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# Absolute continuity of the spectrum of a Schrödinger operator with a potential which is periodic in some directions and decays in others

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## 1 Formulation of the result

There are many papers (see for example [1, 6]) devoted to the question of the absolute continuity of the spectrum of differential operators with coefficients periodic in the whole space. In this article we consider the situation where the coefficients are periodic in some variables and decay very fast (super-exponentially) when the other variables tend to infinity. The corresponding operator describes the scattering of waves on the infinite membrane or filament.

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We denote by  $(x; y)$  points of the space  $\mathbb{R}^{m+d}$ ,  $\Omega = \mathbb{R}^m \times (0, 2\pi)^d$ ,  $\langle x \rangle = \sqrt{x^2 + 1}$ . For real  $a$  introduce the spaces

$$L_{p,a} = \{f : e^{a\langle x \rangle} f \in L_p(\Omega)\}, \quad H_a^2 = \{f : e^{a\langle x \rangle} f \in H^2(\Omega)\},$$

where  $1 \leq p \leq \infty$  and  $H^2(\Omega)$  is the Sobolev space. Our main result is

**Theorem 1.1.** *Consider in  $L_2(\mathbb{R}^{m+d})$  the self-adjoint operator*

$$Hu = -\operatorname{div}(g\nabla u) + Vu \tag{1}$$

*and assume that real scalar functions  $g \geq c_0 > 0$  and  $V$  satisfy following conditions:*

1.  $g(x, y + 2\pi l) = g(x, y)$ ,  $V(x, y + 2\pi l) = V(x, y) \quad \forall l \in \mathbb{Z}^d$ ;
2.  $(g - g_0), \Delta g, V \in L_{\infty,a}$  for any  $a > 0$ , where  $g_0$  is a positive constant.

*Then, the spectrum of  $H$  is purely absolutely continuous.*

**Remark 1.1.** *Operators with different values of  $g_0$  differ one from another by multiplication by a constant, so without loss of generality we can and, from now on, do assume that  $g_0 = 1$ .*

**Remark 1.2.** *If  $V \equiv 0$ , (1) is acoustic operator. If  $g \equiv 1$ , it is the Schrödinger operator with electric potential  $V$ .*

The basic philosophy of our proof is the following. To prove the absolute continuity of the spectrum for periodic operators (i.e. periodic with respect to a non degenerate lattice in  $\mathbb{R}^d$ ), one applies the Floquet-Bloch-Gelfand reduction to the operator and one is left with proving that the Bloch-Floquet-Gelfand eigenvalues must vary with the quasi-momentum i.e. that they cannot be constant on sets of positive measure (see e.g. [6]). If one tries to follow the same line in the case of operators that are only periodic with respect to a sub-lattice, the problem one encounters is that, as the resolvent of the Bloch-Floquet-Gelfand reduction of the operator is not compact, its spectrum may contain continuous components and some Bloch-Floquet-Gelfand eigenvalues may be embedded in these continuous components. The perturbation theory of such embedded eigenvalues (needed to control their behavior in the Bloch quasi-momentum) is more complicated than that of isolated eigenvalues. To obtain a control on these eigenvalues, we use an idea of the theory of resonances (see e.g. [10]): if one analytically dilates Bloch-Floquet-Gelfand reduction of the operator, these embedded eigenvalues become isolated eigenvalues, and thus can be controlled in the usual way.

Let us now briefly sketch our proof. We make the Bloch-Floquet-Gelfand transformation with respect to periodic variables (see section 3) and get a family of operators  $H(k)$  in the cylinder  $\Omega$ . Then, we consider the corresponding resolvent in suitable weighted spaces. It analytically depends on the quasi-momentum  $k$  and the spectral (non real) parameter  $\lambda$ . It turns out that we can extend it analytically with respect to  $\lambda$  from upper half plane to lower one (see below Theorem 5.1) and thus establish the limit absorption principle. It is enough for the absolute continuity of the initial operator (see section 7).

Note that an analytic extension of the resolvent of the operator (1) with coefficients  $g$  and  $V$  which decay in all directions is constructed in the paper [2] (with  $m = 3$ ,  $d = 0$ ; see also [7] for  $g \equiv 1$ ). In the case of a potential decaying in all directions but one (i.e. if  $d = 1$ ), the analytic extension of the resolvent of the whole operator (1) (not only for the operator  $H(k)$  (see section 3)) is investigated in [4] when  $g \equiv 1$ . Note also that our approach has shown to be useful in the investigation of the perturbation of free operator in the half-plane by  $\delta$ -like potential concentrated on a line (see [3]); the wave operators are also constructed there.

In section 2, we establish some auxiliary inequalities. In section 3, we define the Floquet-Gelfand transformation and construct an analytic extension of the resolvent of free operator in the cylinder  $\Omega$ . In sections 4 and 5, we prove a limiting absorption principle for the initial operator in the cylinder. An auxiliary fact from theory of functions is established in section 6. Finally, the proof of Theorem 1.1 is completed in section 7.

