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On Cremona transformations of prime order

Arnaud Beauville

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1. The result

The *Cremona group* $\mathrm{Bir}(\mathbb{P}^2)$ is the group of birational transformations of $\mathbb{P}^2_{\mathbb{C}}$, or equivalently the group of $\mathbb{C}$-automorphisms of the field $\mathbb{C}(x,y)$. There is an extensive classical literature about this group, in particular about its finite subgroups; we refer to the introduction of [dF] for a list of references.

The classification of conjugacy classes of transformations of prime order p has been given a modern treatment in [B-B] for $p=2$ and in [dF] for $p \geq 3$. Let us recall the main results. The linear transformations of given order are contained in a unique conjugacy class. Apart from these there are 3 continuous families of conjugacy classes of involutions (the famous de Jonquières, Bertini and Geiser involutions), and 3 families of transformations of order 3. Every transformation of prime order ≥ 7 is conjugate to a linear automorphism.

We prove in this note the following result, which completes the classification:

Theorem .– *Every Cremona transformation of order* 5 *is conjugate in* $\mathrm{Bir}(\mathbb{P}^2)$ *to a linear automorphism.*

Most of the work has been done already in [dF], where it is proved that such a transformation is conjugate either to a linear automorphism or to the transformation $\gamma : (x,y) \mapsto (1-y, 1-\frac{y}{x})$. What remains to prove is that γ is conjugate to a linear automorphism.

Fix a primitive 5-th root of unity ζ. The problem is readily translated into the existence of a rational function $\mathrm{F}(x,y)$ satisfying the functional equation

$$\mathrm{F}(\zeta x, y)\mathrm{F}(\zeta^{-1}x, y) + \mathrm{F}(x,y) = 1 \qquad (\star)$$

and such that the rational map $(x,y) \mapsto (\mathrm{F}(x,y), 1-\mathrm{F}(\zeta x, y))$ is birational. At this point Don Zagier explained to me a very simple way to construct solutions to $(\star)$; it turns out that essentially the most economical choice satisfies indeed the second condition and thus leads to a proof of the theorem.

2. The proof

Let us first recall why linear automorphisms of the same (finite) order are conjugate:

Proposition .– *Let* n *be a positive integer. Two linear automorphisms of order* n *are conjugate in* $\mathrm{Bir}(\mathbb{P}^2)$.

Proof: Let $\mathrm{T} = \mathbb{C}^* \times \mathbb{C}^*$ be the standard maximal torus of PGL_3. We can view T as a Zariski open subset of $\mathbb{P}^2$, so its automorphism group $\mathrm{GL}(2,\mathbb{Z})$ embeds naturally in $\mathrm{Bir}(\mathbb{P}^2)$: an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{GL}(2,\mathbb{Z})$ corresponds to the Cremona transformation $(x,y) \mapsto (x^a y^b, x^c y^d)$. The image of $\mathrm{GL}(2,\mathbb{Z})$ in $\mathrm{Bir}(\mathbb{P}^2)$ normalizes T, and its action on T by conjugation is just the original action of $\mathrm{GL}(2,\mathbb{Z})$ on T.

We want to prove that two elements of order n of T are conjugate under this action. The kernel of the multiplication by n in T is $(\mathbb{Z}/n)^2$; any element of order n in the $\mathbb{Z}/n$-module $(\mathbb{Z}/n)^2$ is part of a basis, which implies that it is conjugate to $(1,0)$ under the action of $\mathrm{SL}(2,\mathbb{Z}/n)$. Since the natural map $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/n)$ is surjective, this proves our result. ■

Proof of the theorem

We want to prove that $\gamma = h \circ \ell \circ h^{-1}$, with $h \in \mathrm{Bir}(\mathbb{P}^2)$ and $\ell \in \mathrm{PGL}_3$. By the Proposition we can take for ℓ the map $(x,y) \mapsto (\zeta x, y)$, where ζ is a primitive 5-th root of unity. Write $h(x,y) = (\mathrm{F}(x,y), \mathrm{G}(x,y))$. The equality $h = \gamma \circ h \circ \ell^{-1}$ becomes

$$\mathrm{F}(x,y) = 1 - \mathrm{G}(\zeta^{-1}x, y) \qquad \mathrm{G}(x,y) = 1 - \frac{\mathrm{G}(\zeta^{-1}x,y)}{\mathrm{F}(\zeta^{-1}x,y)} \ .$$

The first equality gives $\mathrm{G}(x,y) = 1 - \mathrm{F}(\zeta x, y)$, and the second becomes

$$\mathrm{F}(\zeta x, y)\mathrm{F}(\zeta^{-1}x, y) + \mathrm{F}(x,y) = 1 \ . \qquad (\star)$$

Conversely, if F satisfies $(\star)$ and the map $h : (x,y) \mapsto (\mathrm{F}(x,y), 1 - \mathrm{F}(\zeta x, y))$ is birational, we have $\gamma = h \circ \ell \circ h^{-1}$.

