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# Limit groups as limits of free groups: compactifying the set of free groups

Christophe Champetier, Vincent Guirardel

January 6, 2004

## Abstract

We give a topological framework for the study of Sela's *limit groups*: limit groups are limits of free groups in a compact space of marked groups. Many results get a natural interpretation in this setting. The class of limit groups is known to coincide with the class of finitely generated fully residually free groups. The topological approach gives some new insight on the relation between fully residually free groups, the universal theory of free groups, ultraproducts and non-standard free groups.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>A topology on spaces of marked groups</b>	<b>7</b>
2.1	Definitions . . . . .	7
2.2	Topology on $\mathcal{G}_n$ . . . . .	8
2.3	Changing the marker . . . . .	9
2.4	Examples of convergent sequences . . . . .	10
2.5	Limit groups: first approach . . . . .	12
2.6	Open and closed algebraic properties . . . . .	12
2.7	The isomorphism equivalence relation . . . . .	15
2.8	Subgroups . . . . .	17
2.9	Free and amalgamated products . . . . .	18
2.10	Quotients . . . . .	18
<b>3</b>	<b>Limit groups of Sela</b>	<b>19</b>
3.1	Summary of simple properties of limit groups . . . . .	19
3.2	First examples of limit groups. . . . .	20
3.3	Morphisms to free groups and Makanin-Razborov diagrams. . . . .	22
3.3.1	Finiteness results for limit groups. . . . .	22
3.3.2	Shortening quotients. . . . .	24
3.3.3	Makanin-Razborov diagrams . . . . .	25
3.4	Examples of Makanin-Razborov Diagrams . . . . .	26

<b>4</b>	<b>Constructing limit groups, fully residually free towers.</b>	<b>29</b>
4.1	Generalized double as a subgroup of a double . . . . .	31
4.2	Simple graphs of limit groups . . . . .	33
4.3	Twisting generalized doubles . . . . .	34
4.4	Statement of the key result and characterizations of limit groups . . . . .	36
4.5	Proof of the key result . . . . .	37
	4.5.1 Step 1: cutting surfaces . . . . .	40
	4.5.2 Step 2: pulling centralizers . . . . .	41
4.6	Fully residually free towers . . . . .	42
<b>5</b>	<b>Some logic</b>	<b>44</b>
5.1	The Tarski Problem . . . . .	44
5.2	Universal theory . . . . .	45
<b>6</b>	<b>A little non-standard analysis</b>	<b>47</b>
6.1	definitions . . . . .	47
6.2	Ultraproducts and the topology on the set of marked groups . . . . .	47
6.3	Application to residual freeness . . . . .	48
6.4	Maximal limit quotients . . . . .	49
<b>A</b>	<b>Reading property CSA from a graph of groups.</b>	<b>51</b>

## 1 Introduction

Limit groups have been introduced by Z. Sela in the first paper of his solution of Tarski’s problem [Sel01a]. These groups appeared to coincide with the long-studied class of finitely generated fully residually free groups (definition 2.5, see [Bau67], [Bau62], [KM98a, KM98b], see [BMR00] and [Chi95] and references). In this paper, we propose a new approach of limit groups, in a topological framework, that sheds further light on these groups. We survey the equivalent definitions of limit groups and their elementary properties, and we detail the Makanin-Razborov diagram, and general ways of constructing limit groups. This article is aimed to be self-contained, and some short classical proofs are rewritten for completeness.

**Tarski’s problem.** Tarski’s problem asks whether all the free groups of rank  $\geq 2$  have the same elementary theory. The elementary theory of a group  $G$  is the set of all sentences satisfied in  $G$  (see section 5 for a short introduction, and [CK90, Hod97] for further references). Roughly speaking, a *sentence* in the language of groups is a “usual” logical sentence where one quantifies only on individual elements of a group; to be a little bit more precise, it is a string of symbols made of quantifiers, variables (to be interpreted as elements of a group), the identity element “1”, the group multiplication and inverse symbols (“.” and “ $-1$ ”), equality “=”, and logical connectives “ $\neg$ ” (not), “ $\wedge$ ” (and), “ $\vee$ ” (or) and without free variables. Any sentence is equivalent (assuming the axioms of groups) to a sentence where all quantifiers are placed at the beginning, followed by a disjunction of systems of equations or inequations:

$$\forall x_1, \dots, x_p \exists y_1, \dots, y_q \forall z_1, \dots, z_r \dots \left\{ \begin{array}{l} u_1(x_i, y_j, z_k, \dots) \neq 1 \\ \vdots \\ u_n(x_i, y_j, z_k, \dots) \neq 1 \end{array} \right. \vee \left\{ \begin{array}{l} v_1(x_i, y_j, z_k, \dots) \neq 1 \\ \vdots \\ v_m(x_i, y_j, z_k, \dots) \neq 1 \end{array} \right. \vee \dots$$

Such a sentence is a *universal sentence* if it can be written  $\forall x_1, \dots, x_p \varphi(x_1, \dots, x_p)$  for some quantifier free formula  $\varphi(x_1, \dots, x_p)$ . For example,  $\forall x, y \ xy = yx$  is a universal sentence in the language of groups. This sentence is satisfied in a group  $G$  if and only if  $G$  is abelian. The universal theory  $\text{Univ}(G)$  of a group  $G$  is the set of universal sentences satisfied by  $G$ .

**From equations to marked groups.** The first step in the study of the elementary theory of a group is the study of systems of equations (without constant) in that group. Such a system is written

$$\left\{ \begin{array}{l} w_1(x_1, \dots, x_n) = 1 \\ \vdots \\ w_p(x_1, \dots, x_n) = 1 \end{array} \right.$$

where each  $w_i(x_1, \dots, x_n)$  is a reduced word on the variables  $x_1, \dots, x_n$  and their inverses, i. e. an element of the free group  $F_n = \langle x_1, \dots, x_n \rangle$ .

Solving this system of equations in a group  $G$  consists in finding all tuples  $(a_1, \dots, a_n) \in G^n$  such that for all index  $i$ ,  $w_i(a_1, \dots, a_n) = 1$  in  $G$ . There is a natural correspondence between solutions of this system of equations and morphisms  $h : E \rightarrow G$ , where  $E$  is the group presented by  $E = \langle x_1, \dots, x_n \mid w_1(x_1, \dots, x_n), \dots, w_p(x_1, \dots, x_n) \rangle$ : to a solution  $(a_1, \dots, a_n) \in G^n$  corresponds the morphism  $E \rightarrow G$  sending  $x_i$  on  $a_i$ ; and conversely, given a morphism  $h : E \rightarrow G$ , the corresponding solution is the tuple  $(h(x_1), \dots, h(x_n)) \in G^n$ .

Thus, to study the set of solutions of a system of equations in free groups, one has to understand the set of morphisms from a finitely presented group  $E$  to free groups. Any morphism  $h$  from  $E$  to a free group  $F$  is obtained by composing an epimorphism from  $E$  onto the free group  $h(E)$  with a morphism from  $h(E)$  to  $F$ , and morphisms from a free group to an arbitrary group are well-known. The study of all morphisms from  $E$  to  $F$  thus reduces to the study of epimorphisms from  $E$  onto free groups (of rank at most  $n$ ).

The very beginning of Sela's study of equations in free groups may be viewed as a compactification of the set of all epimorphisms from  $E$  onto free groups. This compactification consists in epimorphisms from  $E$  onto possibly non-free groups, which are called *limit groups* by Sela.

An epimorphism  $h$  from the group  $E = \langle x_1, \dots, x_n \mid w_1(x_1, \dots, x_n), \dots, w_p(x_1, \dots, x_n) \rangle$  onto a group  $G$  gives a preferred generating family  $(h(x_1), \dots, h(x_n))$  of  $G$ . In other words,  $h$  defines a marking on  $G$  in the following sense: a *marked group*  $(G, S)$  is a group  $G$  together with an ordered generating family  $S$ . There is a natural topology on the set of marked groups (topology of Gromov-Hausdorff and Chabauty) which makes it compact. This topology can be roughly described as follows: two marked groups  $(G, S)$  and  $(G', S')$  are closed to each other if large balls of their Cayley graphs are isomorphic. Spaces of marked groups have been used in [Gro81], [Gri84], and studied in [Cha00]. Some elementary properties of this topology will be presented in section 2.

**Limit groups.** In this paper, we propose a new definition of limit groups as limit of marked free groups. We give five equivalent characterizations of limit groups. Three of them are well known in model theory (see [Rem89]).

**Theorem 1.1.** *Let  $G$  be a finitely generated group. The following assertions are equivalent:*

1.  $G$  is a limit group in the sense of Sela ([Sel01a])
2. Some marking (or equivalently any marking) of  $G$  is a limit of markings of free groups in a compact space of marked groups.
3.  $G$  has the same universal theory as a free group.
4.  $G$  is a subgroup of a non standard free group.
5.  $G$  is fully residually free.

*Remark.* Fully residually free groups have another interpretation in the language of algebraic geometry over free groups : they are precisely the coordinate groups of irreducible algebraic sets in free groups (see [BMR00] Theorem D2 and [KM99] Lemma 4)

The proof of this theorem will follow from our propositions 3.10, 5.1, 6.5 and 6.6. The equivalence between 1 and 5 is due to Sela ([Sel01a]). The equivalence between 3 and 4 is a particular case of more general results in model theory (see for example [BS69] Lemma 3.8 Chap.9). The equivalence between 3, 4 and 5 is shown in Remeslennikov [Rem89] (groups that have the same universal theory as free groups are called *universally free groups* in [FGRS95] and  $\exists$ -free groups by Remeslennikov in [Rem89]). Assertion 2 is a reformulation of lemma 1.3(iv) in [Sel01a]. The topological point of view allows us to give direct proofs for the equivalences between 2, 3, and 4 (see sections 5.2 and 6.2). More generally, we relate the topology on the set of marked groups to the universal theory of groups and to ultraproducts by the two following propositions (see sections 5.2, 6.2):

**Propositions 5.2 and 5.3.** *If  $\text{Univ}(G) \supset \text{Univ}(H)$ , then for all generating family  $S$  of  $G$ ,  $(G, S)$  is a limit of marked subgroups of  $H$ . Moreover, if a sequence of marked groups  $(G_i, S_i)$  converge to a marked group  $(G, S)$ , then  $\text{Univ}(G) \supset \limsup \text{Univ}(G_i)$ .*

**Proposition 6.4.** *The limit of a converging sequence of marked groups  $(G_i, S_i)$  embeds in any (non-principal) ultraproduct of the  $G_i$ 's. Moreover, any finitely generated subgroup of an ultraproduct of  $G_i$  is a limit of a sequence of markings of finitely generated subgroups of the  $G_i$ 's.*

**Simple properties of limit groups.** The topological point of view on limit groups gives natural proofs of the following (well known) simple properties of limit groups:

**Proposition 3.1.** *Limit groups satisfy the following properties:*

1. A limit group is torsion-free, commutative transitive, and CSA (see Definitions 2.7 and 2.8).
2. Any finitely generated subgroup of a limit group is a limit group.
3. If a limit group is non-trivial (resp. non-abelian), then its first Betti number is at least 1 (resp. at least 2).
4. Two elements of a limit group generate a free abelian group ( $\{1\}$ ,  $\mathbb{Z}$  or  $\mathbb{Z}^2$ ) or a non-abelian free group of rank 2.
5. A limit group  $G$  is bi-orderable: there is a total order on  $G$  which is left and right invariant.

**First examples of limit groups.** Limit groups have been extensively studied as finitely generated fully residually free groups ([Bau62], [Bau67], and [Chi95] and its references). The class of fully residually free groups is clearly closed under taking subgroup and free products. The first non free finitely generated examples of fully residually free groups, including all the non-exceptional surface groups, have been given by Gilbert and Benjamin Baumslag in [Bau62] and [Bau67]. They obtained fully residually free groups by *free extension of centralizers* in free groups (see section 3.2). A free extension of centralizers of a limit group  $G$  is a group of the form  $G *_C (C \times \mathbb{Z}^p)$  where  $C$  is a maximal abelian subgroup of  $G$ . A free (rank  $p$ ) extension of centralizers of a limit group  $G$  is a group of the form  $G *_C (C \times \mathbb{Z}^p)$  where  $C$  is a maximal abelian subgroup of  $G$ . As a corollary, the fundamental group of a closed surface with Euler characteristic at most  $-2$  is a limit group: indeed, such a group embeds in a double of a free group over a maximal cyclic subgroup, and such a double occurs as a subgroup of an extension of centralizers of a free group (see section 3.2).

As seen above, fully residually free groups of rank at most 2 are known to be either one of the free abelian groups  $1, \mathbb{Z}, \mathbb{Z}^2$ , or the non-abelian free group  $F_2$  (see [Bau62]). In [FGM<sup>+</sup>98], a classification of 3-generated limit group is given: a 3-generated limit group is either a free group of rank 3, a free abelian group of rank 3 or a free rank one extension of centralizer in a free group of rank 2.

**Finiteness properties.** The works of Kharlampovich and Myasnikov ([KM98a, KM98b]), and of Sela [Sel01a] show that limit groups can be obtained recursively from free groups, surface groups and free abelian groups by a finite sequence of free products or amalgamations over  $\mathbb{Z}$  (see also [Gui03] where another proof is presented using actions of groups on  $\mathbb{R}^n$ -trees). Such a decomposition of a limit group implies its finite presentation. In the topological context, the finite presentation of limit groups allows to give short proofs of the two following finiteness results.

**Proposition 3.12 ([BMR00] Corollary 19, [Sel01a]).** *Given a finitely generated group  $E$ , there exists a finite set of epimorphisms  $E \twoheadrightarrow G_1, \dots, E \twoheadrightarrow G_p$  from  $E$  to limit groups  $G_1, \dots, G_p$  such that any morphism from  $E$  to a free group factorizes through one of these epimorphisms.*

**Proposition 3.13 ([Raz84], [KM98a], [Sel01a]).** *Consider a sequence of quotients of limit groups*

$$G_1 \twoheadrightarrow G_2 \twoheadrightarrow \dots \twoheadrightarrow G_k \twoheadrightarrow \dots$$

*Then all but finitely many epimorphisms are isomorphisms.*

In sections 5 and 6.4, we give alternative proofs avoiding the use of the finite presentation of limit groups. The first one is an argument by Remeslennikov ([Rem89]). The second result is due to Razborov ([Raz84], see also [KM98b, BMR99]) and the simple argument we give is inspired by a point of view given in [Cha].

These two finiteness results are important steps in the construction of Makanin-Razborov diagrams.

**Makanin-Razborov diagrams.** To understand the set of solutions a given system of equations  $(S)$  in free groups, or equivalently, the set of morphisms from a fixed group  $E$  into free groups, Sela introduces a Makanin-Razborov diagram associated to  $E$  (or equivalently to  $(S)$ ). This diagram is a finite rooted tree whose root is labelled by  $E$ . Its

other vertices are labelled by quotients of  $E$  which are limit groups. Its essential feature is that any morphism  $h$  from  $E$  to a free group can be *read* from this diagram (see section 3.3.3).

**Construction and characterization of limit groups.** The first characterization of limit groups is due to Kharlampovich-Myasnikov ([KM98b]). A finitely generated group is an *iterated extension of centralizers of a free group* if it can be obtained from a free group by a sequence of free extension of centralizers.

**Theorem 4.2 (First characterization of limit groups [KM98b, Th.4]).** *A finitely generated group is a limit group if and only if it is a subgroup of an iterated extension of centralizers of a free group.*

The second characterization we give does not require to pass to a subgroup. It is defined in terms of what we call *generalized double*, which is derived from Sela's strict MR-resolutions.

**Definition 4.4 (Generalized double).** *A generalized double over a limit group  $L$  is a group  $G = A *_C B$  (or  $G = A *_C$ ) such that both vertex groups  $A$  and  $B$  are finitely generated and*

1.  *$C$  is a non-trivial abelian group whose images under both embeddings are maximal abelian in the vertex groups*
2. *there is an epimorphism  $\varphi : G \rightarrow L$  which is one-to-one in restriction to each vertex group (in particular, each vertex group is a limit group).*

The terminology comes from the fact that a genuine double  $G = A *_C \overline{A}$  over a maximal abelian group is an example of a generalized double where  $L = A$  and  $\varphi : G \rightarrow A$  is the morphism restricting to the identity on  $A$  and to the natural map  $\overline{x} \mapsto x$  on  $\overline{A}$ .

**Theorem 4.6 (Second characterization of limit groups).** *(compare Sela's strict MR-resolution). The class of limit groups coincides with the class  $\mathcal{IGD}$  defined as the smallest class containing finitely generated free groups, and stable under free products and under generalized double over a group in  $\mathcal{IGD}$ .*

**Other constructions of limit groups.** There are more general ways to construct limit groups in the flavor of generalized double, and closer to Sela's strict MR-resolution. Our most general statement is given in Proposition 4.21. A slightly simpler statement is the following proposition.

