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The inverse crime

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Abstract: The inverse crime occurs when the same (or very nearly the same) theoretical ingredients are employed to synthesize as well as to invert data in an inverse problem. This act has been qualified as trivial and therefore to be avoided by Colton and Kress [1]. Their judgement is critically examined herein.

Keywords: inverse scattering problems; boundary identification

Abbreviated title: The inverse crime

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1 Introduction

Solving inverse problems related to some physical application (e.g., finding the constitutive constants of a material from the reflectivity [2]), for a N_p-dimensional vector $\mathbf{p}$ of parameters $\{p_1, p_1, ..., p_{N_p}\}$, requires matching a theoretical model (parametrized by a vector of variables of the same nature as the entries of $\mathbf{p}$) of some observable (reflectivity in the above example) to N_d measurements of the latter embodied in the data vector $\mathbf{d} = \{d_1, d_1, ..., d_{N_d}\}$.

Often, tests of the exact or approximate theoretical models employed in inversion schemes are made with *synthetic data*. Generating the latter also requires a theoretical model, which, mathematically speaking, can be identical to, or different from, the one employed in the inversion scheme. In [1], which serves as a reference to many workers in the field of inverse scattering problems, the authors coin the expression "inverse crime" to denote the act of employing the same model to generate, as well as to invert, synthetic data. Moreover, they warn against committing the inverse crime, "in order to avoid trivial inversion" and go on to state: "it is crucial that the synthetic data be obtained by a forward solver which has no connection to the inverse solver".

These assertions raise the following questions: i) what does the term "no connection" mean? ii) provided a definition can be provided, what kind of reconstructions of the unknown parameters can one obtain when there is "no connection" between the forward and inverse solvers? iii) should the inverse crime always be avoided? and iv) are inverse crime inversions always trivial?

These questions are difficult to address when N_p and/or N_d are larger than one. However, the case N_p=N_d=1 is interesting in its own right, and may furnish insight as to what may occur in inverse problems involving multidimensional $\mathbf{p}$ and $\mathbf{d}$.

2 Representations of the forward and inverse solvers (predictor and estimator)

The inverse problem studied in this investigation is to determine from $p := p_1 \in \mathbb{R}$ from $d := d_1 \in \mathbb{C}$. What in [1] is called the "forward solver" is here termed the *predictor*, i.e., it involves predicting the response of a physical system with the help of some mathematical model incorporating the parameter one wishes to recover. We denote the predictor by $\Phi(\varphi)$, wherein φ is a variable, of the same nature as p, which can take on the value

p. What in [1] is called "inverse solver" is here termed the *estimator*, i.e., it involves estimating the response of some physical system with the help of the same (as previously) or another mathematical model incorporating a variable usually having the same physical and mathematical attributes as the parameter one wishes to recover. We call this $E(\varepsilon)$, wherein ε is the variable analogous to φ. Thus, for a single experiment, we synthesize the single piece of data by $d = \Phi(p)$ and seek to recover p with the help of the estimator $E(\varepsilon)$, wherein ε varies over a certain range, hopefully including p.

In fact, we do this for a continuum of experiments involving the single pieces of data $d = \Phi(p)$, wherein φ varies over the range

$$1 > \varphi > 0 \ . \tag{1}$$

We therefore expect ε to lie in the same range

$$1 > \varepsilon > 0 \ , \tag{2}$$

and expand the predictor and estimator in the following series :

$$\Phi^{(N)}(\varphi) = \sum_{n=0}^{N} b_n f_n(\varphi) \quad , \quad E^{(M)}(\varepsilon) = \sum_{m=0}^{M} a_m f_m(\varepsilon) \ , \tag{3}$$

(wherein the $f_m(\zeta)$ are complex, continuous functions of the real variable ζ satisfying $1 > \zeta > 0$). In order to account for the inverse crime in the easiest manner, it is most convenient to take

$$b_m = a_m \quad ; \quad m = 0, 1, 2, \ldots \ . \tag{4}$$

However, we also want to account for the case in which the predictor and estimator are *different* (i.e., "not connected" in the language of [1]); this is done by taking M to be different from N.

3 The comparison equation for the recovery of the parameter from the data

The usual way to solve the inverse problem is to minimize some functional of the discrepancy between the estimator and the predictor. Since, at present,

we seek only one parameter, it is easier to just solve the so-called comparison equation

$$K^{(M,N)}(\varepsilon;\varphi) := E^{(M)}(\varepsilon) - \Phi^{(N)}(\varphi) = 0 \ , \tag{5}$$

for ε, with the understanding that the reconstruction will be perfect if ε turns out to be equal to φ.

Note that this way of handling the inversion is equivalent to the usual way involving minimization, since the zeros of $K^{(M,N)}(\varepsilon;\varphi)$ are identical to the values of ε for which the discrepancy functional attains its minima.

