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Algebraic elimination of ε -transitions

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Abstract

We present here algebraic formulas associating a k -automaton to a k - ε -automaton. The existence depends on the definition of the star of matrices and of elements in the semiring k . For this reason, we present the theorem which allows the transformation of k - ε -automata into k -automata. The two automata have the same behaviour.

Keywords :

Automata with multiplicities, ε -transitions, behaviour, star of matrices.

1 Introduction

Automata with multiplicities (or weighted automata) are a versatile class of transition systems which can modelize as well classical (boolean), stochastic, transducer automata and be applied to various purposes such as image compression, speech recognition, formal linguistic (and automatic treatment of natural languages too) and probabilistic modelling. For generalities over automata with multiplicities see (1) and (10), problems over identities and decidability results on these objects can be found in (13), (12) and (11). A particular type of these automata are the automata with ε -transitions denoted by k - ε -automata that are the result, for example, of the application of Thompson method to transform a weighted regular expression in a weighted automaton (14). The aim of this paper is to study the equivalence between k - ε -automata and k -automata. Indeed, we will present here an algebraic method in order to compute, for any weighted automaton with ε -transitions

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an equivalent weighted automaton without ε -transitions which has the same behaviour. Here, the closure of ε -transitions depends only on the existence of the star of transition matrix of ε . Its running time complexity is deduced from that of the matrix multiplication in $k^{n \times n}$. In the case of well-known semiring (boolean and tropical), the closure is computed in $O(n^3)$ (15). We fit the running time complexity when k is a ring.

The structure of the paper is the following. We first recall (in Section 2) the notion of a semiring and the computation of the star of matrices. After introducing (in Section 3) the notion of k -automaton and k - ε -automaton, we present (in Section 4 and 5) our principal result which is a method of elimination of ε -transitions. In Section 6, we give the equivalence between two types of automata and discuss its validity. A conclusion section ends the paper.

2 Semirings

In the following, a semiring $(k, \oplus, \otimes, 0_k, 1_k)$ is a set together with two laws and their neutrals. More precisely $(k, \oplus, 0_k)$ is a commutative monoid with 0_k as neutral and $(k, \otimes, 1_k)$ is a monoid with 1_k as neutral. The product is distributive with respect to the addition and zero is an annihilator ($0_k \otimes x = x \otimes 0_k = 0_k$) (7). For example all rings are semirings, whereas $(\mathbb{N}, +, \times, 0, 1)$, the boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ and the tropical semiring $\mathbb{T} = (\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$ are well-known examples of semirings that are not rings. The star of a scalar is introduced by the following definition:

Definition 1 *Let $x \in k$, the scalar y is a right (resp. left) star of x if and only if $x \otimes y \oplus 1_k = y$ (resp. $y \otimes x \oplus 1_k = y$).*

If $y \in k$ is a left and right star of $x \in k$, we say that y is a star for x and we write $y = x^{\otimes}$.

Example 1

1. For $k = \mathbb{C}$, any complex number $x \neq 1$ has a unique star which is $y = (1-x)^{-1}$. In the case $|x| < 1$, we observe easily that $y = 1+x+x^2+\dots$.
2. Let k be the ring of all linear operators $(\mathbb{R}[x] \rightarrow \mathbb{R}[x])$. Let X and Y_α defined by $X(x^0) = 1$, $X(x^n) = x^n - nx^{n-1}$ with $n > 0$ and $Y_\alpha(x^n) = (n+1)^{-1}x^{n+1} + \alpha$ with $\alpha \in \mathbb{R}$. Then $XY_\alpha + 1 = Y_\alpha$ and

there exists an infinite number of solutions for the right star (which is not a left star if $\alpha \neq 0$).

3. For $k = \mathbb{T}$ (tropical semiring), any number x has a unique star $y = 0$.

We can observe that if the opposite $-x$ of x exists then right (resp. left) stars of x are right (resp. left) inverses of $(1 \oplus (-x))$ and conversely. Any right star $x^{\otimes r}$ equals any left star $x^{\otimes l}$ as $x^{\otimes l} = x^{\otimes l} \otimes ((1 \oplus (-x)) \otimes x^{\otimes r}) = (x^{\otimes l} \otimes (1 \oplus (-x))) \otimes x^{\otimes r} = x^{\otimes r}$. Thus, in this case, the star is unique. This remark explains the expressions right and left star.