**Proposition 4.11 ([Sel01a, Th.5.12]).** *Assume that  $G$  is the fundamental group of a graph of groups  $\Gamma$  with finitely generated vertex groups such that:*

- *each edge group is a non-trivial abelian group whose images under both edge morphisms are maximal abelian subgroups of the corresponding vertex groups;*
- *$G$  is commutative transitive;*
- *there is an epimorphism  $\varphi$  from  $G$  onto a limit group  $L$  such that  $\varphi$  is one-to-one in restriction to each vertex group.*

*Then  $G$  is a limit group.*

Another version of this result has a corollary which is worth noticing (Proposition 4.13 and 4.22). Consider a marked surface group  $(G, S)$  of Euler characteristic at most  $-1$ . The modular group of  $G$  acts on the set of marked quotients of  $(G, S)$  as follows: given  $\varphi : G \twoheadrightarrow H$  and  $\tau$  a modular automorphism of  $G$ ,  $(H, \varphi(S)).\tau = (H, \varphi \circ \tau(S))$ . For any marked quotient  $(H, \varphi(S))$  of  $(G, S)$  such that  $H$  is a non-abelian limit group, the orbit of  $(H, \varphi(S))$  under the modular group of  $G$  accumulates on  $(G, S)$ . In other words, one has the following:

**Corollary.** *Let  $G$  be the fundamental group of a closed surface  $\Sigma$  with Euler characteristic at most  $-1$ . Let  $\varphi$  be any morphism from  $G$  onto a non-abelian limit group  $L$ .*

*Then there exists a sequence of elements  $\alpha_i$  in the modular group of  $\Sigma$  such that  $\varphi \circ \alpha_i$  converges to  $\text{id}_G$  in  $\mathcal{G}(G)$ .*

**Fully residually free towers.** Particular examples of limit groups are the *fully residually free towers* (in the terminology of Z. Sela, see also [KM98a] where towers appear as coordinate groups of particular systems of equations), i. e. the class of groups  $\mathcal{T}$ , containing all the finitely generated free groups and surface groups and stable under the following operations :

- free products of finitely many elements of  $\mathcal{T}$ .
- free extension of centralizers.
- glue on a base group  $L$  in  $\mathcal{T}$  a surface group that retracts onto  $L$  (see section 4.6 for the precise meaning of this operation).

**Theorem 4.30 ([Sel01a]).** *A fully residually free tower is a limit group.*

It follows from Bestvina-Feighn combination theorem that a fully residually free tower is Gromov-hyperbolic if and only if it is constructed without using extension of centralizers.

A positive answer to Tarski's problem has been announced in [Sel01b] and [KM98c]. We state the result as a conjecture since the referring processes are not yet completed.

**Conjecture ([Sel01b, Th.7]).** *A finitely generated group is elementary equivalent to a non abelian free group if and only if it is a non elementary hyperbolic fully residually free tower.*

We would like to warmly thank Frederic Paulin for his great encouragements and his careful reading of the paper. We also thank O. Kharlampovich and A. Myasnikov for their very instructive remarks and suggestions.

## 2 A topology on spaces of marked groups

The presentation given here extends [Cha00]. We give four equivalent definitions for the space of marked groups.

### 2.1 Definitions

**(a). Marked groups.** A *marked group*  $(G, S)$  consists in a group  $G$  with a prescribed family  $S = (s_1, \dots, s_n)$  of generators. Note that the family is ordered, and that repetitions are allowed.

Two marked groups will be identified if they are isomorphic in the natural sense for marked groups: two marked groups  $(G, (s_1, \dots, s_n))$  and  $(G', (s'_1, \dots, s'_n))$  are isomorphic



as marked groups if and only if the bijection that sends  $s_i$  on  $s'_i$  for all  $i$  extends to an isomorphism from  $G$  to  $G'$  (in particular the two generating families must have the same cardinality  $n$ ).

**Definition 2.1.** For any fixed  $n$ , the set of marked groups  $\mathcal{G}_n$  is the set of groups marked by  $n$  elements up to isomorphism of marked groups.

**(b). Cayley graphs.** A marked group  $(G, S)$  has a natural Cayley graph, whose edges are labeled by integers in  $\{1, \dots, n\}$ . Two marked groups are isomorphic as marked groups if and only if their Cayley graphs are isomorphic as labeled graphs. Thus  $\mathcal{G}_n$  may be viewed as the set of labeled Cayley graphs on  $n$  generators up to isomorphism.

**(c). Epimorphisms.** Fix an alphabet  $\{s_1, \dots, s_n\}$  and consider the free group  $F_n = \langle s_1, \dots, s_n \rangle$  marked by the free basis  $(s_1, \dots, s_n)$ . Generating families of cardinality  $n$  for a group  $G$  are in one-to-one correspondence with epimorphisms from  $F_n$  onto  $G$ . In this context, two epimorphisms  $h_1 : F_n \rightarrow G_1$ ,  $h_2 : F_n \rightarrow G_2$  correspond to isomorphic marked groups if and only if  $h_1$  and  $h_2$  are *equivalent* in the following sense: there is an isomorphism  $f : G_1 \rightarrow G_2$  making the diagram below commutative.

$$\begin{array}{ccc} F_n & \xrightarrow{h_1} & G_1 \\ & \searrow h_2 & \downarrow f \\ & & G_2 \end{array}$$

**(d). Normal subgroups and quotients of  $F_n$ .** Two epimorphisms  $h_1 : F_n \rightarrow G_1$ ,  $h_2 : F_n \rightarrow G_2$  represent the same point in  $\mathcal{G}_n$  if and only if they have the same kernel. Thus  $\mathcal{G}_n$  can be viewed as the set of normal subgroups of  $F_n$ . Equivalently  $\mathcal{G}_n$  can be viewed as the set of quotient groups of  $F_n$ . A quotient  $F_n/N$  corresponds to the group  $F_n/N$  marked by the image of  $(s_1, \dots, s_n)$ . As a convention, we shall sometimes use a presentation  $\langle s_1, \dots, s_n \mid r_1, r_2, \dots \rangle$  to represent the marked group  $(\langle s_1, \dots, s_n \mid r_1, r_2, \dots \rangle, (s_1, \dots, s_n))$  in  $\mathcal{G}_n$ .

*Remark.* More generally a quotient of a marked group  $(G, S)$  is naturally marked by the image of  $S$ .

Note that in the space  $\mathcal{G}_n$ , many marked groups have isomorphic underlying groups. An easy example is given by  $\langle e_1, e_2 \mid e_1 = 1 \rangle$  and  $\langle e_1, e_2 \mid e_2 = 1 \rangle$ . These groups are isomorphic to  $\mathbb{Z}$ , but their presentations give non isomorphic marked groups (once marked by the generating family obtained from the presentation). They are indeed different points in  $\mathcal{G}_2$ . However, there is only one marking of free groups of rank  $n$  and of rank 0 in  $\mathcal{G}_n$ , since any generating family of cardinality  $n$  of  $F_n$  is a basis, and two bases are mapped onto one another by an automorphism of  $F_n$ . But there are infinitely many markings of free groups of rank  $k$ , for any  $0 < k < n$ : for instance  $(\langle a, b \rangle, (a, b, w(a, b)))$  give non isomorphic markings of the free group  $\langle a, b \rangle$  for different words  $w(a, b)$ .

## 2.2 Topology on $\mathcal{G}_n$

**(a). The topology in terms of normal subgroups.** The generating set  $S$  of a marked group  $(G, S)$  induces a word metric on  $G$ . We denote by  $B_{(G,S)}(R)$  its ball of radius  $R$  centered at the identity element of  $G$ .

Let  $2^{F_n}$  be the set of all subsets of the free group  $F_n$ . For any subsets  $A, A' \in 2^{F_n}$ , consider the maximal radius of the balls on which  $A$  and  $A'$  coincide:

$$v(A, A') = \max \{R \in \mathbb{N} \cup \{+\infty\} \mid A \cap B_{(F_n, (s_1, \dots, s_n))}(R) = A' \cap B_{(F_n, (s_1, \dots, s_n))}(R)\}.$$

It induces a metric  $d$  on  $2^{F_n}$  defined by  $d(A, A') = e^{-v(A, A')}$ . This metric is ultrametric and makes  $2^{F_n}$  a totally discontinuous metric space, which is compact by Tychonoff's theorem.

The set  $\mathcal{G}_n$  viewed as the set of normal subgroups of  $F_n$  inherits of this metric. The space of normal subgroups of  $F_n$  is easily seen to be a closed subset in  $2^{F_n}$ . Thus  $\mathcal{G}_n$  is compact.

**(b). The topology in terms of epimorphisms** Two epimorphisms  $F_n \rightarrow G_1$  and  $F_n \rightarrow G_2$  of  $\mathcal{G}_n$  are close to each other if their kernels are close in the previous topology.

**(c). The topology in terms of relations and Cayley graphs.** A *relation* in a marked group  $(G, S)$  is an  $S$ -word representing the identity in  $G$ . Thus two marked groups  $(G, S), (G', S')$  are at distance at most  $e^{-R}$  if they have exactly the same relations of length at most  $R$ . This has to be understood under the following abuse of language:

**Convention.** *Marked groups are always considered up to isomorphism of marked groups. Thus for any marked groups  $(G, S)$  and  $(G', S')$  in  $\mathcal{G}_n$ , we identify an  $S$ -word with the corresponding  $S'$ -word under the canonical bijection induced by  $s_i \mapsto s'_i, i = 1 \dots n$ .*

In a marked group  $(G, S)$ , the set of relations of length at most  $2L + 1$  contains the same information as the ball of radius  $L$  of its Cayley graph. Thus the metric on  $\mathcal{G}_n$  can be expressed in term of the Cayley graphs of the groups: two marked groups  $(G_1, S_1)$  and  $(G_2, S_2)$  in  $\mathcal{G}_n$  are at distance less than  $e^{-2L+1}$  if their labeled Cayley graphs have the same labeled balls of radius  $L$ .

## 2.3 Changing the marker

Let  $E$  be a finitely generated group. We introduce three equivalent definitions of the set of groups marked by  $E$ .

**(a). Normal subgroups.** The set  $\mathcal{G}(E)$  of *groups marked by  $E$*  is the set of normal subgroups of  $E$ .

**(b). Epimorphisms.**  $\mathcal{G}(E)$  is the set of equivalence classes of epimorphisms from  $E$  to variable groups.

**(c). Markings.** Given a marking  $S_0$  of  $E$  with  $n$  generators,  $\mathcal{G}(E)$  corresponds to the closed subset of  $\mathcal{G}_n$  consisting of marked groups  $(G, S) \in \mathcal{G}_n$  such that any relation of  $(E, S_0)$  holds in  $(G, S)$ .

In the last definition, the marking of  $E$  corresponds to a morphism  $h_0 : F_n \rightarrow E$ , and the embedding  $\mathcal{G}(E) \hookrightarrow \mathcal{G}_n$  corresponds, in terms of epimorphisms, to the map  $f \mapsto f \circ h_0$ .

The set  $\mathcal{G}(E)$  (viewed as the set of normal subgroups of  $E$ ) is naturally endowed with the topology induced by Tychonoff's topology on  $2^E$ . This topology is the same as the topology induced by the embeddings into  $\mathcal{G}_n$  described above. Therefore  $\mathcal{G}(E)$  is compact.

The following lemma is left as an exercise:

**Lemma 2.2.** *Let  $h_0 : E' \twoheadrightarrow E$  be an epimorphism and let  $h_0^* : \mathcal{G}(E) \rightarrow \mathcal{G}(E')$  be the induced map defined in terms of epimorphisms by  $h_0^* : h \mapsto h \circ h_0$ .*

*Then  $h_0^*$  is a homeomorphism onto its image. Moreover,  $h_0^*$  is open if and only if  $\ker h_0$  is the normal closure of a finite set. In particular, when  $E'$  is a free group,  $h_0^*$  is open if and only if  $E$  is finitely presented.*

Because of this property, we will sometimes restrict to the case where the marker  $E$  is finitely presented. A typical use of lemma 2.2 is to embed  $\mathcal{G}(E)$  into  $\mathcal{G}_n$ , or  $\mathcal{G}_n$  into  $\mathcal{G}_m$  for  $n \leq m$ , as an open-closed subset. For example consider the epimorphism  $h_0 : F_{n+1} = \langle e_1, \dots, e_{n+1} \rangle \twoheadrightarrow F_n = \langle f_1, \dots, f_n \rangle$  that sends  $e_i$  to  $f_i$  for  $i = 1, \dots, n$ , and  $e_{n+1}$  to 1. Then  $h_0^*$  embeds  $\mathcal{G}_n$  into  $\mathcal{G}_{n+1}$  in the following way: a marked group  $(G, (g_1, \dots, g_n))$  of  $\mathcal{G}_n$  will correspond to the marked group  $(G, (g_1, \dots, g_n, 1))$  of  $\mathcal{G}_{n+1}$ .

## 2.4 Examples of convergent sequences

**(a). Direct limits.** An infinitely presented group  $\langle s_1, \dots, s_n \mid r_1, r_2, \dots, r_i, \dots \rangle$ , marked by the generating family  $(s_1, \dots, s_n)$ , is the limit of the finitely presented groups

$$\langle s_1, \dots, s_n \mid r_1, r_2, \dots, r_i \rangle$$

when  $i \rightarrow +\infty$ , since for any radius  $R$ , balls of radius  $R$  in the Cayley graphs eventually stabilize when adding relators.

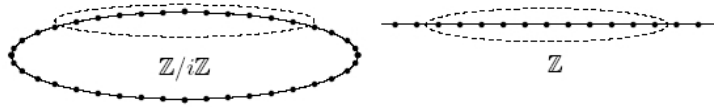


Figure 1:  $\mathbb{Z}/i\mathbb{Z}$  converging to  $\mathbb{Z}$

**(b).  $\mathbb{Z}$  as a limit of finite cyclic groups.** For any  $i$ , the ball of radius  $i/3$  in the marked group  $(\mathbb{Z}/i\mathbb{Z}, (\bar{1}))$  is the same as the ball of radius  $i/3$  in  $(\mathbb{Z}, (1))$ .

In other words, the sequence of marked groups  $(\mathbb{Z}/i\mathbb{Z}, (\bar{1}))$  converges to the marked group  $(\mathbb{Z}, (1))$  when  $i \rightarrow +\infty$

**(c).  $\mathbb{Z}^2$  as a limit of markings of  $\mathbb{Z}$ .** A less trivial example shows that a fixed group with a sequence of different markings may converge to a non isomorphic group. For any  $i \in \mathbb{N}$ , consider the marked group  $(G_i, S_i) = (\mathbb{Z}, (1, i)) \in \mathcal{G}_2$ . For any  $R$ , if  $i \geq 100R$ , the only relations between 1 and  $i$  in the ball of radius  $R$  of  $(G_i, S_i)$  are relations of commutation. Thus the ball of radius  $R$  in  $(G_i, S_i)$  is the same as the ball of radius  $R$  in  $(\mathbb{Z}^2, ((1, 0), (0, 1)))$ .

In other words, the marked groups  $(\mathbb{Z}, (1, i))$  converge when  $i \rightarrow +\infty$  to the marked group  $(\mathbb{Z}^2, ((1, 0), (0, 1)))$ . Of course, the same argument shows that  $\mathbb{Z}^n$  marked by its canonical basis is a limit of markings of  $\mathbb{Z}$  in  $\mathcal{G}_n$ .

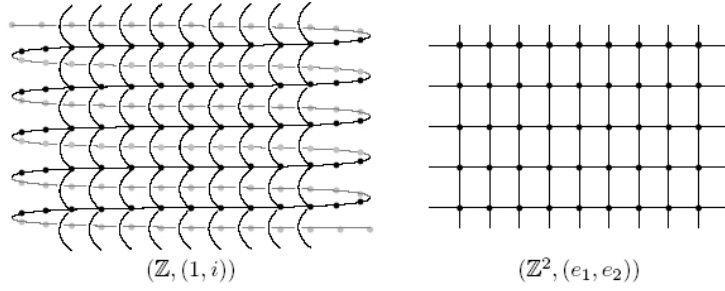


Figure 2:  $\mathbb{Z}^2$  as a limit of markings of  $\mathbb{Z}$

Now consider the marking  $(\mathbb{Z}^k, S)$  in  $\mathcal{G}_n$  ( $k \leq n$ ) defined by taking  $(s_1, \dots, s_k)$  a basis of  $\mathbb{Z}^k$  and  $s_i = 1$  for  $i > k$ . One can deduce from the argument above that there is a sequence of markings  $(\mathbb{Z}, S_i)$  converging to  $(\mathbb{Z}^k, S)$  in  $\mathcal{G}_n$  where the last  $n - k$  generators of  $S_i$  are trivial in  $\mathbb{Z}$ . More conceptually, we use the continuity of the embedding  $\mathcal{G}_h \hookrightarrow \mathcal{G}_n$  defined by  $(G, (s_1, \dots, s_k)) \mapsto (G, (s_1, \dots, s_k, 1, \dots, 1))$  to prove in corollary 2.19 that *any* marking of  $\mathbb{Z}^k$  is a limit of markings of  $\mathbb{Z}$ .

**(d).  $F_k$  as a limit of markings of  $F_2$ .** A similar argument could be used for non abelian free groups to prove that a sequence of markings of  $F_2$  can converge to a free group of rank  $n$  in  $\mathcal{G}_n$ . Consider a free group  $F_2 = \langle a, b \rangle$ . For any large  $L$ , choose random words  $w_1(a, b), \dots, w_{n-2}(a, b)$  of length  $L$  in  $\langle a, b \rangle$ , so that  $w_1(a, b), \dots, w_{n-2}(a, b)$  satisfy small cancellation  $C'(1/100)$  property. Consider the marking  $S_L = (a, b, w_1(a, b), \dots, w_{n-2}(a, b))$  of  $F_2$ . Classical arguments from small cancellation theory show that there are no relations of length less than  $L/2$  between elements of  $S_L$  (since there are no relations of length less than  $L/2$  in the small cancellation group  $\langle a, b \mid w_1(a, b), \dots, w_{n-2}(a, b) \rangle$ ). Therefore, as  $L$  tends to infinity, the sequence  $(F_2, S_L)$  converges to a free group of rank  $n$  marked by a free basis.

The same argument as above shows that there is a marking of  $F_k$  in  $\mathcal{G}_n$  for  $k \leq n$  which is a limit of markings of  $F_2$ . It will be proved in corollary 2.18 that *any* marking of  $F_k$  is a limit of markings of  $F_2$  in  $\mathcal{G}_n$ .

