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► **To cite this version:**

| Arnaud Beauville. On Cremona transformations of prime order. 2004. hal-00001113v3

**HAL Id: hal-00001113**

**<https://hal.science/hal-00001113v3>**

Preprint submitted on 29 Mar 2004 (v3), last revised 21 Sep 2004 (v4)

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# On Cremona transformations of prime order

Arnaud BEAUVILLE

## 1. The result

The *Cremona group*  $\text{Bir}(\mathbb{P}^2)$  is the group of birational transformations of  $\mathbb{P}_{\mathbb{C}}^2$ , or equivalently the group of  $\mathbb{C}$ -automorphisms of the field  $\mathbb{C}(x, y)$ . There is an extensive classical literature about this group, in particular about its finite subgroups; we refer to the introduction of [dF] for a list of references.

The classification of conjugacy classes of elements of prime order  $p$  in  $\text{Bir}(\mathbb{P}^2)$  has been given a modern treatment in [B-B] for  $p = 2$  and in [dF] for  $p \geq 3$ . Let us recall the main results. The linear transformations of given order are contained in a unique conjugacy class. Apart from these there are three continuous families of conjugacy classes of involutions (the famous de Jonquières, Bertini and Geiser involutions), two continuous families of conjugacy classes of order 3, and at least one such family of order 5. Every transformation of prime order  $\geq 7$  is conjugate to a linear automorphism.

The following result completes the classification:

**Theorem.**— *There is a unique conjugacy class of order 5 in the Cremona group apart from the class containing linear automorphisms.*

Most of the work has been done already in [dF], where it is proved that there can be only one more conjugacy class of order 5, namely that of an automorphism  $\sigma$  of order 5 of the Del Pezzo surface  $S_5 \subset \mathbb{P}^5$  (through a birational identification  $\pi : S \xrightarrow{\sim} \mathbb{P}^2$ ). What remains to be proved is that  $\sigma$  is conjugate to a linear automorphism.

In the first version of this note I proposed a proof based on a functional equation which was solved using the “five-terms relation” of the dilogarithm. Unfortunately a sign error in a computation made the argument inconclusive. It can certainly be fixed, perhaps at the cost of a more involved computation. But in the meantime J. Blanc showed me another proof, which is so simple that it definitely settles the question. I will explain below a geometric version of his proof.

## 2. The proof

Let us first recall why linear automorphisms of the same (finite) order are conjugate:

**Proposition.**— *Let  $n$  be a positive integer. Two linear automorphisms of order  $n$  are conjugate in  $\text{Bir}(\mathbb{P}^2)$ .*

*Proof:* Let  $T = \mathbb{C}^* \times \mathbb{C}^*$  be the standard maximal torus of  $\text{PGL}_3$ . We can view  $T$  as a Zariski open subset of  $\mathbb{P}^2$ , so its automorphism group  $\text{GL}(2, \mathbb{Z})$  embeds naturally in  $\text{Bir}(\mathbb{P}^2)$ : an element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  of  $\text{GL}(2, \mathbb{Z})$  corresponds to the Cremona transformation  $(x, y) \mapsto (x^a y^b, x^c y^d)$ . The image of  $\text{GL}(2, \mathbb{Z})$  in  $\text{Bir}(\mathbb{P}^2)$  normalizes  $T$ , and its action on  $T$  by conjugation is just the original action of  $\text{GL}(2, \mathbb{Z})$  on  $T$ .

We want to prove that two elements of order  $n$  of  $T$  are conjugate under this action. The kernel of the multiplication by  $n$  in  $T$  is  $(\mathbb{Z}/n)^2$ ; any element of order  $n$  in the

ccsd-00001113, version 3 - 29 Mar 2004

$\mathbb{Z}/n$ -module  $(\mathbb{Z}/n)^2$  is part of a basis, which implies that it is conjugate to  $(1, 0)$  under the action of  $\mathrm{SL}(2, \mathbb{Z}/n)$ . Since the natural map  $\mathrm{SL}(2, \mathbb{Z}) \rightarrow \mathrm{SL}(2, \mathbb{Z}/n)$  is surjective, this proves our result. ■

*Proof of the theorem* (according to J. Blanc):

Let  $V$  be a 6-dimensional vector space, and  $S \subset \mathbb{P}(V)$  a Del Pezzo surface of degree 5. It admits an automorphism  $\sigma$  of order 5, unique up to conjugation, which comes from a linear automorphism  $s$  of  $V$ .

The automorphism  $\sigma$  has at least one fixed point  $p$  (otherwise the smooth projective surface  $S' := S/\langle\sigma\rangle$  would have  $\chi(\mathcal{O}_{S'}) = \frac{1}{5}$ ). The tangent plane  $\mathbb{P}(T_p)$  to  $S$  at  $p$  corresponds to a 3-dimensional vector subspace  $T_p \subset V$ . Since  $s$  preserves  $T_p$ , it induces a linear automorphism  $\bar{\sigma}$  of  $\mathbb{P}^2 = \mathbb{P}(V/T_p)$ ; we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & S \\ | & & | \\ \pi | & & | \pi \\ \vee & & \vee \\ \mathbb{P}^2 & \xrightarrow{\bar{\sigma}} & \mathbb{P}^2 \end{array}$$

where  $\pi : S \dashrightarrow \mathbb{P}^2$  is the projection from  $\mathbb{P}(T_p)$ .

We claim that  $\pi$  is birational, so that it conjugates  $\sigma$  to the linear automorphism  $\bar{\sigma}$  of  $\mathbb{P}^2$ . Identify  $S$  with  $\mathbb{P}^2$  blown up at 4 points  $p_1, \dots, p_4$  in general position. The map  $S \dashrightarrow \mathbb{P}^2$  is given by the linear system  $|\mathcal{C}|$  of cubics passing through  $p_1, \dots, p_4$  and singular at  $p$ . Note that the 5 points  $p_1, \dots, p_4, p$  are again in general position (that is, no 3 of them are collinear). Indeed this means that  $p$  does not belong to any line in  $S$ . But  $S$  contains 10 lines, on which  $\langle\sigma\rangle$  acts with 2 orbits; if  $p$  lies on a line it must lie on 5 of them, which is impossible.

It follows that the linear system  $|\mathcal{C}|$  defines a morphism from the surface  $S_p$  obtained by blowing up  $S$  at  $p$  onto  $\mathbb{P}^2$ ; since the curves of  $|\mathcal{C}|$  have self-intersection 1 on  $S_p$ , this morphism is birational (in fact it blows down the lines  $\langle p, p_1 \rangle, \dots, \langle p, p_4 \rangle$  and the conic passing through  $p_1, \dots, p_4, p$ ). This achieves the proof of the theorem. ■

## REFERENCES

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