We denote by  $B_\delta(k_0)$  a ball in real space

$$B_\delta(k_0) = \{k \in \mathbb{R}^d : |k - k_0| < \delta\}$$

and by  $k_1$  the first coordinate of  $k$ ,  $k = (k_1, k')$ . We will use the spaces of function in  $\Omega$  with periodic boundary conditions,

$$\begin{aligned} \tilde{H}^2 &= \left\{ f \in H^2(\Omega) : f|_{y_i=0} = f|_{y_i=2\pi}, \frac{\partial f}{\partial y_i}|_{y_i=0} = \frac{\partial f}{\partial y_i}|_{y_i=2\pi}, i = 1, \dots, d \right\}, \\ \tilde{H}_{loc}^2 &= \left\{ f \in H_{loc}^2(\Omega) : f|_{y_i=0} = f|_{y_i=2\pi}, \frac{\partial f}{\partial y_i}|_{y_i=0} = \frac{\partial f}{\partial y_i}|_{y_i=2\pi}, i = 1, \dots, d \right\}. \end{aligned}$$

Finally  $B(X, Y)$  is the space of all bounded operators from  $X$  to  $Y$ , and  $B(X) = B(X, X)$ .

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## 2 Auxiliary estimations

In this section we assume that the pair  $(k_0, \lambda_0) \in \mathbb{R}^{d+1}$  satisfies

$$(k_0 + n)^2 \neq \lambda_0 \quad \forall n \in \mathbb{Z}^d. \quad (2)$$

The constants in all the inequalities in this section may depend on  $(k_0, \lambda_0)$ . The set

$$J = \{n \in \mathbb{Z}^d : (k_0 + n)^2 < \lambda_0\}$$

is finite. In a neighborhood of  $(k_0, \lambda_0)$ , the partition of  $\mathbb{Z}^d$  into  $J$  and  $(\mathbb{Z}^d \setminus J)$  is clearly the same. In other words, there exists  $\delta = \delta(k_0, \lambda_0) > 0$  such that

$$\text{if } k \in B_\delta(k_0), \lambda \in B_\delta(\lambda_0), \text{ then } (k + n)^2 < \lambda \Leftrightarrow n \in J. \quad (3)$$

Choose  $\tilde{k} \in B_\delta(k_0)$  with  $\tilde{k}_1 \notin \mathbb{Z}$  and put

$$k(\tau) = (\tilde{k}_1 + i\tau, \tilde{k}') \in \mathbb{C}^d, \quad \tau \in \mathbb{R}, \quad (4)$$

$$M_1 \equiv M_1(k_0; \lambda_0) = (B_\delta(k_0) \cup \{k(\tau)\}_{\tau \in \mathbb{R}}) \times B_\delta(\lambda_0). \quad (5)$$

**Lemma 2.1.** *For all  $\zeta \in \mathbb{R}^m$ ,  $(k, \lambda) \in M_1$ ,  $n \in \mathbb{Z}^d \setminus J$ ,  $\tau \in \mathbb{R}$ , we have*

$$\zeta^2 + (k + n)^2 - \lambda \neq 0,$$

$$|\zeta^2 + (k(\tau) + n)^2 - \lambda| \geq c|\tau|.$$

*Proof.* By virtue of (3)

$$\zeta^2 + (k + n)^2 - \lambda > 0 \quad \forall k \in B_\delta(k_0).$$

The second inequality is an immediate corollary of the equality

$$\text{Im}(\zeta^2 + (k(\tau) + n)^2 - \lambda) = 2(\tilde{k}_1 + n_1)\tau.$$

□

In the rest of the section we assume  $\lambda_0 > 0$ . In this case, we will need to change the integration path in the Fourier transformation. Fix  $\eta > \sqrt{\lambda_0}$  and, let  $\gamma$  be the contour in the complex plane defined as

$$\gamma = \{-\xi + i\eta\}_{\xi \in [\eta, \infty)} \cup \{\alpha(1 - i)\}_{\alpha \in [-\eta, \eta]} \cup \{\xi - i\eta\}_{\xi \in [\eta, \infty)}. \quad (6)$$

Two following assertions are clear.

**Lemma 2.2.** If  $g \in L_2(\gamma)$  and  $\eta_0 > \eta$  then the function

$$h(t) = e^{-\eta_0|t|} \int_{\gamma} e^{itz} g(z) dz$$

belongs to  $L_2(\mathbb{R})$ .

**Lemma 2.3.** Denote by  $\Gamma$  the open set between real axis and  $\gamma$  (it consists of two connected components). Let  $g$  be an analytic function in  $\Gamma$ ,  $g \in C(\overline{\Gamma})$  and  $|g(z)| \leq C(1 + |\operatorname{Re} z|)^{-2}$ . Then,

$$\int_{\mathbb{R}} e^{itz} g(z) dz = \int_{\gamma} e^{itz} g(z) dz \quad \forall t \in \mathbb{R}.$$

Establish an analogue of Lemma 2.1 for  $n \in J$ .

**Lemma 2.4.** Let  $\lambda_0 > 0$ ,  $\eta > \sqrt{\lambda_0}$  and  $\gamma$  be defined by (6). For all  $\zeta \in \gamma^m$  (i.e.  $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$ ,  $\zeta_j \in \gamma$ ),  $(k, \lambda) \in M_1$ ,  $n \in J$ ,  $\tau \in \mathbb{R}$ , we have

$$\zeta^2 + (k + n)^2 - \lambda \neq 0,$$

$$|\zeta^2 + (k(\tau) + n)^2 - \lambda| \geq c|\tau|.$$

*Proof.* If  $k \in B_{\delta}(k_0)$  then  $\operatorname{Im}(\zeta^2 + (k + n)^2 - \lambda) \neq 0$  for  $\zeta \neq 0$ . So it remains to prove the second inequality. We can write

$$\zeta^2 = -2i \sum_p \alpha_p^2 + \sum_q (\xi_q - i\eta)^2,$$

where the indexes  $p$  correspond to the coordinates of  $\zeta$  which are in the middle part of  $\gamma$  (i.e.  $|\operatorname{Re} \zeta_p| < \eta$ ) and the indexes  $q$  correspond to the extreme parts of  $\gamma$  (i.e.  $|\operatorname{Re} \zeta_q| \geq \eta$ ); it is possible that there are only indexes  $p$  or only  $q$ . Without loss of generality, we suppose that, for all  $q$ ,  $\xi_q \geq 0$ . Thus,

$$\begin{aligned} \zeta^2 + (k(\tau) + n)^2 - \lambda &= \sum_q (\xi_q^2 - \eta^2) + (\tilde{k} + n)^2 - \tau^2 - \lambda \\ &\quad + 2i \left( - \sum_p \alpha_p^2 - \sum_q \xi_q \eta + (\tilde{k}_1 + n_1) \tau \right). \end{aligned}$$