**(e). A residually finite group is a limit of finite groups.** Let  $G$  be a residually finite group, and  $(G, S)$  a marking of  $G$ . For any  $i$ , there is a finite quotient  $G_i$  of  $G$  in which the ball of radius  $i$  of  $(G, S)$  embeds. Denote by  $S_i$  the image of  $S$  in  $G_i$ . Thus  $(G_i, S_i)$  has the same ball of radius  $i$  as  $(G, S)$ . Therefore,  $(G, S)$  is the limit of the marked finite groups  $(G_i, S_i)$ .

There is a partial converse of this result. First note the following easy lemma:

**Lemma 2.3 (Neighbourhood of a finitely presented group).** *Let  $(G, S)$  be a marking of a finitely presented group. There exists a neighbourhood of  $(G, S)$  containing only marked quotients of  $(G, S)$ .*

*Proof.* Let  $\langle s_1, \dots, s_n \mid r_1(s_1, \dots, s_n), \dots, r_k(s_1, \dots, s_n) \rangle$  be a finite presentation of  $(G, S)$ . If  $(G', S')$  is close enough to  $(G, S)$ , these two groups have sufficiently large isomorphic balls to show that  $r_1(s'_1, \dots, s'_n), \dots, r_k(s'_1, \dots, s'_n)$  are trivial in  $G'$ .  $\square$

As a corollary, a finitely presented group which is a limit of finite groups is residually finite.

**Problem.** Describe the closure of the set of finite groups in  $\mathcal{G}_n$ .

## 2.5 Limit groups: first approach

We propose the following definition of limit groups. It follows from lemma 1.3 (iv) of [Sel01a] that this definition is equivalent to the original definition of Sela.

**Definition 2.4.** A marked group in  $\mathcal{G}_n$  is a limit group if it is a limit of marked free groups.

We shall see in corollary 2.18 that being a limit group does not depend on the marking, nor of the space  $\mathcal{G}_n$  where this marking is chosen. As first examples, we have seen that finitely generated free abelian groups are limit groups as limit of markings of  $\mathbb{Z}$ .

Example (e) concerning residual finiteness can be generalized to other residual properties. In particular, residual freeness will play a central role, giving first a criterion for being a limit group (proposition 2.6).

**Definition 2.5 (Residual freenesses).** A group  $G$  is residually free (or 1-residually free) if for any element  $x \in G \setminus \{1\}$ , there exist a morphism  $h$  from  $G$  to a free group such that  $h(x) \neq 1$ .

A group  $G$  is fully residually free if for any finite set of distinct elements  $x_1, \dots, x_i$ , there exist a morphism  $h$  from  $G$  to a free group such that  $h(x_1), \dots, h(x_i)$  are distinct.

Since subgroups of free groups are free, we could assume in this definition that the morphisms are onto.

A residually finite group is fully residually finite, since finite direct products of finite groups are finite. For freeness, the two notions are different. The group  $F_2 \times \mathbb{Z}$  is residually free, since any non trivial element  $(g_1, g_2)$  of  $F_2 \times \mathbb{Z}$  has at least one coordinate  $g_1$  or  $g_2$  which is non trivial. But  $F_2 \times \mathbb{Z}$  is not fully residually free because it is not commutative transitive (see definition 2.7 and corollary 2.10).

As in example (e) above, the following property is immediate:

**Proposition 2.6.** A marked fully residually free group  $(G, S)$  is a limit group.

It results from lemma 2.3 that finitely presented limit groups are fully residually free. A theorem of Kharlampovich-Myasnikov and Sela ([KM98a], [Sel01a]) shows that limit groups are finitely presented. Thus the converse of the proposition 2.6 is true: the limit groups are precisely the finitely generated fully residually free groups. This result will be proved without the use of the finite presentation of limit groups in section 6.3.

## 2.6 Open and closed algebraic properties

In this paragraph, we investigate whether a given property of groups defines a closed or open subset of  $\mathcal{G}_n$ .

**(a). Finite groups are isolated in  $\mathcal{G}_n$ .** Indeed, if  $(G, S)$  is a finite group of cardinal  $R$ , the ball of radius  $R$  of  $(G, S)$  determines the group law of  $G$ . Therefore any marked group with a ball of radius  $R$  isomorphic to this ball is isomorphic to  $G$ .

Thus finiteness is an open property. But it is not closed: the group  $\mathbb{Z}$  is a limit of markings of the finite groups  $\mathbb{Z}/i\mathbb{Z}$ .

**(b). Being abelian is open and closed.** A group generated by  $S$  is abelian if and only if the elements of  $S$  commute. In other words, a marked group  $(G, S)$  is abelian if and only if a certain finite collection of words of length 4 (the commutators in the generators) define the identity element in  $G$ . As soon as two marked groups are close enough (at distance less than  $e^{-4}$ ) in the space  $\mathcal{G}_n$ , they are either both abelian or both non-abelian. Thus being abelian is an open and closed property.

**(c). Nilpotence.** By the same argument, the property of being nilpotent of class less than a given  $k$  is also both open and closed in  $\mathcal{G}_n$ , since this property is satisfied if and only if a finite number of words in the generators are trivial. On the other hand, being solvable, or nilpotent (of any class) are closed properties but not open.

**(d). Torsion.** The property of having torsion is open. Indeed, suppose  $g^i = 1$  in  $G$ , for some  $g \neq 1$ . Then in any marked group  $(G', S')$  close enough to  $(G, S)$ , the element  $g'$  in  $G'$  corresponding to  $g$  (in isometric balls of their Cayley graphs) is non trivial and verifies the relation  $g'^i = 1$  (as soon as the balls are large enough to “contain” this relation). Having torsion is not a closed property: the group  $\mathbb{Z}$  is a limit of finite groups  $\mathbb{Z}/i\mathbb{Z}$ .

**(e). Kurosh rank.** “*Being generated by at most  $k$  elements*” is an open property in  $\mathcal{G}_n$ . Indeed, consider generators  $a_1, \dots, a_k$  of a marked group  $(G, S)$  and write each  $s_i \in S$  as a word  $w_i(a_1, \dots, a_k)$ . In a ball of radius large enough, one reads the relation  $s_i = w_i(a_1, \dots, a_k)$ . In any marked group  $(G', S')$  close enough to  $(G, S)$ , consider  $a'_1, \dots, a'_k$  the elements corresponding to  $a_1, \dots, a_k$  under the bijection between their balls in the Cayley graphs. One can read the relation  $s'_i = w_i(a'_1, \dots, a'_k)$  in  $(G', S')$ , so that  $a'_1, \dots, a'_k$  generate  $G'$ .

In other words, the property “*being generated by less than  $k$  elements*” can be read in a finite ball of the Cayley graph of  $(G, S)$ .

**(f). Commutative transitivity and CSA.** Commutative transitivity has been introduced by B. Baumslag in [Bau67] as a criterium for a residually free group to be fully residually free (see proposition 2.12). CSA-groups (or *Conjugately Separated Abelian* groups) has been defined by A. Myasnikov and V. Remeslennikov in their study of exponential groups ([MR96, GKM95]). These two properties are satisfied by free groups, and they will be shown to be closed, thus satisfied by limit groups.

**Definition 2.7 (Commutative transitivity.).** *A group  $G$  is said to be commutative transitive if commutativity is a transitive relation on  $G \setminus \{1\}$ . In other words:*

$$\forall a, b, c \in G \setminus \{1\}, \quad [a, b] = [b, c] = 1 \Rightarrow [a, c] = 1.$$

A commutative transitive group has the following properties (each one being trivially equivalent to the definition):

- the centralizer of any non trivial element is abelian.
- if two abelian subgroups intersect non trivially, their union generates an abelian subgroup. In other words, different maximal abelian subgroups intersect trivially.

In particular, for any maximal abelian subgroup  $H$  and any element  $g$  of a commutative transitive group  $G$ , the subgroup  $H$  and its conjugate  $gHg^{-1}$  are equal or intersect trivially. A stronger property is given by the following definition:

**Definition 2.8 (CSA).** A group  $G$  is said to be CSA if any maximal abelian subgroup  $H < G$  is malnormal, i. e. for all  $g \in G \setminus H$ ,  $H \cap gHg^{-1} = \{1\}$ .

It is elementary to check that property CSA implies commutative transitivity. The property CSA can be expressed by universal sentences (see section 5.2, and proposition 10 of [MR96]):

**Proposition 2.9.** A group  $G$  is CSA if and only if it satisfies both following properties:

- $\forall a, b, c \in G \setminus \{1\}$ ,  $[a, b] = [b, c] = 1 \Rightarrow [a, c] = 1$  (commutative transitivity),
- $\forall g, h \in G \setminus \{1\}$ ,  $[h, ghg^{-1}] = 1 \Rightarrow [g, h] = 1$ .

The proof is straightforward.

**Corollary 2.10.** Commutative transitivity and CSA are closed properties.

*Remark.* This result extends with the same proof to the fact that any universal formula defines a closed property in  $\mathcal{G}_n$  (see proposition 5.2).

*Proof.* Suppose that elements  $a, b, c$  in  $G \setminus \{1\}$  are such that  $[a, c] \neq 1$ , and  $[a, b] = [b, c] = 1$ . Write  $a, b, c$  as  $S$ -words and let  $L$  be the maximum of their lengths. Consider a ball of radius  $4L$  in the Cayley graph of  $(G, S)$ . Since the relations  $[a, b] = [b, c] = 1$  and the non-relations  $a \neq 1$ ,  $b \neq 1$ ,  $c \neq 1$  and  $[a, c] \neq 1$  can be read in this ball, a marked group close enough to  $(G, S)$  is not commutative transitive. The proof is similar for the property CSA.  $\square$

**Corollary 2.11.** Limit groups are commutative transitive and CSA.

In particular the group  $F_2 \times \mathbb{Z}$ , marked by any generating set, is not a limit group, and hence is not fully residually free.

We conclude by quoting the following theorem of B. Baumslag (theorems 1 and 3 of [Bau67]).

**Theorem 2.12 (B. Baumslag, [Bau67]).** Let  $G$  be a finitely generated group. The following properties are equivalent:

1.  $G$  is fully residually free.
2.  $G$  is residually free and commutative transitive.
3.  $G$  is residually free and does not contain a subgroup isomorphic to  $F_2 \times \mathbb{Z}$ .

**(g). Orderable groups.** A group  $G$  is said to be *left-orderable* (resp. *bi-orderable*) if there is a total order on  $G$  which is left-invariant (resp. left and right-invariant).

**Proposition 2.13.** The property of being a left-orderable (resp. bi-orderable) is closed in  $\mathcal{G}_n$ .

*Proof.* Take a sequence  $(G_i, S_i)$  of ordered marked groups converging to  $(G, S)$ . Let  $R > 0$ , consider the restriction of the total order on the ball of radius  $R$  of  $(G_i, S_i)$ , and let  $\leq_{i,R}$  be the corresponding total order on the ball of radius  $R$  of  $(G, S)$  (for  $i$  large enough). Since there are only finitely many total orders on this ball, a diagonal argument shows that one can take a subsequence so that on each ball of radius  $R$ , the orders  $\leq_{i,R}$  are eventually constant. Thus this defines a total order on  $(G, S)$  by  $g \leq h$  if for  $R \geq \max(|g|, |h|)$ , one has  $g \leq_{i,R} h$  for all but finitely many indexes  $i$ . This order is clearly left-invariant (resp. bi-invariant) if the orders on  $G_i$  are.  $\square$

Free groups are bi-orderable (this non-trivial fact uses the Magnus embedding in a ring of formal series, see for instance [BMR77]). Thus we get the following corollary (we thank T. Delzant who pointed out this fact) :

**Corollary 2.14.** *Limit groups are bi-orderable.*

This result is well-known in the context of model theory, using that limit groups are subgroups of a non-standard free group (see for instance in [Chi95]).

It will be shown in section 3.2 that non-exceptional surface groups are limit groups. Therefore, the corollary implies that these groups are bi-orderable which is not immediate (see [Bau62, RW01]).

**(h). Properties stable under quotient.** Consider a property  $(P)$  stable under taking quotients. Let  $(G, S)$  be a finitely presented marked group satisfying  $(P)$ . Then according to lemma 2.3,  $(P)$  is satisfied in a neighbourhood of  $(G, S)$ .

For example if a sequence of marked groups converges to a finitely presented solvable (resp. amenable, Kazhdan-(T)) group, then all groups in this sequence are eventually solvable (resp. amenable, Kazhdan-(T)).

Shalom has proved in [Sha00] that any finitely generated Kazhdan-(T) group is a quotient of a finitely presented Kazhdan-(T) group. This implies the following result:

**Proposition 2.15.** *Kazhdan's property (T) is open in  $\mathcal{G}_n$ .*

*Proof.* Let  $(G, S)$  be a marked group having property  $(T)$ . Let  $H$  be a finitely presented Kazhdan-(T) group such that  $G$  is a quotient of  $H$ . Write  $(G, S)$  as a direct limit of  $n$ -generated finitely presented groups  $(G_1, S_1) \twoheadrightarrow (G_2, S_2) \twoheadrightarrow \dots$ . Then for  $i$  large enough,  $G_i$  is a quotient of  $H$  and hence has property (T). Now the set of marked quotients of  $(G_i, S_i)$  is an open set (see lemma 2.2) containing  $(G, S)$ , all elements of which have property (T).  $\square$

## 2.7 The isomorphism equivalence relation

The space  $\mathcal{G}_n$  is naturally endowed with the *isomorphism* equivalence relation: two marked groups are equivalent if their underlying group (forgetting about the marking) are isomorphic. We will denote by  $[G]_{\mathcal{G}_n}$  (resp.  $[G]_{\mathcal{G}(E)}$ ) the equivalence class of  $G$  in  $\mathcal{G}_n$  (resp.  $\mathcal{G}(E)$ ). From the definition of  $\mathcal{G}(E)$  by isomorphism classes of epimorphisms,  $[G]_{\mathcal{G}(E)}$  is naturally in bijection with  $\text{Epi}(E \twoheadrightarrow G)/\text{Aut}(G)$  for the natural action of  $\text{Aut}(G)$  on  $\text{Epi}(E \twoheadrightarrow G)$ .

The dynamical properties of this equivalence relation have been studied in [Cha00], in particular on the closure of the set of marked hyperbolic groups. It is shown that this equivalence relation is generated by a pseudo-group of homeomorphisms on  $\mathcal{G}_n$ . This implies the following lemma, but we present here a direct proof.

**Definition 2.16 (saturation).** *A subset  $F \subset \mathcal{G}(E)$  is saturated if it is a union of equivalence classes for the isomorphism relation. The saturation of a subset  $F \subset \mathcal{G}(E)$  is the union of equivalence classes meeting  $F$ .*

**Lemma 2.17.** *Consider a finitely presented group  $E$ . The saturation of an open set  $U \subset \mathcal{G}(E)$  is open in  $\mathcal{G}(E)$ .*

*The closure of a saturated set  $F \subset \mathcal{G}(E)$  is saturated.*



*Proof.* The second statement follows from the first one: consider a saturated set  $F \subset \mathcal{G}(E)$ . Then the interior  $U$  of its complement is the largest open set which does not intersect  $F$ . Since  $F$  is saturated, the saturation  $V$  of  $U$  does not meet  $F$ , and since  $V$  is open,  $V = U$  and  $\overline{F} = \mathcal{G}(E) \setminus V$  is saturated.

We may restrict to the case  $E = F_n$  since  $\mathcal{G}(E)$  is an open and closed subset of  $\mathcal{G}_n$ . Consider an open set  $U \subset \mathcal{G}_n$ ,  $V$  its saturation, and consider  $(G, S) \in U$  and  $(G, S') \in V$ . Consider a radius  $R$  such that any marked group having the same ball of radius  $R$  as  $(G, S)$  lies in  $U$ . We need to prove that there exists a radius  $R'$  such that any marked group  $(H, T')$  having the same ball of radius  $R'$  as  $(G, S')$  has a marking  $(H, T)$  having the same ball of radius  $R$  as  $(G, S)$ .

Express the elements of  $S$  as  $S'$ -words  $s_i = w_i(s'_1, \dots, s'_n)$ . Let  $L$  and  $L'$  be the maximum length of the words  $w_i$  and let  $R' = RL$ . Let  $t_i$  be the element of  $H$  corresponding to the word  $w_i(t'_1, \dots, t'_n)$ . Given a word  $r(e_1, \dots, e_n)$  of length at most  $2R$  in the alphabet  $\{e_1, \dots, e_n\}$ , the following properties are equivalent:

1. the word  $r(t_1, \dots, t_n)$  defines a relation in  $(H, T)$ ,
2. the word  $r(w_1(t'_1, \dots, t'_n), \dots, w_n(t'_1, \dots, t'_n))$  of length at most  $2R'$  defines a relation in  $(H, T')$ ,
3. the word  $r(w_1(s'_1, \dots, s'_n), \dots, w_n(s'_1, \dots, s'_n))$  of length at most  $2R'$  defines a relation in  $(G, S')$ ,
4. the word  $r(s_1, \dots, s_n)$  defines a relation in  $(G, S)$ .

Thus  $(H, T)$  has the same ball of radius  $R$  as  $(G, S)$ . □

**Corollary 2.18.** *Being a limit group does not depend on the marking, nor of the set  $\mathcal{G}_n$  (or  $\mathcal{G}(E)$  for  $E$  finitely presented) in which this marking is chosen.*

*Proof.* The lemma shows that the set of limit groups is saturated, i. e. that if a marking of a group  $G$  in  $\mathcal{G}(E)$  is a limit of markings of free groups, then any other marking of  $G$  is also a limit of markings of free groups in  $\mathcal{G}(E)$ .

