

$$\varphi(y) = y + \left(\int_{x_0}^{x_1} \omega\right) y^k + (\text{higher order terms})$$

where $(x_i, 0) := T_i \cap \Omega$, $i = 0, 1$. Since

$$\Psi^* X_0 = \partial_x - \frac{\psi_x}{\psi + y\psi_y} y\partial_y \quad \text{and} \quad \Psi^* Y_0 = \Psi^* X_0 + \frac{g_0}{\psi + y\psi_y} y^k \partial_y,$$

(ψ_x and ψ_y are partial derivatives of ψ) we derive in restriction to Ω

$$f(x,0) = -\frac{\psi_x(x,0)}{\psi(x,0)} \quad \text{and} \quad g_0(x,0) = \psi(x,0)\cdot g(x,0)$$

yielding the formula for the local invariant of our conjugacy problem

$$\omega = \frac{g(x,0)}{\exp(-\int_{x_0}^{x} f(\zeta,0)d\zeta)} dx.$$

□

5. Complements

Let X be a germ of analytic vector field at the origin of $\mathbb{C}^2$

$$X = f(x,y)\partial_x + g(x,y)\partial_y, \quad f,g \in \mathbb{R}\{x,y\} \text{ or } \mathbb{C}\{x,y\}$$

having a singularity at 0, $f(0) = g(0) = 0$, and consider its linear part

$$X_1 = (ax+by)\partial_x + (cx+dy)\partial_y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(written in matrix form) where $X = X_1 + \cdots$ (quadratic terms). Given an analytic diffeomorphism $\Phi = (\phi_1, \phi_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, we have

$$\Phi^* X = D\Phi^{-1} \cdot X \circ \Phi = \begin{pmatrix} \phi_{1x} & \phi_{1y} \\ \phi_{2x} & \phi_{2y} \end{pmatrix}^{-1} \cdot \begin{pmatrix} f \circ \Phi \\ g \circ \Phi \end{pmatrix}.$$

In particular, the linear part of $\Phi^* X$ is given by

$$D\Phi(0)^* X_1 = \begin{pmatrix} \phi_{1x}(0) & \phi_{1y}(0) \\ \phi_{2x}(0) & \phi_{2y}(0) \end{pmatrix}^{-1} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \phi_{1x}(0) & \phi_{1y}(0) \\ \phi_{2x}(0) & \phi_{2y}(0) \end{pmatrix}$$

and the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ of (the linear part of) the vector field X are invariant under analytic (or formal) change of coordinates. When one of the eigenvalues is not zero, say λ_1, we denote by $\lambda := \lambda_2/\lambda_1 \in \mathbb{C}$ the eigenratio. Since $h \cdot X$ defines the same foliation for any $h \in \mathbb{C}\{x,y\}$, $h(0) \neq 0$, the eignevalues λ_i are no more invariants for the foliation but the eigenratio λ is. In this situation, a famous result of H. Poincaré, improved in the dimension 2 case by H. Dulac, yield

***Theorem* (Poincaré-Dulac).** — *If $\lambda \notin \mathbb{R}^-$, then, up to a (real or complex) analytic change of coordinates, the vector field becomes either linear diagonal*

$$X = \lambda_1 x\partial_x + \lambda_2 y\partial_y$$

or, in the resonant node case $\{\lambda_1, \lambda_2\} = \{\lambda, k\lambda\}$, $k \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}^*$, possibly of the form (including the non diagonal linear case for $n = 1$)

$$X = \lambda(kx + y^k)\partial_x + \lambda y\partial_y$$

or, in the real hyperbolic focus case $\{\lambda_1, \lambda_2\} = \{a \pm ib\}$, $a, b \in \mathbb{R}^*$

$$X = (ax - by)\partial_x + (bx + ay)\partial_y.$$

Those normal forms are unique (up to a permutation of λ_1 and λ_2) so that this statement cannot be improved. In the case $\lambda \in \mathbb{R}^-$, analogous simple polynomial models are obtained for the subjacent foliation after a formal change of coordinates, that is a pair $\Phi = (\phi_1, \phi_2)$ where $\phi_1, \phi_2 \in \mathbb{C}[[x, y]]$ and $\det D\Phi(0) \neq 0$.