If n is a positive integer then the set $k^{n \times n}$ of square matrices with coefficients in k has a natural structure of semiring with the usual operations (sum and product). The (right) star of $M \in k^{n \times n}$ (when there exists) is a solution of the equation $MY + 1_{n \times n} = Y$ (where $1_{n \times n}$ is the identity matrix). Let $M \in k^{n \times n}$ given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} \in k^{p \times p}$, $a_{12} \in k^{p \times q}$, $a_{21} \in k^{q \times p}$ and $a_{22} \in k^{q \times q}$ such that $p+q = n$. Let $N \in k^{n \times n}$ given by

$$N = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with

$$A_{11} = (a_{11} + a_{12}a_{22}^*a_{21})^* \quad (1)$$

$$A_{12} = a_{11}^*a_{12}A_{22} \quad (2)$$

$$A_{21} = a_{22}^*a_{21}A_{11} \quad (3)$$

$$A_{22} = (a_{22} + a_{21}a_{11}^*a_{12})^* \quad (4)$$

Theorem 1 *If Formulas (1), (2), (3) and (4) are defined, the matrix M admits N as a right star.*

Proof. We show that N is a solution of the equation $My + 1_{n \times n} = y$. By computation, one has

$$\begin{aligned} MN + 1 &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} 1_{p \times p} & 0_{p \times q} \\ 0_{q \times p} & 1_{q \times q} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}A_{11} + a_{12}A_{21} + 1_{p \times p} & a_{11}A_{12} + a_{12}A_{22} \\ a_{21}A_{11} + a_{22}A_{21} & a_{21}A_{12} + a_{22}A_{22} + 1_{q \times q} \end{pmatrix} \end{aligned}$$

where $0_{p \times q}$ is the zero matrix in $k^{p \times q}$. We verify the relations (1), (2), (3) and (4) by:

$$\begin{aligned} a_{11}A_{11} + a_{12}A_{21} + 1_{p \times p} &= a_{11}A_{11} + a_{12}a_{22}^*a_{21}A_{11} + 1_{p \times p} \\ &= A_{11}(a_{11} + a_{12}a_{22}^*a_{21}) + 1_{p \times p} \\ &= A_{11} \end{aligned}$$

$$\begin{aligned} a_{11}A_{12} + a_{12}A_{22} &= a_{11}a_{11}^*a_{12}A_{22} + a_{12}A_{22} \\ &= (a_{11}a_{11}^* + 1)a_{12}A_{22} \\ &= a_{11}^*a_{12}A_{22} \\ &= A_{12} \end{aligned}$$

$$\begin{aligned} a_{21}A_{11} + a_{22}A_{21} &= a_{21}A_{11} + a_{22}a_{22}^*a_{21}A_{11} \\ &= (1 + a_{22}a_{22}^*)a_{21}A_{11} \\ &= a_{22}^*a_{21}A_{11} \\ &= A_{21} \end{aligned}$$

$$\begin{aligned} a_{21}A_{12} + a_{22}A_{22} + 1_{q \times q} &= a_{21}a_{11}^*a_{12}A_{22} + a_{22}A_{22} + 1_{q \times q} \\ &= (a_{22}a_{21}a_{11}^*a_{12})A_{22} + 1_{q \times q} \\ &= A_{22} \end{aligned}$$

□

Similar formulas can be stated in the case of the left star. The matrix N is the left star of M with

$$\begin{aligned} A_{11} &= (a_{11} + a_{12}a_{22}^*a_{21})^* \\ A_{12} &= A_{11}a_{12}a_{22}^* \\ A_{21} &= A_{22}a_{21}a_{11}^* \\ A_{22} &= (a_{22} + a_{21}a_{11}^*a_{12})^* \end{aligned}$$

In (8) and (16), analog formulas are expressed for the computation of the inverse of matrices when k is a division ring (it can be extended in the case of rings).