Consider an embedding  $\mathcal{G}(E) \subset \mathcal{G}_n$  given by a marking of  $E$ . It is clear that a limit of a marked free groups in  $\mathcal{G}(E)$  is a limit of marked free groups in  $\mathcal{G}_n$ . For the converse, if a group  $(G, S)$  in  $\mathcal{G}(E)$  is a limit of marked free groups  $(G_i, S_i) \in \mathcal{G}_n$ , then for  $i$  large enough,  $(G_i, S_i)$  lies in the open set  $\mathcal{G}(E)$ , so  $(G, S)$  is a limit of marked free groups in  $\mathcal{G}(E)$ . □

*Remark.* The characterization of limit groups as finitely generated fully residually free groups allows to drop the restriction on the finite presentation of  $E$ .

As a consequence of the previous results, we also get the following remark:

**Corollary 2.19 (closure of markings of a free abelian group).** *For  $k = 1, \dots, n$ , the closure of all the markings of the free abelian group  $\mathbb{Z}^k$  in  $\mathcal{G}_n$  is the set of all markings of the groups  $\mathbb{Z}^k, \mathbb{Z}^{k+1}, \dots, \mathbb{Z}^n$ :*

$$\overline{[\mathbb{Z}^k]}_{\mathcal{G}_n} = [\mathbb{Z}^k]_{\mathcal{G}_n} \cup [\mathbb{Z}^{k+1}]_{\mathcal{G}_n} \cup \dots \cup [\mathbb{Z}^n]_{\mathcal{G}_n}$$

*Proof.* Consider  $p \in \{k, \dots, n\}$ . As in example (c), it is easy to construct sequences of markings of  $\mathbb{Z}^k$  converging to some particular marking of  $\mathbb{Z}^p$ . Now since the closure of all markings of  $\mathbb{Z}^k$  is saturated, all the markings of  $\mathbb{Z}^p$  are limits of markings of  $\mathbb{Z}^k$ . Conversely, if a marked group  $(G, S)$  is a limit of markings of  $\mathbb{Z}^k$ , then it is abelian and torsion-free since these are closed properties. Moreover, its Kurosh rank is at least  $k$  according to property (e) in section 2.6. □

## 2.8 Subgroups

In this section, we study how subgroups behave when going to the limit.

**Proposition 2.20 (Marked subgroups).** *Let  $(G_i, S_i)$  be a sequence of marked groups converging to a marked group  $(G, S)$ . Let  $H$  be a subgroup of  $G$ , marked by a generating family  $T = (t_1, \dots, t_p)$ .*

*Then for  $i$  large enough, there is a natural family  $T_i = (t_1^{(i)}, \dots, t_p^{(i)})$  of elements of  $G_i$ , such that the sequence of subgroups  $H_i = \langle T_i \rangle \subset G_i$  marked by  $T_i$  converges to  $(H, T)$  in  $\mathcal{G}_n$ .*

$$\begin{array}{ccc} (G_i, S_i) & \xrightarrow{i \rightarrow \infty} & (G, S) \\ \uparrow \text{dotted} & & \uparrow \\ \exists (H_i, T_i) & \xrightarrow{i \rightarrow \infty} & (H, T) \end{array}$$

Since subgroups of free groups are free, we get the following corollary:

**Corollary 2.21.** *A finitely generated subgroup of a limit group is a limit group.*

*Proof of the proposition.* Consider  $R > 0$  such that the ball  $B$  of radius  $R$  in  $(G, S)$  contains  $T$ . Let  $i$  be large enough so that the ball  $B_i$  of radius  $R$  in  $(G_i, S_i)$  is isomorphic to  $B$ . Let  $T_i$  be the family of elements corresponding to  $T$  under the canonical bijection between  $B$  and  $B_i$ . Then any  $T$ -word is a relation in  $H$  if and only if for  $i$  large enough, the corresponding  $T_i$ -word is a relation in  $H_i$ .  $\square$

*Remark.* The proposition claims that for any  $R$ , the ball  $B_{(H_i, T_i)}(R)$  converges to the ball  $B_{(H, T)}(R)$  for  $i$  large enough. But one should be aware that the trace of the group  $H_i$  in a ball of  $(G_i, S_i)$  might not converge to the trace of  $H$  in a ball of  $(G, S)$ . For example, take  $(G, S) = (\mathbb{Z}^2, ((1, 0), (0, 1)))$ ,  $T = ((1, 0))$  and  $(G_i, S_i) = (\mathbb{Z}, (1, i))$ . Then the trace of  $H = \mathbb{Z} \times \{0\}$  in a ball of  $(G, S)$  is a small subset of this ball. But on the other hand, since  $H_i = G_i$ , the trace of  $H_i$  in any ball of  $(G_i, S_i)$  is the entire ball. This is due to the fact that elements of  $H_i$  may be short in the word metric associated to  $S_i$ , but long in the word metric associated to  $T_i$ .

This phenomenon, occurring here with the subgroup generated by a finite set, does not occur if one consider the centralizer of a finite set. This is the meaning of the next lemma. It will be used in the proof of Proposition 3.7 to build examples of limit groups.

**Definition (Hausdorff convergence of subgroups).** *Let  $(G_1, S_1)$  and  $(G_2, S_2)$  be two marked groups, and  $H_1, H_2$  two subgroups of  $G_1, G_2$ . We say that  $H_1, H_2$ , or more precisely that the pairs  $((G_1, S_1), H_1)$  and  $((G_2, S_2), H_2)$ , are  $e^{-R}$ -Hausdorff close if*

1. *the balls of radius  $R$  of  $(G_1, S_1)$  and  $(G_2, S_2)$  coincide*
2. *the traces of  $H_1$  and  $H_2$  on these  $R$ -balls coincide.*

Denote by  $Z_G(x)$  the centralizer of an element  $x$  in a group  $G$ .

**Lemma 2.22 (Hausdorff convergence of centralizers).** *Consider a sequence of marked groups  $(G_i, S_i)$  converging to  $(G, S)$  and fix any  $x \in G$ . For  $i$  large enough, consider the element  $x_i \in G_i$  corresponding to  $x$  under the natural bijection between balls of Cayley graphs.*

*Then  $Z_{G_i}(x_i)$  Hausdorff-converge to  $Z_G(x)$ .*

*Proof.* The commutation of an element of length  $R$  with  $x$  is read in the ball of length  $2(R + |x|)$ .  $\square$

## 2.9 Free and amalgamated products

Given two families  $S = (s_1, \dots, s_n)$  and  $S' = (s'_1, \dots, s'_{n'})$ , we denote by  $S \vee S'$  the family  $(s_1, \dots, s_n, s'_1, \dots, s'_{n'})$ .

**Lemma 2.23.** *Let  $(G_i, S_i) \in \mathcal{G}_n$  and  $(G'_i, S'_i) \in \mathcal{G}_{n'}$  be two sequences of marked groups converging respectively to  $(G, S)$  and  $(G', S')$ . Then the sequence  $(G_i * G'_i, S_i \vee S'_i)$  converges to  $(G * G', S \vee S')$  in  $\mathcal{G}_{n+n'}$ .*

This is clear using normal forms in a free product. To generalize this statement for amalgamated products, we first need a definition.

**Definition (convergence of gluings).** *Let  $(A, S)$  and  $(A', S')$  be two marked groups, and  $C, C'$  two subgroups of  $A, A'$ .*

*A gluing between the pairs  $((A, S), C)$  and  $((A', S'), C')$  is an isomorphism  $\varphi : C \rightarrow C'$ .*

*We say that two gluings  $\varphi_1$  and  $\varphi_2$  between the pairs  $((A_1, S_1), C_1)$  and  $((A'_1, S'_1), C'_1)$ ,  $((A_2, S_2), C_2)$  and  $((A'_2, S'_2), C'_2)$ , are  $e^{-R}$ -close if :*

1.  $C_1$  (resp.  $C'_1$ ) is  $e^{-R}$ -Hausdorff close to  $C_2$  (resp. to  $C'_2$ )
2. the restrictions  $\varphi_1|_{C_1 \cap B_R(A_1, S_1)}$  and  $\varphi_2|_{C_2 \cap B_R(A_2, S_2)}$  coincide using the natural identification between  $B_R(A_1, S_1)$  and  $B_R(A_2, S_2)$

*In other words, this means that the following “diagram” commutes on balls of radius  $R$*

$$\begin{array}{ccccccc} A_1 & \supset & C_1 & \xrightarrow{\varphi_1} & C'_1 & \subset & A'_1 \\ \wr_R & & \wr_R & & \wr_R & & \wr_R \\ A_2 & \supset & C_2 & \xrightarrow{\varphi_2} & C'_2 & \subset & A'_2 \end{array}$$

**Proposition 2.24 (Convergence of amalgamated products).** *Consider a sequence of groups  $G_i = A_i \underset{C_i = \varphi_i(C_i)}{*} A'_i$  and  $G = A \underset{C = \varphi(C)}{*} A'$ , some markings  $S_i, S'_i, S, S'$  of  $A_i, A'_i, A, A'$  such that*

1.  $(A_i, S_i)$  converges to  $(A, S)$
2.  $(A'_i, S'_i)$  converges to  $(A', S')$
3.  $C_i$  converges to  $C$  in the Hausdorff topology
4.  $C'_i$  converges to  $C'$  in the Hausdorff topology
5. the gluing  $\varphi_i$  converges to  $\varphi$ .

*Then  $(G_i, S_i \vee S'_i)$  converges to  $(G, S \vee S')$ .*

The proof is straightforward using normal forms in amalgamated products and is left to the reader. A similar statement holds for the convergence of HNN-extensions.

## 2.10 Quotients

The following proposition shows that limits of quotients are quotients of the limits.

**Proposition 2.25 (Limits of quotients).** *Consider a sequence of marked groups  $(G_i, S_i) \in \mathcal{G}_n$  converging to a marked group  $(G, S)$ . For any  $i$ , let  $H_i$  be a quotient of  $G_i$ , marked by the image  $T_i$  of  $S_i$ . Assume that  $(H_i, T_i)$  converge to  $(H, T)$  (which may always be assumed up to taking a subsequence).*

*Then  $(H, T)$  is a marked quotient of  $(G, S)$ .*

$$\begin{array}{ccc} (G_i, S_i) & \xrightarrow{i \rightarrow \infty} & (G, S) \\ \downarrow & & \downarrow \exists \\ (H_i, T_i) & \xrightarrow{i \rightarrow \infty} & (H, T) \end{array}$$

*Proof.* Up to the canonical bijection between the families  $S_i, T_i, S, T$ , the only thing to check is that any relation between the elements of  $S$  in  $G$  is verified by the corresponding elements of  $T$  in  $H$ . But for  $i$  large enough, the relation is verified by the elements of  $S_i$  in  $G_i$ , and therefore by the elements of  $T_i$  in the quotients  $H_i$ . Thus the relation is verified by the elements of  $T$  in the limit  $H$ .  $\square$

In the case of abelian quotients, this gives the following result:

**Corollary 2.26.** *Consider a sequence of marked groups  $(G_k, S_k)$  which converge to a marked group  $(G, S)$ . Then the abelianization  $(G_k^{ab}, S_k^{ab})$  of  $G_k$  converge to an abelian quotient of  $G$ . In particular, the first Betti number does not decrease at the limit.*

*Proof.* Clear since the rank of a converging sequence of free abelian groups increases when taking a limit.  $\square$

*Remark.* The first Betti number may increase at the limit: one can find small cancellation presentations  $\langle s_1, \dots, s_n | r_1, \dots, r_n \rangle$  with trivial abelianization and with arbitrarily large relators  $r_1, \dots, r_n$ . Thus there are perfect groups arbitrarily close to free groups.

### 3 Limit groups of Sela

#### 3.1 Summary of simple properties of limit groups

**Definition.** *Denote by  $\mathcal{L}_n$  the set of marked limit groups in  $\mathcal{G}_n$ , i. e. the closure of the set of markings of free groups in  $\mathcal{G}_n$ .*

By definition,  $\mathcal{L}_n$  is a compact subset of  $\mathcal{G}_n$ . Remember that the notation  $[G]_{\mathcal{G}_n}$  represents the set of markings of the group  $G$  in  $\mathcal{G}_n$  (see section 2.7). Thus

$$\mathcal{L}_n = \bigcup_{i \in \{0, \dots, n\}} \overline{[F_i]_{\mathcal{G}_n}}.$$

**Abelian vs non-abelian limit groups.** There are actually three kinds of limit groups in  $\mathcal{L}_n$ : the trivial group, non-trivial free abelian groups (as limits of markings of  $\mathbb{Z}$ ), and non-abelian limit groups. Since being abelian is a closed property, the non-abelian limit groups are limits of free groups of rank at least 2 in  $\mathcal{G}_n$ . But the closure of  $[F_2]_{\mathcal{G}_n}$  contains a marking of  $F_k$  for all  $l \in \{2, \dots, n\}$  (see example (d) in section 2.4), and since  $\overline{[F_2]_{\mathcal{G}_n}}$

is saturated, the set of non-abelian limit groups is actually the closure of markings of  $F_2$ . To sum up,

$$\mathcal{L}_n = \{1\} \sqcup \overline{[\mathbb{Z}]_{\mathcal{G}_n}} \sqcup \overline{[F_2]_{\mathcal{G}_n}}.$$

**Properties.** The following proposition summarizes some elementary properties of limit groups that we have already encountered, and which easily result from the topological setting.

**Proposition 3.1.** *Limit groups satisfy the following properties:*

1. *A limit group is torsion-free, commutative transitive, and CSA (see Definitions 2.7 and 2.8).*
2. *Any finitely generated subgroup of a limit group is a limit group.*
3. *If a limit group is non-trivial (resp. non-abelian), then its first Betti number is at least 1 (resp. at least 2).*
4. *Two elements of a limit group generate a free abelian group ( $\{1\}$ ,  $\mathbb{Z}$  or  $\mathbb{Z}^2$ ) or a non-abelian free group of rank 2.*
5. *A limit group  $G$  is bi-orderable: there is a total order on  $G$  which is left and right invariant.*

Properties 2, 3 and 4 are clear in the context of fully residually free groups. Property 1 is also easy to see for fully residually free groups (theorem 2 of [Bau62]).

*Proof.* Properties 1 and 5 are proved in section 2.6, property 2 in cor. 2.21, and property 3 in cor. 2.26. There remains to check property 4.

Using point 2, this reduces to check that any 2-generated limit group is isomorphic to  $F_2$ ,  $\mathbb{Z}^2$ ,  $\mathbb{Z}$  or  $\{1\}$ . So consider a marked group  $(G, \{a, b\})$  in  $\mathcal{G}_2$  which is a limit of free groups  $(G_i, \{a_i, b_i\})$ . Assume that  $a$  and  $b$  satisfy a non-trivial relation. Then so do  $a_i$  and  $b_i$  for  $i$  large enough. Since  $G_i$  is a free group,  $a_i$  and  $b_i$  generate a (maybe trivial) cyclic group. Since  $\overline{[\mathbb{Z}, \{1\}]_{\mathcal{G}_2}} = [\{1\}]_{\mathcal{G}_2} \cup [\mathbb{Z}]_{\mathcal{G}_2} \cup [\mathbb{Z}^2]_{\mathcal{G}_2}$ , the point 4 is proved.  $\square$

## 3.2 First examples of limit groups.

Fully residually free groups have been studied for a long time (see for example [Chi95] and references). In this section, we review some classical constructions of fully residually free groups which provide the first known examples of limit groups (see section 2.5). We use the topological context to generalize the Baumslag's extension of centralizers of free groups to limit groups (see prop. 3.7, or [MR96, BMR02]).

**Definition 3.2.** *Let  $Z$  be the centralizer of an element in a group  $G$  and  $A$  be a finitely generated free abelian group.*

*Then  $G *_Z (Z \times A)$  is said to be a free extension of centralizer of  $G$ .*

Such a group can be obtained by iterating extensions of the form  $G *_Z (Z \times \mathbb{Z})$ , which are called *free rank one extension of centralizer*. In the sequel, we might simply say *extension of centralizer* instead of *free extension of centralizer*. More general extensions of centralizers are studied in [BMR02].

**Free products.** The set of limit groups is closed under taking free products since a limit of free products is the free product of the limit (Lemma 2.23):

**Lemma 3.3.** *The free product of two limit groups is a limit group.*

**Extension of centralizers and double of free groups.** The first non-free finitely generated fully residually free groups have been constructed by Gilbert and Benjamin Baumslag ([Bau67] theorem 8, see also [Bau62] theorem 1) by extension of centralizers.

**Proposition 3.4 (Extension of centralizers of free groups, [Bau67]).** *If  $F$  is a free group and  $C$  a maximal cyclic subgroup of  $F$ , then for any free abelian group  $A$ , the free extension of centralizer  $F *_C (C \times A)$  is fully residually free.*

To prove this result, one shows that the morphisms  $G \rightarrow F$  whose restriction to  $F$  is the identity map and which send each generator  $e_i$  of  $A$  to a power  $c^{k_i}$  of a fixed non-trivial element  $c$  of  $C$  converge to the identity map when the  $k_i$ 's tend to infinity. This convergence is a consequence of the following lemma:

**Lemma 3.5 ( [Bau62, Proposition 1] or [Bau67, Lemma 7] ).** *Let  $a_1, \dots, a_n$  and  $c$  be elements in a free group  $F_k$  such that  $c$  does not commute with any  $a_i$ . Then for any integers  $k_0, \dots, k_n$  large enough, the element  $c^{k_0} a_1 c^{k_1} a_2 \dots c^{k_{n-1}} a_n c^{k_n}$  is non-trivial in  $F_k$ .*

As a corollary of the proposition, we get the following result used to prove that non-exceptional surface groups are limit groups.

