Lagrangian submanifolds foliated by $(n-1)$-spheres in $\mathbb{R}^{2n}$

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Abstract

We study Lagrangian submanifolds foliated by $(n-1)$-spheres in $\mathbb{R}^{2n}$ for $n \geq 3$. We give a parametrization valid for such submanifolds, and refine that description when the submanifold is special Lagrangian, self-similar or Hamiltonian stationary. In all these cases, the submanifold is *centered*, i.e. invariant under the action of $SO(n)$. It suffices then to solve a simple ODE in two variables to describe the geometry of the solutions.

Keywords: Lagrangian submanifold, special Lagrangian, self-similar, Hamiltonian stationary.

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Introduction

Lagrangian submanifolds constitute a distinguished subclass in the set of n-dimensional submanifolds of $\mathbb{R}^{2n}$: a submanifold is *Lagrangian* if it has dimension n and is isotropic with respect to the symplectic form ω of $\mathbb{R}^{2n}$ identified with $\mathbb{C}^n$. Such submanifolds are locally characterized as being graphs of the gradient of a real map on a domain of $\mathbb{R}^n$, however to give a general classification of them is not an easy task.

In the case of surfaces i.e. $n = 2$, powerful techniques such as Weierstrass representation formulas may be used to gain some insight into this question (*cf* [HR]). In higher dimension, some work has been done in the direction of characterizing Lagrangian submanifolds with various intrinsic and extrinsic geometric assumptions, see [Ch] for a general overview and more references.

Recently, D. Blair has studied in [Bl] Lagrangian submanifolds of $\mathbb{C}^n$ which are foliated by $(n-1)$-planes. In the present paper, we are devoted to study those Lagrangian submanifolds of $\mathbb{R}^{2n}$, which are foliated by Euclidean $(n-1)$-spheres. In the following, we shall for brevity denote them by σ*-submanifolds.* We first observe that any isotropic round $(n-1)$-sphere spans a n-dimensional Lagrangian subspace, a condition which breaks down in dimension 2. Hence we restrict our attention to the case $n \geq 3$ and we give a characterization of σ-submanifolds as images of an immersion of a particular form involving as data a planar curve and a $\mathbb{R}^n$-valued curve (see Theorem 1).

Next, we focus our attention on several curvatures equations and characterize those among which are σ-submanifolds. The most classical of these equations is the *minimal submanifold equation,* involving the mean curvature vector $\vec{H}$

$$\vec{H} = 0.$$

A Lagrangian submanifold which is also minimal satisfies in addition a very striking property: it is calibrated (*cf* [HL]) and therefore a minimizer; these submanifolds are called *special Lagrangian.* Many special Lagrangian submanifolds with homogeneity properties have been described in [CU2, Jo]. In this context we recover the characterization of the Lagrangian catenoid as the only (non flat) special Lagrangian σ-submanifold (see [CU1]).

We shall call *self-similar* a submanifold satisfying

$$\vec{H} + \lambda X^{\perp} = 0,$$

where $X^{\perp}$ stands for the projection of the position vector X of the submanifold on its normal space and λ is some real constant (*cf* [An2]). Such a submanifold has the property that its evolution under the mean curvature flow is a homothecy (shrinking to a point if $\lambda > 0$ and expanding to infinity if $\lambda < 0$). Here we shall show that there are no more self-similar σ-submanifolds than the ones described in [An2].

The third curvature equation we shall be interested in deals with the *Lagrangian angle* β (*cf* [Wo] for a definition), a $\mathbb{R}/2\pi\mathbb{Z}$-valued function which is defined up to an additive constant on any Lagrangian submanifold and satisfies $J\nabla\beta = n\vec{H}$, where J is the complex structure in $\mathbb{C}^n$ and ∇ is the gradient of the induced metric on the submanifold. Following [Oh], we call *Hamiltonian stationary* a Lagrangian submanifold which is critical for the volume functional under Hamiltonian deformations (generated by vector fields V such that $V \lrcorner\, \omega$ is exact). It turns out that the corresponding Euler-Lagrange equation is

$$\Delta\beta = 0.$$

In others words a Lagrangian submanifold is Hamiltonian stationary if and only if its Lagrangian angle function is harmonic (for the induced metric). Many Hamiltonian stationary surfaces in $\mathbb{C}^2$ have been described in [HR] and [An1], but in higher dimensions, very few examples were known so far (for example the Cartesian product of round circles $a_1\mathbb{S}^1 \times \ldots \times a_n\mathbb{S}^1$). In this paper we shall describe all Hamiltonian stationary σ-submanifolds.

The paper is organized as follows: the first section gives the proof of characterization of σ-submanifolds (Theorem 1). In the second one, we compute for this kind of submanifold the Lagrangian angle and the mean curvature vector. In Section 3, we obtain as a corollary the fact that special Lagrangian or self-similar σ-submanifolds are centered (i.e. invariant under the standard action of the special orthogonal group $SO(n)$, see Example 1 in Section 1 for a precise definition), thus the only examples of such submanifolds are those described in [CU1] (for special Lagrangian ones) and in [An2] (for self-similar ones). In Section 4, we show that the only Hamiltonian stationary σ-submanifolds are also centered (Theorem 3), and in Section 5 we describe them in details (Corollary 1).

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1 Characterization of Lagrangian submanifolds foliated by $(n-1)$-spheres

On $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $n \geq 3$, with coordinates $\{z_j = x_j + iy_j,\ 1 \leq j \leq n\}$ equipped with the standard Hermitian form $\langle .,. \rangle_{\mathbb{C}^n}$ and its associated symplectic form $\omega := \sum_{j=1}^n dx_j \wedge dy_j = \operatorname{Im} \langle .,. \rangle_{\mathbb{C}^n}$, we consider submanifolds of dimension n foliated by round spheres $\mathbb{S}^{n-1}$. Locally, they can be parameterized by immersions

$$\begin{array}{rccc} \ell: & I \times \mathbb{S}^{n-1} & \to & \mathbb{C}^n \simeq \mathbb{R}^{2n} \\ & (s,x) & \mapsto & r(s)M(s)x + V(s) \end{array}$$

where

- I is some interval,
- $\forall s \in I$, $r(s) \in \mathbb{R}^+, M(s) \in SO(2n)$, $V(s) \in \mathbb{C}^n$,
- $x \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$, where we note $\mathbb{R}^n = \{y_j = 0,\ 1 \leq j \leq n\}$.

Lemma 1 *A necessary condition for the immersion ℓ to be Lagrangian is that for any s in I, the (affine) n-plane containing the leaf $\ell(s, \mathbb{S}^{n-1})$ is Lagrangian; in other words, $M \in U(n)$, where $U(n)$ is embedded as a subgroup of $SO(2n)$.*

Proof. For a fixed $s \in I$, the leaf $\ell(s, \mathbb{S}^{n-1})$ spans exactly the affine n-plane $V + M\mathbb{R}^n$. So we have to show that for two independent vectors $W, W' \in \mathbb{R}^n$, $\omega(MW, MW') = 0$. Since $n \geq 3$, there exists $x \in \mathbb{S}^{n-1}$ such that W and W' are tangent to $\mathbb{S}^{n-1}$ at x. Thus rMW and rMW' are tangent to the leaf $\ell(s, \mathbb{S}^{n-1})$ at $\ell(s,x)$. The tangent space to this leaf is isotropic (being included in the tangent space to $\ell(I \times \mathbb{S}^{n-1})$). Therefore $\omega(MW, MW') = 0$. Finally, M is an isometry mapping a Lagrangian n-plane to another Lagrangian n-plane, hence a unitary transformation. $\square$

Remark 1 *This crucial lemma does not hold in dimension two: we can produce examples of Lagrangian surfaces foliated by circles such that the planes containing the circular leaves are not Lagrangian (cf Example 3).*

Now, we know that for any $\xi \in \mathbb{R}^n$ orthogonal to x, $M\xi$ is tangent to the submanifold at $\ell(s,x)$ (because it is tangent to the leaf) so we have:

$$\omega\big(\ell_s, M\xi\big) = 0,$$

where $\ell_s = \partial\ell/\partial s = \dot{r}Mx + r\dot{M}x + \dot{V}$. Along the paper the dot ' $\dot{}$ ' will be a shorthand notation for the derivative with respect to s. We compute:

$$\begin{aligned} \omega(\dot{r}Mx, M\xi) + \omega(r\dot{M}x, M\xi) + \omega(\dot{V}, M\xi) = 0 \\ \Leftrightarrow \mathrm{Im}\,\langle \dot{r}Mx, M\xi\rangle_{\mathbb{C}^n} + \mathrm{Im}\,\langle r\dot{M}x, M\xi\rangle_{\mathbb{C}^n} + \mathrm{Im}\,\langle \dot{V}, M\xi\rangle_{\mathbb{C}^n} = 0 \\ \Leftrightarrow \dot{r}\mathrm{Im}\,\langle x,\xi\rangle_{\mathbb{C}^n} + r\mathrm{Im}\,\langle M^{-1}\dot{M}x,\xi\rangle_{\mathbb{C}^n} + \mathrm{Im}\,\langle M^{-1}\dot{V},\xi\rangle_{\mathbb{C}^n} = 0 \\ \Leftrightarrow r\langle \mathrm{Im}\,(M^{-1}\dot{M})x,\xi\rangle_{\mathbb{C}^n} + \langle \mathrm{Im}\,(M^{-1}\dot{V}),\xi\rangle_{\mathbb{C}^n} = 0, \end{aligned}$$

because ξ is real.

