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# Quasi-1D Bose-Einstein condensates in the dimensional crossover regime

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We study theoretically the crossover from a three-dimensional elongated condensate to a one-dimensional condensate as the transverse degrees of freedom get frozen by tight confinement. We compute analytically the radially integrated density profile at low temperatures using a local density approximation, and study the behavior of phase fluctuations with the transverse confinement. Previous studies of phase fluctuations in trapped gases have either focused on the 3D elongated regimes or on the 1D regime. The present approach recovers these previous results and is able to interpolate between them. We show in particular that the spatial correlation function of the matter wave field is insensitive to the transverse regime of confinement, pointing out an almost universal behavior of phase fluctuations in elongated traps.

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The experimental study of dilute atomic Bose gases benefits from the tunability of neutral atom traps. In particular, the realization of one-dimensional (1D) systems has been demonstrated in very elongated magnetic [1, 2] or optical traps [3, 4]. In these experiments, the interaction energy per particle  $\epsilon_{\text{int}}$  is smaller than the zero-point energy  $\hbar\omega_{\perp}$  in the tightly confining (transverse) directions, so that atomic motion in two tightly confined directions is frozen to zero-point motion. The equilibrium phase diagram of these trapped 1D gases shows a rich behavior [5], including a strong-coupling regime (“Tonks-Girardeau gas” [5, 6, 7]), and a mean-field regime [5, 8, 9, 10, 11], which exhibits below a characteristic temperature  $T_{\phi}$  a continuous transition from a “quasicondensate” with pronounced phase fluctuations to a true condensate. It was realized in [12] that such quasicondensates could also exist in very elongated, but three-dimensional (3D) traps: although atomic motion is possible in all directions, the lowest-lying excitations of these systems, which dominate the long range decay of the phase correlations, are 1D in character [13]. These 3D quasicondensates were observed in equilibrium [14] and non-equilibrium samples [15]. Further experiments have probed the phase coherence through Bragg spectroscopy [16, 17] or matter-wave interferometry [18], confirming the physical picture of phase-fluctuating condensates with suppressed density fluctuations.

Currently, although the limiting 3D or 1D cases are well understood theoretically, the kinematic crossover from the “3D elongated” to the truly 1D regime of transverse confinement have been less studied, despite being relevant to several recent experiments with “conventional traps” [1, 16], or to current research on the creation and manipulation of coherent atomic ensembles on micro-fabricated substrates [19, 20]. Refs. [21] have studied the density profile in the crossover region through a sep-

arability *ansatz*, which is correct in the 1D limit but not in the 3D case. This limitation has been overcome in [22], where solvable hydrodynamic models that reproduce the 3D and 1D TF limits are introduced, and in [23] by numerical integration of the transverse dynamics. In the present paper, we follow a different route and introduce an accurate analytical approximation for the “equation of state” of the quasi-1D gas in the mean-field regime. In the local density approximation introduced in [23], this allows to work out analytically the 1D (integrated over radial degrees of freedom) density profile  $n_1(z)$ , from which many properties of the quasi-1D gas can be calculated. Our results connect smoothly to the two limiting cases, whether 3D or 1D. In a second part, we investigate the influence of the transverse confinement on the crossover from phase-fluctuating to a phase-coherent ensemble. To this aim, we calculate the spatial correlation function of the radially averaged atomic field operator  $\hat{\Psi}$ ,

$$\mathcal{C}^{(1)}(s) = \int d\bar{z} \langle \hat{\Psi}^{\dagger}(\bar{z} + s/2) \hat{\Psi}(\bar{z} - s/2) \rangle, \quad (1)$$

using the local density approach, which is nothing else than a slowly-varying-envelope approximation. We show that this turns out to be a good approximation even if  $T \gtrsim T_{\phi}$ . More importantly, we find the remarkable property that the spatial correlation function (or equivalently the momentum distribution, which is simply the Fourier transform of  $\mathcal{C}^{(1)}$ ) is almost insensitive to the precise regime of transverse confinement in the quasicondensate regime, pointing out the universal character of phase fluctuations in quasi-1D gases.