The formulas described above are applied to matrices of even sizes. But for matrices with odd dimensions, the approach called dynamic peeling is applied (9). More specifically, let $M \in k^{n \times n}$ a matrix given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $n \in 2\mathbb{N} + 1$. The dynamic peeling consists of cutting out the matrix in the following way: a_{11} is a $(n - 1) \times (n - 1)$ matrix, a_{12} is a $(n - 1) \times 1$ matrix, a_{21} is a $1 \times (n - 1)$ matrix and a_{22} is a 1×1 matrix.

Theorem 2 *Let k be a semiring. The right (resp. left) star of a matrix of size $n \in \mathbb{N}$ can be computed in $O(n^\omega)$ operations with:*

- $\omega \leq 3$ if k is not a ring,
- $\omega \leq 2.808$ if k is a ring,
- $\omega \leq 2.376$ if k is a field.

Proof. For $n = 2^m \in \mathbb{N}$, let T_m^+ , T_m^\times and T_m^* denote the number of operations \oplus , \otimes and \circledast in k that the addition, the multiplication and the star of matrix respectively perform on input of size n . Then

$$\begin{aligned} T_0^* &= 1 \\ T_m^* &= 2T_{m-1}^+ + 8T_{m-1}^\times + 4T_{m-1}^* \end{aligned}$$

by Theorem 1, for arbitrary n , we add some zeroes at the matrix. If k is a ring, using Strassen's algorithm for the matrix multiplication (19), it is known that at most $16n^{\log_2(7)}$ operations are necessary. If k is a field, using the Coppersmith and Winograd's algorithm (3), it is known that at most $16n^{2.376}$ operations are necessary. \square

The running time complexity for the computation of the right (resp. left) star of a matrix depends on T_\oplus , T_\otimes and T_\circledast . But it depends also on the representation of coefficients in machine. In the case $k = \mathbb{Z}$ for example, the multiplication of two integers is computed in $O(m \log(m) \log(\log(m)))$, using FFT if m bits is necessary (18).

Theorem 3 *The space complexity of the right (resp. left) star of a matrix of size $n \in \mathbb{N}$ is $O(n^2 \log(n))$.*

Proof. For $n = 2^m \in \mathbb{N}$ and k a semiring, let E_m^* denote the space complexity of operation $*$ that the star of matrix perform on input of size n . Then

$$\begin{aligned} E_0^* &= 1 \\ E_m^* &= 12 \cdot 2^{2m-1} + 4E_{m-1}^* \end{aligned}$$

□

The running of the algorithm needs the reservation of memory spaces for the result matrix (the star of the input matrix) and for intermediate results stored in temporary locations.

If Σ is a finite alphabet, let $k\langle\langle\Sigma\rangle\rangle$ be the set of noncommutative formal series. It is a semiring equipped with $+$ the sum and \cdot the Cauchy product. We denote by $\alpha(?)$ and $(?)\alpha$ the left and right external product respectively. The star $(?)*$ of a formal series is well-defined if and only if the star of the constant term exists (10; 1). The set $\text{RAT}_k(\Sigma)$ is the closure of the alphabet Σ by the sum, the Cauchy product and the star.

3 Automata with multiplicities

Let Σ be a finite alphabet. A weighted automaton (or linear representation) of dimension n on Σ with multiplicities in k is a triplet (λ, μ, γ) where:

- $\lambda \in k^{1 \times n}$ (**the input vector**),
- $\mu : \Sigma \rightarrow k^{n \times n}$ (**the transition function**),
- $\gamma \in k^{n \times 1}$ (**the output vector**).