**Corollary 3.6 (Double of free groups, [Bau67]).** *If  $F$  is a free group and  $u \in F$  is not a proper power, then the double  $F *_u \overline{F}$  of  $F$  over  $u$  is fully residually free*

*Proof.* This double actually embeds in the amalgam  $F *_C (C \times \langle t \rangle)$  as the subgroup generated by  $F$  and  $tFt^{-1}$ .  $\square$

**Surface groups.** The fundamental groups of the torus and of the sphere are trivially limit groups. The fundamental group of the orientable surface of genus 2 can be written as the double  $\langle a, b \rangle *_{[a,b]=[c,d]} \langle c, d \rangle$ , and is thus a limit group. Similarly, the fundamental group of the non-orientable surface of Euler Characteristic  $-2$  is  $\langle a, b \rangle *_{a^2 b^2 = c^2 d^2} \langle c, d \rangle$  and is a limit group. Thus, since finitely generated subgroups of limit groups are limit groups, all non-exceptional surface groups (i. e. distinct from the non-orientable surfaces of Euler characteristic 1,0 or  $-1$ ) are limit groups ([Bau62]).

The fundamental group of the projective plane (resp. of the Klein bottle) is not a limit group since a limit group is torsion free (resp. is commutative transitive); the fundamental group  $G = \langle a, b, c | a^2 b^2 c^2 = 1 \rangle$  of the non-orientable surface of Euler characteristic  $-1$  is not a limit group since three elements in a free group satisfying  $a^2 b^2 c^2 = 1$  must commute ([Lyn59], see also section 3.4 for a topological proof).

**Extension of centralizers and double of limit groups.** Baumslag's constructions for free groups can be generalized to limit groups: the next propositions say that the class of limit groups is stable under extension of centralizers and double over any centralizer. This is proved in [MR96] (see also [BMR02]). This result is an elementary case of a more general construction (MR-resolution) given in [Sel01a], see also section 4.

Recall that limit groups are commutative transitive (Corollary 2.11), so the centralizers of non trivial elements are precisely the maximal abelian subgroups.

**Proposition 3.7** ([MR96, BMR02].). *Let  $G$  be a limit group,  $Z$  a maximal abelian subgroup of  $G$ , and  $A$  be a finitely generated free abelian group.*

*Then the free extension of centralizer  $G *_Z (Z \times A)$  is a limit group.*

*Proof.* Consider a generating family  $S = (s_1, \dots, s_n)$  of  $G$ , and a sequence of free groups  $(G_i, S_i)$  converging to  $(G, S)$ . View  $Z$  as the centralizer of an element  $x$ , and for  $i$  large enough, let  $Z_i$  be the centralizer of the corresponding element  $x_i$  in  $G_i$ . Let  $a_1, \dots, a_p$  be a basis of  $A$  and consider  $\tilde{S} = (s_1, \dots, s_n, a_1, \dots, a_p)$  (resp.  $\tilde{S}_i = (s_1^{(i)}, \dots, s_n^{(i)}, a_1, \dots, a_p)$ ) a generating family of  $\tilde{G} = G *_Z (Z \times A)$  (resp.  $\tilde{G}_i = G_i *_Z (Z_i \times A)$ ).

By Baumslag's extension of centralizers,  $\tilde{G}_i$  is a limit group so we just need to check that  $(\tilde{G}_i, \tilde{S}_i)$  converge to  $(\tilde{G}, \tilde{S})$ .

If we already know that abelian subgroups of limit groups are finitely generated (see [KM98a, KM98b, Sel01a]) one can apply the convergence of amalgamated products (Proposition 2.24): it is clear that the inclusion maps  $Z_i \rightarrow Z_i \times A$  converge to the inclusion  $Z \rightarrow Z \times A$  in the sense of definition 2.9. If we don't assume the finite generation of centralizers, a direct argument based on the normal forms in an amalgamated product gives a proof of the result.  $\square$

As for free groups, the following corollary is immediate.

**Corollary 3.8.** *The double of a limit group over the centralizer of one of its non trivial element is a limit group.*

*Remark.* The corollary also follows directly from the convergence of amalgamated products even when centralizers are not finitely generated (Proposition 2.24).

### 3.3 Morphisms to free groups and Makanin-Razborov diagrams.

In this section we describe the construction of a Makanin-Razborov diagram of a limit group. Such a diagram encodes all the morphisms from a given finitely generated group to free groups. This construction is due to Sela and uses two deep results in [Sel01a].

#### 3.3.1 Finiteness results for limit groups.

The main finiteness result for limit groups is their finite presentation.

**Theorem 3.9** ([KM98a, KM98b, Sel01a]). *Limit groups are finitely presented. Moreover abelian subgroups of limit groups are finitely generated.*

This result is proved using JSJ-decomposition by Z. Sela (the *analysis lattice* of limit groups), and using embeddings into free  $\mathbb{Z}[X]$ -groups by O. Kharlampovich and A. Myasnikov. An alternative proof is given in [Gui03] by studying free actions on  $\mathbb{R}^n$ -trees.

As seen in section 2.5, an elementary consequence of the fact that limit groups are finitely presented is the following corollary:

**Corollary 3.10.** *A finitely generated group is a limit group if and only if it is fully residually free.*

A proof of this fact which does not use the finite presentation of limit groups is given in propositions 6.5 and 6.6.

Following Sela, for any finitely generated marked group  $(E, S)$ , there is a natural partial order on the compact set  $\mathcal{G}(E)$ : we say that  $(G_1, S_1) \leq (G_2, S_2)$  if and only if the marked epimorphism  $E \rightarrow G_1$  factorizes through the marked epimorphism  $E \rightarrow G_2$ , i. e.  $G_1$  is a marked quotient of  $G_2$ .

**Lemma 3.11.** *Let  $E$  be a finitely generated group. Any compact subset  $K$  of  $\mathcal{G}(E)$  consisting of finitely presented groups has at most finitely many maximal elements, and every element of  $K$  is a marked quotient of one of them.*

The main ingredient in this result is Lemma 2.3 claiming that a finitely presented marked group has a neighbourhood consisting of marked quotients of this group.

*Proof.* First, for any  $(G, S)$  in  $K$ , there exists a maximal element  $(G', S') \in K$  such that  $(G, S) \leq (G', S')$ . Indeed, apply Zorn Lemma to the set of marked groups  $(G', S') \in K$  such that  $(G, S) \leq (G', S')$  (this uses only the compactness of  $K$ ).

Now, for each maximal element  $(G, S) \in K$ , the set of marked quotients of  $(G, S)$  is a neighbourhood of  $(G, S)$  since  $G$  is finitely presented. This gives a covering of the compact  $K$  from which one can extract a finite subcovering.  $\square$

The next result is the first step of the construction of a Makanin-Razborov diagram. It claims the existence of finitely many maximal limit quotients of any finitely generated group.

**Proposition 3.12 (Sela [Sel01a]).** *Let  $E$  be a finitely generated group. Then there exists a finite set of epimorphisms  $E \rightarrow G_1, \dots, E \rightarrow G_p$  from  $E$  to limit groups  $G_1, \dots, G_p$  such that every morphism from  $E$  to a free group factorizes through one of these epimorphisms.*

*Proof.* Let  $E$  be a finitely generated group. Let  $\mathcal{L}(E)$  be the closure in  $\mathcal{G}(E)$  of the set of epimorphisms from  $E$  to free groups. Thus  $\mathcal{L}(E)$  is compact, and since every limit group is finitely presented, Lemma 3.11 applies.  $\square$

*Remark.* We will give a proof of Proposition 3.12 which does not use the finite presentation of limit groups in section 6.4.

In some sense, the next finiteness result means that if  $G$  is a limit group, and  $H$  is a limit group which is a strict quotient of  $G$ , then  $H$  is *simpler* than  $G$ .

**Proposition 3.13.** *Consider a sequence of quotients of limit groups*

$$G_1 \twoheadrightarrow G_2 \twoheadrightarrow \dots \twoheadrightarrow G_k \twoheadrightarrow \dots$$

*Then all but finitely many epimorphisms are isomorphisms.*

A proof which does not use the finite presentation of limit groups will be given in section 6.4.

*Proof.* Let  $S_1$  be a marking of  $G_1$  in  $\mathcal{G}_n$ , and let  $S_i$  the image of  $S_1$  in  $G_i$ . The sequence  $(G_i, S_i)$  clearly converges (balls eventually stabilize) so consider  $(G, S)$  the limit of this sequence ( $G$  is the direct limit of  $G_i$ ). As a limit of limit groups,  $(G, S)$  is itself a limit group. Thus  $G$  is finitely presented, which implies that all but finitely many epimorphisms are isomorphisms.  $\square$



### 3.3.2 Shortening quotients.

We now introduce the second deep result of [Sel01a], namely the fact that shortening quotients are strict quotients ([Sel01a, Claim 5.3]). We won't give a proof of this result here.

Let  $(G, S)$  be a freely indecomposable marked limit group, and let  $\Gamma$  be a splitting of  $G = \pi_1(\Gamma)$  over abelian groups. A vertex  $v$  of  $\Gamma$  is said to be of *surface-type* if  $G_v$  is isomorphic to the fundamental group of a compact surface  $\Sigma$  with boundary, such that the image in  $G_v$  of each edge group incident on  $v$  is conjugate to the fundamental group of a boundary component of  $\Sigma$  (and not to a proper subgroup), and if there exists a two-sided simple closed curve in  $\Sigma$  which is not nullhomotopic and not boundary parallel. In other words, this last condition means that there is a non-trivial refinement of  $\Gamma$  at a surface-type vertex corresponding to such a curve. This excludes the case where  $\Sigma$  is a sphere with at most three holes or a projective plane with at most two holes. Note that a surface with empty boundary is allowed only if no edge is incident on  $v$ .

A homeomorphism  $h$  of  $\Sigma$  whose restriction to  $\partial\Sigma$  is the identity naturally induces an outer automorphism of  $G$  whose restriction to the fundamental group of each component of  $\Gamma \setminus \{v\}$  is a conjugation (see for instance [Lev]). We call any element of  $\text{Aut}(G)$  inducing such an outer automorphism of  $G$  a *surface-type modular automorphism* of  $\Gamma$ . Similarly, if  $G_v$  is abelian, any automorphism of  $G_v$  which fix the incident edge groups extends naturally to an outer automorphism of  $G$ , and we call *abelian-type modular automorphism* of  $\Gamma$  any element of  $\text{Aut}(\Gamma)$  inducing such an outer automorphism of  $G$ . Let  $\text{Mod}(\Gamma) \subset \text{Aut}(G)$  be the *modular group* of  $\Gamma$ , i. e. the subgroup generated by inner automorphisms, the preimages in  $\text{Aut}(G)$  of Dehn twists along edges of  $\Gamma$ , and by abelian-type and surface-type modular automorphisms.

Of course,  $\text{Mod}(\Gamma)$  depends on the splitting  $\Gamma$  considered. To define a modular group  $\text{Mod}(G) \subset \text{Aut}(G)$ , one could think of looking at the modular group of a JSJ decomposition of  $G$ . But the JSJ decomposition of  $G$  is not unique and it may be not invariant under  $\text{Aut}(G)$ . However, the *tree of cylinders* of a JSJ decomposition of  $G$  (defined in appendix A) does not depend on the JSJ splitting considered, and is thus invariant under automorphisms of  $G$  (see [GL]). Therefore, we call *canonical splitting* of  $G$  the splitting of  $G$  corresponding to the tree of cylinders of any JSJ decomposition of  $G$ . We denote this canonical splitting by  $\Gamma_{\text{can}}$ , and we let  $\text{Mod}(G) = \text{Mod}(\Gamma_{\text{can}})$ . This modular group is maximal in the following sense: for any abelian splitting  $\Gamma$  of  $G$ , one has  $\text{Mod}(\Gamma) \subset \text{Mod}(\Gamma_{\text{can}})$  (see [GL]).

There is a natural action of  $\text{Mod}(G)$  on  $\mathcal{G}(G)$  by precomposition. Given a morphism  $h$  from  $G$  onto a free group, denote by  $[h]_{\text{Mod}}$  its orbit in  $\mathcal{G}(G)$  under  $\text{Mod}(G)$ . For every such orbit  $[h]_{\text{Mod}}$ , Sela introduces some preferred representants called *shortest morphisms*. We give a slightly different definition.

For any morphism  $h$  from  $G$  onto a free group  $F$ , we define the *length*  $l(h)$  of  $h$ :

$$l(h) = \min_B \max_{s \in S} |h(s)|_B$$

$B$  basis of  $F$

where  $|\cdot|_B$  is the word metric on  $(F, B)$ .

**Definition 3.14 (Shortest morphisms, shortening quotients).** *A morphism  $h \in \mathcal{G}(G)$  is called shortest if*

$$l(h) = \inf_{h' \in [h]_{\text{Mod}}} l(h').$$

The closure of the set of shortest morphisms in  $\mathcal{G}(G)$  is called the set of shortening quotients.

**Theorem 3.15** ([Sel01a, Claim 5.3]). *Let  $G$  be a freely indecomposable limit group. Every shortening quotient is a strict quotient of  $G$ .*

The closure of shortest morphisms is called the set of *shortening quotients*.

*Remark.* Our definition slightly differs from the definition by Sela. In our definition, the length is a well defined function on the subset of  $\mathcal{G}(G)$  consisting of marked free groups (however this function depends on the choice of a generating set  $S$  of  $G$ ). In other words, if  $h : G \rightarrow F$  is a morphism, and  $\tau$  is an automorphism of  $F$ ,  $h$  and  $\tau \circ h$  represent the same element of  $\mathcal{G}(G)$ . Thus, if  $F$  has a preferred basis  $B$ ,  $l(h) = \min_{\tau \in \text{Aut } F} \max_{s \in S} |\tau \circ h(s)|_B$ . On the other hand, Sela does not take the infimum on the set all automorphisms of  $F$ , but only on the set of inner automorphisms. But the limit of a sequence of morphisms depends only on the marked group they induce. Hence the theorem stated below follows from the one proved by Sela using Sela's definition of length.

**Corollary 3.16.** *Let  $G$  be a freely indecomposable limit group.*

*Then there are finitely many maximal shortening quotients, and any shortening quotient is a quotient of one of them.*

*Proof.* This follows from lemma 3.11 since the set of shortening quotients of  $G$  is a compact of finitely presented groups in  $\mathcal{G}(G)$ .  $\square$

### 3.3.3 Makanin-Razborov diagrams

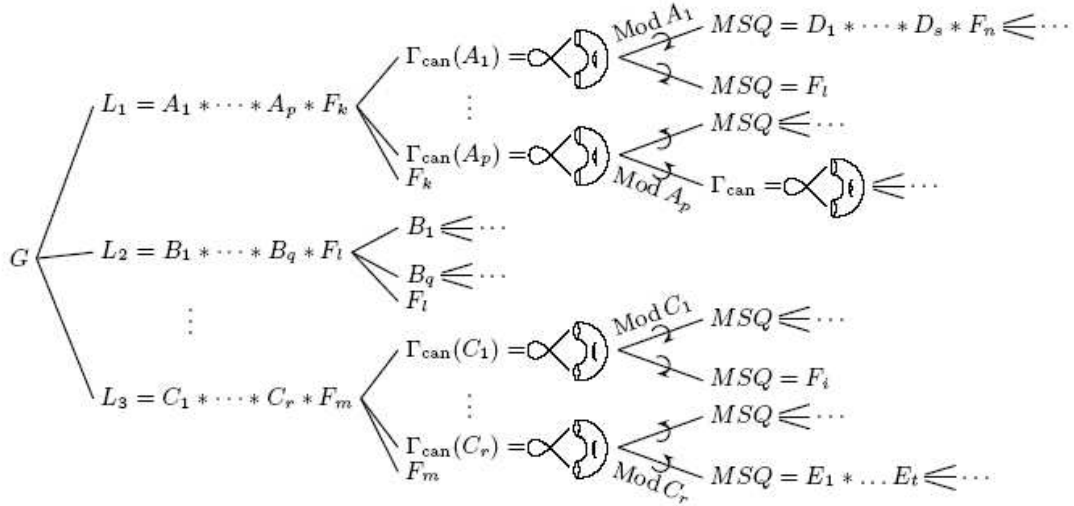


Figure 3: Construction of a Makanin-Razborov Diagram

The main application of Theorem 3.15 (see also Theorem 2 and 3 of [KM98b]) is the construction of a Makanin-Razborov diagram  $\mathcal{D}(G)$  of a finitely generated group  $G$  (see figure 3).

This diagram is a labeled rooted tree where the root vertex is labeled by  $G$ , each other vertex is labeled by a limit group, and each non-oriented edge is labeled by a morphism

(which may go upwards or downwards). Recall that in a rooted tree  $\mathcal{T}$ , a child of a vertex  $v \in \mathcal{T}$  is a vertex  $u \in \mathcal{T}$  adjacent to  $v$ , which is further from the root than  $v$  is.

The children of the root vertex are labeled by the maximal limit quotients of  $G$  in  $\mathcal{G}(G)$  (Prop. 3.12), and the edges originating from the root are labeled by the natural morphisms from  $G$  to its quotients. We now construct inductively  $\mathcal{D}(G)$  by describing the children of any non-root vertex  $v$  of the diagram. Let  $G_v$  be the vertex group at  $v$ .

If  $G_v$  is freely decomposable and is not a free group, write a Grushko decomposition  $G_v = H_1 * \cdots * H_k * F_l$  we define the children of  $v$  to be  $H_1, \dots, H_k, F_l$ . We take as (upwards) edge morphisms the inclusions of  $H_1, \dots, H_k, F_l$  into  $G_v$ .

If  $G_v$  is freely indecomposable, we define the children of  $v$  to be its maximal shortening quotients. The edges originating from  $v$  are labeled by the natural morphisms from  $G_v$  to the corresponding maximal shortening quotient.

If  $G_v$  is a free group, then  $v$  is a leaf of  $\mathcal{D}(G)$ .

Since  $\mathcal{D}(G)$  is locally finite (Prop. 3.12) and has no infinite ray (Prop. 3.13),  $\mathcal{D}(G)$  is finite.