***Theorem* (Poincaré-Dulac).** — *If $\lambda \in \mathbb{R}^-$, then, up to a formal change of coordinates and multiplication by a non vanishing formal power series, the vector field becomes either linear*

$$X = x\partial_x + \lambda y\partial_y$$

or, in the resonant saddle case $\lambda = -\frac{p}{q} \in \mathbb{Q}_-^*$, possibly of the form

$$X = px\partial_x - \left(q + (x^p y^q)^k + \mu(x^p y^q)^{2k}\right) y\partial_y$$

or, in the saddle-node case $\lambda = 0$, of the form

$$X = x\partial_x + \left(y^k + \mu y^{2k}\right) y\partial_y$$

for unique positive integer $k \in \mathbb{N}^*$ and scalar $\mu \in \mathbb{C}$.

See [**19**] or [**24**] for formal models of the corresponding vector fields. When λ belongs to the set $\mathcal{B} \cap \mathbb{R}^-$ of Brjuno numbers, A. D. Brjuno proved in 1971 (see [**1**]) that the vector field X is actually analytically linearisable. The remaining exceptional cases split into diophantine saddles ($\lambda \in \mathbb{R}^- \setminus (\mathbb{Q}^- \cup \mathcal{B})$) resonant saddles ($\lambda \in \mathbb{Q}^- \setminus \{0\}$) and saddle-nodes ($\lambda = 0$). In the resonant case ($\lambda \in \mathbb{R}^-$), Yu. S. Ilyashenko proved in 1981 (see [**13**]) that the formal change of coordinates leading to the Poincaré-Dulac normal form above is divergent as a rule. The complete analytic classification of saddle-nodes and resonant saddles has been given by J. Martinet and J.-P. Ramis in 1982 and 1983 (see [**16, 17**]) giving rise to infinitely many invariants additional to the formal ones $(-\frac{p}{q}, k, \mu)$ above. See [**12, 11**] and [**20, 24**] for the corresponding classification of vector fields. For diophantine saddles, although a complete classification is still missing, J.-C. Yoccoz proved in 1988 (see [**25**]) that the moduli space between analytic and formal classification is also very huge (infinite dimension).

Let us now consider the following family of saddle-nodes ($\varepsilon > 0$)

$$X_\varepsilon = \frac{x^2}{1+\mu x}\partial_x + y\partial_y + \varepsilon f(x,y)y\partial_y, \quad f = \sum_{m\leq 0,\ n\leq -1} f_{m,n}x^m y^n \in \frac{1}{y}\mathbb{C}\{x,y\}$$

with formal invariants $k = 1$ and $\mu \in \mathbb{C}$: $f_{0,0} = f_{0,1} = f_{1,1} = 0$. Consider also the associate Martinet-Ramis' invariants (depending on ε)

$$\varphi_0(\zeta) = e^{2i\pi\mu}\zeta + \sum_{n>0}\varphi_n\zeta^{n+1} \quad \text{and} \quad \varphi_\infty(\zeta) = \zeta + t$$

Then, the main result of [**9**] reads

***Theorem* (Elizarov).** — *The derivative (in the sense of Gâteau) of Martinet-Ramis'moduli at $\varepsilon = 0$ is given by*

$$\frac{d\varphi_n}{\partial\varepsilon}|_{\varepsilon=0} = n^{\mu n-1}e^{-2i\pi n\mu}\sum_{m>0}\frac{m}{\Gamma(1+m+\mu n)}f_{m,n}(-n)^m$$

$$\textit{and} \quad \frac{dt}{\partial\varepsilon}|_{\varepsilon=0} = (-1)^{-\mu}e^{2i\pi\mu}\sum_{m>0}\frac{m}{\Gamma(1+m-\mu)}f_{m,-1}(-n)^m$$

where Γ is the Euler's Gamma Function.

Finally, the analytic form announced in [**2**] is the following

Given a saddle-node $\mathcal{F}$ with formal invariants $k = 1$ and $\mu \in \mathbb{C}$ and a slope $0 < s \leq +\infty$, one can find analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form

$$x^2\partial_x + y\partial_y + x\left(f_0 + \mu y + \sum_{(m,n)\in E_s} f_{m,n}x^m y^{n+1}\right)\partial_y$$

provided that the set of exponents $E_s = \{(m,n); n > 0, \frac{n}{s}+1 \leq m < \frac{n}{s}+2\}$ does not intersect the resonances $\{(m,n); m+\mu n \in -\mathbb{N}\}$.

Références

[1] A. D. Bruno, The analytical form of differential equations. Trans. Mosc. Math. Soc. 25 (1971), 131–288; 26 (1972), 199–239.

[2] A. D. Bruno, Analytic invariants of a differential equation. Soviet Math. Dokl. 28 (1983), no. 3, 691–695.

