Square-tiled surfaces in H(2)

Pascal Hubert, Samuel Lelièvre

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Abstract. This is a study of square-tiled translation surfaces in the stratum $\mathcal{H}(2)$ and their $\mathrm{SL}(2,\mathbf{R})$-orbits or Teichmüller discs, which are arithmetic. We prove that for prime $n > 3$ translation surfaces tiled by n squares fall into two Teichmüller discs, only one of them with elliptic points, and that the genus of these discs has a cubic growth rate in n.

Keywords: rational billiards, Teichmüller discs, square-tiled surfaces, Weierstrass points.

MSC: 32G15 (37C35 37D50 30F30 14H55 30F35)

Contents

1. Introduction

In his fundamental paper of 1989, Veech studied the finite-volume Teichmüller discs. Surfaces with such discs, called Veech surfaces, enjoy very interesting dynamical properties: their directional flows are either completely periodic or uniquely ergodic. An abundant literature

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exists on Veech surfaces: Veech [Ve89, Ve92], Gutkin–Judge [GuJu96, GuJu00], Vorobets [Vo], Ward [Wa], Kenyon–Smillie [KeSm], Hubert–Schmidt [HuSc00, HuSc01], Gutkin–Hubert–Schmidt [GuHuSc], Calta [Ca], McMullen [Mc] ...

The simplest examples of Veech surfaces are translation coverings of the torus, called square-tiled surfaces. They are those translation surfaces whose stabilizer in $\mathrm{SL}(2,\mathbf{R})$ is arithmetic (commensurable with $\mathrm{SL}(2,\mathbf{Z})$), after a theorem by Gutkin and Judge. These surfaces (and many more!) were introduced by Thurston [Th] and studied on the dynamical aspect by Gutkin [Gu], Veech [Ve87] and Gutkin–Judge [GuJu96, GuJu00]. Square-tiled surfaces have been used for explicit computations of volumes of strata by Zorich [Zo] and Eskin–Okounkov [EsOk]: square-tiled surfaces correspond to integer points of the moduli space of holomorphic differential forms; the volume of a stratum is computed from the asymptotic number of integer points in a large ball.

1.1. **Main results.** In this paper, we study square-tiled surfaces in the stratum $\mathcal{H}(2)$. This stratum is the moduli space of forms with a unique zero on a surface of genus 2. Our study restricts to the case of a prime number of squares. We show:

Theorem 1.1. *For any prime $n \geqslant 5$, the $\mathrm{SL}(2,\mathbf{R})$-orbits of n-square-tiled surfaces in $\mathcal{H}(2)$ form two Teichmüller discs $D_A(n)$ and $D_B(n)$.*

Theorem 1.2. *$D_A(n)$ and $D_B(n)$ can be seen as the unit tangent bundles to orbifold surfaces with the following asymptotic behavior:*

- *genus $\sim c\,n^3$, with $c_A = c_B = (3/8)(1/12)$*
- *area $\sim c\,n^3$, with $c_A = c_B = (3/8)(\pi/3)$*
- *number of cusps $\sim c\,n^2$, with $c_A = 1/24$ and $c_B = 1/8$*
- *number of elliptic points $O(n)$, one of them having none*

Proposition 1.3. *All these discs arise from L-shaped billiards.*

1.2. **Side results.** We find the following as side results of our study:

- One-cylinder directions

Proposition 1.4. *All surfaces in $\mathcal{H}(2)$ tiled by a prime number of squares have one-cylinder directions i.e. directions in which they decompose into one single cylinder.*

- Discs without elliptic points

During some time, the quest for new Veech surfaces was focused on examples arising from billiards in rational-angled polygons. Such surfaces necessarily have elliptic elements in their Veech groups as soon as all angles are not multiples of the right angle. Surfaces with all angles

multiples of the right angle have however recently re-entered the scene, especially L-shaped billiards (see [Mc]).

- Discs of (arbitrary high) positive genus

When a surface has a positive genus Teichmüller disc, the subgroup of its Veech group generated by parabolic elements has infinite index, hence one could not conclude that the Veech group is a lattice only by looking for parabolic elements. This makes Gutkin and Judge's theorem (see section 2.2 and appendix C) all the more powerful.

One could probably show that the Veech groups of the surfaces arising from billiards in regular polygons, studied by Veech in [Ve92] have positive genus when the number of sides is large enough, however Veech does not state this explicitly. Furthermore, our examples give families of Teichmüller discs of arbitrarily high genus while the translation surfaces in these discs stay in genus 2, whereas the genus of surfaces of regular polygonal billiards tends to infinity.

- Veech groups that are non-congruence subgroups of $\mathrm{SL}(2,\mathbf{Z})$

Since we deal with families of subgroups of $\mathrm{SL}(2,\mathbf{Z})$, it is natural to check whether they belong to the well-known family of congruence subgroups. Appendix A provides an example of Veech group that is a non-congruence subgroup of $\mathrm{SL}(2,\mathbf{Z})$.

- Deviation from the mean order

Proposition 1.5. *The number of n-square-tiled surfaces in $\mathcal{H}(2)$ for prime n is asymptotically $1/\zeta(4)$ times the mean order of the number of n-square-tiled surfaces in $\mathcal{H}(2)$.*

1.3. **Methods.** We parameterize square tiled surfaces in $\mathcal{H}(2)$ by using separatrix diagrams as in [KoZo], [Zo] and [EsMaSc]. These coordinates bring the study of Teichmüller discs of n-square-tiled surfaces down to a combinatorial problem.

We want to describe the $\mathrm{SL}(2,\mathbf{Z})$ orbits of these surfaces. Using the fact that $\mathcal{H}(2)$ is a hyperelliptic stratum, the combinatorial representation of Weierstrass points allows us to show there are at least two orbits for odd $n \geqslant 5$. Showing there are only two is done for prime n in a combinatorial way, by a careful study of the action of generators of $\mathrm{SL}(2,\mathbf{Z})$ on square-tiled surfaces.

For the countings, we use generating functions.

1.4. **Remarks.** Our results on countings are very close to the formulae that can be found in [EsMaSc]. Eskin–Masur–Schmoll calculate Siegel–Veech constants for torus coverings in genus 2. In $\mathcal{H}(2)$, these calculations are based on counting the square-tiled surfaces with a given

number of squares. The originality of our work is to count square-tiled surfaces disc by disc.

There are also analogies with Schmoll's work [Schmo]. He computes the explicit Veech groups of tori with two marked points and the quadratic asymptotics for theses surfaces. Some of the methods he uses are intimately linked to those used in our work. The Veech groups he exhibits are all congruence subgroups.

It seems difficult to adapt our methods to obtain analogous results for nonprime n or in other strata. In section 8 we give conjectures for the nonprime n case, supported by strong numerical evidence.

A computer program allows to give all the geometric information on Teichmüller discs of square-tiled surfaces in $\mathcal{H}(2)$. We learned that Schmithüsen [Schmi] has a program to compute the Veech group of any given square-tiled surface. She also found positive genus discs as well as non-congruence Veech groups. Möller [Mö] is able to provide algebraic equations of square-tiled surfaces.

1.5. **Acknowledgements.** We thank Anton Zorich for stating questions and some conjectures. We thank the Institut de Mathématiques de Luminy and the Max-Planck-Institut für Mathematik for excellent welcome and working conditions. We thank Joël Rivat, Martin Schmoll and other participants of the conference 'Dynamique dans l'espace de Teichmüller et applications aux billards rationnels' at CIRM in 2003.

2. Background

2.1. **Translation surfaces, Veech surfaces.** A translation structure on a genus g orientable compact surface S consists in a set of points $\{P_1, \ldots, P_n\}$ and a maximal atlas on $S' = S \setminus \{P_1, \ldots, P_n\}$ with translation transition functions.

A holomorphic 1-form on S induces a translation structure by considering its natural parameters, and its zeros as points $P_1, \ldots, P_n$. A translation structure induced by a holomorphic 1-form is called admissible. All translation structures in this paper are admissible. Slightly abusing vocabulary and notation, we refer to a translation surface (S, ω), or sometimes just S or ω.

A translation structure defines: a complex structure, since translations are conformal; a flat metric with cone-type singularities of angle $2(k_i+1)\pi$ at degree k_i zeros of the 1-form; and directional flows $\mathcal{F}_\theta$ for $\theta \in]-\pi, \pi]$.

Define the singularity type of a 1-form ω to be the multiplicity vector $\sigma = (k_1, \ldots, k_n)$ (recall $k_1 + \ldots + k_n = 2g-2$, all $k_i > 0$). It is invariant by orientation-preserving diffeomorphisms.

The moduli space $\mathcal{H}_g$ of holomorphic 1-forms on S is the quotient of the set of translation structures by the group $\mathrm{Diff}^+(S)$ of orientation-preserving diffeomorphisms. $\mathcal{H}_g$ is stratified by singularity types, the strata are denoted by $\mathcal{H}(\sigma)$.

$\mathrm{SL}(2,\mathbf{R})$ acts on holomorphic 1-forms: if ω is a 1-form, $\{(U,f)\}$ the translation structure given by its natural parameters, and $A \in \mathrm{SL}(2,\mathbf{R})$, then $A\cdot\omega = \{(U, A\circ f)\}$. As is well known, this action (to the left) commutes with that (to the right) of $\mathrm{Diff}^+(S)$ and preserves singularity types. Each stratum $\mathcal{H}(\sigma)$ thus inherits an $\mathrm{SL}(2,\mathbf{R})$ action. The dynamical properties of this action have been extensively studied by Masur and Veech [Ma, Ve82, etc.].

From the behavior of the $\mathrm{SL}(2,\mathbf{R})$-orbit of ω in $\mathcal{H}(\sigma)$ one can deduce properties of directional flows $\mathcal{F}_\theta$ on the translation surface (S,ω). The Veech dichotomy expressed below is a remarkable illustration of this.

Call affine diffeomorphism of (S,ω) an orientation-preserving homeomorphism of S such that the following two conditions hold
- f is a diffeomorphism of S' (and hence permutes points $P_1,\dots,P_n$);
- the derivative of f computed in the natural charts of ω is constant.

The derivative can then be shown to be an element of $\mathrm{SL}(2,\mathbf{R})$.

Affine diffeomorphisms of (S,ω) form its affine group $\mathrm{Aff}(S,\omega)$, their derivatives form its Veech group $V(S,\omega) < \mathrm{SL}(2,\mathbf{R})$, a noncocompact fuchsian group.

Theorem (Veech dichotomy). *If $V(S,\omega)$ is a lattice in $\mathrm{SL}(2,\mathbf{R})$ (i.e. $\mathrm{vol}\big(V(S,\omega)\backslash\mathrm{SL}(2,\mathbf{R})\big) < \infty$) then for each direction θ, either the flow $\mathcal{F}_\theta$ is uniquely ergodic, or all orbits of $\mathcal{F}_\theta$ are compact and S decomposes into a finite number of cylinders of commensurable moduli.*

Cylinder decompositions are further discussed in section 2.3. Translation surfaces with lattice Veech group are called a Veech surfaces.

2.2. **Square-tiled surfaces, lattice of periods.** A translation covering is a map $f : (S_1,\omega_1) \longrightarrow (S_2,\omega_2)$ such that
- f is topologically a ramified covering;
- f maps zeros of ω_1 to zeros of ω_2;
- f is locally a translation in the natural parameters of ω_1 and ω_2.

The simplest examples of Veech surfaces are coverings of the standard torus ramified above the origin. These surfaces are tiled by squares and hence called square-tiled. The Gutkin–Judge theorem states:

Theorem (Gutkin–Judge)**.** *A translation surface (S,ω) is square-tiled if and only if $V(S,\omega)$ is arithmetic (i.e. $\exists H < \mathrm{SL}(2,\mathbf{Z})$, $H < V(S,\omega)$ such that $[\mathrm{SL}(2,\mathbf{Z}) : H] < \infty$ and $[V(S,\omega) : H] < \infty$).*

For this reason, translation surfaces with this property have also been called arithmetic surfaces. Another name for such surfaces is origami.

We give a proof of this theorem in appendix C, very different from the original proof.

Denote by $\Lambda(\omega)$ the subgroup of $\mathbf{R}^2$ generated by connection vectors.