The main feature of this diagram  $\mathcal{D}(G)$  is that any morphism from  $G$  to a free group can be read in this diagram inductively in terms of morphisms of free groups to free groups and of modular automorphisms in the following manner. We call a *Makanin-Razborov Diagram* such a diagram:

**Definition (Makanin-Razborov Diagram).** *Given a finitely generated group  $G$ , consider a finite rooted tree  $\mathcal{D}(G)$  whose root vertex is labeled by  $G$ , and whose other vertices are labeled by limit groups, and such that each edge joining a vertex  $u$  to one of its children  $v$  is labeled either by an epimorphism  $G_u \twoheadrightarrow G_v$  (downwards edge) or by a monomorphism  $G_v \hookrightarrow G_u$  (upwards edge).*

*We say that  $\mathcal{D}(G)$  is a Makanin-Razborov diagram of  $G$  if for any vertex  $v$ , any morphism  $h : G_v \rightarrow F$  to a free group  $F$  can be understood in terms of morphisms from its children groups to  $F$  in one of the four following ways:*

1.  $v$  is the root vertex, all edges originating at  $v$  are downwards, and  $h$  factorizes through one of the epimorphisms labeling these edges;
2.  $G_v$  is freely indecomposable, all edges from  $v$  to its children are downwards, and there exists a modular automorphism  $\tau \in \text{Mod}(G)$  such that  $h \circ \tau$  factors through one of the epimorphisms labeling the edges between  $v$  and its children.
3.  $G_v$  is freely decomposable but not free, all edges from  $v$  to its children are upwards, and  $G_v$  has a non-trivial Grushko decomposition of the form  $G_v = i_1(H_1) * \cdots * i_k(H_k) * i_{k+1}(F_l)$  where  $H_1, \dots, H_k, F_l$  are the groups labeling the children of  $v$ , and  $i_1 : H_1 \hookrightarrow G_v, \dots, i_k : H_k \hookrightarrow G_v, i_{k+1} : F_l \hookrightarrow G_v$  are the edge monomorphisms. In that case, one has  $\text{Hom}(G_v, F) \simeq \text{Hom}(H_1, F) \times \cdots \times \text{Hom}(H_k, F) \times \text{Hom}(F_l, F)$  by the natural map  $h \mapsto (h \circ i_1, \dots, h \circ i_{k+1})$ , therefore  $h$  can be understood in terms of morphisms from its children groups  $H_1, \dots, H_k, F_l$  to  $F$ .
4.  $G_v$  is a free group and  $v$  has no child. Note that a morphism  $h : G_v \rightarrow F$  is just a “substitution”.

### 3.4 Examples of Makanin-Razborov Diagrams

In this section, we give some examples of Makanin-Razborov diagrams. Except in the first few cases, we won't actually describe the result of Sela's construction (in particular, we

will not make explicit the set of shortening quotients), but we will rather describe another Makanin-Razborov diagram.

**Trivial examples.** if  $G$  is a free group, then  $G$  is its only maximal limit quotient, and its Makanin-Razborov Diagram is reduced to  $G \rightarrow G$ . If  $G$  has finite abelianization, then its only limit quotient is the trivial group  $L = \{1\}$ , and the Makanin-Razborov Diagram of  $G$  is reduced to  $G \rightarrow \{1\}$ .

**Abelian groups.** We now consider the case where  $G$  is abelian. In this case,  $G$  has a unique maximal limit quotient  $L$  obtained by killing torsion elements. If  $G$  is virtually cyclic, then  $L$  is a free group and the Makanin-Razborov diagram of  $G$  is the segment  $G \rightarrow L$ . Otherwise, any two kernels of epimorphisms from  $L$  to  $\mathbb{Z}$  differ by an automorphism of  $L$ . Since the modular group of  $L$  is its full automorphism group, this means that the diagram  $G \rightarrow L \rightarrow \mathbb{Z}$  is a Makanin-Razborov Diagram of  $G$ .

For  $G = \mathbb{Z}^p$  endowed with its standard marking  $(e_1, \dots, e_p)$ , we can easily work out the output of Sela's construction by computing maximal shortening quotients. If  $h : G \rightarrow \mathbb{Z}$  is an epimorphism, its length is  $l(h) = \max\{|h(e_i)| \mid i = 1, \dots, p\}$  (see section 3.3.2). But there is an automorphism  $\tau$  of  $G$  such that  $h \circ \tau(e_1) = 1$  and  $h \circ \tau(e_i) = 0$  for  $i = 2, \dots, p$ . It means that all shortest morphisms of  $(G, (e_1, \dots, e_p))$  have length 1. Thus, a shortest morphism consists in sending each generator  $e_i$  to 0, 1, or  $-1$  in  $\mathbb{Z}$ . In particular, there are finitely many shortest morphisms  $h_1, \dots, h_n$ . Therefore, every shortening quotient is a shortest morphism. Moreover, each of them is maximal. Since there are several maximal shortening quotients, the output of Sela's construction of the Makanin Razborov diagram has several terminal vertices labeled by  $\mathbb{Z}$ , where the edge morphisms correspond to  $h_1, \dots, h_n$ . The fact that all the morphisms  $h_1, \dots, h_n$  are all in the same orbit under  $\text{Aut } L$  means that we can keep only one of them to get a Makanin-Razborov diagram.

**Surface groups.** We now describe a Makanin-Razborov diagram of a surface group  $G = \pi_1(\Sigma)$  (but without describing the maximal shortening quotients of  $G$ ). This problem has been studied by many authors (see [Pio86], [Sta95], [CE89], [GK90]). Let's first introduce a definition.

**Definition 3.17.** A pinching of a surface  $\Sigma$  is a family  $\mathcal{C}$  of finitely many disjoint simple closed curves such that

- each curve in  $\mathcal{C}$  is two-sided;
- $\Sigma \setminus \mathcal{C}$  is connected;

Corresponding to a pinching  $\mathcal{C}$  of  $\Sigma$ , there is natural free quotient of  $G = \pi_1(\Sigma)$ : the quotient  $F_{\mathcal{C}}$  of  $G$  by the normal subgroup  $N_{\mathcal{C}}$  generated by the fundamental group of the connected component of  $\Sigma \setminus \mathcal{C}$  is free of rank  $\#\mathcal{C}$ . As a matter of fact, let  $G = \pi_1(\Gamma_{\mathcal{C}})$  be the graph of groups decomposition corresponding to  $\mathcal{C}$ . The graph  $\Gamma_{\mathcal{C}}$  has one vertex corresponding to the connected component of  $\Sigma \setminus \mathcal{C}$ . Edges of  $\Gamma_{\mathcal{C}}$  correspond to the connected components of  $\mathcal{C}$ . Vertex and edge groups are the fundamental groups of the corresponding subsets of  $\Sigma$ . The underlying graph  $\mathcal{G}_{\mathcal{C}}$  of  $\Gamma_{\mathcal{C}}$  is thus a rose having one edge for each curve of  $\mathcal{C}$ , and  $F_{\mathcal{C}}$  is the fundamental group of this rose.

A pinching is *maximal* if it cannot be enlarged into a pinching. Clearly, if  $\mathcal{C} \subset \mathcal{C}'$  are pinchings, then  $h_{\mathcal{C}}$  factors through  $h_{\mathcal{C}'}$ . This is why we will only need to consider maximal pinchings of  $\Sigma$ .

**Proposition 3.18** ([Pio86], [CE89], [GK90],[Sta95]). *Let  $G$  be the fundamental group of a closed compact surface  $\Sigma$ . Let  $h : G \rightarrow F$  be a morphism to a free group. Then there exists a maximal pinching  $\mathcal{C}$  of  $\Sigma$ , such that  $h$  factors through  $h_{\mathcal{C}}$ .*

*Moreover, there are only finitely many maximal pinchings up to homeomorphism of  $\Sigma$ .*

*Remark.* Actually, if  $\Sigma$  is orientable, or if  $\Sigma$  has odd Euler characteristic, then there is exactly one maximal pinching up to homeomorphism of  $\Sigma$ . For an orientable surface of genus  $g$ , view this surface as the boundary of a handlebody  $H$ , then the corresponding morphism to  $F_g$  is induced by the inclusion  $\partial H \subset H$ .

Also note that this proposition implies Lyndon’s result that the fundamental group of the non orientable closed surface  $\Sigma$  of Euler characteristic  $-1$  is not a limit group since any maximal pinching  $\mathcal{C}$  in  $\Sigma$  consists of only one curve so  $F_{\mathcal{C}}$  is cyclic.

This proposition means that there exist pinchings  $\mathcal{C}_1, \dots, \mathcal{C}_n$  such that, for every morphism  $h$  from  $G$  to a free group, there exists a modular automorphism  $\tau$  of  $G$  such that  $h \circ \tau$  factors through one of the morphisms  $h_{\mathcal{C}_i}$ . Thus, if the Euler characteristic of  $\Sigma$  is at most  $-2$  (so that  $G$  is a limit group), the diagram

$$\begin{array}{ccc}
 & & F_{\mathcal{C}_1} \\
 & \nearrow h_{\mathcal{C}_1} & \\
 G & \longrightarrow & G \\
 & \searrow h_{\mathcal{C}_n} & \\
 & & F_{\mathcal{C}_n}
 \end{array}$$

is a Makanin-Razborov diagram for  $G$ . For surfaces of characteristic at least  $-1$ , the only maximal limit quotient of  $G$  is the torsion free part of its abelianization.

*Proof of the proposition.* We assume that  $G$  is endowed with its standard generating set so that  $G = \langle a_i, b_i \mid \prod [a_i, b_i] = 1 \rangle$  in the orientable case, or  $G = \langle a_i \mid \prod a_i^2 = 1 \rangle$  in the non-orientable case.

Consider a morphism  $h$  from  $G$  to a free group  $F$ . We want to represent  $h$  by a topological map. Let  $\Sigma$  be the Cayley 2-complex corresponding to the presentation of  $G$  above. Note that  $\Sigma$  is a surface, and its a cellulation has only one 0-cell  $*$ , and one 2-cell. Subdivide the 2-cell into triangles to obtain a one-vertex “triangulation” of  $\Sigma$ . Identify  $F$  with the fundamental group of a rose  $\Gamma$ , and denote by  $*$  the only vertex of  $\Gamma$ . For each 1-cell  $e$  of  $\Sigma$ , we still denote by  $e$  the corresponding element of  $\pi_1(\Sigma, *)$ . We define  $f : \Sigma \rightarrow \Gamma$  as follows. Send  $*$  to  $*$ , for each 1-cell  $e$  of  $\Sigma$ , we let  $f$  send  $e$  to the reduced path in  $\Gamma$  representing  $h(e)$ . For each 2-cell  $\sigma$ , we can define  $f$  so that the preimage of any point  $x \in \Gamma \setminus \{*\}$  is a disjoint union of finitely many disjoint arcs, such that the endpoints of each arc lies in the interior of two distinct sides of  $\sigma$  (this is a track à la Dunwoody). This can be achieved by lifting  $f$  to the universal covering, and by extending  $\tilde{f}$  on 2-cells according to the model shown on figure 4.

Now, let  $D \subset \Gamma$  be the set of midpoints of edges of  $\Gamma$ , and let  $\mathcal{C}_0 = f^{-1}(D)$ . By construction,  $\mathcal{C}$  is the disjoint union of finitely many two-sided simple closed curves of  $\Sigma$  (however,  $\Sigma \setminus \mathcal{C}_0$  may be disconnected). Let  $G = \pi_1(\Gamma_{\mathcal{C}_0})$  be the corresponding graph of groups decomposition of  $G$ . Clearly, each vertex group of  $\Gamma_{\mathcal{C}_0}$  lies in the kernel of  $h$ . Denote by  $\mathcal{G}_{\mathcal{C}_0}$  the graph underlying  $\Gamma_{\mathcal{C}_0}$ , let  $F_{\mathcal{C}_0}$  be the free group  $F_{\mathcal{C}_0} = \pi_1(\mathcal{G}_{\mathcal{C}_0})$ , and let  $h_0 : G \twoheadrightarrow F_0$  be the natural map consisting in killing vertex groups. Thus  $h$

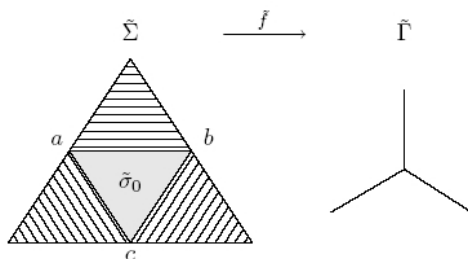


Figure 4: Fibers of  $\tilde{f}$

factors through  $h_0$ . Now let  $T$  be a maximal subtree of  $\Gamma_0$ , and let  $\mathcal{C}_1 \subset \mathcal{C}_0$  be the set of curves corresponding to edges outside  $T$ . Clearly,  $\mathcal{C}_1$  is a pinching of  $\Sigma$ . Denote by  $h_1 : G \rightarrow F_{\mathcal{C}_1} = \pi_1(\mathcal{G}_{\mathcal{C}_1})$  the corresponding morphism. Clearly,  $h_1$  and  $h_0$  have the same kernel so  $h$  factors through  $h_1$ . Now let  $\mathcal{C}$  be a maximal pinching containing  $\mathcal{C}_1$ . Then  $h$  factors through the corresponding morphism  $h_{\mathcal{C}}$ .

To conclude the proof of the proposition, there remains to check that the set of maximal pinching is finite modulo homeomorphisms of  $\Sigma$ .

Indeed, to recover  $\mathcal{C}$  up to homeomorphism, it suffices to consider the surface  $\Sigma_{\mathcal{C}}$  obtained by cutting  $\Sigma$  along  $\mathcal{C}$ , and to know the gluing homeomorphisms between the boundary components of  $\Sigma_{\mathcal{C}}$ . Note that the parity of the Euler characteristic of  $\Sigma_{\mathcal{C}}$  is the same as the one of  $\Sigma$ . Since  $\mathcal{C}$  is a maximal pinching, every two-sided curve of  $\Sigma$  disconnects  $\Sigma$ . Therefore,  $\Sigma_{\mathcal{C}}$  is either a sphere or a projective plane with an even number of holes (the fact that we get a sphere or a projective plane depends only on the parity of the Euler characteristic of  $\Sigma$ , not on  $\mathcal{C}$ ). Since any permutation of the boundary components can be realized by an homeomorphism of  $\Sigma_{\mathcal{C}}$ , up to homeomorphism there is only one way to gather boundary components of  $\Sigma_{\mathcal{C}}$  into pairs. There remains to choose a gluing homeomorphism between the boundary components in each pair. If  $\Sigma_{\mathcal{C}}$  is a punctured sphere, there are two choices for each pair: either the gluing homeomorphism preserves the orientation, or not. In particular, if  $\Sigma$  is orientable, there is exactly one choice for the gluing homeomorphisms. If  $\Sigma$  is not orientable, there are exactly  $c$  choices where  $c = \#\mathcal{C} = 1 - \chi(\Sigma)/2$  (choose the number of orientation-reversing homeomorphisms, it has to be between one and  $\#\mathcal{C}$ ). If  $\Sigma$  is a punctured projective plane, there is a homeomorphism of  $\Sigma_{\mathcal{C}}$  fixing all the boundary components of  $\Sigma_{\mathcal{C}}$  except one, and which restricts to an orientation reversing homeomorphism on the last one. Thus in this case, the two obvious choices differ by an homeomorphism of  $\Sigma_{\mathcal{C}}$ . Finally, there are finitely many possible maximal families  $\mathcal{C}$  up to homeomorphism of  $\Sigma$  (and even exactly one in the orientable case, or when  $\Sigma$  has odd Euler characteristic). □

## 4 Constructing limit groups, fully residually free towers.

Following Sela, the goal of this section is to describe how to construct inductively any limit group as a graph of simpler limit groups. We show two main ways of doing this: the first one (due to Kharlampovich and Myasnikov) claims that any limit groups occurs as a subgroup of a group obtained from a free group by a finite sequence of extension of centralizers ([KM98b, Th.4]). The proof we give is different from the one by Kharlampovich and

Myasnikov since it relies on Sela's techniques. The second way of building any limit group (without passing to a subgroup) is by iterating a construction which we call *generalized double* (see definition below). This characterization is derived from Sela's characterization of limit groups as *strict MR-resolutions* ([Sel01a, Th.5.12]). The arguments of this section follow the proof of Theorem 5.12 of [Sel01a] up to some technical adjustments (see the remarks following Proposition 4.21 and Proposition 4.22).

**Definition 4.1.** *A group is an iterated extension of centralizers of a free group if it is obtained from a finitely generated free group by a finite sequence of free extensions of centralizers.*

*We denote by sub-ICE the class of finitely generated subgroups of iterated extensions of centralizers of a free group.*

*Remark.* In general, one cannot obtain an iterated extension of centralizers of a free group by performing on a free group all the extensions of centralizers simultaneously.

Clearly, the class sub-ICE contains only limit groups. Furthermore, the class sub-ICE is closed under taking finitely generated subgroups, under free product, under free extension of centralizers, and in particular under double over a maximal abelian subgroup.