Such automaton is usually drawn by a directed valued graph (see Figure 1). A transition $(i, a, j) \in \{1, \dots, n\} \times \Sigma \times \{1, \dots, n\}$ connects the state i with the state j . Its weight is $\mu(a)_{ij}$. The weight of the initial (final) state i is λ_i (respectively γ_i). The mapping μ induces a morphism of monoid from Σ^* to $k^{n \times n}$. The behaviour of the weighted automaton \mathcal{A} belongs to $k\langle\langle\Sigma\rangle\rangle$. It is defined by:

$$\text{behaviour}(\mathcal{A}) = \sum_{u \in \Sigma^*} (\lambda \mu(u) \gamma) u.$$

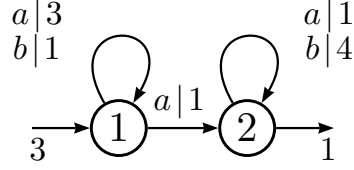


Figure 1: A \mathbb{N} -automaton

More precisely, the weight $\langle \text{behaviour}(\mathcal{A}), u \rangle$ of the word u in the formal series $\text{behaviour}(\mathcal{A})$ is the weight of u for the k -automaton \mathcal{A} (this is an accordance with the scalar product denotation $\langle S|u \rangle := S(u)$ for any function $S : \Sigma^* \rightarrow k$ (2)).

Example 2 The behaviour of the automaton \mathcal{A} of Figure 1 is

$$\text{behaviour}(\mathcal{A}) = \sum_{u,v \in \Sigma^*} 3^{|u|+1} 4^{|v|} uav$$

Let $u = aba$. Then, its weight in \mathcal{A} is:

$$\begin{aligned} \lambda\mu(u)\gamma &= \lambda\mu(a)\mu(b)\mu(a)\gamma \\ &= \begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 21. \end{aligned}$$

The set $\text{REC}_k(\Sigma)$ is known to be equal to the set of series which are the behaviour of a k -automaton. We recall the celebrated result of Schützenberger (17):

$$\text{REC}_k(\Sigma) = \text{RAT}_k(\Sigma).$$

A k - ε -automaton \mathcal{A}_ε is a k -automaton over the alphabet $\Sigma_\varepsilon = \Sigma \cup \tilde{\varepsilon}$ (see Figure 2). We must keep the reader aware that $\tilde{\varepsilon}$ is considered here as a new letter and that there exists an empty word for $\Sigma_\varepsilon^* = (\Sigma \cup \tilde{\varepsilon})^*$ denoted here by ε . The transition matrix of $\tilde{\varepsilon}$ is denoted $\mu_{\tilde{\varepsilon}}$.

Example 3 In Figure 2, the behaviour of the automaton \mathcal{A}_ε is

$$\text{behaviour}(\mathcal{A}_\varepsilon) = 18\tilde{\varepsilon} \left(\sum_{i \in \mathbb{N}} 2^i (a\tilde{\varepsilon})^i \right) \tilde{\varepsilon} = 18\tilde{\varepsilon}(2a\tilde{\varepsilon})^* \tilde{\varepsilon}.$$

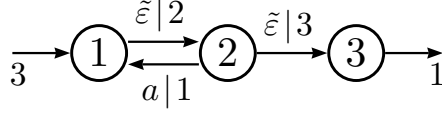


Figure 2: A \mathbb{N} - ε -automaton

4 Algebraic elimination

Let Φ be the morphism from Σ_ε^* to Σ^* induced by

$$\begin{cases} \Phi(x) = x & \text{if } x \in \Sigma, \\ \Phi(\tilde{\varepsilon}) = \varepsilon \end{cases}$$

We remark that the set of antecedents of $u = a_1 a_2 \dots a_n \in \Sigma^*$ by Φ can be written $\tilde{\varepsilon}^* a_1 \tilde{\varepsilon}^* a_2 \dots \tilde{\varepsilon}^* a_n \tilde{\varepsilon}^*$. For $S \in k\langle\langle \Sigma_\varepsilon \rangle\rangle$, define

$$\Phi(S) = \sum_{u \in \Sigma^*} \left(\sum_{\Phi(v)=u} \langle S|v \rangle \right) u.$$

if $\sum_{\Phi(v)=u} \langle S|v \rangle$ is defined.