Fix some  $\sigma \in (\eta^{-1}\sqrt{\lambda_0}, 1)$ . If  $\sum_q \xi_q \geq \sigma|\tau|$  then

$$|\operatorname{Im}(\zeta^2 + (k(\tau) + n)^2 - \lambda)| \geq 2 \left( \sigma\eta - |\tilde{k}_1 + n_1| \right) |\tau| > 2(\sigma\eta - \sqrt{\lambda_0})|\tau|,$$

as  $(\tilde{k} + n)^2 < \lambda_0$ . If  $\sum_q \xi_q \leq \sigma|\tau|$  then  $\sum_q \xi_q^2 \leq \sigma^2 \tau^2$  and

$$|\operatorname{Re}(\zeta^2 + (k(\tau) + n)^2 - \lambda)| \geq \tau^2 + \lambda - (\tilde{k} + n)^2 - \sigma^2 \tau^2 > (1 - \sigma^2) \tau^2$$

again by virtue of (3). □

### 3 The resolvent of free operator in the cylinder

Let us consider the Floquet-Gelfand transformation

$$(Uf)(k, x, y) = \sum_{l \in \mathbb{Z}^d} e^{-2\pi i \langle k, l \rangle} f(x, y + 2\pi l).$$

It is an unitary operator

$$U : L_2(\mathbb{R}^{m+d}) \rightarrow \int_{[0,1)^d}^{\oplus} L_2(\Omega) dk.$$

Introduce the family of operators  $H(k)$  in the cylinder  $\Omega$

$$\begin{aligned} H(k) &= (i\nabla - (0, \bar{k}))^* g(x, y) (i\nabla - (0, k)) + V(x, y), \\ \text{Dom } H(k) &= \tilde{H}^2, \quad k \in \mathbb{C}^d. \end{aligned} \quad (7)$$

Then, the Schrödinger operator (1) is unitarily equivalent to the direct integral of these operators in  $\Omega$ :

$$UHU^* = \int_{[0,1)^d}^{\oplus} H(k) dk. \quad (8)$$

In this section we investigate the free operator

$$A(k) = -\Delta_x + (i\nabla_y - \bar{k})^* (i\nabla_y - k) \quad (9)$$

(which corresponds  $H(k)$  with  $g \equiv 1$ ,  $V \equiv 0$ ). For  $k \in \mathbb{R}^d$ ,  $\text{Im } \lambda \neq 0$  its resolvent can be expressed as

$$((A(k) - \lambda)^{-1} f)(x, y) = \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^m} \frac{e^{i\zeta x + iny} (Ff)(\zeta, n) d\zeta}{\zeta^2 + (k + n)^2 - \lambda}, \quad (10)$$

where  $F$  denotes the Fourier transformation in the cylinder

$$(Ff)(\zeta, n) = (2\pi)^{-m-d} \int_{\Omega} e^{-i\zeta x - iny} f(x, y) dx dy.$$

Let  $(k_0, \lambda_0) \in \mathbb{R}^{d+1}$  satisfy (2),  $J$  and  $M_1$  be defined by formulas (3), (4), (5) in previous section.

**Lemma 3.1.** *There exists a neighborhood of the set  $M_1$  in  $\mathbb{C}^{d+1}$  such that, for  $(k, \lambda) \in M_1$ , the operator  $R_1(k, \lambda)$  given by*

$$(R_1(k, \lambda)f)(x, y) = \sum_{n \in \mathbb{Z}^d \setminus J} \int_{\mathbb{R}^m} \frac{e^{i\zeta x + iny}(Ff)(\zeta, n)d\zeta}{\zeta^2 + (k+n)^2 - \lambda}$$

*is well defined and is bounded from  $L_2(\Omega)$  to  $H^2(\Omega)$ . The  $B(L_2(\Omega), H^2(\Omega))$ -valued function  $(k, \lambda) \mapsto R_1(k, \lambda)$  is analytic in this neighborhood. The estimate*

$$\|R_1(k(\tau), \lambda)\|_{B(L_2(\Omega))} \leq C|\tau|^{-1}$$

*holds.*

*Proof.* It immediately follows from Lemma 2.1.  $\square$

**Lemma 3.2.** *Let  $\lambda_0 > 0$ ,  $\eta > \sqrt{\lambda_0}$ ,  $a > \eta\sqrt{m}$  and the contour  $\gamma$  be defined by (6). Then, there exists a neighborhood of the set  $M_1$  such that the operator  $R_2(k, \lambda)$  given by*

$$(R_2(k, \lambda)f)(x, y) = \sum_{n \in J} \int_{\gamma} \cdots \int_{\gamma} \frac{e^{i\zeta x + iny}(Ff)(\zeta, n)}{\zeta^2 + (k+n)^2 - \lambda} d\zeta_1 \cdots d\zeta_m \quad (11)$$