Our first goal in this section will be the following theorem:

**Theorem 4.2 (First characterization of limit groups [KM98b, Th.4]).** *A finitely generated group is a limit group if and only if it is a subgroup of an iterated extension of centralizers of a free group.*

**Corollary 4.3 ([KM98b, Cor. 6]).** *Any limit group has a free action on a  $\mathbb{Z}^n$ -tree (where  $\mathbb{Z}^n$  has the lexicographic ordering).*

*Any limit group has a free properly discontinuous action (maybe not cocompact) on a CAT(0) space.*

*Proof of the corollary.* It follows from [Bas91, Th. 4.16] that if a group  $G$  has a free action on a  $\mathbb{Z}^n$ -tree, then a free rank one extension of centralizers of  $G$  has a free action on a  $\mathbb{Z}^{n+1}$ -tree. Similarly, if a group  $G$  has a free properly discontinuous action on a CAT(0) space, then so does a free extension of centralizers of  $G$  [BH99]. The corollary is then clear since both properties claimed in the corollary pass to subgroups.  $\square$

**Definition 4.4 (Generalized double).** *A generalized double over a limit group  $L$  is a group  $G = A *_C B$  (or  $G = A *_C$ ) such that both vertex groups  $A$  and  $B$  are finitely generated and*

1.  $C$  is a non-trivial abelian group whose images under both embeddings are maximal abelian in the vertex groups
2. there is an epimorphism  $\varphi : G \twoheadrightarrow L$  which is one-to-one in restriction to each vertex group (in particular, each vertex group is a limit group).

We will also say that  $G$  is a generalized double over  $\varphi$ .

*Remark.* The double considered in corollary 3.8 is a particular case of generalized double: if  $G = A *_C \overline{A}$  where  $C$  a maximal abelian subgroup of  $A$ , one can take  $L = A$ , and  $\varphi$  is the natural morphism sending  $A$  and  $\overline{A}$  on  $A$ .

Free rank one extension of centralizers is also a particular case of a generalized double:  $A *_C (C \times \mathbb{Z})$  is isomorphic to the HNN extension  $G = A *_C$  (where the two embeddings

of  $C$  are the inclusion), and one can take  $L = A$ , and  $\varphi : G \rightarrow A$  the morphism killing the stable letter of the HNN extension.

We will prove in next section that a generalized double over a limit group is a limit group. More general constructions of limit groups are given in Definition 4.10 and Proposition 4.11, and in Definition 4.20 and Proposition 4.21.

**Definition 4.5 (Iterated generalized double).** *A group is an iterated generalized double if it belongs to the smallest class of groups  $\mathcal{IGD}$  containing finitely generated free groups, and stable under free products and generalized double over a group in  $\mathcal{IGD}$ .*

The second goal of this section is the following Theorem deriving from Sela's work:

**Theorem 4.6 (Second characterization of limit groups).** *(Compare Sela's MR-resolution). A group is a limit group if and only if it is an iterated generalized double.*

The argument is structured as follows. First, in section 4.1, we prove that a generalized double over a limit group  $L$  is a subgroup of an extension of centralizers of  $L$ . In particular, a generalized double is a limit group. Then, in sections 4.2 and 4.3, we extend this result to a more general situation: *simple graphs of limit groups*, and we show that those simple graphs of limit groups can be obtained using iteratively the generalized double construction or by iteratively taking subgroups of extensions of centralizers. Thanks to the finiteness results mentioned in section 3.3.1, to conclude, it will suffice to prove the following *key result*: any non-trivial, freely indecomposable limit group can be written as a simple graph of limit groups over a strict quotient. This is proved in section 4.5 using the fact that shortening quotient are strict quotients.

## 4.1 Generalized double as a subgroup of a double

**Proposition 4.7.** *Let  $G = A *_C B$  (resp.  $G = A *_C$ ) be a generalized double over a limit group  $L$ .*

*Then  $G$  is a limit group. More precisely,  $G$  is a subgroup of a double of  $L$  over a maximal abelian subgroup of  $L$  (resp.  $G$  is a subgroup of a free rank one extension of centralizers of  $L$ ).*

*In particular, if  $L$  is a subgroup of an iterated extension of centralizers of a free group, then so is  $G$ .*

We first prove the following simple lemma. Remember that a  $G$ -tree is  $k$ -acylindrical if the set of fix points of any element of  $G \setminus \{1\}$  has diameter at most  $k$ . Accordingly, a graph of groups  $\Gamma$  is  $k$ -acylindrical if the action of  $\pi_1(\Gamma)$  on the Bass-Serre tree of  $\Gamma$  is  $k$ -acylindrical. We will also say that  $\Gamma$  is *acylindrical* if it is  $k$ -acylindrical for some  $k$ .

**Lemma 4.8.** *If  $G = A *_C B$  is a generalized double over a limit group  $L$ , then this splitting is necessarily 1-acylindrical.*

*If  $G = A *_C$  is a generalized double over a limit group  $L$ , then either this splitting is 1-acylindrical or it can be rewritten so that the two embedding of  $C$  into  $A$  coincide. In the latter case, this splitting is not  $k$ -acylindrical for any  $k$ , and  $G$  is isomorphic to a free rank one extension of centralizer of  $A$ .*

*Proof.* Since vertex groups are CSA (they embed into the limit group  $L$ ), each edge group is malnormal in the neighboring vertex groups since it is maximal abelian. Acylindricity follows in the case of an amalgamated product.



Consider now the case of an HNN extension. Denote by  $C_1$  and  $C_2$  the images of  $C$  in  $A$ , and by  $t$  the stable letter of the HNN extension. If this splitting is not 1-acylindrical, then there exists  $a \in A$  such that  $C_1 \cap aC_2a^{-1}$  is non-trivial. Since  $A$  embeds into  $L$ ,  $A$  is commutative transitive so  $C_1 = aC_2a^{-1}$ . Let  $c_1 \in C_1$  and  $c_2 = tc_1t^{-1} \in C_2$  and let  $c'_1 = (at)c_1(at)^{-1} \in C_1$ . Since  $\varphi(c'_1)$  and  $\varphi(c_1)$  commute and  $L$  is CSA,  $\varphi(at)$  commutes with  $\varphi(c_1)$  and  $\varphi(c_1) = \varphi(c'_1)$ . Since  $\varphi$  one-to-one in restriction to  $A$ , one gets that  $at$  commutes with  $C_1$ . Therefore, changing  $t$  to  $at$ , the HNN extension can be rewritten as  $G = \langle A, t \mid tct^{-1} = c, c \in C \rangle$ . The lemma follows.  $\square$

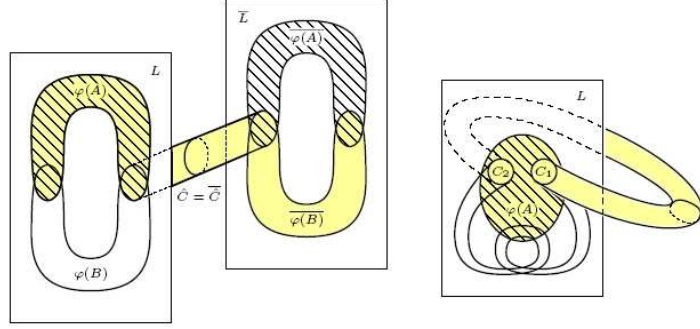


Figure 5: A generalized double inside a double or an extension of centralizers

*Proof of the proposition.* Suppose first that  $G = A *_C B$  with  $\varphi : G \rightarrow L$  one to one in restriction to  $A$  and  $B$ , and  $L$  a limit group. We identify  $A, B$  and  $C$  with their natural images in  $G$ . Let  $\hat{C}$  be the maximal abelian subgroup of  $L$  containing  $\varphi(C)$ . One has  $\hat{C} \cap \varphi(A) = \hat{C} \cap \varphi(B) = \varphi(C)$ . Consider the double  $D = L *__{\hat{C}=\bar{C}} \bar{L}$ . Then the map  $\psi : G \rightarrow D$  whose restriction to  $A$  and  $B$  is  $\varphi$  and  $\bar{\varphi}$  respectively (with obvious notations) is one-to-one. In other words,  $G \simeq \varphi(A) *__{\varphi(C)=\bar{\varphi}(C)} \varphi(B)$ . In particular, if  $L$  lies in sub- $\mathcal{IC}\mathcal{E}$ , so does its double  $D$ , so  $G \in \text{sub-}\mathcal{IC}\mathcal{E}$ .

Suppose now that  $G = A *_C$ . If the HNN extension is not acylindrical, then  $G$  is a free rank one extension of centralizers  $G = A *_C (C \times \mathbb{Z})$ . Let  $\hat{C}$  be the maximal abelian subgroup of  $L$  containing  $\varphi(C)$ , and let  $D$  be the free rank one extension of centralizers  $D = L *__{\hat{C}} (\hat{C} \times \mathbb{Z})$ . Then the map  $\psi : G \rightarrow D$  whose restriction to  $A$  is  $\varphi$  and sending  $\mathbb{Z}$  to  $\mathbb{Z}$  is one-to-one. Thus, if  $L$  lies in sub- $\mathcal{IC}\mathcal{E}$ , then so does  $G$ .

Suppose finally that the HNN extension is acylindrical. Let  $C_1$  and  $C_2$  be the two images of  $C$  in  $A$ , and still identify  $A$  with its natural image in  $G$ . Let  $\hat{C}_1$  and  $\hat{C}_2$  be the maximal abelian subgroups of  $L$  containing  $\varphi(C_1)$  and  $\varphi(C_2)$ . Since  $C_1$  and  $C_2$  are conjugate in  $G$ , and since  $L$  is commutative transitive,  $\hat{C}_1$  and  $\hat{C}_2$  are also conjugate in  $L$  by an element  $t$ . Consider the group  $D = L *__{\hat{C}_1}$  where one embedding of  $\hat{C}_1$  is the inclusion, and the second embedding is the conjugation by  $t$ . Since this HNN extension is not acylindrical,  $D$  is a free rank one extension of  $L$ . Finally, the map  $\psi : G \rightarrow D$  whose restriction to  $A$  is  $\varphi$  and sending the stable letter of  $G$  to the stable letter of  $D$  is one-to-one. In other words,  $G \simeq \varphi(A) *__{C_1} \subset D$ .  $\square$

The following result will be used in the next section. It controls how centralizers grow in a generalized double.

**Lemma 4.9.** *Let  $G = A *_C B$  (resp.  $G = A *_C$ ) be a splitting satisfying the hypothesis of the generalized double. Let  $C'$  be a maximal abelian subgroup of a vertex group. If  $G$  is an amalgamated product or an acylindrical HNN extension, then  $C'$  is maximal abelian in  $G$ . If  $G$  is an HNN extension which is not acylindrical,  $C'$  is maximal abelian in  $G$  if and only if  $C'$  is not conjugate to the edge group  $C$ .*

The proof is straightforward and left to the reader.

## 4.2 Simple graphs of limit groups

In this section, we extend the notion of generalized double to some more general graphs of limit groups to give more general constructions of limit groups.

**Definition 4.10 (Simple graph of limit groups).** *A group  $G$  is a simple graph of limit groups over a limit group  $L$  if  $G$  is the fundamental group of a graph of groups  $\Gamma$  such that:*

- *each vertex group is finitely generated;*
- *each edge group is a non-trivial abelian group whose images under both edge morphisms are maximal abelian subgroups of the corresponding vertex groups;*
- *$G$  is commutative transitive;*
- *there is an epimorphism  $\varphi : G \rightarrow L$  such that  $\varphi$  is one-to-one in restriction to each vertex group.*

We will also say that  $G$  is a simple graph of limit groups over  $\varphi$ .

*Remark.* Corollary A.8 in Appendix A shows that the requirement that  $G$  is commutative transitive would be implied by the stronger hypothesis that  $\Gamma$  is acylindrical.

**Proposition 4.11.** *A simple graph of limit groups over a limit group  $L$  is a limit group.*

*Moreover, if  $L$  is a subgroup of an iterated extension of centralizers of a free group, then so is  $G$ .*

*In other words, the class of groups sub-ICE is stable under simple graph of limit groups.*

*Proof.* We proceed by induction on the number of edges of the graph of groups  $\Gamma$ . We denote by  $\varphi : G \rightarrow L$  the corresponding morphism. If there is no edge, the proposition is trivial. Assume first that  $\Gamma$  contains an edge  $e$  with distinct endpoints (i. e. if  $\Gamma$  has at least two vertices). Let  $H = A *_C B$  be the amalgam carried by  $e$ .

Assume first that  $\Gamma \setminus e$  has two connected components, and denote by  $\Gamma_A$  and  $\Gamma_B$  the components containing the vertex group  $A$  and  $B$  respectively. Consider the double  $D = L *__{\hat{C}=\bar{C}} \bar{L}$  where  $\hat{C}$  is the maximal abelian subgroup of  $L$  containing  $\varphi(C)$ . By Proposition 4.7, the map  $\psi : H \rightarrow D$  whose restriction to  $A$  is  $\varphi$ , and whose restriction to  $B$  is  $\bar{\varphi}$  (with obvious notations) is one-to-one. The map  $\psi$  has a natural extension to  $G$  which coincide with  $\varphi$  (resp.  $\bar{\varphi}$ ) on the fundamental group of  $\Gamma_A$  (resp.  $\Gamma_B$ ). One can then apply induction hypothesis to the graph of groups  $\Gamma_0$  obtained by collapsing  $e$ , together with the morphism  $\psi : G \rightarrow D$ :  $\psi$  is one to one in restriction to the new vertex group  $H$ , and each edge group of  $\Gamma_0$  is maximal abelian in its neighbouring vertex groups because of Lemma 4.9.

If  $\Gamma \setminus e$  is connected, write  $G$  as the HNN extension  $\pi_1(\Gamma \setminus e) *_C$  obtained from  $\Gamma$  by collapsing  $\Gamma \setminus e$ . Denote by  $s$  the stable letter of this HNN extension. Consider the HNN extension  $D = L *__{\hat{C}}$  where  $\hat{C}$  is the maximal abelian subgroup of  $L$  containing  $\varphi(C)$  and

where both edge embeddings are the inclusion. Denote by  $t$  the stable letter of this HNN extension. Let  $\psi : G \rightarrow D$  whose restriction to  $\pi_1(\Gamma \setminus e)$  is  $\varphi$ , and sending  $s$  on  $t$ . One easily checks as in the proof of the previous proposition that  $\psi$  is one-to-one in restriction to  $H$ . As above, using Lemma 4.9, one can apply the induction hypothesis to the graph of groups  $\Gamma_0$  obtained by collapsing  $e$ .

Assume now that  $\Gamma$  has only one vertex and assume that there is an edge  $e$  in  $\Gamma$  such that the HNN extension  $A*_C$  carried by  $e$  is acylindrical. Similarly, write  $G$  as the HNN extension  $G = \pi_1(\Gamma \setminus e)*_C$ , define the morphism  $\psi : G \rightarrow D = L*_C$  whose restriction to  $\pi_1(\Gamma \setminus e)$  is  $\varphi$  and sending the stable letter of  $\pi_1(\Gamma \setminus e)*_C$  to the stable letter of  $D$ . The previous proposition shows that  $\psi$  is one-to-one in restriction to  $H = A*_C$ , and one can use induction hypothesis thanks to Lemma 4.9.

Finally, if  $\Gamma$  has only one vertex and all the edges of  $\Gamma$  carry a non-acylindrical HNN extension, then  $G$  is an iterated extension of centralizers of the vertex group of  $\Gamma$ , which is a subgroup of  $L$ .  $\square$

### 4.3 Twisting generalized doubles

In this section, we give an alternative proof (due to Sela) that a generalized double is a limit group. Actually, we prove the more precise result (which will be needed in the sequel) that given a generalized double  $G$  over  $\varphi : G \rightarrow L$ , there is a sequence of Dehn twists  $\tau_i$  such that the markings of  $L$  defined by  $\varphi \circ \tau_i$  converge to  $G$ .

**Proposition 4.12 ([Sel01a, Th.5.12]).** *Consider a generalized double  $G = A*_C B$  (resp.  $G = A*_C$ ) over  $\varphi : G \rightarrow L$ .*

*Then there exists a sequence of Dehn twists  $(\tau_i)$  on  $G$  such that  $\varphi \circ \tau_i$  converge to  $\text{id}_G$  in  $\mathcal{G}(G)$ .*

*Proof.* Using the fact that  $L$  is fully residually free, consider a sequence of morphisms  $\psi_i$  from  $L$  to free groups  $F^{(i)}$  converging to  $\text{id}_L$ , so that  $\varphi_i = \psi_i \circ \varphi$  converge to  $\varphi$  in  $\mathcal{G}(G)$ . We prove that for any finite set  $g_1, \dots, g_k$  of non-trivial elements of  $G$ , there exist  $i$  such that for  $n$  large enough, the images of  $g_1, \dots, g_k$  under  $\varphi_i \circ \tau^n$  are all non-trivial in the free groups  $F^{(i)}$ . Since each  $\varphi_i$  factorizes through  $\varphi$ , this will imply that  $\varphi \circ \tau^n(g_j)$  is non-trivial for any  $j$ , which will prove the convergence of  $\varphi \circ \tau^n$  to  $\text{id}_G$ . To save notation, we will treat only the case of one element  $g \in G \setminus \{1\}$ , the case  $k > 1$  being identical.

We first consider the case of an amalgamated product. Take  $c$  an element of  $C$  with non-trivial image in  $L$  under  $\varphi$  and denote by  $\tau$  the Dehn twist along  $c$ . Write  $g$  as a reduced form  $g = a_1 b_1 \dots a_p b_p$ , with  $a_j \in A \setminus C$  and  $b_j \in B \setminus C$  (except the maybe trivial elements  $a_1$  and  $b_p$ ). In particular,  $C$  being maximal abelian in both  $A$  and  $B$ , for all  $j$ ,  $c$  does not commute with  $a_j$  nor  $b_j$ . For  $i$  large enough,  $\varphi_i(c)$  does not commute with  $\varphi_i(a_j)$ , nor  $\varphi_i(b_j)$  for all  $j$ . Baumslag's Lemma 3.5 shows that for  $n$  large enough, the image  $\varphi_i(a_1)\varphi_i(c)^n\varphi_i(b_1)\dots\varphi_i(a_p)\varphi_i(c)^n\varphi_i(b_p)$  of  $g$  under  $\varphi_i \circ \tau^n$  is non-trivial.