Note also that, on account of (3) and (4), $\Phi^{(N)}(\varphi)$ can be replaced by $E^{(N)}(\phi)$ in (5).

4 General features of solutions of the comparison equation in the context of the inverse crime

As stated previously, the inverse crime corresponds to $M = N$, so that

$$K^{(M,M)}(\varepsilon;\varphi) := E^{(M)}(\varepsilon) - \Phi^{(M)}(\varphi) = 0 \ . \tag{6}$$

It may be thought that this equation possesses *at least one solution*, i.e., the correct ("trivial inverse" in the language of [1]) solution

$$\varepsilon = \varphi \ , \tag{7}$$

but this is not necessarily so (inspite of what is stated in [1]), when, for example, $M = 0$ and f_0 is a constant (i.e., does not depend on its argument)).

To go deeper into the features of the inverse crime, we must be more specific about the functions f_m. We make two choices, one of which is abstract and the other a feature of a real-life inverse problem.

The first choice is:

$$f_m(\zeta) = \zeta^m \ , \tag{8}$$

by means of which we obtain

$$K^{(M,M)}(\varepsilon;\varphi) = \sum_{m=0}^{M} a_m(\varepsilon^m - \varphi^m) = 0 \ . \tag{9}$$

This generally non-linear comparison equation has at least one solution (i.e., the "trivial inverse" $\varepsilon = \varphi$) as long as $M \geq 1$. The solution is unique when and only if $M = 1$ (we are excluding the absurd choice $a_1 = 0$). Otherwise (i.e., $M \geq 2$) the polynomial equation (9) possesses M roots which are usually not degenerate. For instance, when $M = 2$, the two roots are:

$$\varepsilon = \varphi, \quad , \quad \varepsilon = -\varphi - a_1/a_2 \ . \tag{10}$$

Since the second root is not generally equal to the "trivial inverse", *the inverse crime is not trivial in this case.* Of course, this argument and conclusion carry over to situations in which higher-order models are employed.

Consider next the second choice whereby $M = 0$, $a_0 \neq 0$, and

$$f_0(\zeta) = \exp(-2ik\zeta) \ , \tag{11}$$

with k a real constant and $i := \sqrt{-1}$. This is the exact model of reflection of a plane wave $\exp(-ikx_3)$ normally-incident on a flat mirror $x_3 = \varphi$; $\forall x_1 \in \mathbb{R}$, $\forall x_2 \in \mathbb{R}$. For instance, if the boundary condition is of the Dirichlet type (corresponding to the case in which the electric field is wholly tangential to the boundary and the latter covers a perfectly conducting medium), then $a_0 = -1$ [2]. With this choice, and the observable being the reflectivity, the comparison equation (6) becomes

$$K^{(0,0)}(\varepsilon; \varphi) = a_0[\exp(-2ik\varepsilon) - \exp(-2ik\varphi)] = 0 \ . \tag{12}$$

This is equivalent to the non-linear equation $\sin[k(\varepsilon - \varphi)] = 0$ the solutions of which are:

$$\varepsilon = \varphi + n\pi/k \ ; \ n \in \mathbb{Z} \ . \tag{13}$$

Once again, we obtain not only the "trivial inverse" $\varepsilon = \varphi$, but a number (infinite) of other solutions. The latter betray the fact that, even in the context of the inverse crime, the inverse problem of locating the height φ of the mirror from one measurement of the reflectivity is an ill-posed problem (due to the non-uniqueness of the solutions [3],[4]). This finding is at odds with what is written in [1] concerning a boundary recovery problem solved by committing the inverse crime: "Hence, it is no surprise...that the surface ∂D is recovered pretty well" (here ∂D is the boundary of the mirror, or in particular its location), since, in our example, not only do we recover the location of the mirror, but also the locations of other planes on which the boundary condition is satisfied (this being a surprise).

The latter example illustrates another feature of the inverse crime: its *usefulness*. To be more specific, (13) tells us that we can actually obtain the location of the mirror unambiguously from *two or more* experiments conducted for two or more values of k (i.e., for ≥ 2 values of the frequency of the incident wave), since the only value of ε in (13) that is independent of the frequency is the correct value $\varepsilon = \varphi$. This idea has been put to use recently in a more complicated shape identification problem [5].

5 The "no connection" issue

As stated previously, in the framework of this paper, an estimator that is different from the predictor means that $M \neq N$ so that the comparison equation is:

$$K^{(M,N)}(\varepsilon;\varphi) := \sum_{m=0}^{M} a_m f_m(\varepsilon) - \sum_{m=0}^{N} a_m f_m(\varphi) = 0 \ . \tag{14}$$

The term "no connection" can be understood to mean that M be very different from N, but this may not be necessary for certain ranges of the estimator and predictor.

To begin the discussion, consider again the case $M = 0$ such that f_0 is a constant (i.e., does not depend on its argument). Then ε does not appear in (14) which means the non-existence of a solution.