*is well defined as bounded operator from  $L_{2,a}$  to  $H_{-a}^2$ . The  $B(L_{2,a}, H_{-a}^2)$ -valued function  $(k, \lambda) \mapsto R_2(k, \lambda)$  is analytic in this neighborhood. The estimate*

$$\|R_2(k(\tau), \lambda)\|_{B(L_{2,a}, L_{2,-a})} \leq C|\tau|^{-1}$$

*holds.*

*Proof.* If  $f \in L_{2,a}$  then the function  $(Ff)(\cdot, n)$  is square integrable on  $\gamma^m$ . By Lemma 2.4, the denominator in (11) never vanishes for  $(k, \lambda) \in M_1$ , and therefore in some neighborhood of  $M_1$ . So

$$|(\zeta^2 + (k+n)^2 - \lambda)^{-1} e^{i\zeta x + iny}| \leq C|e^{i\zeta x}|$$

where the constant does not depend on  $\zeta \in \gamma^m$  and on  $x$ ; the same is true for the second derivatives of  $(\zeta^2 + (k+n)^2 - \lambda)^{-1} e^{i\zeta x + iny}$  with respect to  $x, y$ . Hence  $R_2(k, \lambda) \in B(L_{2,a}, H_{-a}^2)$  by virtue of Lemma 2.2. Lemma 2.4 yields the estimation for the norm of  $R_2(k(\tau), \lambda)$ .  $\square$

Now, we construct an analytic extension of the resolvent of  $A(k)$ .



**Theorem 3.1.** *Let  $(k_0, \lambda_0) \in \mathbb{R}^{d+1}$  satisfy (2) and the set  $M_1$  be defined by formulas (3), (4), (5). Then, there exist a neighborhood  $M_0$  in  $\mathbb{C}^{d+1}$  of  $M_1$ , a real number  $a$  and an analytic  $B(L_{2,a}, H_{-a}^2)$ -valued function, say  $(k, \lambda) \mapsto R_A(k, \lambda)$ , defined in  $M_0$ , such that*

$$R_A(k, \lambda)f = (A(k) - \lambda)^{-1}f \quad \text{for } (k, \lambda) \in M_0, k \in \mathbb{R}^d, \operatorname{Im} \lambda > 0, f \in L_{2,a} \quad (12)$$

and

$$\|R_A(k(\tau), \lambda)\|_{B(L_{2,a}, L_{2,-a})} \leq C|\tau|^{-1}. \quad (13)$$

*Proof.* If  $\lambda_0 \leq 0$ , we can take  $R_A = R_1$  ( $R_1$  is constructed in Lemma 3.1; here  $J = \emptyset$  and  $a = 0$ ).

If  $\lambda_0 > 0$  then, we put  $R_A = R_1 + R_2$ , where  $R_1$ ,  $R_2$  and  $a$  are defined in Lemmas 3.1 and 3.2, and,  $M_0$  is the intersection of two corresponding neighborhoods of  $M_1$ . If  $f \in L_{2,a}$  then  $(Ff)(\cdot, n)$  is an analytic function in the domain  $\{\zeta : |\operatorname{Im} \zeta| < a\}$  and is uniformly bounded on  $\{\zeta : |\operatorname{Im} \zeta| \leq \eta\sqrt{m}\}$ . If  $\zeta \in \overline{\Gamma}^m$  where  $\Gamma$  is the open set between  $\mathbb{R}$  and  $\gamma$  (see Lemma 2.3), then  $\operatorname{Im} \zeta^2 \leq 0$  and therefore the integrand in (11) has no poles when  $\operatorname{Im} \lambda > 0$ . Hence the integral in right hand side of (10) for  $n \in J$  coincides with the corresponding integral in (11) due to Lemma 2.3, and (12) holds.

The estimate (13) is a simple corollary of the estimations of Lemmas 3.1 and 3.2.  $\square$

## 4 Invertibility of operators of type $(I + WR_A)$

**Lemma 4.1.** *Let  $W \in L_{\infty,b}$  for  $b > 2a > 0$ . Then, the operator of multiplication by  $W$  (we will denote it by the same letter) is*

- 1) *bounded as an operator from  $L_{2,-a}$  to  $L_{2,a}$ ;*
- 2) *compact as an operator from  $H_{-a}^2$  to  $L_{2,a}$ .*

*Proof.* The first assertion is evident. In order to prove the second it is enough to introduce functions

$$W_\rho(x, y) = \begin{cases} W(x, y), & |x| < \rho, \\ 0, & |x| > \rho, \end{cases}$$

and note that the multiplication by  $W_\rho$  is a compact operator from  $H_{-a}^2$  to  $L_{2,a}$  and

$$\|W - W_\rho\|_{B(L_{2,-a}, L_{2,a})} \rightarrow 0$$

when  $\rho \rightarrow \infty$ .  $\square$

The next lemma is a well known result from analytic Fredholm theory (see e.g. [5, 8]).

**Lemma 4.2.** *Let  $U$  be a domain in  $\mathbb{C}^p$ ,  $z_0 \in U$ . Let  $T(z)$  be an analytic function with the values in the set of compact operators in some Hilbert space  $\mathcal{H}$ . Then, there exists a neighborhood of the point  $z_0$  and a scalar analytic function  $h$  defined in this neighborhood such that*

$$(I + T(z))^{-1} \text{ exists if and only if } h(z) \neq 0.$$

Now we can establish the existence of the inverse operator for  $(I + WR_A)$ .