$\Lambda(\omega)$ is the lattice of relative periods of ω.

Lemma 2.1. *A translation surface (S,ω) is square-tiled if and only if $\Lambda(\omega)$ is a rank 2 sublattice of $\mathbf{Z}^2$.*

Proof. If (S,ω) is square-tiled, connection vectors are obviously integer vectors, so they span a sublattice of $\mathbf{Z}^2$. Conversely, let

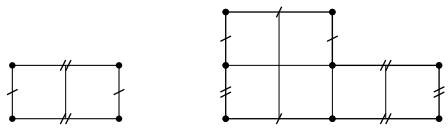
$$\begin{array}{rccl} f : (S,\omega) & \to & \mathbf{R}^2/\Lambda(\omega), & \text{where } z_0 \text{ is a given point of } (S,\omega). \\ z & \mapsto & \int_{z_0}^{z} \omega \bmod \Lambda(\omega), & \end{array}$$

The integral being well-defined modulo the lattice of absolute periods, f is a fortiori well-defined. f is a covering since it is holomorphic and onto. Since relative periods are integer-valued, it is clear that zeros of ω project to the origin. So, given a point $P \neq 0$ on the torus, preimages of P are all regular points, so P is not a branch point. Hence the covering is ramified only above the origin. Composing f with the covering $g : \mathbf{R}^2/\Lambda(\omega) \longrightarrow \mathbf{R}^2/\mathbf{Z}^2$, we get that (S,ω) is square-tiled. □

A square-tiled surface (S,ω) is called primitive if $\Lambda(\omega) = \mathbf{Z}^2$.

Lemma 2.2. *Let (S,ω) be an n-square-tiled surface of genus $g > 1$. If n is prime then $\Lambda(\omega) = \mathbf{Z}^2$.*

Proof. Lemma 2.1 shows that (S,ω) is a ramified cover of $\mathbf{R}^2/\Lambda(\omega)$. Let d be the degree of the covering. Then $n = d \cdot [\mathbf{Z}^2 : \Lambda(\omega)]$. So obviously if n is prime then $\Lambda(\omega) = \mathbf{Z}^2$. □



Note that $\Lambda(\omega)$ is not always $\mathbf{Z}^2$, as shown by the examples in the figure (a torus and a genus 2 cover of this torus, both with $\Lambda(\omega) = 2\mathbf{Z} \times \mathbf{Z}$).

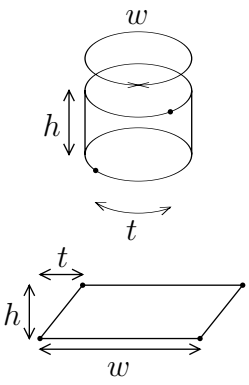
The following lemma was explained to us first by Martin Schmoll then by Anton Zorich.

Lemma 2.3. *Let (S,ω) be a square-tiled surface, then $V(S,\omega)$ is a subgroup in $V(\mathbf{R}^2/\Lambda(\omega)\,, dz)$. In particular, if (S,ω) is primitive, then $V(S,\omega) < \mathrm{SL}(2,\mathbf{Z})$.*

Proof. Let $\phi : \begin{array}{ccc} V(S,\omega) & \to & V(\mathbf{R}^2/\Lambda(\omega)\,, dz), \\ A = df,\ f \in \mathrm{Aff}(S,\omega) & \mapsto & A. \end{array}$

The only difficulty is to show that ϕ is well-defined i.e. that any element A in $V(S,\omega)$ preserves $\Lambda(\omega)$. Since any element of the affine group maps a connection to a connection, hence A maps a connection vector to a connection vector (i.e. an element in $\Lambda(\omega)$). □

2.3. **Cylinders of square-tiled surfaces.** A square-tiled surface decomposes into maximal horizontal cylinders; each cylinder is bounded above and below by unions of saddles and saddle connections. Each saddle connection appears once on the top of a cylinder and once on the bottom of a cylinder, determining the way the cylinders are glued together to make up the surface.



A cylinder on a translation surface is isometric to $\mathbf{R}/w\mathbf{Z} \times [0,h]$, for some h and w.

Convention. We will refer to these dimensions as height and width respectively, whether the 'horizontal direction of the cylinder' coincides with the horizontal direction of the surface or not.

An additional twist parameter t is needed, measuring the distance along the 'horizontal direction of the cylinder' between some (arbitrary) reference points on the bottom and top of the cylinder, for instance some ends of saddle connections.

2.4. **Action of $\mathrm{SL}(2,\mathbf{Z})$ on square-tiled surfaces.**

Lemma 2.4. *Let (S,ω) be a primitive n-square-tiled surface, then its $\mathrm{SL}(2,\mathbf{Z})$-orbit is made of those surfaces in its $\mathrm{SL}(2,\mathbf{R})$-orbit that are also n-square-tiled.*

Proof. $\mathrm{SL}(2,\mathbf{Z})$ preserves $\mathbf{Z}^2$ so it preserves $\Lambda(\omega)$ $(= \mathbf{Z}^2)$, which implies that the $\mathrm{SL}(2,\mathbf{Z})$-orbit of (S,ω) is made of square-tiled surfaces.

Conversely, let $(\Sigma,\alpha) = A(S,\omega)$ with $A \in \mathrm{SL}(2,\mathbf{R})$ be a square-tiled surface. The connection vectors of (Σ,α) are the Av where v are the connection vectors of (S,ω). So A preserves $\mathbf{Z}^2$, so $A \in \mathrm{SL}(2,\mathbf{Z})$. □

Remark. The number of squares, n, is preserved by $\mathrm{SL}(2,\mathbf{R})$ because it is the area of the surface. Consequently $\mathrm{SL}(2,\mathbf{Z}) \cdot (S,\omega)$ is finite.

Notation. We denote by U and R the following matrices: $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (U and R generate $\mathrm{SL}(2,\mathbf{Z})$). We denote by $\mathcal{U}$ the subgroup $\mathcal{U} = \langle U \rangle = \left\{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right),\ n \in \mathbf{Z} \right\}$ of $\mathrm{SL}(2,\mathbf{Z})$ generated by U.

Here is the action of U and R on squares.

The action on square-tiled surfaces is obtained by applying the same to all square tiles. The new horizontal cylinder decomposition is then recovered by cutting and gluing (see example in section 3.4).

2.5. **Hyperelliptic surfaces, Weierstrass points.** Recall that a Riemann surface Σ of genus g is hyperelliptic if there exists a degree 2 meromorphic function on Σ. Such a function induces a holomorphic involution on Σ. This involution has $2g+2$ fixed points called Weierstrass points. The set of these points is invariant by all automorphisms of the complex structure. A translation surface is called hyperelliptic if the underlying Riemann surface is hyperelliptic.

Hyperelliptic translation surfaces have been studied by Veech [Ve95]. He shows that a hyperelliptic surface in genus g is obtained from a centrosymmetric polygon with $4g$ or $4g+2$ sides by pairwise identifying opposite sides (these sides are parallel and have same lengths).

The hyperelliptic involution is in these coordinates the reflection in the center of the polygon; the Weierstrass points are the center of the polygon, the midpoints of its sizes, and the vertices (identified into one point in the $4g$ case, two points in the $4g+2$ case).

2.6. **Cusps.** Let Γ be a fuchsian group. A parabolic element of Γ is a matrix of trace 2 (or -2). A point of the boundary at infinity is parabolic if it is fixed by a parabolic element of Γ. A cusp is a conjugacy class under Γ of primitive parabolic elements (primitive meaning not powers of other parabolic elements of Γ).

Recall that a lattice admits only a finite number of cusps.

Geometrically the neighborhood of a cusp in $\Gamma\backslash\mathbf{H}^2$ is isometric to the quotient of the strip $\{0 < |\operatorname{Re} z| < \lambda,\ \operatorname{Im} z > M\}$ by the translation $z \mapsto z+\lambda$ for some positive λ and M.

On a Veech surface (S,ω), any 'periodic' direction is fixed by a parabolic element of the Veech group (we call such directions parabolic directions). Conversely the eigendirection of a parabolic in the Veech group is a 'periodic' direction. Thus parabolic limit points of $V(S,\omega)$ are cotangents of periodic directions.

When (S,ω) is a square-tiled surface, the set of parabolic limit points is $\mathbf{Q}$. Cusps are therefore equivalence classes of rationals under the homographic action of $V(S,\omega)$. The following lemma gives a combinatorial description of cusps for a square-tiled surface.

Lemma 2.5 (Zorich)**.** *Let (S,ω) be a primitive n-square-tiled surface and $E = \mathrm{SL}(2,\mathbf{Z})\cdot(S,\omega)$ the set of n-square-tiled surfaces in its orbit. The cusps of (S,ω) are in bijection with the $\mathcal{U}$-orbits of E.*

Proof. Let φ : $\begin{array}{ccccc} \mathrm{SL}(2,\mathbf{Z}) & \xrightarrow{f} & \mathbf{Q} & \xrightarrow{\pi} & \mathcal{C}, \\ A & \mapsto & A^{-1}\infty & \mapsto & A^{-1}\infty \bmod V(S,\omega). \end{array}$ where $\mathcal{C}$ is the set of cusps of (S,ω). Note that ∞ corresponds to the horizontal direction in (S,ω) because the projective action is the action on co-slopes and not on slopes. $A^{-1}\infty$ corresponds to the direction on (S,ω) that is mapped by A to the horizontal direction of $A\cdot(S,\omega)$.

φ pulls down as ψ : $\begin{array}{ccc} E & \to & \mathcal{C}, \\ A\cdot(S,\omega) & \mapsto & A^{-1}\infty \bmod V(S,\omega). \end{array}$

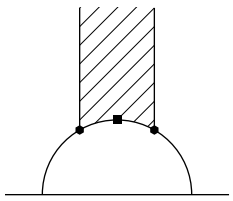
ψ is well-defined: if $A\cdot(S,\omega) = B\cdot(S,\omega)$, then $\exists P \in V(S)$, $B = AP$, so setting $A^{-1}\infty = \alpha$, $B^{-1}\infty = \beta$, we have $\beta = B^{-1}\infty = (B^{-1}A)A^{-1}\infty = P^{-1}\alpha$, so α and β correspond to the same cusp. Further, ψ is surjective because f is. Indeed, $\forall \alpha = p/q$, $\exists A \in \mathrm{SL}(2,\mathbf{Z})$ s.t. $A^{-1}\infty = \alpha$. (The orbit of ∞ under $\mathrm{SL}(2,\mathbf{Z})$ is $\mathbf{Q}$.)

Recall that the stabilizer of ∞ for the homographic action of $\mathrm{SL}(2,\mathbf{Z})$ is $\mathcal{U}$. If $\psi(S_1,\omega_1) = \psi(S_2,\omega_2)$, where $(S_1,\omega_1) = A\cdot(S,\omega)$ and $(S_2,\omega_2) = B\cdot(S,\omega)$, then $\varphi(A) = \varphi(B)$.

Let $\alpha = f(A) = A^{-1}\infty$ and $\beta = f(B) = B^{-1}\infty$. Since α and β correspond to the same cusp, $\exists P \in V(S)$ s.t. $\beta = P\alpha$. So $\infty = A\alpha = AP^{-1}\beta = AP^{-1}B^{-1}\infty$ which implies $AP^{-1}B^{-1} \in \mathcal{U}$ i.e. $\exists U^k \in \mathcal{U}$ s.t. $AP^{-1} = U^kB$ i.e. $AP^{-1}\cdot(S,\omega) = A\cdot(S,\omega) = U^kB\cdot(S,\omega)$, so that (S_1,ω_1) and (S_2,ω_2) are in the same $\mathcal{U}$-orbit.

Conversely: if $(S_2,\omega_2) = U^k(S_1,\omega_1)$ with $U^k \in \mathcal{U}$, and $(S_2,\omega_2) = B\cdot(S,\omega)$ and $(S_1,\omega_1) = A\cdot(S,\omega)$, then $\psi(S_2,\omega_2) = B^{-1}\infty = A^{-1}U^{-k}\infty = A^{-1}\infty = \psi(S_1,\omega_1)$. □

2.7. **Elliptic points.** Recall that in a fuchsian group Γ, any elliptic element has finite order and is conjugate to a rational rotation.