Theorem 4 *Let $S \in RAT_k(\Sigma_\varepsilon)$. If $\Phi(S)$ exists then $\Phi(S) \in RAT_k(\Sigma)$.*

The following result (whose proof is straightforward) will be useful for the proof of Theorem 4:

Lemma 1 *Let $x \in k$ and $\varphi : k \rightarrow k'$ a morphism of semirings. If x^\otimes exists then $\varphi(x^\otimes) = \varphi(x)^\otimes$.*

In Lemma 1, we should replace \otimes by \otimes_l or \otimes_r .

Proof of Theorem 4. Let (λ, μ, γ) be a linear representation with S as behaviour. Let $u = a_1 a_2 \cdots a_n \in \Sigma^*$. The weight of u is:

$$\begin{aligned} \langle \Phi(S)|u \rangle &= \sum_{\Phi(v)=u} \langle S|v \rangle \\ &= \sum_{\Phi(v)=u} \lambda \mu(v) \gamma \\ &= \lambda \sum_{\Phi(v)=u} \mu(v) \gamma \\ &= \lambda \mu_{\tilde{\varepsilon}}^* \mu(a_1) \mu_{\tilde{\varepsilon}}^* \mu(a_2) \cdots \mu_{\tilde{\varepsilon}}^* \mu(a_n) \mu_{\tilde{\varepsilon}}^* \gamma \end{aligned}$$

because Lemma 1 gives $\mu(\tilde{\varepsilon}^*) = \mu_{\tilde{\varepsilon}}^*$. Therefore $\Phi(S)$ is the behaviour of the automaton $(\lambda', \mu', \gamma')$ with $\lambda' = \lambda$, $\mu'(a) = \mu_{\tilde{\varepsilon}}^* \mu(a)$ and $\gamma' = \mu_{\tilde{\varepsilon}}^* \gamma$. \square

At this moment, we can study under which conditions the star of $\mu_{\tilde{\varepsilon}}$ exists.

5 Φ -finite series

Let $S \in k\langle\langle \Sigma_{\varepsilon} \rangle\rangle$ be a formal series. The support of S is given by:

$$\text{support}(S) = \{v \in \Sigma_{\varepsilon}^* : \langle S, v \rangle \neq 0\}$$

We will call **(FF)** the following condition:

(FF) For any $u \in \Sigma^*$, the set $\text{support}(S) \cap (\Phi^{-1}(u))$ is finite.

Condition **(FF)** is equivalent to say that there exists an integer n such that any word $v \in \text{support}(S) \cap (\Phi^{-1}(u))$, which verifies $|v| > n$, does not contain $\tilde{\varepsilon}$. If the formal series S satisfies **(FF)**, we say that it is Φ -finite. The set of Φ -finite series in $k\langle\langle \Sigma_{\varepsilon} \rangle\rangle$ is denoted $(k\langle\langle \Sigma_{\varepsilon} \rangle\rangle)_{\Phi\text{-finite}}$.

Theorem 5 *The set $(k\langle\langle \Sigma_{\varepsilon} \rangle\rangle)_{\Phi\text{-finite}}$ is stable with $+$, \cdot , $\alpha(?)$ and $(?)\alpha$.*

Proof. As $\text{support}(S_1 + S_2) \subseteq \text{support}(S_1) \cup \text{support}(S_2)$, $\text{support}(\alpha S_1) = \text{support}(S_1)$ and $\text{support}(S_1 \alpha) = \text{support}(S_1)$ for $S_1, S_2 \in k\langle\langle \Sigma_{\varepsilon} \rangle\rangle$ and $\alpha \in k$, the stability is shown for $+$, $\alpha(?)$ and $(?)\alpha$. Now, for the Cauchy product, one has:

$$\text{support}(S_1 S_2) \subseteq \text{support}(S_1) \text{support}(S_2) \quad (5)$$

$$\Phi^{-1}(u) = \bigcup_{u=u_1 u_2} \Phi^{-1}(u_1) \Phi^{-1}(u_2) \quad (6)$$

for any word u such that $|u| > 1$. In this case,

$$\text{support}(S_1 S_2) \cap \Phi^{-1}(u) \subseteq \bigcup_{u=u_1 u_2} (\text{support}(S_1) \cap \Phi^{-1}(u_1)) (\text{support}(S_2) \cap \Phi^{-1}(u_2))$$