**Theorem 4.1.** *Let  $(k_0, \lambda_0)$  satisfy (2),  $R_A(k, \lambda)$  and  $a$  be defined as in Theorem 3.1. Let  $W(x, y, \lambda)$  be a function which belongs to  $L_{\infty, b}$ ,  $b > 2a$ , for all  $\lambda$ , and is analytic with respect to  $\lambda$  i.e.  $W \in \text{Hol}(\mathbb{C}, L_{\infty, b})$ . Then, there exists a number  $\varepsilon > 0$ , a neighborhood  $U$  in  $\mathbb{C}^{d+1}$  of the point  $(k_0, \lambda_0)$  containing the set  $B_\varepsilon(k_0) \times B_\varepsilon(\lambda_0)$ , and a scalar function  $h$  defined and analytic in  $U$ , having the following properties:*

$$\forall \lambda \in B_\varepsilon(\lambda_0) \quad \exists k \in B_\varepsilon(k_0) \quad \text{such that} \quad h(k, \lambda) \neq 0, \quad (14)$$

and for any  $(k, \lambda) \in U$  the operator  $(I + W(\lambda)R_A(k, \lambda))$  is invertible in  $L_{2, a}$  if and only if  $h(k, \lambda) \neq 0$ .

*Proof.* Due to Theorem 3.1 and Lemma 4.1 the operator  $W(\lambda)R_A(k(\tau), \lambda)$  is compact in  $L_{2, a}$  and satisfies the inequality

$$\|W(\lambda)R_A(k(\tau), \lambda)\|_{B(L_{2, a})} \leq C|\tau|^{-1} \quad \forall \lambda \in B_\varepsilon(\lambda_0).$$

Therefore, in the space  $B(L_{2, a})$  the operator  $(I + W(\lambda)R_A(k(\tau), \lambda))^{-1}$  exists for  $|\tau|$  large enough. The operator-valued function  $\lambda \mapsto W(\lambda)R_A(k, \lambda)$  is analytic in  $M_0$  where the set  $M_0$  is defined in Theorem 3.1. The analytic Fredholm alternative yields that for each  $\lambda \in B_\varepsilon(\lambda_0)$  one can find  $k \in B_\varepsilon(k_0)$  such that the operator  $(I + W(\lambda)R_A(k, \lambda))^{-1}$  exists. Now Lemma 4.2 with  $\mathcal{H} = L_{2, a}$ ,  $z = (k, \lambda)$ ,  $T(z) = WR_A$  completes the proof.  $\square$

## 5 The resolvent of the operator $H$

We can reduce the general case of operator (1) with a “metric”  $g$  to the case of “pure” Schrödinger operator due to the following lemma. This identity (for the totally periodic case) is known (see [1]). We include the proof for the convenience of the reader.

**Lemma 5.1.** *Let the operators  $H(k)$  and  $A(k)$  be defined by (7) and (9) respectively and the conditions of Theorem 1.1 be fulfilled with  $g_0 = 1$ . If  $u \in \tilde{H}^2$  then,*

$$(H(k) - \lambda) g^{-1/2} u = g^{1/2} (A(k) + W(\lambda) - \lambda) u,$$

where

$$W(\lambda) = \frac{1}{g} \left( \frac{\Delta g}{2} - \frac{|\nabla g|^2}{4g} + V + \lambda(g - 1) \right). \quad (15)$$

**Remark 5.1.** *If  $g \equiv 1$  then  $W(\lambda) \equiv V$ .*

*Proof.* It is enough to prove the equality

$$(i\nabla - (0, \bar{k}))^* g (i\nabla - (0, k)) (g^{-1/2} u) = g^{1/2} \left( A(k) + \frac{\Delta g}{2g} - \frac{|\nabla g|^2}{4g^2} \right) u. \quad (16)$$

We have

$$(i\nabla - (0, k)) (g^{-1/2} u) = i g^{-1/2} \nabla u - \frac{i}{2} g^{-3/2} \nabla g u - (0, k)(g^{-1/2} u).$$

Therefore the left hand side of (16) is equal to

$$\begin{aligned} & (i\nabla - (0, \bar{k}))^* \left( i g^{1/2} \nabla u - \frac{i}{2} g^{-1/2} \nabla g u - (0, k)(g^{1/2} u) \right) \\ &= -g^{1/2} \Delta u + \frac{1}{2} \operatorname{div}(g^{-1/2} \nabla g) u - i \langle k, \nabla_y (g^{1/2} \bar{u}) \rangle_{\mathbb{C}} \\ & \quad - i g^{1/2} \langle \nabla_y u, \bar{k} \rangle_{\mathbb{C}} + \frac{i}{2} g^{-1/2} \langle \nabla_y g, \bar{k} \rangle_{\mathbb{C}} u + k^2 g^{1/2} u \\ &= g^{1/2} \left( -\Delta_x u + (i\nabla_y - \bar{k})^* (i\nabla_y - k) u + \frac{1}{2} g^{-1/2} \operatorname{div}(g^{-1/2} \nabla g) u \right). \end{aligned}$$

□

In the following theorem, we describe the possibility of analytic extension of the resolvent of  $H(k)$ .