In the following, we shall note $b := \mathrm{Im}\,(M^{-1}\dot{V}) \in \mathbb{R}^n$ and $B := \mathrm{Im}\,(M^{-1}\dot{M})$. The complex matrix $M^{-1}\dot{M}$ is skew-Hermitian so B is symmetric. Then we have the following equation:

$$(*) \quad \langle b,\xi\rangle_{\mathbb{R}^n} + r\langle Bx,\xi\rangle_{\mathbb{R}^n} = 0, \quad \text{for all } (x,\xi) \in \mathbb{S}^{n-1}\times\mathbb{R}^n \text{ such that } \langle x,\xi\rangle_{\mathbb{R}^n} = 0$$

From now on, we denote the real scalar product $\langle .,.\rangle_{\mathbb{R}^n}$ by $\langle .,.\rangle$. Next we have three steps:

Step 1: *b vanishes or is an eigenvector for B.*

If $b \neq 0$, take x collinear to b, so that $x = \lambda b$ for some real non zero constant λ. From $(*)$, we have

$$\forall \xi \in b^\perp, \langle B\lambda b,\xi\rangle = 0.$$

Therefore Bb is collinear to b, so there exists μ such that $Bb = \mu b$.

Step 2: *b vanishes.*

Suppose $b \neq 0$, then set $\xi = b$ and x orthogonal to b, so $\langle b,x\rangle = 0$ and $\langle Bb,x\rangle = 0$ (using step 1). Using the fact that B is symmetric, we write:

$$\langle b, Bx\rangle = 0,$$

so

$$\langle \xi, Bx\rangle = 0.$$

From $(*)$ we deduce that $\langle b,\xi\rangle = ||b||^2 = 0$. So b vanishes.

Step 3: *B is a homothecy.*

Now, $(*)$ becomes

$$\langle Bx,\xi\rangle = 0, \quad \forall x,\xi \in \mathbb{S}^{n-1} \text{ such that } \langle x,\xi\rangle = 0.$$

This implies that any vector x is an eigenvector for B, so B is a homothecy.

We deduce that $M^{-1}\dot{V} \in \mathbb{R}^n$, and that $M^{-1}\dot{M} \in \mathfrak{so}(n) \oplus i\mathbb{R}Id$. This implies that $M \in SO(n).U(1)$. In other words, we may write $M = e^{i\phi}N$, where $N \in SO(n)$. Moreover, we observe that N can be fixed to be Id: this does not change the image of the immersion. Then $M = e^{i\phi}Id$, so $\dot{V} = e^{i\phi}W$, where $W \in \mathbb{R}^n$.

Finally, we have shown:

Theorem 1 *Any Lagrangian submanifold of $\mathbb{R}^{2n}, n \geq 3$, which is foliated by round $(n-1)$-spheres is locally the image of an immersion of the form:*

$$\begin{array}{cccc} \ell: & I \times \mathbb{S}^{n-1} & \longrightarrow & \mathbb{C}^n \simeq \mathbb{R}^{2n} \\ & (s,x) & \longmapsto & r(s)e^{i\phi(s)}x + \int_{s_0}^{s} e^{i\phi(t)}W(t)dt, \end{array}$$

where

- *$s \mapsto \gamma(s) := r(s)e^{i\phi(s)}$ is a planar curve,*
- *W is a curve from I into $\mathbb{R}^n$,*
- *$s_0 \in I$.*

Geometrically, $r(s)$ is the radius of the spherical leaf and $V(s) := \int_{s_0}^{s} e^{i\phi(t)}W(t)dt$ its center. The fiber lies in the Lagrangian plane $e^{i\phi(s)}\mathbb{R}^n$.

Example 1 If $V = W = 0$, the center of each leaf is fixed. In the following, we shall simply call these submanifolds *centered.* In this case, the submanifold is $SO(n)$-equivariant (for the following action $z \mapsto Az, A \in SO(n)$ where $SO(n)$ is seen as a subgroup of $U(n)$, itself a subgroup of $SO(2n)$).

Example 2 Assume the curve γ is a straight line passing through the origin. Then up to reparametrization we can write $\gamma(s) = se^{i\phi_0}$, where ϕ_0 is some constant. This implies that

$$\ell(x,s) = e^{i\phi_0}\left[sx + \int_{s_0}^{s} W(t)dt\right],$$

thus the immersion is totally geodesic and the image is simply an open subset of the Lagrangian subspace $e^{i\phi_0}\mathbb{R}^n$.

Example 3 Assume the centers of the leaves lie on some straight line. Then there exists some function $u(s)$ such that $V(s) = u(s)a + c$ with $a, c \in \mathbb{C}^n$. Differentiating, we obtain

$$\dot{u}(s)a = e^{i\phi(s)}W(s).$$

As a is constant and W real, it implies that ϕ is constant, so we have $a = e^{i\phi}b$, where $b \in \mathbb{R}^n$. Then

$$\ell(x,s) = e^{i\phi}(r(s)x + u(s)b) + c$$

so again the immersion is totally geodesic. We notice that this situation is in contrast with the case of dimension 2, where there exists a Lagrangian flat cylinder, which is foliated by round circles whose centers lie on a line.

Example 4 Epicycloids: Assume the centers of the leaves lie on a circle contained in a complex line. Then there exists some function $u(s)$ such that $V(s) = e^{iu(s)}a + c$ with $a, c \in \mathbb{C}^n$. Differentiating, we obtain

$$W(s) = i\dot{u}(s)e^{i(u(s)-\phi(s))}a.$$

As in the previous example, it implies that $\phi - u$ is constant; without loss of generality, we can take $u = \phi$ and $a = -ib$, where $b \in \mathbb{R}^n$. Then

$$\ell(x, s) = e^{i\phi(s)}(r(s)x - ib) + c.$$

2 Computation of the Lagrangian angle and of the mean curvature vector

2.1 The Lagrangian angle

We may assume, and we shall do so from now on, that γ is parameterized by arclength, so there exists θ such that $\dot{\gamma} = e^{i\theta}$, and the curvature of γ is $k = \dot{\theta}$. We also introduce, as in [An2], $\alpha := \theta - \phi$, so that $\dot{r} = \cos\alpha$ and $\dot{\phi} = \frac{\sin\alpha}{r}$.

Let $(v_2, ..., v_n)$ be an orthonormal basis of $T_x\mathbb{S}^{n-1}$. Here and in the following, the indices j, k and l (and the sums) run from 2 to n, unless specified. We have:

$$\ell_s = e^{i\theta}x + e^{i\phi}W \text{ and } \ell_* v_j = re^{i\phi}v_j.$$

The induced metric g on $I \times \mathbb{S}^{n-1}$ has the following components with respect to the basis $(\partial_s, v_2, ..., v_n)$:

$$g_{11} = 1 + |W|^2 + 2\cos\alpha\langle W, x\rangle,$$

$$g_{1j} = g_{j1} = r\langle W, v_j\rangle,$$
$$g_{jk} = g_{kj} = \delta_{jk}r^2.$$

We consider the orthonormal basis (for g) $(e_1, ..., e_n)$ defined as follows:

$$e_j := \frac{v_j}{r}, \quad e_1 := A\partial_s + \sum B_j v_j,$$

where the real numbers A and B_j are uniquely determined by the orthonormality condition:

$$A = \left(1 + 2\cos\alpha\langle W, x\rangle + \langle W, x\rangle^2\right)^{-1/2} \quad \text{and} \quad B_j = -\frac{A\langle W, v_j\rangle}{r} ,$$

so

$$e_1 = A\left(\partial_s - \sum \frac{\langle W, v_j\rangle}{r} v_j\right).$$

This yields an orthonormal basis of the tangent space to the submanifold and we may compute the Lagrangian angle by means of the following formula: $e^{i\beta} = \det_{\mathbb{C}}(\ell_* e_1, ..., \ell_* e_n)$. Indeed we have

$$\ell_* e_j = e^{i\phi} v_j,$$

and

$$\begin{aligned} \ell_* e_1 &= A(\ell_s - \sum \frac{\langle W, v_j \rangle}{r} \ell_* v_j) = A(e^{i\theta} x + e^{i\phi} W - \sum \langle W, v_j \rangle e^{i\phi} v_j) \\ &= A(e^{i\theta} + e^{i\phi} \langle W, x \rangle) x. \end{aligned}$$