which is a finite set if $S_1, S_2 \in (k\langle\langle\Sigma_\varepsilon\rangle\rangle)_{\Phi\text{-finite}}$. When $u = \varepsilon$, there exists n_1 such that, for all $n > n_1$, $\tilde{\varepsilon}^n \notin \text{support}(S_1)\text{support}(S_2)$. When $u = a \in \Sigma$, there exists m_1 and n_1 such that, for all $m > m_1$ and $n > n_1$, $\tilde{\varepsilon}^m a \tilde{\varepsilon}^n \notin \text{support}(S_1)\text{support}(S_2)$. In these two previous cases, the set $\text{support}(S_1 S_2) \cap \Phi^{-1}(u)$ is finite by Formula (5). \square

Remark 1

- Every polynomial is Φ -finite,
- The star S^* need not be Φ -finite even if S is Φ -finite. The simple example is provided by $S = \tilde{\varepsilon}$.

Next we show that $\Phi : (k\langle\langle\Sigma_\varepsilon\rangle\rangle)_{\Phi\text{-finite}} \rightarrow k\langle\langle\Sigma\rangle\rangle$ is a polymorphism.

Theorem 6 For any $S, T \in (k\langle\langle\Sigma_\varepsilon\rangle\rangle)_{\Phi\text{-finite}}$,

$$\begin{aligned} \Phi(S + T) &= \Phi(S) + \Phi(T) \\ \Phi(ST) &= \Phi(S)\Phi(T) \\ \Phi(\alpha S) &= \alpha\Phi(S) \\ \Phi(S\alpha) &= \Phi(S)\alpha \end{aligned}$$

and if the terms are well-defined

$$\Phi(S^*) = (\Phi(S))^*$$

Proof. For the sum and the Cauchy product, we obtain the result by the following relations:

$$\begin{aligned} \sum_{v \in \Phi^{-1}(u)} \langle S + T, v \rangle &= \sum_{v \in \Phi^{-1}(u)} \langle S, v \rangle \oplus \sum_{v \in \Phi^{-1}(u)} \langle T, v \rangle \\ \sum_{v \in \Phi^{-1}(u)} \langle ST, v \rangle &= \sum_{u=u_1 u_2} \left(\sum_{v \in \Phi^{-1}(u_1)} \langle S, v \rangle \otimes \sum_{v \in \Phi^{-1}(u_2)} \langle T, v \rangle \right) \end{aligned}$$

Then $\Phi(S^*)$ is a solution of the equation $Y = \varepsilon + \Phi(S)Y$ as $S^* = \varepsilon + SS^*$, and $\Phi(S^*) = \Phi(S)^*$. \square

The image of Φ -finite series always exists. By Theorem 4:

Theorem 7 *Let $S \in (k\langle\langle\Sigma\rangle\rangle)_{\Phi\text{-finite}}$. If $S \in \text{RAT}_k(\Sigma)$ then $\Phi(S) \in \text{RAT}_k(\Sigma)$.*

We can observe easily that a formal series of $\Phi^{-1}(S)$ may be not rational and $S \in \text{RAT}_k(\Sigma)$.

Example 4 Consider the series in $\mathbb{N}\langle\langle\Sigma\rangle\rangle$

$$S = \sum_{|u|_a = |u|_\varepsilon} u.$$

It is not rational and it is Φ -finite.

We recall that a matrix $M \in k^{n \times n}$ is nilpotent if there exists a positive integer $\ell \geq n$ such that $M^\ell = 0$.

Proposition 1 *Let $\mathcal{A}_\varepsilon = (\lambda, \mu, \gamma)$ be a weighted ε -automaton. If μ_ε is nilpotent then $\text{behaviour}(\mathcal{A}_\varepsilon)$ satisfies **(FF)**.*

Proof. One has $\langle \text{behaviour}(\mathcal{A}_\varepsilon), \tilde{\varepsilon}^i \rangle = \lambda \mu_\varepsilon^i \gamma$. \square

6 Equivalence

Theorem 4 deals with an algebraic method to eliminate the ε -transitions from a weighted ε -automaton \mathcal{A}_ε . The result is a weighted automaton with $\Phi(S)$ as behaviour if $\text{behaviour}(\mathcal{A}_\varepsilon) = S$.