**Theorem 5.1.** *Let the conditions of Theorem 1.1 be fulfilled, the operator  $H(k)$  be defined by (7) and  $(k_0, \lambda_0) \in \mathbb{R}^{d+1}$  satisfy (2). Then, there exists numbers  $a \geq 0$ ,  $\varepsilon > 0$ , a neighborhood  $U$  of  $(k_0, \lambda_0)$  in  $\mathbb{C}^{d+1}$  containing the set  $B_\varepsilon(k_0) \times B_\varepsilon(\lambda_0)$ , a function  $h \in \operatorname{Hol}(U)$  satisfying (14) and an operator-valued function  $R_H(k, \lambda)$  having the following properties:*

1.  $R_H$  is defined on the set  $\{(k, \lambda) \in U : h(k, \lambda) \neq 0\}$  and is analytic there;

2.  $R_H(k, \lambda) \in B(L_{2,a}, L_{2,-a})$ ;

3. for  $(k, \lambda) \in U$ ,  $k \in \mathbb{R}^d$ ,  $\text{Im } \lambda > 0$ ,  $f \in L_{2,a}$

$$R_H(k, \lambda)f = (H(k) - \lambda)^{-1}f. \quad (17)$$

**Remark 5.2.** *It will be seen from the proof that  $R_H(k, \lambda) \in B(L_{2,a}, H_{-a}^2)$  though we do not need this fact.*

*Proof.* One can see that  $\nabla g \in L_{\infty,b} \forall b$ . So if we define  $W(\lambda)$  by (15) then  $W(\lambda) \in L_{\infty,b} \forall b$  too, and we can apply Theorem 4.1. Let  $U$ ,  $h$ ,  $a$  and  $R_A$  be as in this theorem. On the set where  $h(k, \lambda) \neq 0$ , we put

$$R_H(k, \lambda) = g^{-1/2} R_A(k, \lambda) (I + W(\lambda) R_A(k, \lambda))^{-1} g^{-1/2}.$$

By Theorem 4.1,  $R_H(k, \lambda) \in B(L_{2,a}, H_{-a}^2)$ . Let  $f \in L_{2,a}$ . Then

$$(I + W(\lambda) R_A(k, \lambda))^{-1} g^{-1/2} f \in L_{2,a} \quad (18)$$

and we can apply Lemma 5.1 to the function

$$u = R_A(k, \lambda) (I + W(\lambda) R_A(k, \lambda))^{-1} g^{-1/2} f \in H_{-a}^2, \quad (19)$$

so

$$(H(k) - \lambda) R_H(k, \lambda) f = g^{1/2} (A(k) + W(\lambda) - \lambda) u. \quad (20)$$

For real  $k$  and non real  $\lambda$  with we have by (12) and (18)

$$(A(k) - \lambda) u = (I + W(\lambda) R_A(k, \lambda))^{-1} g^{-1/2} f,$$

hence by (19)

$$(A(k) + W(\lambda) - \lambda) u = g^{-1/2} f,$$

and finally by (20)

$$(H(k) - \lambda) R_H(k, \lambda) f = f.$$

The operators  $(H(k) - \lambda)^{-1}$  and  $(A(k) - \lambda)^{-1}$  are well defined in  $L_2(\Omega)$  because  $\text{Im } \lambda > 0$ , so  $R_H(k, \lambda) f \in L_2(\Omega)$  and  $R_H(k, \lambda) f = (H(k) - \lambda)^{-1} f$ .  $\square$

## 6 One fact from the theory of functions

**Lemma 6.1.** *Let  $V$  be an open subset of  $\mathbb{R}^d$ . Let  $f$  be a real-analytic function in a rectangle  $(c, d) \times V$ . Let  $\Lambda$  be a subset of  $V$  of measure zero,  $\text{mes } \Lambda = 0$ . Then,*

$$\text{mes}\{k \in (c, d) : f(k, \lambda) = 0, \partial_k f(k, \lambda) \neq 0 \text{ for some } \lambda \in \Lambda\} = 0. \quad (21)$$

*Proof.* The Implicit Function Theorem implies that, for any point  $(k_0, \lambda_0)$  such that  $f(k_0, \lambda_0) = 0 \neq \partial_k f(k_0, \lambda_0)$ , we can find rational numbers  $\tilde{k}_0$ ,  $\tilde{r}_0 > 0$ , a vector  $\tilde{\lambda}_0 = (\tilde{\lambda}_0^1, \dots, \tilde{\lambda}_0^d)$  with rational coordinates, and a cube  $C_{\tilde{r}_0}(\tilde{k}_0, \tilde{\lambda}_0)$  where

$$(k_0, \lambda_0) \in C_{\tilde{r}_0}(\tilde{k}_0, \tilde{\lambda}_0) = (\tilde{k}_0 - \tilde{r}_0, \tilde{k}_0 + \tilde{r}_0) \times C_{\tilde{r}_0}(\tilde{\lambda}_0) \subset (c, d) \times V$$

$$C_{\tilde{r}_0}(\tilde{\lambda}_0) = (\tilde{\lambda}_0^1 - \tilde{r}_0, \tilde{\lambda}_0^1 + \tilde{r}_0) \times \dots \times (\tilde{\lambda}_0^d - \tilde{r}_0, \tilde{\lambda}_0^d + \tilde{r}_0)$$

and a real analytic function  $\theta : C_{\tilde{r}_0}(\tilde{\lambda}_0) \rightarrow (\tilde{k}_0 - \tilde{r}_0, \tilde{k}_0 + \tilde{r}_0)$  such that

1.  $\theta(\lambda_0) = k_0$ ;
2.  $f(k, \lambda) = 0 \Leftrightarrow \theta(\lambda) = k$  if  $(k, \lambda) \in C_{\tilde{r}_0}(\tilde{k}_0, \tilde{\lambda}_0)$ .