From the above remarks we deduce that:

$$\begin{aligned} \beta &= \operatorname{Arg}(e^{i\theta} + e^{i\phi} \langle W, x \rangle) + (n-1)\phi \\ &= \operatorname{Arg}(e^{i\alpha} + \langle W, x \rangle) + n\phi. \end{aligned}$$

In the centered case $W = 0$ (Example 1), we find that $\beta = n\phi + \alpha$. This is the only case where the Lagrangian angle is constant on the leaves. We recall that a necessary and sufficient condition for a Lagrangian submanifold to be a special Lagrangian one is that the Lagrangian angle be locally constant. So our computation shows that a special Lagrangian σ-submanifold must be centered. Moreover, it has been shown in [An2] that the only such submanifolds are pieces of the Lagrangian catenoid (*cf* [CU1] for a complete description), so we recover one of the theorems of [CU1]:

Theorem *A (non flat) special Lagrangian submanifold which is foliated by $(n-1)$-spheres is congruent to a piece of the Lagrangian catenoid.*

2.2 Computation of the mean curvature vector

We first calculate the second derivatives of the immersion.

$$\ell_{ss} = ike^{i\theta} x + \left(\frac{i \sin \alpha}{r} W + \dot{W} \right) e^{i\phi},$$

$$v_j(\ell_s) = e^{i\theta} v_j,$$

$$\ell_{v_j v_k} = -\delta_{jk} r e^{i\phi} x.$$

Then we obtain the following expressions:

$$\langle \ell_{ss}, J\ell_s \rangle = k + k \cos \alpha \langle W, x \rangle + \frac{\sin \alpha \cos \alpha}{r} \langle W, x \rangle + \sin \alpha \langle \dot{W}, x \rangle + \frac{\sin \alpha}{r} |W|^2,$$

$$\langle \ell_{ss}, J\ell_* v_j \rangle = \langle v_j(\ell_s), J\ell_s \rangle = \sin \alpha \langle W, v_j \rangle,$$

$$\langle \ell_{v_j v_j}, J\ell_s\rangle = \langle v_j(\ell_s), J\ell_* v_j\rangle = r\sin\alpha,$$
$$\langle \ell_{v_j v_k}, J\ell_* v_k\rangle = 0.$$

Thus we have, using the property that the tensor $C := \langle h(.,.), J.\rangle$ is totally symmetric for any Lagrangian immersion:

$$\begin{aligned}
C_{111} &= \langle h(e_1,e_1), J\ell_* e_1\rangle \\
&= A^3 \left\langle h\left(\partial_s - \frac{1}{r}\sum \langle W, v_j\rangle v_j, \partial_s - \frac{1}{r}\sum\langle W, v_j\rangle v_j\right), J\ell_s - \frac{\sum\langle W, v_j\rangle J\ell_* v_j}{r}\right\rangle \\
&= A^3 \Big(k + k\cos\alpha\langle W, x\rangle + \frac{\sin\alpha\cos\alpha}{r}\langle W, x\rangle + \sin\alpha\langle \dot W, x\rangle \\
&\quad + \frac{\sin\alpha}{r}|W|^2 + \frac{\sin\alpha}{r}\sum\langle W, v_j\rangle^2 - 2\frac{\sin\alpha}{r}\sum\langle W, v_j\rangle^2\Big) \\
&= A^3\left(k + k\cos\alpha\langle W, x\rangle + \frac{\sin\alpha\cos\alpha}{r}\langle W, x\rangle + \sin\alpha\langle\dot W, x\rangle + \frac{\sin\alpha}{r}\langle W, x\rangle^2\right),
\end{aligned}$$

$$C_{jj1} = \langle h(e_j, e_j), J\ell_* e_1\rangle = \frac{A}{r^2}\left\langle h(v_j, v_j), J\ell_s - \frac{1}{r}\sum\langle W, v_k\rangle J\ell_* v_k\right\rangle = \frac{A\sin\alpha}{r},$$

$$\begin{aligned}
C_{11j} &= \langle h(e_1, e_1), J\ell_* e_j\rangle = \frac{A^2}{r}\left(\sin\alpha\langle W, v_j\rangle - \frac{2}{r}\sum_k \langle W, v_k\rangle r\sin\alpha\delta_{jk}\right) \\
&= -\frac{A^2\sin\alpha\langle W, v_j\rangle}{r},
\end{aligned}$$

$$C_{jjk} = \langle h(e_j, e_j), J\ell_* e_k\rangle = 0.$$

So finally we have the following:

$$\begin{aligned}
n\vec{H} &= A\Big[A^2\Big(k + k\cos\alpha\langle W, x\rangle + \frac{\sin\alpha\cos\alpha}{r}\langle W, x\rangle \\
&\quad + \sin\alpha\langle\dot W, x\rangle + \frac{\sin\alpha}{r}\langle W, x\rangle^2\Big) + \frac{(n-1)\sin\alpha}{r}\Big] J\ell_* e_1 \\
&\quad - \sum\frac{A^2\sin\alpha\langle W, v_j\rangle}{r} J\ell_* e_j.
\end{aligned}$$

In Section 5, we shall use the following notations: $nJ\vec{H} = ae_1 + \sum a_j e_j$, where

$$(H_0)\quad a = -A^3\left(k + \left(k\cos\alpha + \frac{\sin\alpha\cos\alpha}{r}\right)\langle W, x\rangle + \sin\alpha\langle\dot W, x\rangle + \frac{\sin\alpha}{r}\langle W, x\rangle^2\right) - (n-1)\frac{A\sin\alpha}{r}$$

$$(H_j)\quad a_j = A^2\frac{\sin\alpha\langle W, v_j\rangle}{r}.$$

3 Application to self-similar equations

Theorem 2 *In the class of Lagrangian submanifolds which are foliated by $(n-1)$-spheres, there are no self-translators and the only self-shrinkers/expanders are the centered ones described in [An2].*

Proof. The self-translating equation is $\vec{H} = V^\perp$ for some fixed vector $V \in \mathbb{C}^n$. In particular,

$$\begin{aligned} &\langle \vec{H}, J\ell_* e_j\rangle = \langle V, J\ell_* e_j\rangle \\ \iff & -\frac{A^2 \sin\alpha}{nr}\langle W, v_j\rangle = \langle V, Jre^{i\phi}e_j\rangle \\ \iff & -\frac{A^2 \sin\alpha}{nr}\langle W, v_j\rangle = \langle V, Je^{i\phi}v_j\rangle \\ \iff & -\frac{\sin\alpha}{nr}\langle W, v_j\rangle = \left(1 + 2\cos\alpha\langle W, x\rangle + \langle W, x\rangle^2\right)\langle V, Je^{i\phi}v_j\rangle. \end{aligned}$$

Differentiating this last equation with respect to v_k, $k \neq j$ (this is possible since $n \geq 3$), we obtain

$$0 = \Big(1 + 2\langle W, v_k\rangle(\cos\alpha + \langle W, x\rangle)\Big)\langle V, Je^{i\phi}v_j\rangle.$$

We now observe that the set of points (x, v_k) for which the first factor in the r.h.s. term of the above equation vanishes is of codimension 1 at most in the unit sphere bundle over $\mathbb{S}^{n-1}$ and apart from this set, $\langle V, Je^{i\phi}v_j\rangle$ vanishes. Coming back to the third equality of the above equivalences, this yields that either $\sin\alpha$ vanishes or $\langle W, v_j\rangle$ does. In the first case the curve γ is a line passing through the origin. Then we know by Example 2 that the immersion is totally geodesic so the image is a Lagrangian subspace. In the other case, we know that for $j \neq k$, $\langle W, v_j\rangle$ vanishes on some dense open subset of points (x, v_k) in the unit sphere bundle over $\mathbb{S}^{n-1}$. This shows that W vanishes identically. Then it is clear that so does $V^\perp$. This implies that the only solution of the equation is for vanishing $\vec{H}$, which is the trivial, minimal case.

The self-shrinking/expanding equation is $\vec{H} + \lambda X^\perp = 0$, where λ is some real constant. This implies

$$\begin{aligned} &\langle \vec{H}, J\ell_* e_j\rangle + \lambda\langle \ell, J\ell_* e_j\rangle = 0 \\ \Leftrightarrow & -\frac{A^2\sin\alpha}{nr}\langle W, v_j\rangle + \lambda\left\langle \int e^{i\phi}W, Je^{i\phi}v_j\right\rangle = 0 \\ \Leftrightarrow & \frac{A^2\sin\alpha}{nr}\langle W, v_j\rangle = \lambda\left\langle \operatorname{Im}(e^{-i\phi}\int e^{i\phi}W), v_j\right\rangle. \end{aligned}$$

The quantity $\operatorname{Im}(e^{-i\phi}\int e^{i\phi}W)$ depends only on s, so the same argument as above holds, and we deduce that either γ is a line passing through the origin (totally geodesic case) or W vanishes, which is the centered case treated in [An2]. $\square$

4 Hamiltonian stationary σ-submanifolds

The purpose of this section is to prove the following

Theorem 3 *Any Hamiltonian stationary Lagrangian submanifold foliated by $(n-1)$-spheres must be centered.*

Proof. Let ℓ be a parametrization of such a submanifold. We follow the same notations than in Section 2. We are going to show that if $\Delta\beta$ vanishes, then either γ is a straight line, so as we have seen in Example 2, we are in the totally geodesic case, which is in particular centered, or W vanishes (centered case). The proof is based on an analysis of the quantity $f(x) := A^{-6}\Delta\beta$ which turns to be polynomial. Its expression is given in the next lemma and the computation is detailed in Appendix.