A fixed point in $\mathbf{H}^2$ of an elliptic element of Γ is called elliptic. Its projection to the quotient $\Gamma\backslash\mathbf{H}^2$ is a cone point, with a curvature default. For instance the modular surface $\mathrm{SL}(2,\mathbf{Z})\backslash\mathbf{H}^2$ has two cone points, of angles π and $2\pi/3$.

Suppose that Γ is the Veech group of a translation surface and has an elliptic point. By applying a convenient element of $\mathrm{SL}(2,\mathbf{R})$, we can suppose that this point is i. The corresponding elliptic element is a rational rotation. The translation surfaces which project to i have this rotation in their Veech group. This roughly means that they have an apparent symmetry. At the Riemann surface level, the rotation is an automorphism of the complex structure (it modifies the vertical direction but not the metric). For genus 1, the cone point i (resp. $e^{i\pi/3}$) of the modular surface corresponds to the square (resp. hexagonal) torus, which has a symmetry of projective order 2 (resp. 3).

One should note that the translation surfaces obtained from rational polygonal billiards always have elliptic elements in their Veech group.

If one writes the angles of a simple polygon as $(k_1\pi/r, \ldots, k_q\pi/r)$, with $k_1 \wedge \ldots \wedge k_q \wedge r = 1$, the covering translation surface is obtained by gluing $2r$ copies by symmetry. The rotation of angle $2\pi/r$ is in the Veech group (this rotation is minus the identity if $r = 2$). Many explicit calculations of lattice Veech groups make use of this remark (see [Ve89], [Vo], [Wa]). Our method is completely different.

2.8. **The Gauss–Bonnet Formula.** Let Γ be a finite index subgroup of $\mathrm{SL}(2, \mathbf{Z})$ containing $-\mathrm{Id}$. The quotient of $\Gamma\backslash\mathrm{SL}(2, \mathbf{R})$ is the unit tangent bundle to an orbifold surface with cusps S_Γ. Algebraic information on the group is related to the geometry of the surface.

Let d be the index $[\mathrm{SL}(2, \mathbf{Z}) : \Gamma]$ of Γ in $\mathrm{SL}(2, \mathbf{Z})$, e_2 (resp. e_3) the number of conjugacy classes of elliptic elements of order 2 (resp. 3) of Γ, e_∞ the number of conjugacy classes of cusps of Γ.

Then the surface S_Γ has hyperbolic area $d\frac{\pi}{3}$, e_2 cone points of angle π, e_3 cone points of angle $\frac{2\pi}{3}$, e_∞ cusps, and its genus g is given by:

The Gauss–Bonnet Formula. $g = 1 + d/12 - e_2/4 - e_3/3 - e_\infty/2$.

3. Specific Tools

In this section we give specific properties of the stratum $\mathcal{H}(2)$, and a combinatorial coordinate system for square-tiled surfaces in $\mathcal{H}(2)$.

3.1. **Hyperellipticity.** First recall that any genus 2 Riemann surface is hyperelliptic. Given a genus 2 Riemann surface X and its hyperelliptic involution τ, any 1-form ω on X satisfies $\tau^*\omega = -\omega$.

In the moduli space of holomorphic 1-forms of genus 2, $\mathcal{H}(2)$ is the stratum of 1-forms with a degree 2 zero (a cone point of angle 6π).

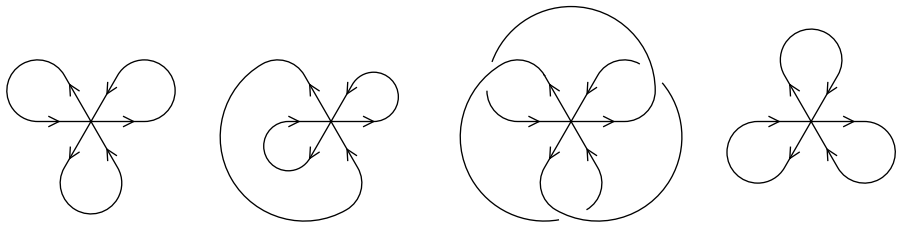
As said in section 2.5, any translation surface in $\mathcal{H}(2)$ can be represented as a centro-symmetric octagon. The six Weierstrass points are the center of the polygon, the middles of the sides and the cone-type singularity. The position of the Weierstrass points in a surface decomposed into horizontal cylinders is described in section 5.1.

3.2. **Separatrix diagrams.** In the stratum $\mathcal{H}(2)$, there is a single zero of degree 2, which is geometrically a cone point of angle 6π. There are hence three outgoing separatrices and three incoming ones at this point in any direction.

Recall that the horizontal direction of a square-tiled surface is completely periodic; the horizontal separatrices are saddle connections. The combinatorics of these connections is called a separatrix diagram in

[KoZo]. The surface is obtained from this diagram by gluing cylinders along the saddle connections.

Each outgoing horizontal separatrices returns to the saddle making an angle π, 3π or 5π with itself. Four separatrix diagrams are combinatorially possible (up to rotation by 2π around the cone point), corresponding to return angles (π, π, π), $(\pi, 3\pi, 5\pi)$, $(3\pi, 3\pi, 3\pi)$, $(5\pi, 5\pi, 5\pi)$:



The first and last diagrams cannot be realized since there is no consistent way of gluing cylinders along their saddle connections to obtain a translation surface. The second diagram is possible with the condition that the saddle connections that return with angles π and 5π have the same length; this diagram corresponds to surfaces with two cylinders. The third diagram corresponds to surfaces with one cylinder, with no restriction on the lengths of the saddle connections.

3.3. **Parameters for square-tiled surfaces in $\mathcal{H}(2)$.** Here we give complete combinatorial coordinates for square-tiled surfaces in $\mathcal{H}(2)$.

3.3.1. 1-*cylinder surfaces.* A one-cylinder surface is parameterized by the height of the cylinder, the lengths of the three horizontal saddle connections (a triple of integers up to cyclic order), and the twist parameter. If all three horizontal saddle connections have the same length, the twist parameter is taken to be less than that length; otherwise, less than the sum of the three lengths.

If we restrict our attention to primitive square-tiled surfaces, a 1-cylinder surface must have height 1. It is therefore determined by 4 integers (the lengths of the 3 saddle connections, whose sum is the area n of the surface, and the twist).

The horizontal saddle connections appear in some (cyclic) order on the bottom of the cylinder, and in reverse order on the top.

3.3.2. 2-*cylinder surfaces.* On a two-cylinder surface, the horizontal saddle connections have return angle π, 3π, 5π. Call ℓ_2 the length of the one that returns with 3π, and ℓ_1 the common length of the other two. One of the cylinders is bounded below by the π saddle connection and above by the 5π one. The other cylinder is bounded below by the

5π and the 3π saddle connections, and above by the π and 3π saddle connections.

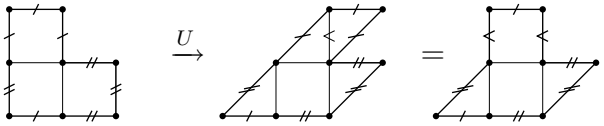
A two-cylinder surface is determined by the heights h_1, h_2 and widths $w_1 = \ell_1$, $w_2 = \ell_1 + \ell_2 > w_1$ of the cylinders as well as two twist parameters t_1, t_2 satisfying $0 \leqslant t_1 < w_1$, $0 \leqslant t_2 < w_2$. The area of the surface is $h_1 w_1 + h_2 w_2 = n$. When n is prime, this implies $h_1 \wedge h_2 = 1$.

Notation. *We use the notation $a \wedge b$ for* $\gcd(a, b)$.

3.4. **Action of** $\mathrm{SL}(2, \mathbf{Z})$**.** The action of R (rotation by $\pi/2$) does not preserve separatrix diagrams in general. The horizontal cylinder decomposition of $R \cdot S$ is the vertical cylinder decomposition of S.

In this paragraph we focus on the action of $\mathcal{U}$ on primitive square-tiled surfaces. U is the primitive parabolic element in $\mathrm{SL}(2, \mathbf{Z})$ that preserves the horizontal direction. We will call U the 'horizontal parabolic'. Acting on surfaces, it preserves separatrix diagrams, as well as heights h_i and widths w_i of horizontal cylinders C_i, and only changes twist parameters t_i to $(t_i + h_i) \bmod w_i$.

Here is an example of how U acts on a surface.



For prime n, given integers a, b, c with $a+b+c = n$, all surfaces made of one horizontal cylinder bounded on the bottom by saddle connections of lengths (a, b, c) (up to cyclic permutation) are in the same $\mathcal{U}$-orbit, and so by Lemma 2.5 in the same cusp.

Two 2-cylinder surfaces are in the same cusp if and only if they have same heights and widths, and their twists t_i are equal modulo $w_i \wedge h_i$.

In the case when n is prime, the heights and widths of two-cylinder surfaces are such that $w_1 \wedge h_1$ and $w_2 \wedge h_2$ are relatively prime, so each cusp has a (unique) representative with twists satisfying $0 \leqslant t_1 < w_1 \wedge h_1$ and $0 \leqslant t_2 < w_2 \wedge h_2$.

4. Results

This section expands the results summarized in the introduction, detailed proofs are postponed to the next sections. Additional conjectures appear in section 8.

4.1. **Two orbits.** Theorem 1.1 can be reformulated as follows:

Proposition 4.1. *Given a prime number $n \geqslant 5$, the square-tiled surfaces in $\mathcal{H}(2)$ tiled by exactly n squares decompose into two orbits under the action of* $\mathrm{SL}(2, \mathbf{Z})$.

The idea for proving this is first to give an invariant that can take two different values and thus prove that there are at least two orbits (see section 4.2 below, and 5.1), then proceed to prove that there are exactly two orbits by showing that each orbit contains a one-cylinder surface (see section 5.2), and that all one-cylinder surfaces with the same invariant are indeed in the same orbit (section 5.3).

We will call these orbits A and B.

Remark. An extension of this result in some components of higher-dimensional strata is presented in appendix B.

4.2. **Invariant.** We present a geometric invariant that can easily be computed for any square-tiled surface in $\mathcal{H}(2)$ (for instance presented in its decomposition into cylinders in the horizontal direction of the tiling.)

Let (S, ω) be an surface in $\mathcal{H}(2)$. Its Weierstrass points are

- the saddle (6π-angle cone point),
- and five regular points.

Lemma 4.2. *The number of integer Weierstrass points of a square-tiled surface is invariant under the action of* $\mathrm{SL}(2, \mathbf{Z})$.

By integer point we mean a vertex of the square tiling. The proof of the lemma is obvious, since $\mathrm{SL}(2, \mathbf{Z})$ preserves $\mathbf{Z}^2$.

Our invariant can be shown to take the following values for primitive square-tiled surfaces.

Proposition 4.3. *The number of integer Weierstrass points of primitive n-square-tiled surfaces in $\mathcal{H}(2)$ is*

- *for even n, exactly 2,*
- *for odd n, either 1 or 3. When $n = 3$ it is always 1, and for greater odd n both values occur.*

4.3. **Elliptic affine diffeomorphisms.**

Proposition 4.4. *A translation surface in $\mathcal{H}(2)$ has no nontrivial translation in its affine group. Hence the derivation from its affine group to its Veech group is an isomorphism.*

Proposition 4.5. *A translation surface in $\mathcal{H}(2)$ can have no elliptic element of order 3 in its Veech group.*

Lemma 4.6. *Any R-invariant Veech surface in $\mathcal{H}(2)$ can be represented as a R-invariant octagon.*

Proposition 4.7. *For any given prime n, there exist R-invariant n-square-tiled $\mathcal{H}(2)$ surfaces. All of them have the same invariant, namely, A if $n = -1 \mod 4$ and B if $n = 1 \mod 4$.*

Remark. This proposition contains the following interesting fact: there are finite-covolume Teichmüller discs with no elliptic points. This differs from the billiard case which has been the main source of explicit examples of lattice Veech groups.