Therefore,

$$\begin{aligned} \text{mes}\{k : (k, \lambda) \in C_{\tilde{r}_0}(\tilde{k}_0, \tilde{\lambda}_0), f(k, \lambda) = 0 \text{ for some } \lambda \in \Lambda\} \\ \leq \text{mes } \theta(\Lambda \cap C_{\tilde{r}_0}(\tilde{\lambda}_0)) = 0. \end{aligned}$$

The set

$$\{(k, \lambda) : f(k, \lambda) = 0, \partial_k f(k, \lambda) \neq 0\}$$

can be covered by a countable number of cubes  $C_{\tilde{r}_i}(\tilde{k}_i, \tilde{\lambda}_i)$  constructed as above, say  $(C_{\tilde{r}_i}(\tilde{k}_i, \tilde{\lambda}_i))_{i \in \mathbb{N}}$ ; hence, the measure of the set under consideration in (21) is also equal to zero as

$$\begin{aligned} \{k \in (c, d) : f(k, \lambda) = 0, \partial_k f(k, \lambda) \neq 0 \text{ for some } \lambda \in \Lambda\} \\ \subset \bigcup_{i \in \mathbb{N}} \text{mes}\{k : (k, \lambda) \in C_{\tilde{r}_i}(\tilde{k}_i, \tilde{\lambda}_i), f(k, \lambda) = 0 \text{ for some } \lambda \in \Lambda\} \end{aligned}$$

□

Lemma 6.1 has a multidimensional analogue.

**Lemma 6.2.** *Let  $U$  be an open subset of  $\mathbb{R}^d$ , and  $V$  be an open subset of  $\mathbb{R}^{d'}$ . Let  $f$  be a real-analytic function on the set  $U \times V$ , and pick  $\Lambda \subset V$  such that  $\text{mes } \Lambda = 0$ . For  $k \in U$ , we write  $k = (k_1, k')$  where  $k_1$  is real and  $k' \in \mathbb{R}^{d-1}$ . Then,*

$$\text{mes}\{k \in U : f(k, \lambda) = 0, \partial_{k_1} f(k, \lambda) \neq 0 \text{ for some } \lambda \in \Lambda\} = 0. \quad (22)$$

*Proof.* Cover  $U$  with countably many sets of the form  $(a, b) \times \tilde{U}$  i.e.

$$U = \bigcup_{i \in \mathbb{N}} (a_i, b_i) \times \tilde{U}_i.$$

For  $i \in \mathbb{N}$ , one has

$$\begin{aligned} & \{k \in (a_i, b_i) \times \tilde{U}_i : f(k, \lambda) = 0, \partial_{k_1} f(k, \lambda) \neq 0 \text{ for some } \lambda \in \Lambda\} \\ & \subset \{k_1 \in (a_i, b_i) : f(k_1, k', \lambda) = 0, \partial_{k_1} f(k_1, k', \lambda) \neq 0 \\ & \quad \text{for some } (k', \lambda) \in \tilde{U}_i \times \Lambda\} \times \tilde{U}_i \end{aligned} \quad (23)$$

By Lemma 6.1, the set in the right hand side of equation (23) has measure 0 (as  $\tilde{U}_i \times \Lambda$  has measure zero in  $\mathbb{R}^{d+d'-1}$ ). As

$$\begin{aligned} & \{k \in U : f(k, \lambda) = 0, \partial_{k_1} f(k, \lambda) \neq 0 \text{ for some } \lambda \in \Lambda\} \\ & = \bigcup_{i \in \mathbb{N}} \{k \in (a_i, b_i) \times \tilde{U}_i : f(k, \lambda) = 0, \partial_{k_1} f(k, \lambda) \neq 0 \text{ for some } \lambda \in \Lambda\}, \end{aligned}$$

(22) holds.  $\square$

**Theorem 6.1.** *Let  $U$  be a region in  $\mathbb{R}^d$ ,  $\Lambda$  be a subset of an interval  $(a, b)$ ,  $\text{mes } \Lambda = 0$ . Let  $h$  be a real-analytic function defined on the set  $U \times (a, b)$  and suppose that*

$$\forall \lambda \in \Lambda \quad \exists k \in U \quad \text{such that} \quad h(k, \lambda) \neq 0.$$

*Then*

$$\text{mes}\{k \in U : h(k, \lambda) = 0 \text{ for some } \lambda \in \Lambda\} = 0.$$

*Proof.* For any  $k \in U$  and  $\lambda \in \Lambda$ , by assumption, there exists a multi-index  $\alpha \in \mathbb{Z}_+^d$  such that  $\partial_k^\alpha h(k, \lambda) \neq 0$ . Therefore

$$\begin{aligned} & \{k \in U : h(k, \lambda) = 0 \text{ for some } \lambda \in \Lambda\} \\ & \subset \bigcup_{j=1}^d \bigcup_{\alpha \in \mathbb{Z}_+^d} \{k \in U : \partial_k^\alpha h(k, \lambda) = 0, \partial_{k_j} \partial_k^\alpha h(k, \lambda) \neq 0 \text{ for some } \lambda \in \Lambda\}. \end{aligned}$$

Reference to Lemma 6.2 then completes the proof of Theorem 6.1.  $\square$

## 7 The proof of Theorem 1.1

The following lemma is well known (see for example [9]).