Lemma 2 *For any fixed s, $f := A^{-6}\Delta\beta$ is a polynomial in the three variables $\langle W,x\rangle, \langle \dot W,x\rangle$ and $\langle \ddot W,x\rangle$. Indeed we have: $f=(I)+(II)+(III)+(IV)$ with*

$$
\begin{aligned}
(I) \;&:=\; 3B\Big(\cos\alpha\langle \dot W,x\rangle - \dot\alpha\sin\alpha\langle W,x\rangle + \langle W,x\rangle\langle \dot W,x\rangle\Big)\\
&\quad -A^{-2}\left[\dot B-(n-1)\frac{\sin\alpha}{r}\Big(\cos\alpha\langle \dot W,x\rangle - \dot\alpha\sin\alpha\langle W,x\rangle + \langle W,x\rangle\langle \dot W,x\rangle\Big)\right]\\
&\quad -A^{-4}(n-1)\partial_s\left(\frac{\sin\alpha}{r}\right),\\
(II) \;&:=\; -3B\frac{\cos\alpha+\langle W,x\rangle}{r}(|W|^2-\langle W,x\rangle^2)\\
&\quad +\frac{A^{-2}}{r}\left[\left(k\cos\alpha-(n-2)\frac{\sin\alpha\cos\alpha}{r}-(n-3)\frac{\sin\alpha}{r}\langle W,x\rangle\right)(|W|^2-\langle W,x\rangle^2)\right.\\
&\quad \left.+\sin\alpha\Big(\langle \dot W,W\rangle-\langle \dot W,x\rangle\langle W,x\rangle\Big)\right],\\
(III) \;&:=\; -A^{-2}\frac{\sin\alpha}{r^2}(\cos\alpha+\langle W,x\rangle)\Big(|W|^2-\langle W,x\rangle^2\Big)-A^{-4}(n-1)\frac{\sin\alpha}{r^2}\langle W,x\rangle,\\
(IV) \;&:=\; -A^{-2}B\frac{n-1}{r}(\cos\alpha+\langle W,x\rangle)-A^{-4}\frac{(n-1)^2\sin\alpha}{r^2}(\cos\alpha+\langle W,x\rangle)
\end{aligned}
$$

and

$$B := k+\left(k\cos\alpha+\frac{\sin\alpha\cos\alpha}{r}\right)\langle W,x\rangle+\sin\alpha\langle \dot W,x\rangle+\frac{\sin\alpha}{r}\langle W,x\rangle^2.$$

Proof. See Appendix. ☐

Lemma 3

$$f(x) = -\Big(\sin\alpha+2\sin\alpha\cos\alpha\langle W,x\rangle+\sin\alpha\langle W,x\rangle^2\Big)\langle \ddot W,x\rangle + R\Big(\langle W,x\rangle,\langle \dot W,x\rangle\Big),$$

where $R(.,.)$ is a polynomial in its two variables.

Proof. In the previous expression of f, we see that there are no contributions to the terms in $\langle \ddot{W}, x\rangle$ apart from the $\dot{B}$ term in (I). Then we compute

$$\dot{B} = \sin\alpha\langle \ddot{W}, x\rangle + S\Big(\langle W, x\rangle, \langle \dot{W}, x\rangle\Big),$$

where $S(.,.)$ is a polynomial in its two variables. We deduce that the only term of $\langle \ddot{W}, x\rangle$ in f is the following:

$$-A^{-2}\sin\alpha\langle \ddot{W}, x\rangle,$$

so we have our claim. $\square$

Lemma 4 *The polynomial f has total degree at most 5, and in that case its leading term is $\frac{-n^2+n+2}{r^2}\sin\alpha\langle W, x\rangle^5$.*

Proof. Remembering that A^{-2}, B and $\dot{B}$ are polynomials of degree at most 2 in $\langle W, x\rangle$ and $\langle \dot{W}, x\rangle$, we first observe that there are no terms of order 5 in (I). Then we check that in (II), (III) and (IV) there are no terms in $\langle \ddot{W}, x\rangle\langle W, x\rangle^4$ (this has already been observed in the previous lemma) or in $\langle \dot{W}, x\rangle\langle W, x\rangle^4$. Finally, the coefficient of $\langle W, x\rangle^2$ in A^{-2} and B are respectively 1 and $\frac{\sin\alpha}{r}$, so summing up, we obtain that the coefficients of $\langle W, x\rangle^5$ in (II), (III) and (IV) are respectively

$$3\frac{\sin\alpha}{r^2} + (n-3)\frac{\sin\alpha}{r^2}$$

$$\frac{\sin\alpha}{r^2} - (n-1)\frac{\sin\alpha}{r^2}$$

$$-(n-1)\frac{\sin\alpha}{r^2} - (n-1)^2\frac{\sin\alpha}{r^2},$$

from which we conclude the proof. $\square$

Lemma 5 *Let $P(x_1, x_2, x_3)$ be an irreducible polynomial with real coefficients. Assume the set of zeroes of P is non empty and that $f \in \mathbb{R}[x_1, x_2, x_3]$ vanishes on it. Then f is of the form PQ with $Q \in \mathbb{R}[x_1, x_2, x_3]$.*

Proof. We embed $\mathbb{R}^3$ into $\mathbb{C}^3$. Assume by contradiction that f is not of the form PQ with $Q \in \mathbb{R}[x_1, x_2, x_3]$. Then also f is of not the form PQ with $Q \in \mathbb{C}[x_1, x_2, x_3]$, so the set $Y := \{x/f(x) = P(x) = 0\} = \{f(x) = 0\} \cap \{P(x) = 0\}$ has complex codimension 2, so real dimension $6 - 4 = 2$. Now our assumption is that $Y \cap \mathbb{R}^3$ contains $\{P = 0\} \cap \mathbb{R}^3$, so in particular, Y which has real dimension 2 contains $\{P = 0\} \cap \mathbb{R}^3$, also of real dimension 2. We conclude that it is an irreducible component of Y, a contradiction because $\mathbb{R}^3$ does not contain any complex curve. $\square$

End of the proof of Theorem 3.
First case: the three vectors W, $\dot{W}$ and $\ddot{W}$ do not span $\mathbb{R}^n$ (it is in particular always the case when $n > 3$).

In this case, there exist coordinates $(y_1, \dots, y_n)$ on $\mathbb{R}^n$ such that f does not depend on y_n. In particular $\{y, f(y) = 0\}$ contains some straight line $\{y_j = \text{Const}, 1 \leq j \leq n-1\}$. However by assumption $\{y, f(y) = 0\}$ contains also the hypersphere $\mathbb{S}^{n-1}$. As it is an algebraic set, it is necessarily the whole $\mathbb{R}^n$, thus f vanishes identically. This implies that either W vanishes (this is the centered case), or that so does f as a polynomial of *independent* variables of $\mathbb{R}^n$: the vectors $W, \dot{W}$ and $\ddot{W}$ might be non independent and in this case we should rewrite f as a polynomial of less variables. Anyway, we know from Lemma 4 that the only term of order 5 in f is $\sin\alpha$ times a non negative constant, thus we deduce in any case that $\sin\alpha$ should vanish, which means that the curve γ would be a line passing through the origin. Then we know that by Example 2 the immersion is totally geodesic.

Second case: $n = 3$ and $\text{Span}(W, \dot{W}, \ddot{W}) = \mathbb{R}^3$.

Here we apply the algebraic Lemma 5 with $P(x_1, x_2, x_3) = |x|^2 - 1$, thus obtaining that either f is a multiple of $|x|^2 - 1$, or it vanishes. In the first case, there should be at least one term of degree 2 in $\langle \ddot{W}, x\rangle$, which is not the case (*cf* Lemma 3). So we deduce again that f vanishes and the conclusion is the same as above. □

5 Centered Hamiltonian stationary σ-submanifolds

5.1 The differential system

In this case, the induced metric is diagonal: $g = \text{diag}(1, r^2, ..., r^2)$, and $\det g = r^{2(n-1)}$. Moreover, β depends only on $s : \beta = \alpha + n\phi$, thus

$$\Delta\beta = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial s}\left(\sqrt{\det g}\,\frac{\partial\beta}{\partial s}\right).$$

Hence the submanifold is Hamiltonian stationary if and only if

$$\sqrt{\det g}\,\frac{\partial\beta}{\partial s} = C,$$

which amounts to

$$r^{n-1}(n\dot{\phi} + \dot{\alpha}) = C,$$

for some non vanishing constant C (the case of vanishing C implies β to be constant, so this is the special Lagrangian case). We add that we could have used the computations of the previous section as well with $W = 0$ to find the same equation.