[3] C. Camacho, P. Sad, Invariant varieties through singularities of holomorphic vector fields. Ann. of Math. (2) 115 (1982), no. 3, 579–595.

[4] M. Canalis-Durand, F. Michel, M. Teisseyre, Algorithms for formal reduction of vector field singularities. J. Dynam. Control Systems 7 (2001), no. 1, 101–125.

[5] M. Canalis-Durand, R. Schäfke, On the normal form of a system of differential equations with nilpotent linear part. C. R. Acad. Sci. Paris, Ser. I 336 (2003), 129–134.

[6] D. Cerveau, R. Moussu, Groupes d'automorphismes de $(C, 0)$ et équations différentielles $ydy + \cdots = 0$. Bull. Soc. Math. France 116 (1988), no. 4, 459–488 (1989).

[7] J. Écalle, Les fonctions résurgentes. Tome III. L'équation du pont et la classification analytique des objects locaux. Publications Mathématiques d'Orsay, 85-5. Université de Paris-Sud, Département de Mathématiques, Orsay, (1985).

[8] P. M. Elizarov, Usp. Mat. Nauk. 38(5) (1983) 144–145.

[9] P. M. Elizarov, Tangents to moduli maps. Nonlinear Stokes phenomena, 107–138, Adv. Soviet Math., 14, Amer. Math. Soc., Providence, RI, (1993).

[10] W. Fischer, H. Grauert, Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten (German). Nachr. Akad. Wiss. Gttingen Math.-Phys. Kl. II (1965) 89–94.

[11] A. A. Grintchy, Analytical classification of saddle resonant singular points of holomorphic vector fields on the complex plane (Russian), Vinity, w1690-B96 (1996).

[12] A. A. Grintchy, S. M. Voronin, An analytic classification of saddle resonant singular points of holomorphic vector fields in the complex plane. J. Dynam. Control Systems 2 (1996), no. 1, 21–53.

[13] Yu. S. Ilyashenko, In the theory of normal forms of analytic differential equations violating the conditions of A. D. Bryuno divergence is the rule and convergence the exception. Moscow Univ. Math. Bull. 36 (1981), no. 2, 11–18.

[14] F. Loray, Réduction formelle des singularités cuspidales de champs de vecteurs analytiques. J. Differential Equations 158 (1999), no. 1, 152–173.

[15] -, A preparation theorem for codimension one singular foliations. Preprint IRMAR, http://hal.ccsd.cnrs.fr/view/ccsd-00001190/fr/

(2004). See also Analytic normal forms for non degenerate singularities of planar vector fields. Preprint CRM 545, Bellaterra (2003). http://www.crm.es/Publications/preprints2003.htm

[16] J. Martinet, J.-P. Ramis, Problèmes de modules pour des équations différentielles non linéaires du premier ordre. Inst. Hautes Études Sci. Publ. Math. No. 55 (1982), 63–164.

[17] -, Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. Ann. Sci. École Norm. Sup. (4) 16 (1983), no. 4, 571–621.

[18] J.-F. Mattei, R. Moussu, Holonomie et intégrales premières. Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 4, 469–523.

[19] Yu. I. Meshcheryakova, Formal classification of degenerate elementary singular points. (Russian) 197–206, Chelyabinsk. Gos. Univ., Chelyabinsk, (2002).

[20] Yu. I. Meshcheryakova, S. M. Voronin, Analytic classification of generic degenerate elementary singular points of germs of holomorphic vector fields on the complex plane. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. (2002), , no. 1, 13–16.

[21] R. Pérez Marco, J.-C. Yoccoz, Germes de feuilletages holomorphes à holonomie prescrite. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992). Astérisque No. 222, (1994), 7, 345–371.

[22] V. I. Savelev, Zero-type imbedding of a sphere into complex surfaces. Moscow Univ. Math. Bull. 37 (1982), no. 4, 34–39.

[23] E. Strozyna, H. Zoladek, The analytic and formal normal form for the nilpotent singularity. J. Differential Equations 179 (2002), no. 2, 479–537.

[24] L. Teyssier, Analytical classification of singular saddle-node vector field Classification analytique des champs de vecteurs nœud-cols, C. R. Acad. Sci. Paris, Ser. I 336 (2003), p.619-624; Jour. of Dyn. and Control Syst. (to appear).

[25] J.-C. Yoccoz, Linéarisation des germes de difféomorphismes holomorphes de $(C, 0)$. C. R. Acad. Sci. Paris Sér. I Math. 306 (1988), no. 1, 55–58.

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