Theorem 8 *Let $\mathcal{A}_\varepsilon = (\lambda, \mu, \gamma)$ be a weighted ε -automaton then there exists a weighted automaton $\mathcal{A} = (\lambda', \mu', \gamma')$ such that*

$$\text{behaviour}(\mathcal{A}) = \Phi(\text{behaviour}(\mathcal{A}_\varepsilon)).$$

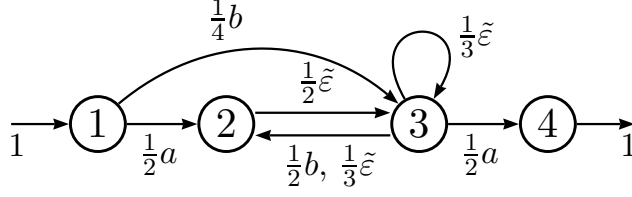


Figure 3: A \mathbb{Q} - ε -automaton

Theorem 2 gives the lower bounds if the set of coefficients is a semiring (resp. ring, field).

Proposition 2 *Let k be a semiring. The elimination of ε -transitions is computed in $O((|\Sigma| + 1) \times n^\omega)$ if n is the dimension of the weighted ε -automaton.*

Proof. First we compute the matrix μ_ε^* . Then set $\lambda' = \lambda$, $\gamma' = \mu_\varepsilon^* \gamma$ and $\mu'(a) = \mu_\varepsilon^* \mu(a)$ for each letter $a \in \Sigma$. \square

Remark 2 One could also with the same result set $\lambda' = \lambda \mu_\varepsilon^*$, $\mu'(a) = \mu(a) \mu_\varepsilon^*$ for each letter $a \in \Sigma$ and $\gamma' = \gamma$.

In the next example, we will apply our algebraic method on a \mathbb{Q} - ε -automaton.

Example 5 The linear representation of Figure 3 is:

$$\lambda = (1 \ 0 \ 0 \ 0), \quad \mu_\varepsilon = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mu(a) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mu(b) = \begin{pmatrix} 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and } \gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

By computation:

$$\lambda' = (1 \ 0 \ 0 \ 0), \quad \gamma' = \mu_\varepsilon^* \gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mu_\varepsilon^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 1 & 0 \\ 0 & \frac{2}{3} & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

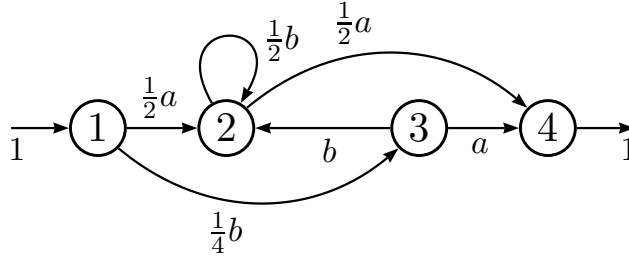


Figure 4: A \mathbb{Q} -automaton

$$\mu'(a) = \mu_{\varepsilon}^* \mu(a) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mu'(b) = \mu_{\varepsilon}^* \mu(b) = \begin{pmatrix} 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The resulting automaton is presented in Figure 4 and its linear representation is $(\lambda', \mu', \gamma')$.

7 Conclusion

Algebraic elimination for ε -automata has been presented. The problem of removing the ε -transitions is originated from generic ε -removal algorithm for weighted automata (15) using Floyd-Warshall and generic single-source shortest distance algorithms. Here, we have the same objective but the methods and algorithms are different. In (15), the principal characteristics of semirings used by the algorithm as well as the complexity of different algorithms used for each step of the elimination are detailed. The case of acyclic and non acyclic automata are analysed differently. Our algorithm here works with any semiring and the complexity is unique for the case of acyclic or non acyclic automata. It is more efficient when the considered semiring is a ring.

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