Thus we are reduced to studying the following differential system (using that $\dot{\phi} = \frac{\sin\alpha}{r}$):

$$\begin{cases} \dot{\alpha} = \frac{C}{r^{n-1}} - \frac{n\sin\alpha}{r} \\ \dot{r} = \cos\alpha. \end{cases}$$

There are fixed points ($\bar{\alpha} = \pm\frac{\pi}{2}$ mod $2\pi, \bar{r} = \left(\frac{|C|}{n}\right)^{1/(n-2)}$), corresponding to the case of γ being a circle centered at the origin. Up to a sign change of parameter s, we may assume that C is non negative and then $\overline{\alpha} = \frac{\pi}{2}$ mod 2π. Moreover, the system admits a first integral: $E = 2r^n \sin\alpha - Cr^2$, so we can draw the phase portrait easily. As the system is periodic in the variable α, it is sufficient to study integral lines in a strip of length 2π. Moreover integral lines are symmetric with respect to the vertical lines $\alpha = \pi/2$ mod π. The energy of the fixed points is $E_0 := \left(\frac{C}{n}\right)^{n/(n-2)}(2-n)$. There is a critical integral line of energy E_0 with bounded pieces connecting two fixed points and unbounded pieces starting from (or ending to) a fixed point and having a branch asymptotic to a vertical line $\alpha = k\pi, k \in \mathbb{Z}$. We observe that the

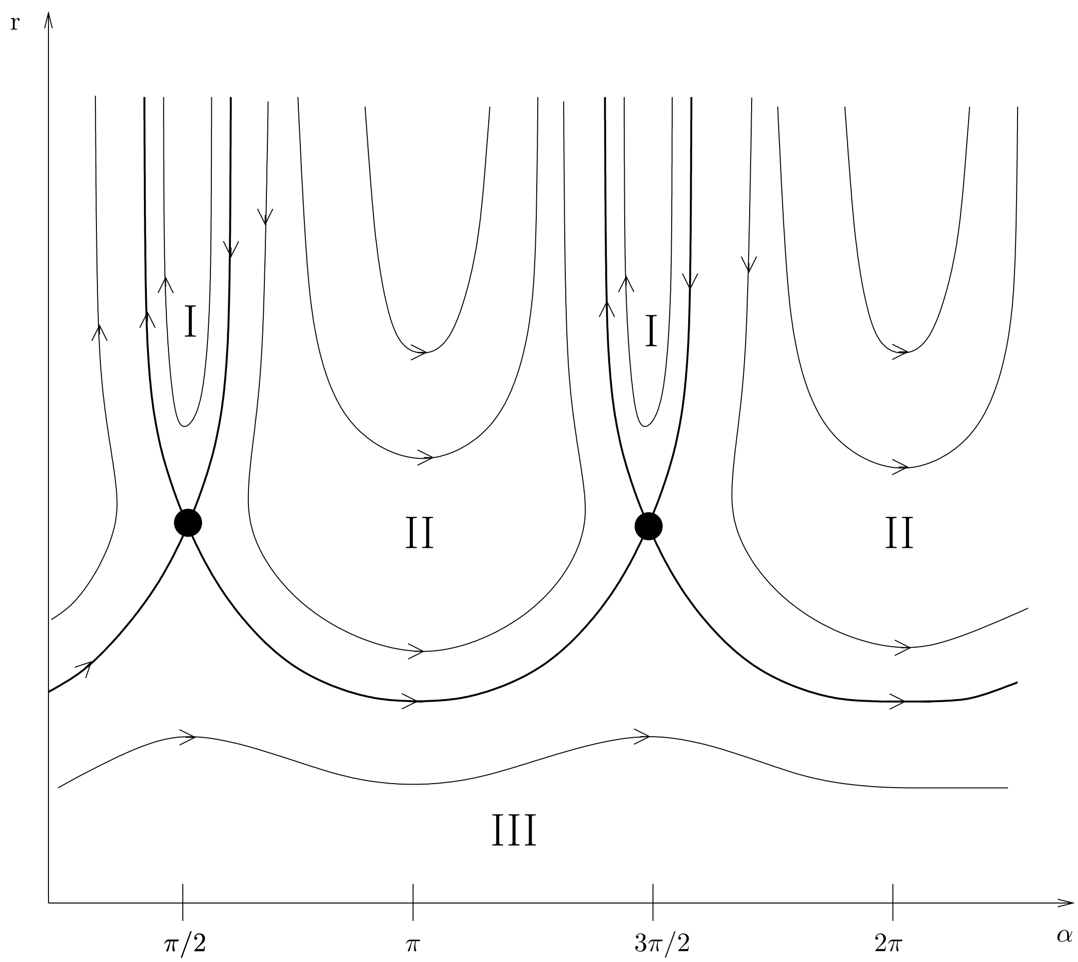

Figure 1: Phase diagram

integral lines corresponding to a non negative energy (resp. strictly less than E_0) are unbounded, we call them *type I* (resp. *type II*) curves. On the other hand, integral lines of energy $E \in (E_0, 0)$ have two connected components (up to periodicity), one of them being bounded in the variable r; this will be called in the following a *type III* curve. As the unbounded component shares the same features as the curves of non negative energy, it is also called a *type I* curve.

In order to have a better picture of the corresponding curves γ and of Hamiltonian stationary Lagrangian submanifolds they generate, we shall discuss in the next two paragraphs the inflection points of γ and the quantity $\Phi(E) := \int \dot{\phi} = \int \frac{\sin\alpha}{r}$ that we call *total variation of phase.*

5.2 Inflection points of γ

They correspond to the vanishing of the curvature $k = \dot{\theta} = \frac{C}{r^{n-1}} - \frac{(n-1)\sin\alpha}{r}$. This implies the relation $r = \left(\frac{C}{(n-1)\sin\alpha}\right)^{1/(n-2)}$.

We observe that in the case of dimension 3, this is exactly the equation of the integral curve of level $E = 0$: so in this dimension there is a solution which is of curvature zero, i.e. a straight line, and the other ones don't have any inflection point and are locally convex.

In higher dimension, let us compute the energy E at points of vanishing curvature. At such a point (α, r), we have $r^{n-2} = \frac{C}{(n-1)\sin\alpha}$ so we deduce that $E = Cr^2(\frac{2}{n-1} - 1)$, which has range $(-\infty, E_1)$ where E_1 is the energy level defined by $E_1 = 2r_1^n - Cr_1^2$ (r_1 is the least radius on the curve of points (α, r) corresponding to vanishing curvature: $r_1 = \left(\frac{C}{(n-1)}\right)^{1/(n-2)}$). As $r \mapsto E(r) = 2r^n - Cr^2$ is increasing when $r > \overline{r}$, E_1 belongs to the interval $(E_0, 0)$. We conclude that when $n > 3$ every type II curve has two (symmetric) inflection points. This is also the case of some type I curves, while the remaining ones are locally convex.

5.3 Study of the total variation of phase

We first compute $\Phi(E)$ for type I curves. Let r_0 be the minimal value taken by r. We have the relation $E = 2r_0^n - Cr_0^2$. We shall use the fact that on the curve $\sin\alpha > 0$; moreover, by symmetry we may restrict ourselves to half of the curve, thus we may also assume that $\cos\alpha > 0$. Thus

$$\Phi = \int \frac{\dot{\phi}}{\dot{r}} dr = 2\int_{r_0}^{\infty} \frac{\sin\alpha}{r\cos\alpha} dr = 2\int_{r_0}^{\infty} \frac{dr}{r}\left(\frac{1}{\sin^2\alpha} - 1\right)^{-1/2}$$

$$= 2\int_{r_0}^{\infty} \frac{dr}{r}\left(\left(\frac{2r^n}{Cr^2 + E}\right)^2 - 1\right)^{-1/2}$$

$$= 2\int_{r_0}^{\infty} \frac{dr}{r}\left(\left(\frac{2r^n}{C(r^2 - r_0^2) + 2r_0^n}\right)^2 - 1\right)^{-1/2}$$

Making the change of variable $r = xr_0$, we infer

$$\Phi = 2\int_1^\infty \frac{dx}{x}\left(\left(\frac{x^n}{\lambda(x^2-1)+1}\right)^2 - 1\right)^{-1/2},$$

where $\lambda = \frac{C}{2r_0^{n-2}}$. We observe that this integral equals π/n when λ vanishes, is divergent for $\lambda = n/2$, *i.e.* for $E = E_0$ and is decreasing in the variable r_0, so $\Phi(E)$ has range $(\pi/n, +\infty)$. Notice that the sign of $\dot{\phi} = \sin\alpha/r$ does not change, so that $\gamma = re^{i\phi}$ is embedded whenever the total variation of phase is small enough, i.e. whenever E is large enough.