**Lemma 7.1.** *Let  $B$  be a self-adjoint operator in  $L_2(\Omega)$ . Suppose that  $R_B$  is an analytic function defined in a complex neighborhood of an interval  $[\alpha, \beta]$  except a finite number of points  $\{\mu_1, \dots, \mu_N\}$ , the values of  $R_B$  are in  $B(L_{2,-b}, L_{2,b})$ ,  $b > 0$ , and that*

$$R_B(\lambda)\varphi = (B - \lambda)^{-1}\varphi \quad \text{if } \text{Im } \lambda > 0, \varphi \in L_{2,b}.$$

Then, the spectrum of  $B$  in the set  $[\alpha, \beta] \setminus \{\mu_1, \dots, \mu_N\}$  is absolutely continuous. If  $\Lambda \subset [\alpha, \beta]$ ,  $\text{mes } \Lambda = 0$  and  $\mu_j \notin \Lambda$ ,  $j = 1, \dots, N$ , then  $E_B(\Lambda) = 0$ , where  $E_B$  is the spectral projector of  $B$ .

*Proof of Theorem 1.1.* By Theorem 5.1, the set of all points  $(k, \lambda) \in \mathbb{R}^{d+1}$  satisfying (2) can be represented as the union

$$\bigcup_{j=1}^{\infty} B_{\varepsilon_j}(k_j) \times B_{\varepsilon_j}(\lambda_j),$$

where, for every  $j$  there exist

- a number  $a_j \geq 0$ ,
- an analytic scalar function  $h_j$  defined in a complex neighborhood of  $\overline{B_{\varepsilon_j}(k_j) \times B_{\varepsilon_j}(\lambda_j)}$  with the property
$$\forall \lambda \in B_{\varepsilon_j}(\lambda_j) \quad \exists k \in B_{\varepsilon_j}(k_j) \quad \text{such that} \quad h_j(k, \lambda) \neq 0,$$
- an analytic  $B(L_{2,a_j}, L_{2,-a_j})$ -valued function  $R_H^{(j)}$  defined on the set where  $h_j(k, \lambda) \neq 0$  and satisfying (17).

Now, pick  $\Lambda \subset \mathbb{R}$  such that  $\text{mes } \Lambda = 0$ . Set

$$\begin{aligned} K_0 &= \{k \in [0, 1]^d : (k + n)^2 = \lambda \text{ for some } n \in \mathbb{Z}^d, \lambda \in \Lambda\}, \\ K_1 &= \{k \in [0, 1]^d : h_j(k, \lambda) = 0 \text{ for some } j \in \mathbb{N}, \lambda \in \Lambda\}. \end{aligned}$$

Thanks to Theorem 6.1, we know

$$\text{mes } K_0 = \text{mes } K_1 = 0. \tag{24}$$

For  $k \notin K_0$  denote

$$\Lambda_j(k) = \{\lambda \in \Lambda : (k, \lambda) \in B_{\varepsilon_j}(k_j) \times B_{\varepsilon_j}(\lambda_j)\}.$$

It is clear that  $\Lambda_j(k) \subset (\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j)$ ,  $\text{mes } \Lambda_j(k) = 0$  and

$$\Lambda = \bigcup_{j=1}^{\infty} \Lambda_j(k) \quad \forall k \notin K_0.$$

If  $k \notin (K_0 \cup K_1)$  and  $\Lambda_j(k) \neq \emptyset$  then  $h_j(k, \lambda) \neq 0$  for  $\lambda \in \Lambda_j(k)$  and  $h_j(k, \lambda)$  as function of  $\lambda \in [\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j]$  has at most a finite number of zeros. So we can apply Lemma 7.1, therefore

$$E_{H(k)}(\Lambda_j(k)) = 0 \quad \forall j \quad \Rightarrow \quad E_{H(k)}(\Lambda) = 0.$$

Finally

$$E_H(\Lambda) = \int_{[0,1]^d} E_{H(k)}(\Lambda) dk = \int_{[0,1]^d \setminus K_0 \setminus K_1} E_{H(k)}(\Lambda) dk = 0$$

by virtue of (24). So we proved that the spectral resolution of  $H$  vanishes on any set of Lebesgue measure 0, which means, by definition, that the spectrum of the operator  $H$  is purely absolutely continuous.

## References

- [1] M. Sh. Birman, T. A. Suslina, *Periodic magnetic Hamiltonian with variable metrics. Problem of absolute continuity*, Algebra i Analiz, vol. 11 (1999), 2, p. 1–40 (Russian). English translation in St. Petersburg Math. J. 11 (2000), no. 2, 203–232.
- [2] D. M. Eidus, *Some boundary value problems in infinite domains*, Izvestiya AN SSSR 27 (1963), 5, p. 1055–1080 (Russian).
- [3] R. Frank, R. Shterenberg *On the scattering theory of the Laplacian with a periodic boundary condition. Additional channels of scattering*, Preprint.
- [4] Ch. Gérard, *Resonance theory in atomic-surface scattering*, Comm. Math. Phys., 126 (1989), 263–290.
- [5] T. Kato. *Perturbation Theory for Linear Operators*. Springer Verlag, Berlin, 1980.
- [6] P. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser, Basel (1993).
- [7] O. A. Ladyzhenskaya, *On the principle of limit amplitude*, Uspekhi Mat. Nauk, XII N3, 75 (1957), p. 161–164 (Russian).
- [8] M. Reed and B. Simon. *Methods of Modern Mathematical Physics, Vol I: Functional Analysis*. Academic Press, New-York, 1980.
- [9] D. R. Yafaev, *Mathematical scattering theory*, Providence, Rhode Island, AMS, 1992.
- [10] M. Zworski, *Quantum resonances and partial differential equations*, Proceedings of the ICM, Beijing 2002, vol. 3, 243–254.