Lest this type of estimator appear to be too extreme, consider the choice of f_m given in (8), and the case in which the estimator is linear and the predictor quadratic. Then (14) yields

$$\varepsilon = \varphi + (a_2/a_1)\varphi^2 \ . \tag{15}$$

What is remarkable about this result is the *uniqueness* of the solution (due to the linearity of the estimator). However, the relative error of the reconstruction $\delta = |(\varepsilon - \varphi)/\varphi|$ (=0 for the "trivial inverse") is

$$\delta = |(a_2/a_1)\varphi| \ , \tag{16}$$

and this is small only if $|(a_2/a_1)|$ and/or $|\varphi|$ are/is small. This shows that it may be impossible to obtain a solution, whose relative error lies below some prescribed threshold, when employing an estimator (linear in this example)

that has "no connection" to the predictor (quadratic in this example). A similar finding can, of course, be obtained for other unconnected estimator/predictor pairs. In other words, if one wants to recover the unknown parameter very accurately he shouldn't follow the recommendation [1] of choosing an estimator that has "no connection" with the predictor. In general, the larger is the functional difference between the estimator and the predictor, the larger is the relative error of the inversion [6].

6 Discussion

Let us return to the four questions raised in section 1. Concerning the meaning of the term "no connection", we proposed that this be materialized by a difference in the number of terms in a power series representation of the predictor and estimator. Of course, we could have proposed more radical differences, but it is hardly conceivable that they would yield a result that is better than the one ((16)) of section 5.

The response to the question concerning the kind of reconstructions one can obtain when there is "no connection" between the predictor and estimator is provided eloquently by (16), i.e., in such a case one can obtain reconstructions of $\mathbf{p}$ that have "no connection" to the actual $\mathbf{p}$.

The reponse to the question as to whether the inverse crime should always be avoided is, negative, since committing this crime can, at the very least, reveal the non-uniqueness of the inverse problem. It could be argued that this non-uniqueness (certainly a negative feature) is somehow induced by the inverse crime, with the implication that abolishing this crime would make the solution unique. However, non-uniqueness is neither a necessary, nor specific, feature of the inverse crime as is illustrated by the fact that by taking a linear estimator and linear predictor one obtains one and only one solution, the so-called "trivial solution" $\varepsilon = \varphi$.

A last comment on this question: it is rather strange to employ the disparaging term "trivial" to qualify what is, after all, the very objective of solving an inverse problem, i.e., reconstructing as accurately as possible the unknown parameters (in our case, obtaining ε as close as possible to φ).

The response to the question as to whether inverse crime solutions are *always trivial*, is provided by (10) for the abstract example, and by (13) for the concrete example, i.e., not only does committing the inverse crime lead to the "trivial solution" $\varepsilon = \varphi$, but also to one or more other solutions

whose existence might be observed in a full-blown numerical procedure, but not necessarily revealed by a non-inverse-crime analysis.

The material provided herein, based essentially on three examples, does not pretend to be universal. Obtaining one parameter from one piece of data does not resume the way things are usually done in the field of inverse problem solving. In fact, often one relies on more data than the number of unknown parameters, but even this does not always resolve the non-uniqueness issue [7]. This is particularly so in an abstract setting (as for the first example in section 4), but, as we have shown, when there are physical reasons why a solution should be invariant to the change in some physical parameter (e.g., the location of a mirror should be invariant to the frequency of the probe radiation) then two measurements (at different frequencies) are certainly better than one measurement to distinguish the correct solution from false solutions, provided, of course that the error (experimental, or with respect to the estimator when employing synthetic data) of the two (or more) pieces of data are of the same order.

When, as is usually the case, one wishes to recover more than one parameter from more than one pieces of data, then a mathematical analysis of the type proposed herein rapidly becomes very complicated, but would, if it were carried out, certainly provide a useful contribution to the field, although it can hardly be expected that the last sentence of section 5 would be contradicted in situations more complicated than the ones analyzed herein.

A final question, not raised in section 1, is whether there is a risk (if such is one's concern) of committing the inverse crime in practical problems, this meaning problems appealing to *real* (as opposed to synthetic) data. We see no such risk since *the predictor for real data is unknown.* The *worst* (if one apprehends punishment for committing a crime, but the *best* otherwise) that can happen is that the experiment be done very properly and the model employed in the estimator very nearly mimic the physics of the experiment, in which case the chances are great that one will accurately recover the sought-for parameters of the object under study. But, as we have shown herein, one might also recover artifacts [8] that will have to be eliminated by some sort of strategy such as performing other well-chosen experiments (this choice could be facilitated by an analysis relying on data synthesized by a predictor that is functionally equivalent to the estimator, i.e., by committing the inverse crime) or incorporating a priori information about the object at the comparison stage (this is usually what is done in the procedure called regularization [9]).

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