On the other hand a unbounded piece of the integral curve of energy level E_0 is singular, with an infinite spiral branch asymptotic to the unit circle.

We now look at the case of type II integral curves. Let r_1 be the value of r at $\alpha = 0 \bmod \pi$, so that $E = -Cr_1^2$. We shall calculate separately the contributions of the curve when the integrand is positive (resp. negative), corresponding to the part of the curve lying in the region $\{\sin\alpha > 0\}$ (resp. $\{\sin\alpha < 0\}$).

$$\begin{aligned}
\Phi_+ &:= 2\int_{r_1}^\infty \frac{\sin\alpha}{r\cos\alpha}dr = 2\int_{r_1}^\infty \frac{dr}{r}\left(\frac{1}{\sin^2\alpha} - 1\right)^{-1/2} \\
&= 2\int_{r_1}^\infty \frac{dr}{r}\left(\left(\frac{2r^n}{Cr^2+E}\right)^2 - 1\right)^{-1/2} \\
&= 2\int_{r_1}^\infty \frac{dr}{r}\left(\left(\frac{2r^n}{C(r^2-r_1^2)}\right)^2 - 1\right)^{-1/2}
\end{aligned}$$

Making the change of variable $r = xr_1$, we obtain

$$\Phi_+ = 2\int_1^\infty \frac{dx}{x}\left(\left(\frac{x^n}{\lambda(x^2-1)}\right)^2 - 1\right)^{-1/2},$$

where $\lambda = \frac{C}{2r_1^{n-2}}$.

One calculates that Φ_+ is increasing in the variable E and that $\lim_{E\to-\infty} = \pi/n$.

In the computation of Φ_-, we may use α as a parameter:

$$\begin{aligned}
\Phi_- &:= \int_\pi^{2\pi} \frac{\dot{\phi}}{\dot{\alpha}}d\alpha = \int_\pi^{2\pi} \frac{\frac{\sin\alpha}{r}}{\frac{C}{r^{n-1}} - \frac{n\sin\alpha}{r}}d\alpha \\
&= \int_\pi^{2\pi} \frac{r^{n-2}\sin\alpha}{C - nr^{n-2}\sin\alpha}d\alpha
\end{aligned}$$

It is an easy computation to show that $-\pi/n < \Phi_- < 0$ and that $\lim_{E\to-\infty}\Phi_- = -\pi/n$.

Next we show that γ has always a self-intersection. Let s_0 be the parameter value corresponding to the point of the curve of least radius (in particular $\alpha(s_0) = 3\pi/2 \bmod 2\pi$). The fact that $\Phi_+ > |\Phi_-|$ and the intermediate value theorem imply that there exists s_1 such that $\phi(s_1) = \phi(s_0)$; moreover by the symmetry of the phase portrait, there exists $s_2 \neq s_1$ with the same property, the corresponding points in the phase portrait satisfying $r(s_1) = r(s_2)$ and $\frac{1}{2}(\alpha(s_1) + \alpha(s_2)) = 3\pi/2 \bmod 2\pi$. Thus $\gamma(s_1) = \gamma(s_2)$.

We end this section by looking at the type III curves. As α is always increasing on the curve, we shall use it as a parameter and consider the piece of curve $\gamma([\pi/2, 2\pi + \pi/2])$.

As ϕ is decreasing on $\alpha \in [\pi, 2\pi]$ and increasing elsewhere, we have:

$$\Phi_+ := \int_{\pi/2}^{\pi} \dot{\phi} + \int_{2\pi}^{2\pi+\pi/2} \dot{\phi} > 0$$

and

$$\Phi_- := \int_{\pi}^{2\pi} \dot{\phi} < 0.$$

It is easy to show (the calculations are left to the Reader) that $\Phi_+ > |\Phi_-|$ and that $\lim_{E\to 0}\Phi_+ = \lim_{E\to 0}\Phi_- = 0$. Moreover, $\lim_{E\to E_0}\Phi_+ = +\infty$ and $\lim_{E\to E_0}\Phi_-$ is finite. This implies that the range of the total variation of phase for type III curves is $(0,\infty)$. In particular, to the limiting case $E = E_0$ corresponds a bounded, complete curve spiraling asymptotically to the unit circle.

With a similar argument as above, we show that here again γ has always self-intersections. We conclude that for a type III curve such that $\Phi(E) \in 2\pi\mathbb{Q}$, the corresponding curve γ is bounded, closed and non-embedded.

5.4 Conclusion (Corollary 1)

We are now in position to describe the whole family of Hamiltonian stationary σ-submanifolds:

Corollary 1 *Any Hamiltonian stationary Lagrangian submanifold which is foliated by $(n-1)$-spheres is locally congruent to one of the following:*

- *The standard embedding*

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{S}^{n-1} & \to & \mathbb{C}^n \\ (e^{is}, x) & \mapsto & e^{is}x. \end{array}$$

- *A singular, bounded immersion of* $\mathbb{R}\times\mathbb{S}^{n-1}$ *"spiraloid" asymptotic to* $\mathbb{S}^1\times\mathbb{S}^{n-1}$.

- *A singular, unbounded "spiraloid" with a smooth end and asymptotic to* $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.
- *A family of smooth "catenoid-type" immersions of* $\mathbb{R} \times \mathbb{S}^{n-1}$*, some of them being embedded. In dimension* 3*, one of them takes the particular following form:*

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{S}^2 & \to & \mathbb{C}^3 \\ (s,x) & \mapsto & (C+is)x, \end{array}$$

where C *is some real constant.*

- *A family of non-standard smooth immersions of* $\mathbb{S}^1 \times \mathbb{S}^{n-1}$*. They always have self-intersections.*

Appendix: computation of $\Delta\beta$ (proof of Lemma 2)

Here we are going to use the same notations as in Section 2 and 4. We first compute the following

$$\langle \ell_{ss}, \ell_s \rangle = \langle W, \dot{W} \rangle + \left(\frac{\sin\alpha}{r} - k \sin\alpha \right) \langle W, x \rangle + \cos\alpha \langle \dot{W}, x \rangle$$

$$\langle \ell_{ss}, \ell_* v_j \rangle = r \langle \dot{W}, v_j \rangle$$

$$\langle v_j(\ell_s), \ell_s \rangle = \cos\alpha \langle W, v_j \rangle$$

$$\langle v_j(\ell_s), \ell_* v_k \rangle = \delta_{jk} r \cos\alpha$$

$$\langle \ell_{v_j v_k}, \ell_s \rangle = -\delta_{jk} r (\cos\alpha + \langle W, x \rangle)$$

$$\langle \ell_{v_j v_k}, \ell_* v_l \rangle = 0.$$

We shall make use of the following formula:

$$\Delta\beta = \mathrm{div} Jn\vec{H} = \langle \nabla_{e_1} Jn\vec{H}, e_1 \rangle + \sum_{j=2}^{n} \langle \nabla_{e_j} Jn\vec{H}, e_j \rangle$$

Writing $Jn\vec{H} = ae_1 + \sum_k a_k e_k$, we have

$$\nabla_{e_j} Jn\vec{H} = e_j(a)e_1 + a\nabla_{e_j} e_1 + \sum_k e_j(a_k) e_k + \sum_k a_k \nabla_{e_j} e_k,$$

and an analogous expression for $\nabla_{e_1} Jn\vec{H}$.

Then we obtain

$$\begin{aligned} \Delta\beta \ &= \ e_1(a) + a\langle \nabla_{e_1} e_1, e_1 \rangle + \sum_k a_k \langle \nabla_{e_1} e_k, e_1 \rangle \\ &\quad + \sum_j \left[a \langle \nabla_{e_j} e_1, e_j \rangle + e_j(a_j) + \sum_k a_k \langle \nabla_{e_j} e_k, e_j \rangle \right]. \end{aligned}$$

We give the expressions of the coefficients of $J\vec{H}$:

$$a_j = A^2 \frac{\sin\alpha \langle W, v_j\rangle}{r}$$

and

$$a = -A^3 B - A(n-1)\frac{\sin\alpha}{r},$$

where

$$A = \big(1 + 2\cos\alpha\langle W, x\rangle + \langle W, x\rangle^2\big)^{-1/2}$$

and

$$B = k + \left(k\cos\alpha + \frac{\sin\alpha\cos\alpha}{r}\right)\langle W, x\rangle + \sin\alpha\langle \dot{W}, x\rangle + \frac{\sin\alpha}{r}\langle W, x\rangle^2.$$

We start with some easy computations:

Lemma 6 **a)** $\dot{A} = -A^3\Big(\cos\alpha\langle\dot{W}, x\rangle - \dot{\alpha}\sin\alpha\langle W, x\rangle + \langle W, x\rangle\langle\dot{W}, x\rangle\Big)$,

b) $\langle\partial_s, e_1\rangle = A^{-1}$,

c) $v_j(A) = -A^3\Big(\cos\alpha + \langle W, x\rangle\Big)\langle W, v_j\rangle$,

d) $v_j(B) = \Big(k\cos\alpha + \frac{\sin\alpha\cos\alpha}{r} + \frac{2\sin\alpha}{r}\langle W, x\rangle\Big)\langle W, v_j\rangle + \sin\alpha\langle\dot{W}, v_j\rangle$,

e) $\nabla_{v_j} v_k = -\delta_{jk} A r(\cos\alpha + \langle W, x\rangle)e_1$,

f) $\nabla_{v_j}\partial_s = \cos\alpha\ e_j$.