We follow the general method introduced in Ref. [5], assuming small density fluctuations (*i.e.* a system far from the Tonks regime). Then, well below the degeneracy temperature, the equilibrium 3D density profile  $n_0$  in the trapping potential  $V_{\text{trap}}(\mathbf{r}) = m\omega_{\perp}^2 \rho^2/2 + V(z)$  is given by the solution of the usual Gross-Pitaevskii equation even if the cloud is a quasi-condensate. Without loss of generality, we introduce the 1D density  $n_1(z) = \int d^{(2)}\rho n_0(\rho, z)$  obtained by integration over the transverse plane, and the radial mode  $f_{\perp}$  through  $n_0(\rho, z) = |f_{\perp}(\rho, z)|^2 n_1(z)$ .

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Now, we assume that the axial confinement is sufficiently shallow that the density derivatives of  $n_1$ ,  $f_\perp$  with respect to  $z$  are negligible. In this limit, the GP equation reduces to

$$-\frac{\hbar^2}{2M} \frac{\Delta_\perp f_\perp}{f_\perp} + \frac{1}{2} M \omega_\perp^2 \rho^2 + U n_1(z) |f_\perp|^2 = \mu_{1.e.}[n_1(z)], \quad (2)$$

Here, we have used cylindrical coordinates, denoting the transverse radius as  $\rho$  and the longitudinal coordinate as  $z$ . The 3D coupling constant  $U$  is related to the s-wave scattering length  $a$  through  $U = 4\pi\hbar^2 a/M$ , where  $M$  is the atomic mass. The local equilibrium chemical potential  $\mu_{1.e.}$  depends on  $z$  through

$$\mu_{1.e.}[n_1(z)] + V(z) = \mu, \quad (3)$$

with  $\mu$  the global chemical potential of the cloud in the trap. It is therefore sufficient to solve (2) locally, *i.e.* for each value of  $n_1(z)$ , or equivalently by considering a geometry with radial harmonic trapping, but homogeneous 1d density  $n_1$  (“cylinder model”). The relation between  $n_1$  and  $\mu_{1.e.}$ , that includes the effect of transverse confinement, can be seen as an effective “equation of state” for the 1D gas [23].

We thus consider a system of  $N$  atoms confined by a transverse harmonic potential, and axially constrained to move between two boundaries at  $\pm L$  (inset in Fig. 1), giving  $n_1 = N/2L$ . The equation of state in this model was found in [23] by numerical integration. Here we follow an alternative route, and derive the equilibrium properties by taking for  $f_\perp[n_1]$  a Gaussian trial wavefunction, whose width  $w_\perp[n_1]$  is a variational parameter, and by minimizing the *chemical potential*  $\mu_{1.e.}$ . This simple calculation yields the optimized width  $w_\perp[n_1] = a_\perp(1 + 4an_1)^{1/4}$ , where  $a_\perp = \sqrt{\hbar/M\omega_\perp}$  is the radial oscillator length, and the local chemical potential

$$\mu_{1.e.}[n_1] = \hbar\omega_\perp \sqrt{1 + 4an_1}. \quad (4)$$

The very good agreement of Eq. (4) with the available numerical results [23], illustrated in Fig. 1a, has been pointed out in [22] on a phenomenological basis. Here we show that this expression follows from the condition that  $\mu_{1.e.}$  is the lowest eigenvalue of the Gross-Pitaevskii equation. Note that this approximation does not correspond to a separability *ansatz*, as  $n_1$  is in general a function of the axial coordinate  $z$ . In this respect, the method used here differs from the work reported in [21].

We now reintroduce the axial trapping potential, assuming for definiteness a harmonic form,  $V(z) = \frac{1}{2}M\omega_z^2 z^2$ . From the local equilibrium condition (3) and the equation of state (4), one finds the density profile of the trapped gas,

$$\begin{aligned} n_1(z) &= \frac{1}{16a} \frac{V(L) - V(z)}{\hbar\omega_\perp} \left[ \frac{V(L) - V(z)}{\hbar\omega_\perp} + 1 \right] \\ &= \frac{\alpha}{4a} (1 - \tilde{z}^2) [\alpha(1 - \tilde{z}^2) + 4]. \end{aligned} \quad (5)$$