4.4. **Countings.** The number of square-tiled surfaces of area bounded by N in $\mathcal{H}(2)$ is given in [Zo] (see also [EsOk] and [EsMaSc]) to be asymptotically $\zeta(4)\frac{N^4}{24}$ for surfaces with 1 cylinder, $\frac{5}{4}\zeta(4)\frac{N^4}{24}$ for surfaces with 2 cylinders, and thus $\frac{9}{4}\zeta(4)\frac{N^4}{24}$ for all surfaces.

The number of square-tiled surfaces of area exactly n in $\mathcal{H}(2)$ therefore has mean order $\zeta(4)\frac{n^3}{6}$ for surfaces with 1 cylinder, $\frac{5}{4}\zeta(4)\frac{n^3}{6}$ for surfaces with 2 cylinders, $\frac{9}{4}\zeta(4)\frac{n^3}{6}$ for all surfaces.

The following proposition, from which Theorem 1.2 follows, states that for prime n, there are in fact asymptotics for these numbers, which are $\zeta(4)$ times smaller than the mean order.

Proposition 4.8. *For prime n, the following countings and asymptotics hold for surfaces, cusps and elliptic points, according to the number of cylinders and to the orbit.*

surfaces:

	1-*cyl*	2-*cyl*	*all*
A	$\frac{n(n-1)(n+1)}{24}$	$\sim \frac{7}{8}\frac{n^3}{6}$	$\sim \frac{9}{8}\frac{n^3}{6}$
B	$\frac{n(n-1)(n-3)}{8}$	$\sim \frac{3}{8}\frac{n^3}{6}$	$\sim \frac{9}{8}\frac{n^3}{6}$
all	$\frac{n(n-1)(n-2)}{6}$	$\sim \frac{5}{4}\frac{n^3}{6}$	$\sim \frac{9}{4}\frac{n^3}{6}$

cusps:

	1-*cyl*	2-*cyl*	*all*
A	$\frac{(n-1)(n+1)}{24}$	$o(n^{3/2+\varepsilon})$	$\sim \frac{n^2}{24}$
B	$\frac{(n-1)(n-3)}{8}$	$o(n^{3/2+\varepsilon})$	$\sim \frac{n^2}{8}$
all	$\frac{(n-1)(n-2)}{6}$	$o(n^{3/2+\varepsilon})$	$\sim \frac{n^2}{6}$

elliptic points: $O(n)$

This proposition gives more detail than Theorem 1.2 by distinguishing one-cylinder and two-cylinder cusps and surfaces. Proposition 1.4 and Proposition 1.5 are corollaries of this proposition.

Remarks. Orbits A and B have asymptotically the same size (same number of square-tiled surfaces). However orbit B has asymptotically 3 times as many one-cylinder surfaces as orbit A.

In each orbit the number of two-cylinder cusps is asymptotically negligible before the number of one-cylinder cusps; however it is not the case for two-cylinder surfaces. This shows that the average width of the two-cylinder cusps grows faster than n, the width of one-cylinder cusps.

4.5. **Non-congruence subgroups of** $\mathrm{SL}(2,\mathbf{Z})$**.** Some of the stabilizers that appear are non-congruence subgroups of $\mathrm{SL}(2,\mathbf{Z})$. See the case of $n = 5$ in appendix A.

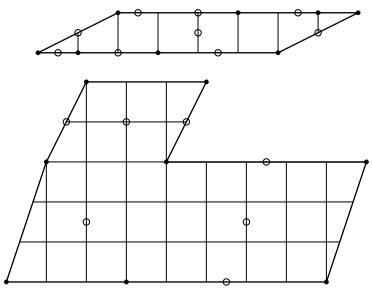
5. Proof of main theorem (two orbits)

In this section we first prove Proposition 4.3, then Proposition 4.1.

Convention for figures. In all figures, a square-tiled surface in $\mathcal{H}(2)$ is represented as an octagon, identification of parallel sides of same length being understood. A black dot represents the saddle, circles are sometimes used to indicate the Weierstrass points.

Throughout this section, these surfaces are decomposed into horizontal cylinders; these cylinders are represented as horizontal parallelograms, identification of their nonhorizontal sides being understood; their horizontal sides are made of horizontal saddle connections, for which the identification pattern is as follows: for one-cylinder surfaces, the saddle connections on the top and on the bottom of the cylinder are identified in opposite cyclic order; for two-cylinder surfaces, with the bottom cylinder wider than the top cylinder, the top and bottom sides of the top cylinder are identified with the left parts of the bottom and top sides of the bottom cylinder, and the right parts of the top and bottom sides of the bottom cylinder are identified with each other.

5.1. **Two values of the invariant.** Here we prove Proposition 4.3, about the possible values of the number of integer Weierstrass points of an primitive square-tiled surface in $\mathcal{H}(2)$.



The saddle is always an integer Weierstrass point. The position of the remaining five points depends on the different possible parities of the parameters.

Recall that the hyperelliptic involution turns the cylinders upside-down. We deduce the position of Weierstrass points in the cylinders (see figure).

• Saddle connections that bound the cylinder both on its top and on its bottom are mapped to themselves with reversed orientation, so that their middle point is fixed. Such Weierstrass points are integer when the length of the saddle connection is even.

• The core circle of a cylinder is also mapped to itself with orientation reversed, so it has two antipodal fixed points. When the height of the cylinder is odd, none of them is integer. When the height is even and the width odd, one of them is integer. When the height and width are even, either both or none is integer, depending on the parity of the twist parameter.

5.1.1. *One-cylinder case.* The core line of the cylinder is at height $1/2$, which gives us two non-integer Weierstrass points. The remaining three Weierstrass points are the middles of the horizontal saddle connections (whose lengths add up to n).

If n is odd, we can have either 3 odd lengths, meaning 3 non-integer Weierstrass points, or 1 odd and 2 even lengths, meaning 2 integer Weierstrass points. Note that for $n = 3$ there is only one possibility (all lengths are 1), while for any greater odd n both values do occur.

If n is even, the three lengths have to be 1 even and 2 odd (if all were even, the surface could not be primitive).

This completes the one-cylinder case.

5.1.2. *Two-cylinder case, odd n.* We use parameters h_1, h_2, w_1, w_2, t_1, t_2 introduced above. We also use ℓ_1 and ℓ_2 the lengths of the horizontal saddle connections. We then have:

$\ell_1 = w_1$, $\ell_1 + \ell_2 = w_2$, $n = w_1h_1 + w_2h_2 = h_1\ell_1 + h_2(\ell_1 + \ell_2)$ $(*)$.

If ℓ_2 is even, then the corresponding saddle connection contains one integer Weierstrass point. Because n is odd, the equation $(*)$ implies that ℓ_1 is odd, thus both cylinders have odd widths, and still by $(*)$ one of the heights must be even. The corresponding cylinder has one integer Weierstrass point on its core line. The total number of integer Weierstrass points is then 3.

If ℓ_2 is odd, the Weierstrass point on the corresponding saddle connection is non-integer; if ℓ_1 is odd (resp. even), then w_2 is even (resp. odd), thus by (*) h_1 has to be odd (resp. h_2 has to be odd), meaning the top (resp. bottom) cylinder contains two non-integer Weierstrass points. The two Weierstrass points in the bottom (resp. top) cylinder are integer if h_2 is even and t_2 is odd (resp. if h_1 and t_1 are even), non-integer otherwise (see figure above). The value of the invariant is accordingly 3 or 1.

Again, for $n = 3$ there is only one possibility (all saddle connection lengths are 1), while for any greater odd n both values do occur.

5.1.3. *Two-cylinder case, even n.* It remains true for even n that h_1 and h_2 have to be relatively prime for the surface to be primitive. In particular one of them at least is odd.

If both heights are odd (which implies the Weierstrass points inside the cylinders are non-integer), then because $n = (h_1 + h_2)\ell_1 + h_2\ell_2$ is even, ℓ_2 has to be even, so the Weierstrass point lying on a horizontal saddle connection is integer, and the invariant is 2.

If h_1 is odd and h_2 even, then by $(*)$ ℓ_1 has to be even. Then if ℓ_2 is odd, the Weierstrass point on the saddle connection is non-integer, one of the Weierstrass points in the bottom cylinder is integer, and the

invariant is 2. If ℓ_2 is even, then t_2 has to be odd for the surface to be primitive, hence the Weierstrass point on the saddle connection is integer, the ones inside the cylinders are not, and the invariant is 2.

The last case to consider is when h_1 is even and h_2 odd. If ℓ_1 is odd, then so is ℓ_2 (by $(*)$), so one Weierstrass point in the top cylinder is integer, and the invariant is 2. If ℓ_1 is even, then ℓ_2 is also even by $(*)$ and t_1 is odd for primitiveness. Thus all Weierstrass points inside cylinders are non-integer, and the one on the saddle connection is integer: the invariant is 2.

This completes the two-cylinder case, and Proposition 4.3 is proved.

h_1	h_2	ℓ_1	ℓ_2	invariant
1	1	1	1	1
1	1	0	1	1
0	1	0	1	t_1 odd: 1; t_1 even: 3
0	1	1	0	3
1	0	0	1	t_2 odd: 3; t_2 even: 1
1	0	1	0	3

We sum up the case study for 2-cylinder primitive surfaces tiled by an odd number n of squares in a table giving the value of the invariant according to the parity of h_1, h_2, ℓ_1, ℓ_2 (recall that $w_1 = \ell_1$ and $w_2 = \ell_1 + \ell_2$).

The other combinations of parities of the parameters cannot happen for odd n and primitive surfaces.

Note that for even n we concluded that all primitive surfaces have the same invariant: 2.

5.2. **Reduction to one cylinder.**

Proposition 5.1. *Each orbit contains a one-cylinder surface. Equivalently, each surface has a direction in which it decomposes in one single cylinder.*

A baby version of this proposition is the following lemma.

Lemma 5.2. *A 2-cylinder surface of height 2 has a one-cylinder direction.*

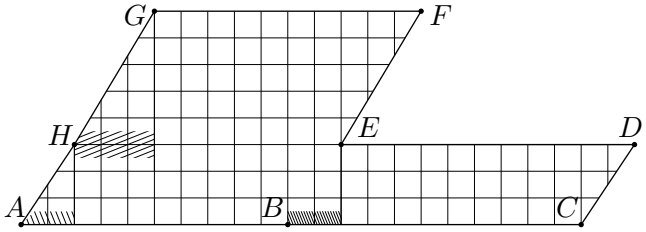
Proof of the lemma. Our surface is made of two cylinders, both of height 1. Since n is prime, the lengths of the two cylinders are relatively prime. By acting by U, we can set the twists to whatever values we wish. Set the top twist to 0 and the bottom twist to 1. Then by considering the vertical flow, we get a one-cylinder surface. □

We prove the proposition by induction on the height of the surface: given a two-cylinder surface, we show that its orbit contains a surface of strictly lesser height.

Consider a two-cylinder square-tiled surface S in $\mathcal{H}(2)$, with a prime number of square tiles. By acting by U we can move to the representative of the same cusp that has the least nonnegative twist parameters, so we will assume these to be less than the heights of the cylinders.

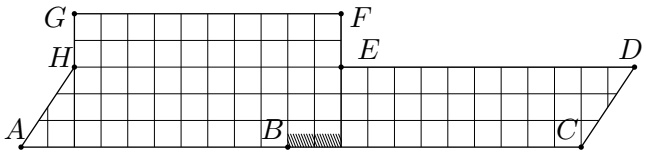
We split our study into four cases according to which twists are zero.

Case 1. Both twists are zero.



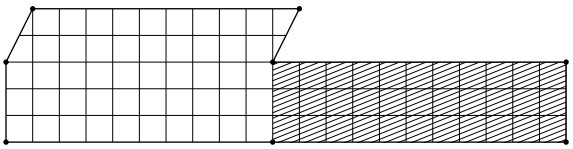
Call h_1, h_2 the heights and t_1, t_2 the twists of the horizontal cylinders of S. Consider the rotated surface RS. Its horizontal cylinders are the vertical cylinders of S. The vertical cylinder to the right of A has height at most t_2, that to the right of B also, and that to the right of H at most t_1. So if RS has two cylinders, the sum of their heights is at most $t_1 + t_2$, so it is less than $h_1 + h_2$.

Case 2. The bottom twist is nonzero but the top twist is zero.