Proof. **a),b),c)** and **d)** are easy and left to the reader.

e) We write $\nabla_{v_j} v_k = b_1\partial_s + \sum_m b_m v_m$. In particular, we have

$$0 = \langle\nabla_{v_j} v_k, v_l\rangle = b_1\langle\partial_s, v_l| > + \sum_m b_m\langle v_m, v_l| >= b_1 g_{1l} + \sum_m b_m g_{ml} = b_1 g_{1l} + b_l r^2,$$

so we obtain the relation $b_l = -\frac{\langle W, v_l\rangle}{r} b_1$.

When $j \neq k$, it yields

$$0 = \langle\nabla_{v_j} v_k, \partial_s\rangle = b_1\left(g_{11} - \sum_l \frac{\langle W, v_l\rangle}{r} g_{1l}\right),$$

so all the coefficients b_1, b_l vanish, and we obtain **e)**.

If $j = k$, we have

$$-r(\cos\alpha + \langle W, x\rangle) = \langle\nabla_{v_j} v_j, \partial_s\rangle = b_1\left(g_{11} - \sum_l \frac{\langle W, v_l\rangle}{r} g_{1l}\right),$$

so $b_1 = -A^2 r(\cos\alpha + \langle W, x\rangle)$. We deduce that

$$\nabla_{v_j} v_j = b_1 \left(\partial_s - \sum_l \frac{\langle W, v_l\rangle}{r} v_l \right) = b_1 \frac{e_1}{A} = -Ar(\cos\alpha + \langle W, x\rangle) e_1,$$

which implies **e)**.

f) We write $\nabla_{v_j}\partial_s = b_1 \partial_s + \sum_l b_l v_l$. Then we have

$$\delta_{jk} r\cos\alpha = \langle \nabla_{v_j}\partial_s, v_k\rangle = b_1 g_{1k} + \sum_l b_l g_{kl}.$$

When $j \neq k$, we deduce that

$$b_k = -\frac{g_{1k}}{g_{kk}} b_1 = -\frac{\langle W, v_k\rangle}{r} b_1$$

When $j = k$, we obtain

$$r\cos\alpha = b_1 r\langle W, v_j\rangle + r^2 b_j,$$

so

$$b_j = r^{-2}(r\cos\alpha - b_1 r\langle W, v_j\rangle) = r^{-1}(\cos\alpha - b_1\langle W, v_j\rangle).$$

From these relations we can write

$$\cos\alpha\langle W, v_k\rangle = \langle \nabla_{v_k}\partial_s, \partial_s\rangle = b_1 g_{11} + \sum_j b_j g_{1j}$$

$$= b_1 \left(g_{11} - \sum_j r\langle W, v_j\rangle \frac{\langle W, v_j\rangle}{r} \right) + r\langle W, v_k\rangle r^{-1}\cos\alpha = b_1 A^{-2} + \cos\alpha\langle W, v_k\rangle$$

We deduce that b_1 vanishes and also b_l for $l \neq k$, and finally $b_k = \frac{\cos\alpha}{r}$, so we obtain **f)**.

□

We now compute the different terms of $\Delta\beta$:

Lemma 7 a)

$$e_1(a) = A\dot{a} - \sum_j \frac{A\langle W, v_j\rangle}{r} v_j(a)$$

with

$$A\dot{a} = 3A^6 B \Big(\cos\alpha\langle \dot{W}, x\rangle - \dot{\alpha}\sin\alpha\langle W, x\rangle + \langle W, x\rangle\langle \dot{W}, x\rangle \Big)$$

$$-A^4\left[\dot{B}-(n-1)\frac{\sin\alpha}{r}\Big(\cos\alpha\langle\dot{W},x\rangle-\dot{\alpha}\sin\alpha\langle W,x\rangle+\langle W,x\rangle\langle\dot{W},x\rangle\Big)\right]$$
$$-A^2(n-1)\partial_s\left(\frac{\sin\alpha}{r}\right)$$

and

$$\sum_j\frac{A\langle W,v_j\rangle}{r}v_j(a)=3A^6B\frac{\cos\alpha+\langle W,x\rangle}{r}(|W|^2-\langle W,x\rangle^2)$$
$$-\frac{A^4}{r}\left[\left(k\cos\alpha-(n-2)\frac{\sin\alpha\cos\alpha}{r}-(n-3)\frac{\sin\alpha}{r}\langle W,x\rangle\right)(|W|^2-\langle W,x\rangle^2)\right.$$
$$\left.+\sin\alpha\Big(\langle\dot{W},W\rangle-\langle\dot{W},x\rangle\langle W,x\rangle\Big)\right],$$

b)

$$\langle\nabla_{e_1}e_1,e_1\rangle=0,$$

c)

$$\textstyle\sum_k a_k\langle\nabla_{e_1}e_k,e_1\rangle+\sum_j e_j(a_j)=-\frac{\sin\alpha}{r^2}\Big(\quad A^4(\cos\alpha+\langle W,x\rangle)(|W|^2-\langle W,x\rangle^2)$$
$$+A^2(n-1)\langle W,x\rangle\quad\Big),$$

d)

$$\sum_j\langle\nabla_{e_j}e_1,e_j\rangle=\frac{A(n-1)}{r}\left(\cos\alpha+\langle W,x\rangle\right),$$

e)

$$\langle\nabla_{e_j}e_k,e_j\rangle=0,\quad\forall j,k.$$

Proof.

a) We compute

$$\begin{aligned}\dot{a}&=-3A^2\dot{A}B-A^3\dot{B}-\dot{A}(n-1)\frac{\sin\alpha}{r}-A\partial_s\left((n-1)\frac{\sin\alpha}{r}\right)\\&=A^3\left(3A^2B+(n-1)\frac{\sin\alpha}{r}\right)\Big(\cos\alpha\langle\dot{W},x\rangle-\dot{\alpha}\sin\alpha\langle W,x\rangle+\langle W,x\rangle\langle\dot{W},x\rangle\Big)\\&\quad-A^3\dot{B}-A(n-1)\partial_s\left(\frac{\sin\alpha}{r}\right)\end{aligned}$$

from which we deduce the first equality. Next we have

$$\begin{aligned}
v_j(a) &= -3A^2 v_j(A)B - A^3 v_j(B) - v_j(A)(n-1)\frac{\sin\alpha}{r}\\
&= A^3\left(3A^2B + (n-1)\frac{\sin\alpha}{r}\right)\Big(\cos\alpha + \langle W,x\rangle\Big)\langle W,v_j\rangle\\
&\quad -A^3\left[\left(k\cos\alpha + \frac{\sin\alpha\cos\alpha}{r} + \frac{2\sin\alpha}{r}\langle W,x\rangle\right)\langle W,v_j\rangle + \sin\alpha\langle\dot W,v_j\rangle\right]\\
&= 3A^5(\cos\alpha + \langle W,x\rangle)B\langle W,v_j\rangle\\
&\quad -A^3\Bigg[\Bigg(k\cos\alpha + \frac{\sin\alpha\cos\alpha}{r} - (n-1)\frac{\sin\alpha\cos\alpha}{r}\\
&\quad +\left(\frac{2\sin\alpha}{r} - (n-1)\frac{\sin\alpha}{r}\right)\langle W,x\rangle\Bigg)\langle W,v_j\rangle + \sin\alpha\langle\dot W,v_j\rangle\Bigg]\\
&= 3A^5(\cos\alpha + \langle W,x\rangle)B\langle W,v_j\rangle\\
&\quad -A^3\left[\left(k\cos\alpha - (n-2)\frac{\sin\alpha\cos\alpha}{r} - (n-3)\frac{\sin\alpha}{r}\langle W,x\rangle\right)\langle W,v_j\rangle\right.\\
&\quad \left.+\sin\alpha\langle\dot W,v_j\rangle\right].
\end{aligned}$$