Here we have introduced the parameter  $\alpha = 2(\mu/\hbar\omega_\perp - 1)$  and  $\tilde{z} = z/L$ . The condensate length  $L$  is defined by the relation  $\mu_{1.e.}[n_1(L) = 0] + \frac{1}{2}M\omega_z^2 L^2 = \mu$ , and is given explicitly by

$$L = \frac{a_z^2}{a_\perp} \sqrt{\alpha}, \quad (6)$$

where  $a_z = \sqrt{\hbar/M\omega_z}$  is the axial oscillator length. Using  $\int_{-L}^L n_1(z) dz = N$ , and the density profile (5), we obtain the equation for the key quantity  $\alpha$ ,

$$\alpha^3(\alpha + 5)^2 = (15\chi)^2. \quad (7)$$

The only parameter of the calculation,  $\chi = Naa_\perp/a_z^2$ , roughly gives the ratio of the interaction energy to the radial zero-point energy [23]. Numerical solution of Eq. (7) is straightforward, and obtains the static properties of the condensate at any confinement strength. In the limit  $\chi \gg 5$ , the mean-field interaction dominates over the transverse confinement, and one recovers the well-known 3D TF result,  $\alpha \approx \alpha_{3D} = (15\chi)^{2/5}$ . Conversely, if  $\chi \ll 5$ , the transverse motion is almost frozen and one finds  $\alpha \approx \alpha_{1D} = (3\chi)^{2/3}$ . The crossover between the two regimes occurs approximately for  $\alpha_{1D} = \alpha_{3D}$ , giving a cross-over value  $\chi_{\text{cross}} = 5^{3/2}/3 \approx 3.73$ . Our analytical results are very accurate even in the crossover region, as shown in Fig. 1b where we compare Eq. (5) to a direct numerical solution of the GP equation.

Equations (5,6,7) are the key results of this paper. The calculation of the density profile for any strength of the confinement in the transverse direction allows to deduce a number of interesting quantities. As an example, we focus in the remainder of the paper on phase fluctuations, a key phenomenon to understand the physics of 1D gases. In general, phase fluctuations originate from the very large population of the 1D excited states of the system, *i.e.* those in the low-energy energy range  $\hbar\omega_z < \epsilon < \hbar\omega_\perp$  (in the following, these excitations will be referred to as “excitations from the axial branch”).

Keeping with the local density approach, we analyze first these excitations in a sample with harmonic radial confinement, but without the axial potential. The axial branch corresponds to excitations characterized by an axial wavenumber  $k$ , and no radial nodes [13]. Following [12], we introduce the operators  $\hat{\phi}$  and  $\delta\hat{n}$  describing fluctuations of the phase and of the density, and their expansion on the set of axial plane waves,  $\delta\hat{n}(z) = \sum_k \delta n_k \mathcal{A}_k(\rho) e^{ikz} \hat{b}_k / \sqrt{2n_1 L} + \text{h.c.}$ , and  $\hat{\phi}(z) = -i \sum_k \phi_k e^{ikz} \hat{b}_k / \sqrt{2n_1 L} + \text{h.c.}$ . The function  $\mathcal{A}_k(\rho)$  describes the radial dependence of density fluctuations, and the operators  $\hat{b}_k$ ,  $\hat{b}_k^\dagger$  destroy and create one quasi-particle with wavevector  $k$ .

Ref. [5] has shown that the elementary excitations of the phase-fluctuating ensemble obey the same Bogoliubov-De Gennes equations than in the usual coherent ensemble. In terms of the Fourier components of

$\hat{\phi}$  and  $\delta\hat{n}$ , they read

$$\hbar\omega_k^{\text{B}}\delta n_k\mathcal{A}_k(\rho) = \frac{\hbar k^2}{M}n_0(\rho)\phi_k, \quad (8)$$

$$\hbar\omega_k^{\text{B}}\phi_k n_0(\rho) = \left( \frac{\hbar^2 k^2}{4M} + Un_0(\rho) \right) \delta n_k\mathcal{A}_k(\rho) \quad (9)$$

$$- \frac{\hbar^2\delta n_k}{4M}\nabla_{\perp} \left[ n_0(\rho)\nabla_{\perp} \left( \frac{\mathcal{A}(\rho)}{n_0(\rho)} \right) \right].$$