In this case the same vertical cylinder is to the right of A and H. If the vertical separatrix going down from H ends in B, there is only one vertical cylinder (one horizontal cylinder for the rotated surface RS); if not, it necessarily crosses the shaded region to the right of B, so there are two vertical cylinders, and the sum of their heights is at most t_2 (the twist of the bottom cylinder of S), hence less than the height of the bottom cylinder of S.

Case 3. The bottom twist is zero but the top twist is nonzero.

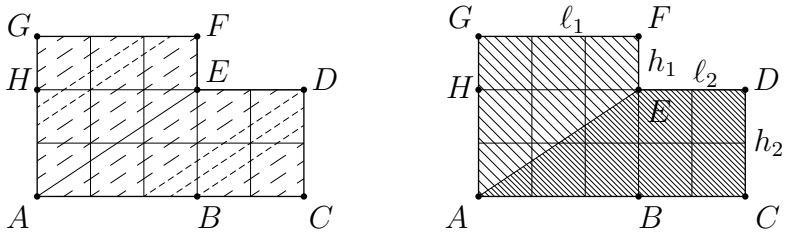


Act by R. The side part of S (shaded on the figure) becomes the top cylinder of the rotated surface RS, with zero twist, while the new bottom cylinder has nonzero twist and a bottom cylinder of height at most t_1, which is less than h_1. The surface in the same cusp with least

nonnegative twists also has zero top twist, so if it has zero bottom twist, conclude by case 4, otherwise apply case 2 to obtain a surface of height less than h_1.

Case 4. The twist parameters are both zero. In this case we end the induction by jumping to a one-cylinder surface directly:

Lemma 5.3. *The diagonal direction for the "base rectangle" of an L surface is a one-cylinder direction.*



Proof. The ascending diagonal $[AE]$ of the base rectangle of our L surface cuts it into two zones. Note that $[AE]$ has no other integer point than A and E because the total number of tiles is prime.

The other two saddle connections parallel to $[AE]$ start from B and H and end in F and D. We want to prove that the one starting from H ends in F and the one issued from B ends in D, meaning each saddle connection returns with angle 3π.

Set the origin in A or E and consider coordinates modulo $\ell_1\mathbf{Z} \times h_2\mathbf{Z}$.

Follow a saddle connection parallel to $[AE]$ from integer point to integer point. While it winds in a same zone, the coordinates of the integer points it reaches remain constant modulo $\ell_1\mathbf{Z} \times h_2\mathbf{Z}$. Changing zone has the following effects for the coordinates of the next integer point:

- from the upper zone to the lower zone: decrease y by h_1 modulo h_2;
- from the lower zone to the upper zone: decrease x by ℓ_2 modulo ℓ_1.

Zone changes have to be alternated. Once inside a zone with the right coordinates modulo $\ell_1\mathbf{Z} \times h_2\mathbf{Z}$, a separatrix reaches the top right corner of the zone with no more zone change.

So we want to prove that starting from B, in the lower zone with coordinates $(0, 0)$, and adding in turn $(-\ell_2, 0)$ and $(0, -h_1)$, coordinates $(\ell_2, 0)$ (point D) will be reached before $(0, h_1)$ (point H).

After k changes from lower to upper zone and k changes from upper to lower zone, the coordinates are final if $k = -1 \mod \ell_1$ and $k = 0 \mod h_2$; that is, if k is $h_2(\ell_1 - 1)$. After $k + 1$ changes from lower to upper zone and k changes from upper to lower zone, the coordinates are final if $k = 0 \mod \ell_1$ and $k = 0 \mod h_2$, which means k is $h_2 \cdot \ell_1$. So the separatrix parallel to $[AE]$ starting from B reaches D. □

5.3. **Linking one-cylinder surfaces of each type.** We call a surface type A (resp. B) if it has 1 (resp. 3) integer Weierstrass points.

Recall that a one-cylinder surface in $\mathcal{H}(2)$ tiled by a prime number n of squares has height one, hence it is determined by the cyclically ordered lengths of the three saddle connections on the bottom of this cylinder (which add up to n), and by a twist parameter.

The repeated action of U can set the twist parameter to any of its n possible values, so for the purpose of linking surfaces of the same type by $\mathrm{SL}(2,\mathbf{Z})$ action, we may already consider surfaces with the same cyclically ordered partition (a,b,c) as equivalent (allowing implicit U-action). We will call them (a,b,c) surfaces.

Partitions into three odd numbers correspond to type A; partitions into two even numbers and one odd number correspond to type B.

We will first show that any one-cylinder surface has a $(1,*,*)$ surface in its orbit; then we will show that $(1,b,c)$ surfaces with b and c odd are in the same orbit as a $(1,1,n-2)$ surface, proving all type A surfaces to be in one same orbit; then that $(1,2a,2b)$ surfaces are in the same orbit as a $(1,2,n-3)$ surface, proving all type B surfaces to be in one same orbit.

Consider a rational-slope direction on a square-tiled surface S; this direction is completely periodic. Say it is given by a vector $(p,q)\in\mathbf{Z}^2$, with $p\wedge q=1$. For any $(u,v)\in\mathbf{Z}^2$ such that $\det\left(\begin{smallmatrix}p&u\\q&v\end{smallmatrix}\right)=1$ our surface can be seen as tiled by parallelograms of sides (p,q), (u,v), whose vertices are the vertices of the square tiling.

These parallelograms are taken to unit squares by $M=\left(\begin{smallmatrix}p&u\\q&v\end{smallmatrix}\right)^{-1}$ (a matrix in $\mathrm{SL}(2,\mathbf{Z})$). We call $M\cdot S$ "the surface seen in direction (p,q)" on S.

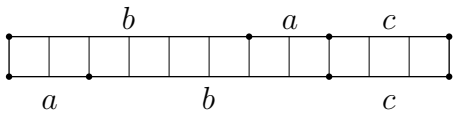
Consider a saddle connection σ on S in direction (p,q); the corresponding saddle connection on $M\cdot S$ is horizontal with an integer length equal to the number of integer points (vertices of the square tiling) σ reaches on S. Abusing vocabulary we also call this the length of σ.

A saddle connection returns at an angle of 3π if and only if it has a Weierstrass point in its middle. If two saddle connections in a given direction return with angle 3π then so does the third, and that direction is one-cylinder; thus two saddle connection lengths give the third.

5.3.1. *First step:any one-cylinder surface has a* $(1,*,*)$ *surface in its orbit.* To show this, we prove that an (a,b,c) surface has a $(\delta,k\delta,\gamma)$ surface in its orbit, where $\delta\,|\,a\wedge b$. Then because n is prime we have $\gamma\wedge\delta=1$, hence applying the argument a second time with γ and δ in place of a and b shows that there is a $(1,*,*)$ one-cylinder surface in the orbit of the surface we started with.

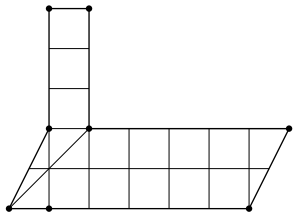
The proof is as follows. Consider the (a, b, c) surface S having saddle connections of lengths a, b, c on the bottom, b, a, c on the top.

RS has two cylinders, the top one of height c and width 1, and the bottom one of height $d = a \wedge b$ and width $\frac{a+b}{d}$, and some twist t.



Now the direction $(1+t, d)$ is a $(\delta, k\delta, \gamma)$ one-cylinder direction with $\delta = (1 + t) \wedge d$. Note that $k = \frac{a+b}{d} - 1$, and that $\gamma \wedge \delta = 1$.

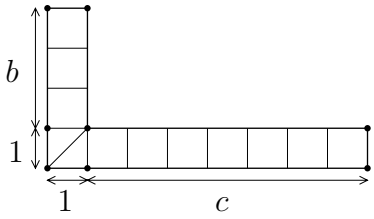
So by applying this procedure twice we see that any surface has a $(1, *, *)$ one-cylinder surface in its orbit.



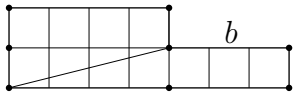
5.3.2. *End of proof for type A surfaces.* There only remains to link any $(1, b, c)$ surface, where b and c are odd, to a $(1, 1, n - 2)$ surface.

Consider the L surface with arms of width 1 and lengths b and c.

Apply U^2 to set the bottom twist to 2. Then rotate by applying R, and obtain a surface with two cylinders of height 1. By applying a convenient power of U the twists can be made both 0.



In the diagonal direction of the base rectangle of this new L surface, we see a $(1, 1, n - 2)$ surface.



5.3.3. *End of proof for type B surfaces.* Here we take the one-cylinder surface with the partition $(1, 2, n - 3)$ as the reference surface, and prove by steps that any type B surface has it in its orbit.

To do this, we first show that any one-cylinder surface has a one-cylinder surface with a $(1, 2a, 2b)$ partition in its orbit. This is done by the first step explained above.

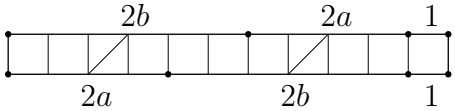
Then we link

•$(1, 2a, 2b)$ where $a \neq b$ with $(d, 2d, *)$, then with $(1, 2, n - 3)$;

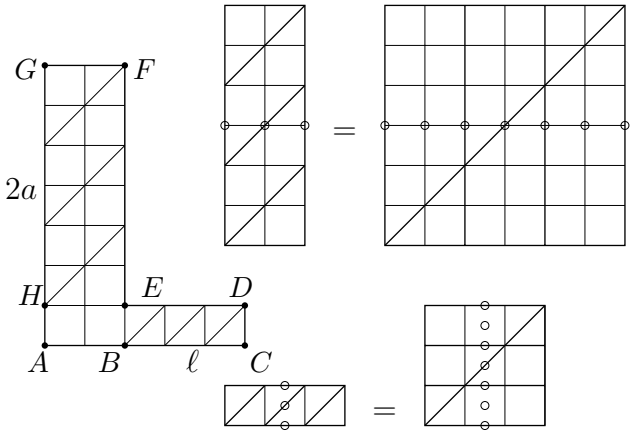
•$(1, 2a, 2b)$ where $a = b$ with $(2, 2, n - 4)$, then with $(1, 2, n - 3)$.

- Linking $(1, 2a, 2b)$ with $(1, 2, *)$ when $a \neq b$.

Without loss of generality, suppose $a < b$. Consider the one-cylinder surface with saddle connections of lengths $2a$, $2b$, 1 on the bottom and $2b$, $2a$, 1 on the top.



In the direction $(b-a,1)$ there is a connection between two integer Weierstrass points, so in this direction we see a two-cylinder surface. Its top cylinder has height $2a$ and width 2 and its bottom cylinder has height 1 and with $2+\ell$ for some ℓ.

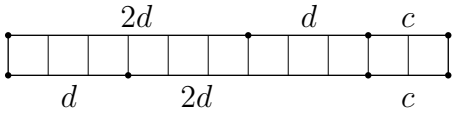


In appropriate directions, the separatrix issued from H winds around the horizontal cylinder $HEGF$. In particular, in any direction (k,a), $k\in\mathbf{N}$, it will run into a Weierstrass point (and into a saddle after twice the distance).

Likewise, in appropriate directions, the separatrix issued from B winds around the vertical cylinder $BCDE$. In particular, in any direction $(\ell/2,k/2)$ (equivalently ℓ,k), $k\in\mathbf{N}$, it will run into a Weierstrass point (and into a saddle after twice the distance).

Consider therefore the direction (ℓ,a). In this direction we get a $(d,2d,*)$ one-cylinder surface, where $d=a\wedge\ell$.

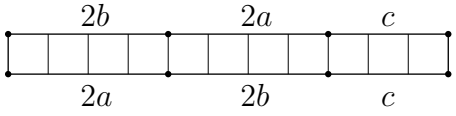
Now there only remains to link $(d,2d,*)$ with $(1,2,*)$, which is easily done: consider the one-cylinder surface with saddle connections d, $2d$, c on the bottom and $2d$, d, c on the top;



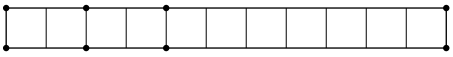
in the $(d,1)$ direction we get a $(1,2,*)$ one-cylinder surface.