Thus

$$\begin{aligned}
A\sum\frac{v_j(a)\langle W,v_j\rangle}{r} &= 3A^6B\frac{\cos\alpha+\langle W,x\rangle}{r}\sum\langle W,v_j\rangle^2\\
-\frac{A^4}{r}\Bigg[\left(k\cos\alpha - (n-2)\frac{\sin\alpha\cos\alpha}{r} - (n-3)\frac{\sin\alpha}{r}\langle W,x\rangle\right)&\sum\langle W,v_j\rangle^2\\
&+\sin\alpha\sum\langle W,v_j\rangle\langle\dot W,v_j\rangle\Bigg]
\end{aligned}$$

b) We first compute

$$\begin{aligned}\nabla_{e_1}e_1 &= A\Bigg[\nabla_{\partial_s}\left(A\partial_s - \frac{A}{r}\sum\langle W,v_j\rangle v_j\right)\\ &\quad -\sum\frac{\langle W,v_j\rangle}{r}\nabla_{v_j}\left(A\partial_s - \frac{A}{r}\sum\langle W,v_k\rangle v_k\right)\Bigg]\\ &= A\Bigg[A\nabla_{\partial_s}\partial_s + \dot{A}\partial_s - \sum\partial_s\left(\frac{A\langle W,v_j\rangle}{r}\right)v_j - \frac{A}{r}\sum\langle W,v_j\rangle\nabla_{\partial_s}v_j\\ &\quad -\sum\frac{\langle W,v_j\rangle}{r}A\nabla_{v_j}\partial_s - \sum\frac{\langle W,v_j\rangle v_j(A)}{r}\partial_s\\ &\quad +\frac{1}{r^2}\sum_{j,k}\langle W,v_j\rangle\Big(v_j(A)\langle W,v_k\rangle v_k + A(-\delta_{jk}\langle W,x\rangle v_k + A\langle W,v_k\rangle\nabla_{v_j}v_k\Big)\Bigg].\end{aligned}$$

Making the scalar product with e_1, many terms vanish and we obtain

$$\begin{aligned}\langle\nabla_{e_1}e_1,e_1\rangle &= A^2\langle\nabla_{\partial_s}\partial_s,e_1\rangle + A\dot{A}\langle\partial_s,e_1\rangle - \frac{A}{r}\Big(\sum\langle W,v_j\rangle v_j(A)\Big)\langle\partial_s,e_1\rangle\\ &\quad -\frac{A^3}{r}(\cos\alpha + \langle W,x\rangle)\sum\langle W,v_j\rangle^2\\ &= A^3\left(\dot{\alpha}\sin\alpha\langle W,x\rangle + \frac{\sin^2\alpha\langle W,x\rangle}{r} - k\sin\alpha\langle W,x\rangle\right)\\ &\quad +\frac{1}{r}\Big(\sum\langle W,v_j\rangle^2A^3(\cos\alpha+\langle W,x\rangle)\Big)\\ &\quad -\frac{A^3}{r}(\cos\alpha+\langle W,x\rangle)\sum\langle W,v_j\rangle^2.\end{aligned}$$

Using the fact that $\dot{\alpha} = \dot{\theta} - \dot{\phi} = k - \frac{\sin\alpha}{r}$ we see that the first term in the last sum vanishes, and so does the last two ones together, so we obtain **b)**.

c) It will be a consequence of the fact that $|W|^2 = \langle W,x\rangle^2 + \sum\langle W,v_j\rangle^2$ and of the two following equalities:

$$(1)\quad \sum e_j(a_j) = -\frac{\sin\alpha}{r^2}\bigg(2A^4(\langle W,x\rangle+\cos\alpha))\sum\langle W,v_j\rangle^2 + (n-1)A^2\langle W,x\rangle\bigg),$$

$$(2)\qquad \sum a_k\langle\nabla_{e_1}e_k,e_1\rangle = \frac{A^4}{r^2}\sin\alpha(\cos\alpha+\langle W,x\rangle)\sum\langle W,v_k\rangle^2.$$

To prove (1) and (2) we proceed as follows:

$$e_j(a_j) = \frac{1}{r}v_j\left(A^2\frac{\sin\alpha\langle W,v_j\rangle}{r}\right) = \frac{\sin\alpha}{r^2}\left(2Av_j(A)\langle W,v_j\rangle - A^2\langle W,x\rangle\right)$$

$$= -\frac{\sin\alpha}{r^2}\left(2A^4(\langle W,x\rangle + \cos\alpha))\langle W,v_j\rangle^2 + A^2\langle W,x\rangle\right),$$

from which we obtain (1). Then we compute

$$\begin{aligned}
\nabla_{e_1}e_k &= A\left(\nabla_{\partial_s}e_k - \sum_l \frac{\langle W,v_l\rangle}{r}\nabla_{v_l}e_k\right)\\
&= A\left(\frac{1}{r}\nabla_{\partial_s}v_k + \partial_s(r^{-1})v_k + \sum_l \frac{\langle W,v_l\rangle}{r^2}\nabla_{v_l}v_k\right)\\
&= A\left(\frac{\cos\alpha}{r}e_k - \frac{\cos\alpha}{r^2}v_k + \sum_l \frac{\langle W,v_l\rangle}{r^2}\nabla_{v_l}v_k\right).
\end{aligned}$$

The first two terms of the last expression vanish and thanks to the **e)** of Lemma 6, we deduce

$$\langle\nabla_{e_1}e_k, e_1\rangle = -\frac{A^2\langle W,v_j\rangle}{r}(\cos\alpha + \langle W,x\rangle),$$

which implies (2).

d) We have

$$\begin{aligned}
\nabla_{e_j}e_1 &= \frac{1}{r}\nabla_{v_j}\left(A\partial_s - \sum_k \frac{A\langle W,v_k\rangle}{r}v_k\right)\\
&= \frac{1}{r}\left[v_j(A)\partial_s + A\nabla_{v_j}\partial_s - \sum_k v_j\left(\frac{A\langle W,v_k\rangle}{r}\right)v_k - \sum_k \frac{A\langle W,v_k\rangle}{r}\nabla_{v_j}v_k\right]\\
&= \frac{1}{r}v_j(A)\partial_s + A\frac{\cos\alpha}{r}e_j - \frac{1}{r}\sum_k v_j\left(\frac{A\langle W,v_k\rangle}{r}\right)v_k\\
&\quad + \frac{A\langle W,v_j\rangle}{r^2}r(\cos\alpha + \langle W,x\rangle)e_1
\end{aligned}$$

so

$$\begin{aligned}
\langle\nabla_{e_j}e_1, e_j\rangle &= \frac{1}{r}v_j(A)\langle\partial_s, e_j\rangle + A\cos\alpha - v_j\left(\frac{A\langle W,v_j\rangle}{r}\right)\\
&= \frac{1}{r}v_j(A)\langle W,v_j\rangle + A\cos\alpha - v_j(A)\frac{\langle W,v_j\rangle}{r} + \frac{A\langle W,x\rangle}{r}
\end{aligned}$$

The first and the third terms cancel out, and we obtain **d)**.

e) This is an obvious consequence of part **e)** the Lemma 6.

□

Summing these computations, and using the formula for $\Delta\beta$ given at the beginning of this Appendix, we conclude that $f := A^{-6}\Delta\beta = (I) + (II) + (III) + (IV)$ with

$$(I) := A^{-5}\dot{a} = 3B\Big(\cos\alpha\langle\dot{W},x\rangle - \dot{\alpha}\sin\alpha\langle W,x\rangle + \langle W,x\rangle\langle\dot{W},x\rangle\Big)$$
$$- A^{-2}\left[\dot{B} - (n-1)\frac{\sin\alpha}{r}\Big(\cos\alpha\langle\dot{W},x\rangle - \dot{\alpha}\sin\alpha\langle W,x\rangle + \langle W,x\rangle\langle\dot{W},x\rangle\Big)\right]$$
$$- A^{-4}(n-1)\partial_s\left(\frac{\sin\alpha}{r}\right)$$

$$\begin{aligned}
(II) &:= -A^{-6}\sum\frac{\langle W,v_j\rangle v_j(a)}{r}\\
&= -3B\frac{\cos\alpha + \langle W,x\rangle}{r}(|W|^2 - \langle W,x\rangle^2)\\
&\quad +\frac{A^{-2}}{r}\left[\left(k\cos\alpha - (n-2)\frac{\sin\alpha\cos\alpha}{r} - (n-3)\frac{\sin\alpha}{r}\langle W,x\rangle\right)(|W|^2 - \langle W,x\rangle^2)\right.\\
&\quad \left. +\sin\alpha\Big(\langle\dot{W},W\rangle - \langle\dot{W},x\rangle\langle W,x\rangle\Big)\right]
\end{aligned}$$

$$\begin{aligned}
(III) &:= A^{-6}\left(\sum_k a_k\langle\nabla_{e_1}e_k,e_1\rangle + \sum_j e_j(a_j)\right)\\
&= -A^{-2}\frac{\sin\alpha}{r^2}(\cos\alpha + \langle W,x\rangle)\Big(|W|^2 - \langle W,x\rangle^2\Big) - A^{-4}(n-1)\frac{\sin\alpha}{r^2}\langle W,x\rangle
\end{aligned}$$

$$\begin{aligned}
(IV) &:= A^{-6}a\sum\langle\nabla_{e_j}e_1,e_j\rangle\\
&= -A^{-2}B\frac{n-1}{r}(\cos\alpha + \langle W,x\rangle) - A^{-4}\frac{(n-1)^2\sin\alpha}{r^2}(\cos\alpha + \langle W,x\rangle).
\end{aligned}$$

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