The terms involving the radial envelope remain small whatever the transverse confinement for the axial branch of interest. Indeed, the transverse envelope changes continuously from a flat profile in the 3D regime to the radial ground state in the 1D regime. In any case, the quantity  $\mathcal{A}_k/n_0$  is almost flat, and the corresponding spatial derivatives in (9) are very small, being of order  $(\hbar\omega_{\perp})^2/\mu$  in the 3D regime, and going to zero in the 1D limit. We thus neglect their contribution, and average over transverse degrees of freedom to get rid of the remaining radial dependence [13]. The equations obtained in this way are of the usual Bogoliubov form. The excitation spectrum is  $\omega_k^{\text{B}} = \sqrt{\omega_k^2 + c_{\text{1D}}^2 k^2}$ , where the 1D speed of sound is  $c_{\text{1D}}^2(k) = Un_0\mathcal{A}_k/M$ , and where the free particle kinetic energy is  $\omega_k = \hbar k^2/2M$ . The density and phase amplitudes are  $\frac{\delta n_k}{n_1} = (\omega_k/\omega_k^{\text{B}})^{1/2}$ ,  $\phi_k = (\omega_k^{\text{B}}/4\omega_k)^{1/2}$ . In general, the speed of sound may depend on  $k$  through  $n_0\mathcal{A}_k = \int d^{(2)}\rho n_0\mathcal{A}_k$ . This originates from the radial dependence of density fluctuations, which is important in 3D gases for  $k \sim R^{-1}$ : this has been suggested in [24] as a possible mechanism contributing to the decrease of the critical velocity. In this paper, we are interested in phase fluctuations only, which are dominated by the lowest-energy modes,  $k \ll R^{-1}$ . We can therefore safely assume that the radial envelope  $\mathcal{A}_k$  is  $k$ -independent in the following.

We now include the effect of the trapping potential by introducing a local density profile  $n_1(z)$  according to (3) and (5). From the above expressions, one deduces the fluctuating phase profile by replacing the local chemical potential and density envelope everywhere (see [17] for further details). Using  $\langle \hat{b}_k^\dagger \hat{b}_k \rangle \approx k_{\text{B}}T/\hbar\omega_k^{\text{B}}$ , the variance of phase fluctuations then reads

$$\Delta\phi^2(\bar{z}, s) = \langle [\hat{\phi}(z) - \hat{\phi}(z')]^2 \rangle \approx \frac{T}{T_{\phi}} \frac{n_1(0)}{n_1(\bar{z})} \frac{|s|}{L}, \quad (10)$$

with the relative distance  $s = z - z'$  and the mean coordinate  $\bar{z} = (z + z')/2$ . The variance of phase fluctuations depends only on the density profile  $n_1(\bar{z})/n_1(0)$ , and of the phase temperature,  $k_{\text{B}}T_{\phi} = \hbar^2 n_1(0)/ML$ . Together with the density envelope (5), the expression (10) is suf-

ficient to find the spatial correlation function [5],

$$\mathcal{C}^{(1)}\left(\frac{s}{L}\right) = \frac{1}{N} \int_{-L}^L d\bar{z} \sqrt{n_1(\bar{z} + s/2)n_1(\bar{z} - s/2)} \exp\left(-\frac{1}{2}\Delta\phi^2(\bar{z}, s)\right). \quad (11)$$

This expression differs slightly from the one used in [17] in the treatment of the overlap term, defined here as  $\sqrt{n_1(\bar{z} + s/2)n_1(\bar{z} - s/2)}$ . The way we write it here yields better agreement with the numerical calculation of the correlation function in the 3D case (see Fig.2b).

Several comments can be made on this expression. First, already for  $T = 4T_{\phi}$ , the LDA expression (11) agrees well with the numerical calculation of the correlation function based on Refs. [5, 12]. This is shown in Fig.2a for the 3D case, and Fig.2b for the 1D case. Second, the expression (11), which depends on the dimensionless space variable  $s/L$ , is a universal function completely determined by two dimensionless parameters,  $\chi$ , which controls the regime of transverse confinement and the functional form of the density profile, and  $T/T_{\phi}$ , which controls the magnitude of phase fluctuations. The third and most important conclusion is that the resulting correlation function is insensitive to a large extent to the transverse regime of confinement (in other words, to the value of  $\chi$ ). This is illustrated in Fig.2b, where we plot the correlation function for  $T = 4T_{\phi}$  and  $\chi = 100$  (dotted),  $\chi = 1$  (solid) and  $\chi = 10^{-2}$  (dashed). Despite the dissimilar transverse profiles, the axial correlation function are almost identical. This very weak dependence on  $\chi$  points out the almost universal nature of thermal phase fluctuations in elongated trapped gases.