- Linking $(1,2a,2b)$ with $(1,2,*)$ when $a=b$.

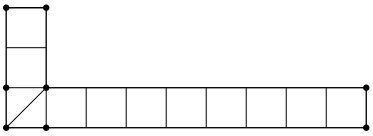
Consider the one-cylinder surface with saddle connections of length $2a$, $2b$, c on the bottom and $2b$, $2a$, c on the top.



In the direction $(a,1)$ we see a $(2,2,*)$ one-cylinder surface.

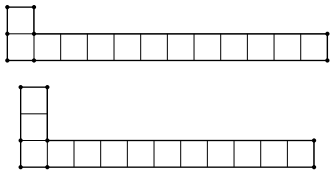


On this surface, in the direction $(2, 1)$, we have a two-cylinder surface with its top cylinder of height 2 and width 1, and its bottom cylinder of height 1. Acting by U we can set the twist parameters to 0.



Then in the direction $(1, 1)$ we see a $(1, 2, n-3)$ one-cylinder surface.

5.4. **L-shaped billiards.** L-shaped billiards give rise to L-shaped translation surfaces by an unfolding process; any L-shaped translation (with zero twists) surface is the covering translation surface of an L-shaped billiard.



Fix some prime $n > 3$, and consider the two-cylinder surfaces S_1 and S_2, both having $h_2 = 1$, $w_1 = 1$ and $t_1 = t_2 = 0$, and S_1 having $h_1 = 1$, $w_2 = n - 1$ and S_2 having $h_2 = 2$, $w_2 = n - 2$. The picture on the side represents S_1 and S_2 for $n = 13$.

For each n, S_1 and S_2 belong to orbit A and B respectively, and arise from L-shaped billiards. This proves Proposition 1.3.

6. Proof of results about elliptic points

Some constructions in this section are inspired by [Ve95].

6.1. **Translations.** Here we prove Proposition 4.4.

Suppose a surface $S \in \mathcal{H}(2)$ has a nontrivial translation f in its affine group. f fixes the saddle and induces a permutation on outgoing horizontal separatrices. Let ε be smaller than the length of the shortest saddle connection of S, and consider the three points at distance ε from the saddle on the three separatrices in a given direction. f cannot fix any of these points, otherwise it would be the identity of S, but it fixes the set of these points, hence it induces a cyclic permutation on them. This implies that except for the saddle, which is fixed, all f-orbits have size 3. However the set of regular Weierstrass points is also fixed (since the translation f is an automorphism of the underlying Riemann surface), and has size 5. This is a contradiction.

6.2. **Elliptic points of order 3.** Proof of Proposition 4.5.

Suppose a surface S in $\mathcal{H}(2)$ has an elliptic element of projective order three in its Veech group. Since the hyperelliptic involution has order 2, S has in fact an elliptic element of order 6 in its Veech group. Conjugate by $\mathrm{SL}(2,\mathbf{R})$ to a surface that has the rotation by $\pi/3$ (hereafter denoted by r) in its Veech group.

The set of Weierstrass points is preserved by r. We use the same notation for an affine diffeomorphism as for its derivative. This is no problem in $\mathcal{H}(2)$ since the derivative is a one-to-one map from the affine group to the Veech group, by Proposition 4.4.

The saddle is fixed; there are five other Weierstrass points, and they are setwise fixed, so at least two of them are also fixed. Consider one Weierstrass point that is fixed, call it W. Consider the shortest saddle connections through W. They come by triples making angles $\pi/3$.

Take one such triple, consider the corresponding regular hexagon (which has these saddle connections as its diagonals).

Claim: No two points of this hexagon are the same except the six corners (the saddle).

Proof of this fact: suppose the contrary. If two points are identified on opposite sides then this means the whole sides are identified but then the rotational symmetry would imply other identifications and mean we have a torus. If some other two points are the same then this would contradict the minimality of length of our saddle connections.

No Weierstrass points other than W lie inside or on this hexagon. (Otherwise there would be three Weierstrass points aligned, which cannot happen: the hyperelliptic involution is a rotation by π around any regular Weierstrass point, so when two of them are linked by a connection, they are linked by another connection on the other side, and they are in fact antipodal points on a closed geodesic.)

We see three saddle connections in one same direction by considering two sides and one diagonal of our hexagon. So this is a completely periodic direction, and we want to see two cylinders in this direction. This implies that two opposite sides are identified, but then the rotational symmetry would imply other identifications and mean we have a torus.

6.3. **Elliptic elements of order 2.**

6.3.1. *A convenient representation of R-invariant Veech surfaces in $\mathcal{H}(2)$: proof of Lemma* 4.6. Consider a $\mathcal{H}(2)$ Veech surface that has R in its Veech group. We denote also by R the corresponding affine automorphism.

The set of Weierstrass points is fixed by R (as by any affine diffeomorphism). The saddle being fixed, at least one of the remaining 5 points must be fixed.

Consider such a point and the shortest saddle connections through this point. They come by orthogonal pairs. Take one such pair. Consider the square having this pair of saddle connections as diagonals.

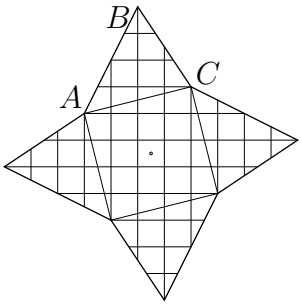
No two points in or on this square are the same except for the four corners (the saddle). Also no Weierstrass point in or on the square except for the corners and the center.

Consider the direction of two (parallel) opposite sides of our square. These sides are saddle connections so this direction is periodic.

These sides are not identified, otherwise by R-symmetry the other two would also be and we would have a torus. So this is a two-cylinder direction and our two saddle connections bound the short cylinder in this direction. This short cylinder lies outside the square and can be represented as a parallelogram with its "top-left" corner in the continuation of the square (i.e. with a "reasonable" twist).

By R-symmetry there also is such a parallelogram in the other direction. To make the picture more symmetric each parallelogram can be cut into two triangles, glued to opposite sides of the square. Thus we get a representation of the surface as an octagon with (parallel) opposite sides identified. Note that the four remaining Weierstrass points are the middle of the sides of this octagon.

6.3.2. *Proof of Proposition* 4.7. Represent the surface as above: an octagon made of a square and four triangles glued to its sides. All vertices lie on integer points.



Let ABC be one of the triangles, labeled clockwise starting from a corner of the square (so AC is a side of the square).

Let (p, q) be the coordinates of $\overrightarrow{AC}$ and (r, s) those of $\overrightarrow{AB}$.

The area of the surface is then $p^2 + q^2 + 2(ps - qr)$.

If n is prime then p and q have to be relatively prime, and of different parity. Then $p^2 + q^2 = 1 \mod 4$. The center of the square lies at the center of a square of the tiling. The condition for two Weierstrass points to lie on integer points is for $(ps - rq)$ to be even.

We conclude by observing that n is 1 (resp. 3) modulo 4 when $(ps - rq)$ is even (resp. odd).

7. Proof of countings

We want to establish the countings and estimates of Proposition 4.8.

7.1. **One-cylinder cusps and surfaces.** For prime $n > 3$, one-cylinder n-square-tiled cusps in $\mathcal{H}(2)$ are in 1-1 correspondence with cyclically ordered 3-partitions of n.

Ordered 3-partitions (a, b, c) of n are in 1-1 correspondence with pairs of integers $\{\alpha, \beta\}$ in $\{1, \dots, n-1\}$: assuming $\alpha < \beta$, the correspondence is given by $a = \alpha$, $a + b = \beta$, $a + b + c = n$. So there are C^2_{n-1} ordered 3-partitions of n. Ordered 3-partitions of n being in 3-1 correspondence with cyclically ordered 3-partitions, there are $\frac{1}{3}C^2_{n-1} = \frac{(n-1)(n-2)}{6}$ cyclically ordered 3-partitions of n.

Thus there are $\frac{(n-1)(n-2)}{6}$ one-cylinder cusps of n-square-tiled translation surfaces in $\mathcal{H}(2)$.

Those in orbit A are those with 3 odd parts $2a-1$, $2b-1$, $2c-1$; these are in 1-1 correspondence with cyclically ordered partitions a, b, c, of $\frac{n+3}{2}$. Their number is hence $\frac{1}{6}(\frac{n+3}{2} - 1)(\frac{n+3}{2} - 2) = \frac{(n+1)(n-1)}{24}$.

The remaining ones are in orbit B, their count is hence the difference, $\frac{(n-1)(n-3)}{8}$.

All one-cylinder cusps discussed here have width n (n possible values of the twist parameter), so the counts of one-cylinder surfaces are n times the corresponding cusp counts.

7.2. **Two-cylinder surfaces.** The total number of 2-cylinder n-square-tiled surfaces (n prime) is

$$S(n) = \sum_{a,b,k,\ell} k\ell,$$

where the sum is over $a, b, k, \ell \in \mathbf{N}^*$ such that $k < \ell$ and $ak + b\ell = n$.

This follows from the parametrization in section 3.3.2; the letters a, b, k, ℓ used here correspond to the parameters h_1, h_2, w_1, w_2 there, and the summand is the number of possible values of the twist parameters, given the heights and widths of the two cylinders.

We want the asymptotic for this quantity as n tends to infinity, n prime. In order to find this, we consider the sum as a double sum: the sum over a and b of the sum over k and ℓ.

Write $S(n) = \sum_{a,b} S(n, a, b)$, where $S(n, a, b) = \sum_{k,\ell} k\ell$.

We study the inner sum by analogy with a payment problem: how many ways are there to pay n units with coins worth a and b units?

This problem is classically solved by the use of generating series: denote the number of ways to pay by $s(n, a, b)$; then

$s(n,a,b) = \mathrm{Card}\{(k,\ell) \in \mathbf{N}^2 : ak + b\ell = n\} = \sum_{k,\ell\in\mathbf{N}:ak+b\ell=n} 1$.

Now notice that $\sum_{k=0}^{\infty} z^{ak} \sum_{\ell=0}^{\infty} z^{b\ell} = \sum_{n=0}^{\infty} s(n,a,b) z^n$, and deduce that the number looked for is the n-th coefficient of the power series expansion of the function $\frac{1}{1-z^a}\frac{1}{1-z^b}$.

We turn back to our real problem, $S(n,a,b) = \sum_{k,\ell\in\mathbf{N}^*:ak+b\ell=n,k<\ell} k\ell$.

We want to show that $S(n) \sim cn^3$ for prime n. For this we will use the dominated convergence theorem: we show that $S(n,a,b)/n^3$ has a limit $c(a,b)$ when n tends to infinity with a and b fixed, and that $S(n,a,b)/n^3$ is bounded by some $g(a,b)$ such that $\sum_{a,b} g(a,b) < \infty$, to conclude that $S(n)/n^3 = \sum_{a,b} S(n,a,b)/n^3$ tends to $c = \sum_{a,b} c(a,b)$, which means $S(n) \sim cn^3$.

The dominated convergence is proved as follows.

Write $S(n,a,b) = \sum_{k,h\in\mathbf{N}^*:(a+b)k+bh=n} k(k+h)$ by introducing $h = \ell - k$. Then split the sum into $\sum k^2$ and $\sum kh$. Write

$$S_1(n,a,b) = \sum_{k,h\in\mathbf{N}^*,\ (a+b)k+bh=n} k^2/n^3 \leqslant \sum_{k\in\mathbf{N}^*,\ h\in\mathbf{Q},\ (a+b)k+bh=n} k^2/n^3$$

$$S_2(n,a,b) = \sum_{k,h\in\mathbf{N}^*,\ (a+b)k+bh=n} kh/n^3 \leqslant \sum_{k\in\mathbf{N}^*,\ h\in\mathbf{Q},\ (a+b)k+bh=n} kh/n^3$$

(In the sums on the right-hand side, h has been allowed to be a rational instead of an integer.) Hence

$$S_1(n,a,b) \leqslant \frac{1}{(a+b)^3}\Big[\frac{a+b}{n} \sum_{k=1}^{\lfloor n/(a+b)\rfloor} \big(\frac{a+b}{n}k\big)^2\Big]$$

$$S_2(n,a,b) \leqslant \frac{1}{(a+b)^2 b}\Big[\frac{a+b}{n} \sum_{k=0}^{n/(a+b)} \big(\frac{a+b}{n}k\big)\big(1 - \frac{a+b}{n}k\big)\Big]$$

The expressions in brackets, Riemann sum approximations to the integrals $\int_0^1 x^2 \,\mathrm{d}\,x$ and $\int_0^1 x(1-x)\,\mathrm{d}\,x$, are uniformly bounded by 1.