The following trick, which was explained to me by Don Zagier, allows to solve equation $(\star)$. We will use for the cross-ratio of distinct elements $x_1, \ldots, x_4$ in a field K the normalization

$$[x_1 : \ldots : x_4] = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_3)(x_2 - x_4)} \ .$$

Lemma.– *Let* K *be a field, and* σ *an automorphism of order* 5 *of* K. *Let* t *be an element of* K *with* $\sigma(t) \neq t$. *The cross-ratio* $r = [\sigma(t) : \ldots : \sigma^4(t)]$ *satisfies*

$$\sigma(r)\sigma^{-1}(r) + r = 1 \ .$$

Proof: Let $t_0, \ldots, t_4$ be distinct elements of $\mathbb{P}^1(\mathrm{K})$. The cross-ratios

$$r_i = [t_{i+1} : t_{i+2} : t_{i+3} : t_{i+4}] \ ,$$

where the index i runs in $\mathbb{Z}/5$, satisfy

$$r_{i-1}r_{i+1} + r_i = 1 \quad \text{for each } i \ ;$$

this is easily seen either by direct computation, or by using the PGL_2-invariance of the cross-ratio to reduce to the case $(t_0, \ldots, t_4) = (\lambda, 0, \infty, 1, \mu)$.

Coming back to the statement of the lemma we put $t_i = \sigma^i(t)$; then $r_i = \sigma^i(r)$, hence the result. ∎

We apply this lemma to the field $\mathbb{C}(x, y)$, with the automorphism σ which fixes y and maps x to ζx. Starting with any rational function $t(x, y)$ with $t(x, y) \neq t(\zeta x, y)$, the cross-ratio $\mathrm{F}(x, y = [t(\zeta x, y : \ldots : t(\zeta^4 x, y))]$ is a solution to our equation $(\star)$; we want moreover the map $h : (x, y) \mapsto (\mathrm{F}(x, y), 1 - \mathrm{F}(\zeta x, y))$ to be birational. Observe that this excludes functions t of the form $x^p a(y) + x^q b(y)$, because for that choice h factors through the map $(x, y) \mapsto x^{q-p} b(y) a(y)^{-1}$. This leads us to try $t(x, y) = x + x^2 + x^3 y$; putting $\omega = \zeta + \zeta^{-1}$ we find:

$$\mathrm{F}(x, y) = \omega \, \frac{1 - x - x^2(y-1) + 2x^3 y - x^4 y^2}{1 - x - \omega^2 x^2 (y-1) + \omega x^3 y - \omega^2 x^4 y^2} \ .$$

We want to invert the system $\mathrm{F}(x, y) = u$, $1 - \mathrm{F}(\zeta x, y) = v$. Together with the functional equation $(\star)$ for F it implies

$$\mathrm{F}(\zeta^i x, y) = u_i \quad (0 \leq i \leq 3), \quad \text{with } u_0 = u \ , \ u_1 = 1 - v \ , \ u_2 = \frac{v}{u} \ , \ u_3 = \frac{u - v}{u(1-v)} \ ;$$

that is, explicitely:

$$x\,\zeta^i(\omega^{-1}u_i - 1) + x^2(y-1)\,\zeta^{2i}(\omega u_i - 1) + x^3 y\,\zeta^{3i}(2 - u_i) + x^4 y^2\,\zeta^{4i}(\omega u_i - 1) = \omega^{-1} u_i - 1 \ .$$

This gives a system of 4 linear equations in $x, x^2(y-1), x^3 y, x^4 y^2$. If the corresponding determinant is nonzero we can express x and $x^3 y$, and thus also y, as rational functions of (u, v), so that h is birational.

Thus it remains to prove that the determinant of the system

$$\det \begin{pmatrix} \omega^{-1}u_0 - 1 & \omega u_0 - 1 & 2 - u_0 & \omega u_0 - 1 \\ \zeta(\omega^{-1}u_1 - 1) & \zeta^2(\omega u_1 - 1) & \zeta^3(2 - u_1) & \zeta^4(\omega u_1 - 1) \\ \zeta^2(\omega^{-1}u_2 - 1) & \zeta^4(\omega u_2 - 1) & \zeta(2 - u_2) & \zeta^3(\omega u_2 - 1) \\ \zeta^3(\omega^{-1}u_3 - 1) & \zeta(\omega u_3 - 1) & \zeta^4(2 - u_3) & \zeta^2(\omega u_3 - 1) \end{pmatrix}$$

is nonzero. It is enough to prove that it does not vanish for particular values of u, v such that $u_0, \ldots, u_3$ are finite. We choose $u = \omega^{-1}$, $v = -\omega$. This gives $u_0 = u_1 = \omega^{-1}$, $u_2 = \omega - 1$, $u_3 = 3\omega - 1$; our determinant becomes:

$$\det \begin{pmatrix} \omega^{-1} & 0 & \omega^2 & 0 \\ \zeta\omega^{-1} & 0 & \zeta^3\omega^2 & 0 \\ -\zeta^2\omega^{-1} & -2\zeta^4\omega & \zeta(3 - \omega) & -2\zeta^3\omega \\ \zeta^3\omega^2 & \zeta(2 - 4\omega) & 3\zeta^4\omega^2 & \zeta^2(2 - 4\omega) \end{pmatrix}$$

$$= -\det\begin{pmatrix} \omega^{-1} & \omega^2 \\ \zeta\omega^{-1} & \zeta^3\omega^2 \end{pmatrix} \det\begin{pmatrix} -2\zeta^4\omega & -2\zeta^3\omega \\ \zeta(2-4\omega) & \zeta^2(2-4\omega) \end{pmatrix}$$

$$= 2\zeta^2\omega^2(4\omega-2)(\omega^2-4) \neq 0$$

which proves the theorem. ∎

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