In conclusion, we have investigated in this paper the crossover from a very elongated, 3D Bose gas to a 1D situation where transverse motion is frozen. By relying on a local density approximation, we have been able to compute the radially integrated density profile for any transverse confinement; we believe these results are simple enough to prove useful for the analysis of time of flight images of very elongated samples, with  $\epsilon_{\text{int}} \sim \hbar\omega_{\perp}$ . We have applied the method to the problem of phase fluctuations arising in such geometry at finite temperatures, and have found the correlation function to be quite insensitive to the regime of transverse confinement. This points out the almost ‘‘universal’’ behavior of quasicondensates in an elongated geometry, related to the essentially classical nature of thermal phase fluctuations.

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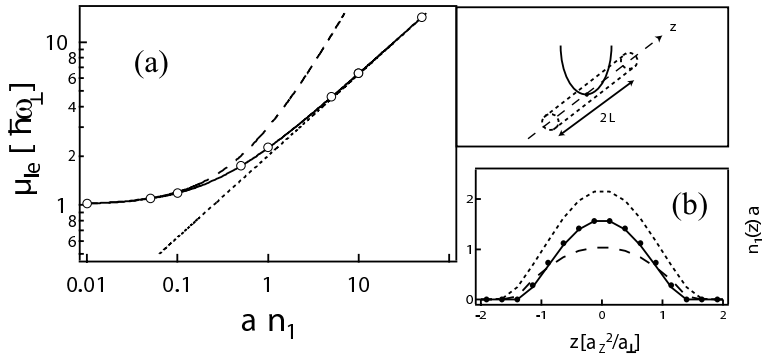


FIG. 1: Accuracy of the local density approximation. **(a)** Local chemical potential in the cylinder geometry near the 3D-1D crossover, as a function of the local 1D density  $n_1$ . The circles show the results of a numerical calculation [23], undistinguishable at the scale of the figure from Eq. (4) [solid line]. The dotted and dashed lines show the 3D and 1D Thomas-Fermi limiting cases. **(b)** Integrated density profiles taking the (harmonic) axial trapping potential into account. To highlight the crossover between shallow and tight transverse confinement, the parameter  $\chi = 1$  has been chosen (corresponding to  $\mu \approx 1.85\hbar\omega_{\perp}$ ). The circles results from a numerical solution of the Gross-Pitaevskii equation, and are almost indistinguishable from the LDA result (solid line) at the scale of the figure. The dotted and dashed lines give the 3D and 1D Thomas-Fermi profiles, extrapolated to  $\chi = 1$  for comparison.

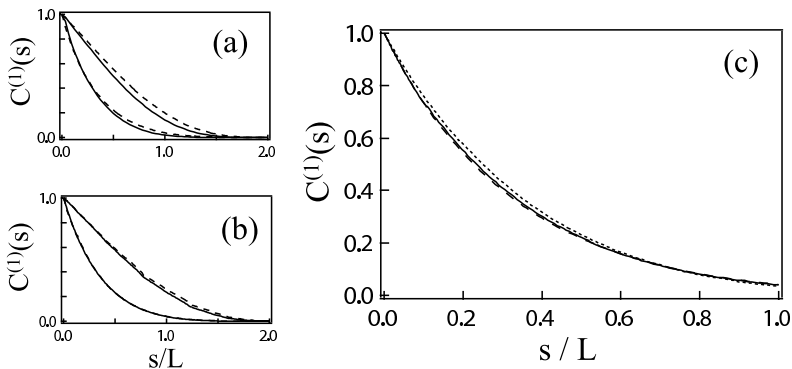


FIG. 2: Spatial correlation function of an elongated, phase-fluctuating condensate for various confinement regimes. In **(a)**, the spatial correlation function is drawn in the 3D case as a function of the reduced distance  $s/L$ , for  $T = T_{\phi}$  (upper curve) and for  $T = 4T_{\phi}$  (lower curve). The solid line follows from a numerical calculation based on the results in [12], and the dashed line is the LDA. Figure **(b)** shows the corresponding curves for the 1D case [5]. In **(c)**, the correlation function of an elongated condensate is plotted for  $T = 4T_{\phi}$  and different regimes of transverse confinement:  $\chi = 100$  (3D case, dotted),  $\chi = 1$  (intermediate case, solid) and  $\chi = 10^{-2}$  (1D case, dashed). Despite 4 orders of magnitude of variations in  $\chi$ , the spatial correlation function is largely unchanged, pointing out to a universal behavior of phase fluctuations in elongated traps.