Now notice that $\sum_{a,b} \frac{1}{(a+b)^3}$ and $\sum_{a,b} \frac{1}{(a+b)^2 b}$ are convergent. This ends the dominated convergence argument.

We can now investigate the limit. For ease of calculation, we drop the condition $k < \ell$. We take care of it by writing $\sum_{k,\ell} = 2\sum_{k<\ell} + \sum_{k=\ell}$. For prime n, $k = \ell$ implies that they are both equal to 1. The sum for $k = \ell$ is hence equal to $n-1$, and we will not need to take it into account since the whole sum will grow as n^3.

Denote by $\tilde{S}(n,a,b)$ the sum over all k and ℓ.

Notice that $\sum_{k=0}^{\infty} kz^{ak} \sum_{\ell=0}^{\infty} \ell z^{b\ell} = \sum_{n=0}^{\infty} \tilde{S}(n,a,b) z^n$.

$\tilde{S}(n,a,b)$ is therefore the n-th coefficient of the power series expansion of the function $f_{a,b} = \frac{z^a}{(1-z^a)^2} \frac{z^b}{(1-z^b)^2}$.

In order to know this coefficient, we decompose $f_{a,b}$ into partial fractions. This function has poles at a-th and b-th roots of 1. Since n is prime, we are only interested in relatively prime a and b, for which the only common root of 1 is 1 itself, which is hence a 4-th order pole of $f_{a,b}$, while other poles have order 2.

The n-th coefficient of the power series expansion of $f_{a,b}$ is a polynomial of degree 3 in n, whose leading term is $c_{a,b}\frac{n^3}{6}$, where $c_{a,b}$ is the coefficient of $\frac{1}{(1-z)^4}$ in the decomposition of $f_{a,b}$ into partial fractions. This coefficient is computed to be $\frac{1}{a^2b^2}$.

We want the sum over relatively prime a and b. We relate it to the sum over all a and b by sorting the latter according to $d = a \wedge b$.

$$\sum_{a,b} \frac{1}{a^2b^2} = \sum_{d} \sum_{a,b,\ a\wedge b=d} \frac{1}{a^2b^2} = \sum_{d} \frac{1}{d^4} \sum_{a,b,\ a\wedge b=1} \frac{1}{a^2b^2}$$

By observing that $\sum_{a,b} \frac{1}{a^2b^2} = (\sum_a \frac{1}{a^2})^2 = \zeta(2)^2 = \frac{\pi^4}{36}$ and that $\sum_d \frac{1}{d^4} = \zeta(4) = \frac{\pi^4}{90}$ we get that the sum $\sum_{a,b,\ a\wedge b=1} \frac{1}{a^2b^2}$ is equal to $5/2$. Divide by 2 to get back to $k < \ell$, and find that $S(n) \sim \frac{5}{4}\frac{n^3}{6}$.

7.3. **Two-cylinder surfaces by orbit.** We will now compute the number of 2-cylinder surfaces in orbit A.

2-cylinder surfaces in orbit A are those for which both heights are odd, and one half of those for which both widths are odd (see the table in section 5.1); the factor one half comes from the conditions on the twists when both widths are odd.

First compute

$$S'(n) = \sum_{\substack{a,b,k,\ell \\ ak+b\ell=n \\ a,b \text{ odd} \\ a\wedge b=1 \\ k<\ell}} k\ell.$$

For $a \wedge b = 1$,

$$\tilde{S}'_{a,b}(n) = \sum_{\substack{k,\ell \\ ak+b\ell=n}} k\ell \sim \frac{1}{a^2b^2}\frac{n^3}{6}.$$

$$\sum_{a,b \text{ odd}} \frac{1}{a^2b^2} = \sum_{d \text{ odd}} \frac{1}{d^4} \sum_{\substack{a,b \text{ odd} \\ a\wedge b=1}} \frac{1}{a^2b^2}.$$

Now

$$\sum_{a,b \text{ odd}} \frac{1}{a^2b^2} = \Big(\sum_{a \text{ odd}} \frac{1}{a^2}\Big)^2 = ((1-1/2^2)\zeta(2))^2 = \frac{9}{16}\frac{\pi^4}{36}$$

and

$$\sum_{d \text{ odd}} \frac{1}{d^4} = (1-1/2^4)\zeta(4) = \frac{15}{16}\frac{\pi^4}{90}$$

so

$$\sum_{\substack{a,b \text{ odd} \\ a\wedge b=1}} \frac{1}{a^2b^2} = 3/2.$$

We deduce that $S'(n) \sim (3/4)(n^3/6)$ (the condition $k < \ell$ is responsible for a factor $1/2$).

Similarly if

$$S''(n) = \sum_{\substack{a,b,k,\ell \\ ak+b\ell=n \\ k,\ell \text{ odd} \\ a\wedge b=1 \\ k<\ell}} k\ell,$$

put

$$\tilde{S}''_{a,b}(n) = \sum_{\substack{k,\ell \text{ odd} \\ ak+b\ell=n}} k\ell.$$

Notice that $\sum_{k \text{ odd}} kz^{ak} \sum_{\ell \text{ odd}} \ell z^{b\ell} = \sum \tilde{S}''_{a,b}(n) z^n$.

Because $\sum kz^k = \frac{z}{(1-z)^2}$, $\sum 2kz^{2k} = \frac{2z^2}{(1-z^2)^2}$, and the difference is $\sum(2k+1)z^{2k+1} = \frac{z(1+z^2)}{(1-z^2)^2}$.

$\tilde{S}''_{a,b}(n)$ is now the n-th coefficient of the power series expansion of $\frac{z^a(1+z^{2a})}{(1-z^{2a})^2}\frac{z^b(1+z^{2b})}{(1-z^{2b})^2}$. When $a \wedge b = 1$, this rational function has two order 4 poles at 1 and -1 and its other poles have order 2; the coefficients of $\frac{1}{(1-z)^4}$ and $\frac{1}{(1+z)^4}$ in its decomposition into partial fractions are respectively $\frac{1}{4a^2b^2}$ and $\frac{(-1)^{a+b}}{4a^2b^2}$.

Because n is odd, and k and ℓ are odd, a and b have to have different parities, so $a+b$ is odd. So $\frac{1}{4a^2b^2} + \frac{(-1)^{a+b}(-1)^n}{4a^2b^2} = \frac{1}{2a^2b^2}$.

Now

$$\sum_{\substack{a\wedge b=1\\ a\neq b \mod 2}} \frac{1}{a^2b^2} = \sum_{a\wedge b=1} \frac{1}{a^2b^2} - \sum_{\substack{a\wedge b=1\\ a,b \text{ odd}}} \frac{1}{a^2b^2} = 5/2 - 3/2 = 1.$$

The condition $k < \ell$ brings a factor $1/2$, and we said that only half of the surfaces with k and ℓ odd were in orbit A, thus we get $S''(n) \sim (1/8)(n^3/6)$.

Putting pieces together, we get that the number of n-square-tiled 2-cylinder surfaces of type A, n prime, is equivalent to $(3/4+1/8)(n^3/6) = (7/8)(n^3/6)$.

7.4. **Two-cylinder cusps.** For n prime, the number of 2-cylinder cusps (in both orbits) is given by

$$S(n) = \sum_{\substack{a,b,k,\ell\in\mathbf{N}^*\\ ak+b\ell=n\\ k<\ell}} (a\wedge k)(b\wedge \ell).$$

(See counting of two-cylinder surfaces in section 7.2 and discussion of cusps in section 3.4.)

$S(n)$ is less than

$$\tilde{S}(n) = \sum_{\substack{a,b,k,\ell\in\mathbf{N}^*\\ ak+b\ell=n}} (a\wedge k)(b\wedge \ell).$$

where the condition $k < \ell$ is dropped.

We will show that for any $\varepsilon > 0$, $\tilde{S}(n) \ll n^{3/2+\varepsilon}$ when n tends to infinity.

This will imply that the number of two-cylinder cusps of n-square-tiled surfaces is sub-quadratic, thus negligible before the (quadratic) number of one-cylinder cusps in each orbit.

$$\tilde{S}(n) = \sum_{\substack{A,B,u,v\in\mathbf{N}^*\\ Au^2+Bv^2=n}} uvf(A)f(B), \quad \text{where } f(m) = \sum_{\substack{rs=a\\ r\wedge s=1}} 1.$$

Note that $f(m) \leqslant \mathrm{d}(m) \ll m^{\varepsilon}$, where $\mathrm{d}(m)$ is the number of divisors of m. The factors $f(A)f(B)$ therefore contribute less than an n^{ε}.

$$\sum_{\substack{A,B,u,v\in\mathbf{N}^*\\ Au^2+Bv^2=n}} uv = \sum_{u} u \sum_{A\leqslant n/u^2} \Big(\sum_{v^2\mid n-Au^2} v\Big).$$

The sum in parentheses has less than $\mathrm{d}(n-Au^2)$ summands, each of which is bounded by $\sqrt{n-Au^2}$, so

$$\tilde{S}(n) \ll n^{1/2+2\varepsilon} \sum_u n/u \ll n^{3/2+3\varepsilon}.$$

This proves the estimate.

We thank Joël Rivat for contributing this estimate.

7.5. **Elliptic points.** The discussion in section 6.3.2 implies that their number is less than the number of integer-coordinate vectors in a quarter of a circle of radius $\sqrt{n}$, so it is $O(n)$.

8. Strong numerical evidence

There is strong numerical evidence that the theorems stated in sections 1 and 4 extend as follow for prime and nonprime n:

Conjecture 8.1. *For odd $n \geqslant 5$, there are exactly two Teichmüller discs of primitive n-square-tiled surfaces in $\mathcal{H}(2)$, one of them with elliptic points and the other without.*

For even $n \geqslant 4$, there is exactly one Teichmüller disc of primitive n-square-tiled surfaces in $\mathcal{H}(2)$; this disc contains elliptic points if and only if n is a multiple of 4.

Conjecture 8.2. *Countings are given by the following functions:*

Odd n: *orbit A:* $\frac{3}{16}(n-1)n^2\prod_{p|n}(1-\frac{1}{p^2})$

orbit B: $\frac{3}{16}(n-3)n^2\prod_{p|n}(1-\frac{1}{p^2})$

Even n: $\frac{3}{8}(n-2)n^2\prod_{p|n}(1-\frac{1}{p^2})$

The formulae in Conjecture 8.2 were suggested by M. Schmoll, who has similar formulae in counting problems for marked tori and torus coverings. These formulae give degree 3 polynomials when restricted to prime n, for which Theorem 1.2 gives the leading term. These polynomials are expressed in the table below.

	one cylinder	2 cylinders	all
A	$\frac{1}{24}(n^3-n)$	$\frac{1}{48}(7n^3-9n^2-7n+9)$	$\frac{3}{16}(n^3-n^2-n+1)$
B	$\frac{1}{8}(n^3-4n^2+3n)$	$\frac{1}{16}(n^3-n^2-9n+9)$	$\frac{3}{16}(n^3-3n^2-n+3)$
all	$\frac{1}{6}(n^3+3n^2+2n)$	$\frac{1}{24}(5n^3-6n^2-17n+18)$	$\frac{3}{8}(n^3-2n^2-n+2)$

On the other hand, the counting functions for 2-cylinder cusps are not polynomials.

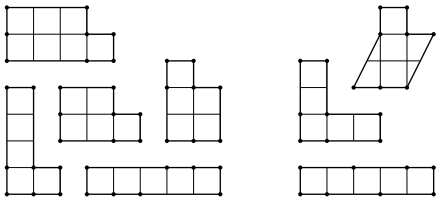
Conjecture 8.3. *For prime n, the number of elliptic points is k where $n = 4k \pm 1$.*

9. Appendix A: $n = 3$ and $n = 5$

$n = 3$. For $n = 3$, we have the following three surfaces.

If we call S_1 the one-cylinder surface, and S_2 and S_3 the two-cylinder surfaces, the generators of $\mathrm{SL}(2, \mathbf{Z})$ act as follow: $US_1 = S_1$, $US_2 = S_3$, $US_3 = S_2$, $RS_1 = S_3$, $RS_2 = S_2$, $RS_3 = S_1$. So there is only one orbit, containing $d = 3$ surfaces, the number of cusps is $c = 2$, the number of elliptic points (R-invariant surfaces) is $e = 1$, so the genus is $g = 0$ by the Gauss-Bonnet formula.

$n = 5$. For $n = 5$, we have 27 surfaces forming 8 cusps, a representative of which appears on the following picture.



Computing the $\mathrm{SL}(2, \mathbf{Z})$ action shows that they fall into two orbits, orbit A being made of the surfaces on the left and orbit B of those on the right.

The data for orbit A is $d = 18$ surfaces, $c = 5$ cusps, $e = 0$ elliptic point, so the genus is $g = 0$ by the Gauss-Bonnet formula.

The data for orbit B is $d = 9$ surfaces, $c = 3$ cusps, $e = 1$ elliptic point, so the genus is $g = 0$ by the Gauss-Bonnet formula.

By inspection of the congruence subgroups of genus 0 of $\mathrm{SL}(2, \mathbf{Z})$ (see for example [CuPa]), the stabilizers of orbits A and B are noncongruence subgroups of $\mathrm{SL}(2, \mathbf{Z})$.

10. Appendix B: other hyperelliptic components of strata of abelian differentials

For all hyperelliptic square-tiled surfaces, one can count the number of Weierstrass points with integer coordinates. This provides an invariant for the action of $\mathrm{SL}(2, \mathbf{Z})$ on square-tiled surfaces in all hyperelliptic components of strata of moduli spaces of abelian differentials.

The strata with hyperelliptic components are $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$, for $g > 1$.

Proposition 10.1. *In $\mathcal{H}(2g-2)^{hyp}$ and $\mathcal{H}(g-1, g-1)^{hyp}$, for large enough odd n there are at least g orbits containing one-cylinder surfaces.*

This is proved by the following reasoning.

Completely periodic surfaces in $\mathcal{H}(2g-2)$ or $\mathcal{H}(g-1,g-1)$, for $g>1$, have respectively $2g-1$ and $2g$ saddle connections.

For one-cylinder primitive surfaces (necessarily of height 1), the lengths of the saddle connections add up to n, and the Weierstrass points are two points on the circle at half-height of this cylinder (these do not have integer coordinates), the saddle in the $\mathcal{H}(2g-2)^{\text{hyp}}$ case, and the midpoints of the saddle connections that bound the cylinder (these have integer coordinates for exactly those saddle connections of even length).

If n is odd, the sum of the lengths is odd. So the number of odd-length saddle connections has to be odd, and is between 1 and $2g-1$. There are g possibilities for that. Since the value of the invariant is the number of even-length saddle connections, it can take g different values.

11. Appendix C: the theorem of Gutkin and Judge

Theorem (Gutkin–Judge)**.** *(S,ω) has an arithmetic Veech group if and only if (S,ω) is parallelogram-tiled.*

Up to conjugating by an element of $\mathrm{SL}(2,\mathbf{R})$, it suffices to show:

Theorem. *(S,ω) is a square-tiled surface if and only if $V(S,\omega)$ is commensurable to $\mathrm{SL}(2,\mathbf{Z})$.*

(i.e. these two groups share a common subgroup of finite index in each.)

Remark. In this theorem, the size of the square tiles is not assumed to be 1. One can always act by a homothety to make this true, and we will suppose that in the proof of the direct way of this theorem.

11.1. **A square-tiled surface has an arithmetic Veech group.** Fix a square-tiled surface (S,ω). We saw in Lemma 2.3 that $V(S,\omega) < V(\mathbf{R}^2/\Lambda(\omega)\,, dz)$.

Case 1. Let us first assume that $\Lambda(\omega)=\mathbf{Z}^2$, i.e. (S,ω) is a primitive square-tiled surface.

Lemma 2.4 implies that $\mathrm{SL}(2,\mathbf{Z})$ acts on the set E of square-tiled surfaces contained in its $\mathrm{SL}(2,\mathbf{R})$-orbit. The set E is finite and the stabilizer of this action is $V(S,\omega)$. The class formula then implies that $V(S,\omega)$ has finite index in $\mathrm{SL}(2,\mathbf{Z})$.

Case 2. Suppose that $\Lambda(\omega)$ is a strict sublattice of $\mathbf{Z}^2$. Consider $P_1,\dots,P_k$ the preimages of the origin on S. Denote by $\mathrm{Aff}(P_1,\dots,P_k)$ the stabilizer of the set of these points in the affine group of (S,ω), and $V(P_1,\dots,P_k)$ the associated Veech group. The translation surface $(S,\omega,\{P_1,\dots,P_k\})$ where $\{P_1,\dots,P_k\}$ are artificially marked is a

primitive square-tiled surface. From Case 1 above, its Veech group $V(P_1, \dots, P_k)$ is therefore a lattice contained in the discrete group $V(S, \omega)$, hence of finite index in this group.

Thus $V(P_1, \dots, P_k)$ is a finite index subgroup in both $V(S, \omega)$ and $\mathrm{SL}(2, \mathbf{Z})$.

11.2. **A surface with an arithmetic Veech group is square-tiled.** This part is inspired by ideas of Thurston [Th] and Veech [Ve89, §9], and appeared in [Hu, appendix B].

Let S be a translation surface with an arithmetic Veech group Γ.

If Γ is commensurable to $\mathrm{SL}(2, \mathbf{Z})$ only in the wide sense, we move to the case of strict commensurability. This conjugacy on Veech groups is obtained by $\mathrm{SL}(2, \mathbf{R})$ action on surfaces.

We prove the following propositions.

Proposition 11.1. *A group Γ commensurable with $\mathrm{SL}(2, \mathbf{Z})$ contains two elements of the form $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ for some $m, n \in \mathbf{N}^*$.*

Proposition 11.2. *If the Veech group Γ of a translation surface S contains two elements of the form $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ for some $m, n \in \mathbf{N}^*$, then S is square-tiled.*

Proposition 11.1 follows from the following lemma.

Lemma 11.3. *If $H \leq G$ is a finite index subgroup then every $g \in G$ of infinite order has a power in H.*

Proof of the lemma. If H has finite index there is a partition of G into a finite number of classes modulo H. The powers of g, in countable number, are distributed in these classes, so there exist distinct integers i and j such that g^i and g^j are in the same class, and then $g^{j-i} \in H$. □

Apply this lemma to $G = \mathrm{SL}(2, \mathbf{Z})$ and H the common subgroup to G and Γ, of finite index in both G and Γ, and $g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ or $g = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

We now prove Proposition 11.2.

Since $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, the horizontal direction is parabolic, so S decomposes into horizontal cylinders C_i^{h} of rational moduli. Replacing $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ with one of its powers if necessary, suppose it fixes the boundaries of these cylinders. This means their moduli are multiples of $1/m$. Calling w_i^{h}, h_i^{h} the widths and heights of these cylinders, we have relations $h_i^{\mathrm{h}}/w_i^{\mathrm{h}} = k_i/m$ for some integers k_i.

By a similar argument, since $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right) \in \Gamma$, the vertical direction is also parabolic, and S decomposes into vertical cylinders C_j^{v} of rational moduli $h_j^{\mathrm{v}}/w_j^{\mathrm{v}} = k_j'/n$ for some integers k_j'.

Combining these two decompositions yields a decomposition of S into rectangles of dimensions $h_j^{\mathrm{v}} \times h_i^{\mathrm{h}}$ (these rectangles are the connected components of the intersections of the horizontal and vertical cylinders). Here we keep on with the convention of section 2.3 about heights and widths of cylinders.

What we want to show is that these rectangles have rational dimensions (up to a common real scaling factor), in order to prove that S is a covering of a square torus; indeed, if the rectangles are such, then they can be divided into equal squares, so we obtain a covering of a square torus. Since singular points of S lie on the edges both of horizontal and of vertical cylinders, they are at corners of rectangles and hence of squares of the tiling, so that the covering is ramified over only one point.

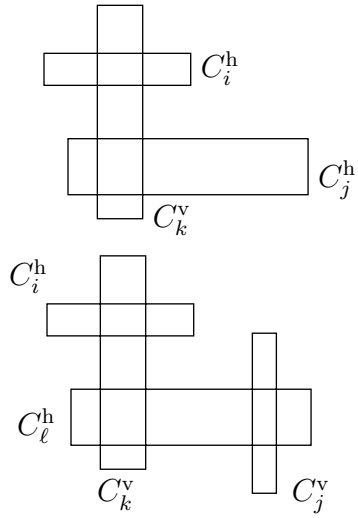
Because the cylinders in the decompositions above are made up of these rectangles, we have $w_i^{\mathrm{h}} = \sum m_{ij} h_j^{\mathrm{v}}$ and $w_j^{\mathrm{v}} = \sum n_{ji} h_i^{\mathrm{h}}$, where $m_{ij}, n_{ji} \in \mathbf{N}$.

Combining equations, $m h_i^h = \sum k_i m_{ij} h_j^v$ and $n h_j^v = \sum k'_j n_{ji} h_i^h$.

Then, setting $X^{\mathrm{h}} = (h_i^{\mathrm{h}})$, $X^{\mathrm{v}} = (h_j^{\mathrm{v}})$, $M = (k_i m_{ij})_{ij}$, $N = (k'_j n_{ji})_{ji}$, we have $mX^{\mathrm{h}} = MX^{\mathrm{v}}$ and $nX^{\mathrm{v}} = NX^{\mathrm{h}}$, so that $MNX^{\mathrm{h}} = mnX^{\mathrm{h}}$ and $NMX^{\mathrm{v}} = nmX^{\mathrm{v}}$.

M, N and their products are matrices with nonnegative integer coefficients.

In view of applying the Perron–Frobenius theorem, we show that MN and NM have powers with all coefficients positive.



This results from the connectedness of S and the following observation: $M_{ij} \neq 0$ if and only if C_i^{h} and C_j^{v} intersect; $(MN)_{ij} \neq 0$ if and only if there exists a cylinder C_k^{v} which intersects both C_i^{h} and C_j^{h}; more generally the element i, j of a product of alternately M and N matrices is nonzero if and only if there exists a corresponding sequence of alternately horizontal and vertical cylinders such that two successive cylinders intersect. So MN and NM do have powers with all coefficients positive.

X^{h} (resp. X^{v}) is an eigenvector for the eigenvalue nm of the square matrix MN (resp. NM). By the Perron–Frobenius theorem, there exists a unique eigenvector associated with the real positive eigenvalue nm for the matrix NM (resp.

MN). Since both matrices have rational coefficients and the eigenvalue is rational, there exist eigenvectors with rational coefficients. Up to scaling, they are unique by the Perron–Frobenius theorem. This allows to conclude that X^{h} is a multiple of a vector with rational coordinates. From the equation $nX^{\mathrm{v}} = NX^{\mathrm{h}}$, we then conclude that the rectangles have rational moduli and can be tiled by identical squares.

This completes the proof of the theorem.

11.3. **A corollary.** The following result of [GuHuSc] arises as a corollary of section 11.1 and Proposition 11.2.

Corollary 11.4. *If a subgroup* $\Gamma < \mathrm{SL}(2,\mathbf{Z})$ *contains two elements* $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix}1 & 0\\ n & 1\end{smallmatrix}\right)$ *and has infinite index in* $\mathrm{SL}(2,\mathbf{Z})$*, then* Γ *cannot be realized as the Veech group of a translation surface.*

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Pascal Hubert
Institut de Mathématiques de Luminy
Campus Luminy, case 907
F-13288 Marseille cedex 09
France
hubert@iml.univ-mrs.fr

Samuel Lelièvre, case 51
Université Montpellier 2
Place Eugène Bataillon
F-34095 Montpellier cedex 5
France
samuel.lelievre@m4x.org