Let us now consider the slightly more subtle case of an HNN extension. Denote by  $C_1$  and  $C_2$  the images of  $C$  in  $A$  under the two embeddings  $j_1$  and  $j_2$ . We write  $G = \langle A, t \mid tj_1(c)t^{-1} = j_2(c), c \in C \rangle$ .

We first assume that  $C_1 \cap aC_2a^{-1} = \{1\}$  for all  $a \in A$  (this means that the HNN extension is 1-acylindrical). We prove that for all  $c_1 \in C_1 \setminus \{1\}$ ,  $\varphi(c_1)$  does not commute with any element of the form  $\varphi(at)$  for any  $a \in A$ . Indeed, if  $\varphi([at, c_1]) = 1$ , then  $\varphi(a^{-1}c_1a) = \varphi(tc_1t^{-1}) = \varphi(c_2)$  where  $c_2 = j_2(j_1^{-1}(c_1))$ . Since  $\varphi$  is one-to-one in restriction to  $A$ , one gets  $a^{-1}c_1a = c_2$ , a contradiction.

Let  $c_1$  be a non-trivial element of  $C_1$ , and let  $\tau$  the Dehn twist along  $c_1$ :  $\tau$  restricts to the identity on  $A$  and  $\tau(t) = tc_1$ . Consider  $g \in G \setminus \{1\}$ , and let us prove that there exist  $i$  such that for  $n$  large enough,  $\varphi_i \circ \tau^n(g) \neq 1$  in the free group  $F^{(i)}$ . The element  $g$  can be written as a reduced form  $g = a_0 t^{\varepsilon_1} a_1 t^{\varepsilon_2} a_2 \dots t^{\varepsilon_p} a_p$  with  $\varepsilon_j = \pm 1$ , and where  $a_j \notin C_1$  if  $\varepsilon_j = -\varepsilon_{j+1} = 1$  (resp.  $a_j \notin C_2$  if  $\varepsilon_j = -\varepsilon_{j+1} = -1$ ).

Choose  $i$  large enough so that  $\varphi_i(c)$  does not commute with the image of any  $\varphi_i(a_j t)$ . We have

$$\tau^n(g) = a_0 (tc_1^n)^{\varepsilon_1} a_1 (tc_1^n)^{\varepsilon_2} a_2 \dots (tc_1^n)^{\varepsilon_p} a_p$$

so the words  $w_j$  appearing between two powers of  $c_1$  are of one of the following forms:  $a_j$ ,  $a_j t$ ,  $t^{-1} a_j$  and  $t^{-1} a_j t$ . The reduced form guarantees that  $\varphi_i(w_j)$  do not commute with  $\varphi_i(c_1)$ . Baumslag's Lemma 3.5 then concludes, the case where  $\varphi_i(a_0)$  or  $\varphi_i(a_p)$  commutes with  $\varphi_i(c_1)$  being easy to handle.

In the case when the HNN extension is non-acylindrical, then it can be rewritten as  $G = \langle A, t \mid tct^{-1} = c, c \in C \rangle$  (see Lemma 4.8). One checks as above that if an element of the form  $t^k a$  with  $a \in A$  commutes with an element  $c_1 \in C$ , then  $a \in C$ . Now choose a non-trivial element  $c_1 \in C$ , and  $\tau$  the Dehn twist along  $c_1$  sending  $t$  to  $tc_1$ . The argument above can be adapted to this case by writing each element of  $G$  as a reduced word of the form  $a_0 t^{k_1} a_1 t^{k_2} \dots t^{k_p} a_p$  where  $a_0, \dots, a_p \in A$ ,  $a_1, \dots, a_{p-1} \notin C$ , and  $k_2, \dots, k_p \neq 0$ . The argument above concludes the proof.  $\square$

**Proposition 4.13** ([Sel01a, Th.5.12]). *Consider  $G = \pi_1(\Gamma)$  a simple graph of limit groups over  $\varphi : G \rightarrow L$ .*

*Then there exists a sequence of multiple Dehn twists  $\tau_i$  on  $\Gamma$  such that  $\varphi \circ \tau_i$  converges to the identity in  $\mathcal{G}(G)$ .*

*Moreover, if  $\varphi$  is not one-to-one, then  $G$  can be written as a generalized double over an epimorphism  $\varphi' : G \rightarrow L'$  which is not one-to-one.*

*Remark.* The *moreover* part of the proposition will be used to prove that every limit group is an iterated generalized double.

*Proof.* We argue by induction on the number of edges of  $\Gamma$ . If there is only one edge, then we are in the situation of a generalized double and the proposition results from Proposition 4.12.

Consider an edge  $e$  of  $\Gamma$ , and let  $H = A *_C B$  (resp.  $H = A *_C$ ) be the subgroup of  $G$  corresponding to the amalgam or HNN extension carried by  $e$ . By Proposition 4.12, there exists Dehn twists  $\tau_i$  along  $e$  such that  $\varphi \circ \tau_i|_H$  converges to  $\text{id}_H$  in  $\mathcal{G}(H)$ .

In the compact  $\mathcal{G}(G)$ , extract a subsequence of  $\varphi \circ \tau_i$  converging to an epimorphism  $\psi : G \rightarrow L_0$ . The group  $L_0$  is a limit group as a limit of markings of the limit group  $L$ . Let  $\bar{\Gamma}$  be the graph of groups obtained from  $\Gamma$  by collapsing  $e$ . The map  $\psi$  is one-to-one in restriction to  $H$  since  $\varphi \circ \tau_i|_H$  converges to  $\text{id}_H$ , and  $\psi$  is one-to-one in restriction to any other vertex group  $G_v$  since  $\varphi \circ \tau_i$  is one-to-one on  $G_v$  for all  $i$ .

For the first part of the proposition, there remains to check that the edge groups of  $\bar{\Gamma}$  are maximal abelian in their neighbouring vertex group to conclude using induction hypothesis. This is true if the endpoints of  $e$  are distinct, or if the HNN extension carried by  $e$  is 1-acylindrical, since the maximal abelian subgroups of the vertex groups are maximal abelian in  $H$  (Lemma 4.9).

Therefore, we can assume that no edge of  $\Gamma$  holds an amalgam or a 1-acylindrical HNN extension. This means that  $\Gamma$  is a multiple HNN extension of the form  $G =$

$\langle A, t_1, \dots, t_n \mid t_i c t_i^{-1} = c, c \in G_{e_i} \rangle$  where  $e_1, \dots, e_n$  are the edges of  $\Gamma$ . In this case, we take  $e = e_1$ ,  $H = \langle A, t_1 \mid t_1 c t_1^{-1} = c, c \in G_{e_1} \rangle$ , and  $\bar{\Gamma}$  the graph of groups obtained by collapsing  $e_1$  as above. The fact that  $G$  is commutative transitive implies that for  $i \neq 1$ ,  $G_{e_i}$  is maximal abelian in  $H$ . Indeed, if  $g \in H$  commutes with  $G_{e_i}$ , then  $g$  commutes with  $t_i$ , so  $g$  must act by translation on the axis of  $t_i$  in the Bass-Serre tree of  $\bar{\Gamma}$ . Since  $g \in H$ ,  $g$  is elliptic, so  $g$  fixes the axis of  $t_i$ . In particular,  $g \in G_{e_i}$ , so  $G_{e_i}$  is maximal abelian in  $H$ . Thus the induction hypothesis concludes the proof of the first part of the proposition.

To check the *moreover* part, we just need to take care of the case where  $\psi$  is one-to-one. In this case, consider a connected component  $\Gamma'$  of  $\Gamma \setminus e$ . We claim that  $\varphi$  is one-to-one in restriction to the fundamental group  $G'$  of  $\Gamma'$ . Indeed,  $\tau_i$  restricts to a conjugation on  $G'$ , hence for all  $g \in G'$ , if  $\varphi(g) = 1$ , then  $\varphi(\tau_i(g)) = 1$  for all  $i$ , and since  $\varphi \circ \tau_i$  converges to  $\psi$  which is one-to-one, one gets  $g = 1$ . Therefore, by collapsing the connected components of  $\Gamma \setminus e$ , we obtain a 1-edge graph of groups such that  $\varphi$  is one-to-one in restriction to its vertex groups. If the edge group is maximal abelian in both neighbouring vertex groups, then  $G$  is a generalized double over  $\varphi$ , and we are done. Otherwise, the following claim concludes since a free rank one extension of centralizers is a generalized double over a strict quotient.  $\square$

**Claim 4.14.** *Let  $G$  be a group which decomposes as a graph of groups  $\Gamma$  with finitely generated vertex groups and non-trivial abelian edge groups where each edge group is maximal abelian in its neighbouring vertex groups. Assume that  $G$  is a limit group (in other words,  $G = \pi_1(\Gamma)$  is a simple graph of limit groups over the identity). Assume that there exists a maximal abelian subgroup  $C$  of a vertex group of  $\Gamma$  such that  $C$  is not maximal abelian in  $G$ .*

*Then  $G$  can be written as a free rank one extension of centralizers.*

*Proof.* We proceed by induction on the number of edges of  $\Gamma$ . If  $\Gamma$  has no edge, then the claim holds as the hypothesis is impossible.

Assume now that  $\Gamma$  contains an edge  $e$  such that the 1-edge subgraph of groups  $\Gamma_e$  of  $\Gamma$  containing  $e$  is acylindrical. By Lemma 4.9, every maximal abelian subgroup of a vertex group of  $\Gamma_e$  is maximal abelian in  $\pi_1(\Gamma_e)$ . Therefore, the graph of groups  $\bar{\Gamma}$  obtained from  $\Gamma$  by collapsing  $e$  satisfies the hypotheses of the claim and induction hypothesis conclude.

Otherwise, for every edge  $e$  of  $\Gamma$ , the HNN extension  $\Gamma_e$  is a free rank one extension of centralizers, and the result follows.  $\square$

#### 4.4 Statement of the key result and characterizations of limit groups

The following key result will be proved in next section.

**Theorem 4.15 (Key result (see [Sel01a, Th.5.12])).** *Any non-trivial freely indecomposable limit group  $G$  is a simple graph of limit groups over a strict quotient, i. e. over a morphism  $\varphi : G \rightarrow L$  which is not one-to-one.*

The key result allows to deduce the characterizations of limit groups:

**Theorem 4.2 (First characterization of limit groups [KM98b, Th.4]).** *A finitely generated group is a limit group if and only if it is a subgroup of an iterated extension of centralizers of a free group.*

**Theorem 4.6 (Second characterization of limit groups).** *(Compare Sela's MR-resolution). A group is a limit group if and only if it is an iterated generalized double.*

*Proof of the two characterization theorems.* We have already seen that the classes  $\text{sub-IC}\mathcal{E}$  and  $\text{IGD}$  consist of limit groups.

Let  $G$  be a non-trivial limit group. We are going to construct inductively a labeled rooted tree  $\mathcal{T}$ , where each vertex is labeled by a non-trivial limit group, and where the root is labeled by  $G$ . If a vertex  $v$  of  $\mathcal{T}$  holds a group  $H$  which is freely decomposable, we define its children to be its freely indecomposable free factors. In particular, if  $v$  is labeled by a free group, then  $v$  is a leaf of  $\mathcal{T}$  (remember that  $\mathbb{Z}$  is freely decomposable so free groups have no freely indecomposable free factors). If a vertex  $v$  of  $\mathcal{T}$  holds a non-trivial freely indecomposable limit group  $H$ , the key result provides a strict quotient  $L$  of  $H$  such that  $H$  is a simple graph of limit groups over  $L$ . In this case, we attach a single child to  $v$  labeled by  $L$ .

This tree  $\mathcal{T}$  is locally finite, and has no infinite ray by the finiteness property in Proposition 3.13. Thus  $\mathcal{T}$  is finite.

Since labels of leaves of  $\mathcal{T}$  are free groups, they belong to  $\text{sub-IC}\mathcal{E}$ , and since  $\text{sub-IC}\mathcal{E}$  is stable under free products and simple graphs of limit groups (Prop. 4.11), we deduce that  $G$  belongs to  $\text{sub-IC}\mathcal{E}$ .

To prove that  $G$  belongs to  $\text{IGD}$ , we consider a tree  $\mathcal{T}'$ , which similar to  $\mathcal{T}$  except in the case of freely indecomposable groups: if a vertex  $v$  of  $\mathcal{T}'$  holds a non-trivial freely indecomposable limit group  $H$ , the key result provides a strict quotient  $L$  of  $H$  such that  $H$  is a simple graph of limit groups over  $L$ , and Proposition 4.13 gives another strict quotient  $L'$  of  $H$  such that  $H$  is a generalized double over  $L'$ . We then attach a single child to  $v$  labeled by  $L'$ . The same finiteness argument concludes that  $\mathcal{T}'$  is finite and that  $G \in \text{IGD}$  since  $\text{IGD}$  is stable under generalized double and under free product.  $\square$

## 4.5 Proof of the key result

Our aim in this section is to prove the key result (Th. 4.15) i. e. that any non-trivial freely indecomposable limit group  $G$  is a simple graph of groups over a strict quotient (definition 4.10).

Since  $G$  is a limit group, consider a sequence of epimorphisms  $\varphi_i$  from  $G$  to free groups converging to the identity in  $\mathcal{G}(G)$ . For each index  $i$ , consider  $\tau_i$  in  $\text{Mod}(G)$  such that  $\sigma_i = \varphi_i \circ \tau_i$  a shortest morphism in  $[\varphi_i]_{\text{Mod}}$  (see section 3.3.2). Up to taking a subsequence,  $\sigma_i$  converges to a shortening quotient  $\sigma : G \rightarrow L$ . Since shortening quotients are strict quotients (Theorem 3.15),  $\sigma$  is not one-to-one.

Next proposition will gather some properties of this morphism  $\sigma$ . We first need a definition.

**Definition 4.16 (Elliptic abelian neighbourhood).** *Consider a graph of groups  $\Gamma$  over abelian groups whose fundamental group  $G$  is commutative transitive.*

*Consider a non-trivial elliptic subgroup  $H \subset G$ . The elliptic abelian neighbourhood of  $H$  is the subgroup  $\hat{H} \subset G$  generated by all the elliptic elements of  $G$  which commute with a non-trivial element of  $H$ .*

*Remark.* If  $H$  is abelian (in particular, if  $H$  is an edge group of  $\Gamma$ ), the elliptic abelian neighbourhood  $\hat{H}$  of  $H$  is precisely the set of elliptic elements of  $G$  commuting with  $H$  (since this set is a group). In particular, if  $\Gamma$  is acylindrical, then the elliptic abelian neighbourhood of an abelian group is its centralizer.

**Claim 4.17.** *For a vertex  $v$  of  $\Gamma$ ,  $\hat{G}_v$  is the subgroup of  $G$  generated by  $G_v$  and all the groups  $\hat{G}_e$  for  $e$  incident on  $v$ .*

*Proof.* Let  $g \in G_v$ , and  $h \in G$  and elliptic element commuting with  $g$ . We just need to prove that if  $g$  does not fix an edge then  $h \in G_v$ . But since  $h$  commutes with  $g$ ,  $h$  preserves  $\text{Fix } g$ , and  $\text{Fix } g = \{v\}$  by hypothesis.  $\square$

**Proposition 4.18.** *Let  $G$  be a non-trivial freely indecomposable limit group. Let  $\Gamma_{\text{can}}$  be the canonical splitting of  $G$  (see section 3.3.2). Then, either  $G$  can be written as a non-trivial free extension of centralizers, or there exists an epimorphism  $\sigma$  from  $G$  to a limit group  $L$  which is not one-to-one, and such that*

- $\sigma$  is one-to-one in restriction to each edge group;
- for each vertex  $v \in \Gamma$  of surface type,  $\sigma(G_v)$  is non-abelian;
- for each non surface type vertex  $v$ ,  $\sigma$  is one-to-one in restriction to the elliptic abelian neighbourhood  $\hat{G}_v$  of  $G_v$ .

To prove the proposition, we will use the following simple lemma.

**Lemma 4.19.** *Consider a sequence of morphisms  $\varphi_i \in \mathcal{G}(G)$  converging to  $\text{id}$  and let  $\tau_i$  be a sequence of endomorphisms of  $G$  such that  $\varphi_i \circ \tau_i$  converge to  $\psi$ .*

*Assume that there is a subgroup  $H \subset G$  such that for all index  $i$ ,  $\tau_i|_H$  coincides with the conjugation by an element of  $G$ .*

*Then  $\psi$  is one-to-one in restriction to  $H$ .*

*Proof.* Let  $h \in H$ , and assume that  $\psi(h) = 1$ . Then for  $i$  large enough,  $\varphi_i \circ \tau_i(h) = 1$ , therefore  $\varphi_i(h) = 1$ . At the limit, one gets  $\text{id}_G(h) = 1$  and  $h = 1$ .  $\square$

*Proof of the proposition.* If  $\Gamma_{\text{can}}$  contains an abelian vertex group  $G_v$  such that the group generated by incident edge groups is contained in a proper free factor of  $G_v$ , then  $G$  can clearly be written as a non-trivial free extension of stabilizers. Thus, from now on, we can assume that for each abelian vertex group  $G_v$ , the subgroup generated by incident edge groups has finite index in  $G_v$ .

Therefore, each element  $\tau$  of the modular group of  $\Gamma$  coincides with a conjugation in restriction to each non-surface type vertex group and to each edge group of  $\Gamma$ .

Consider a morphism  $\sigma$  as defined above:  $\sigma$  is a limit of shortest morphisms  $\varphi_i \circ \tau_i$  where  $\varphi_i$  is a sequence of morphisms to free groups converging to the identity.

Let  $G_v$  be a non-surface type vertex group of  $\Gamma_{\text{can}}$ . In view of Lemma 4.19, to prove that  $\sigma$  is one-to-one in restriction to  $\hat{G}_v$ , we just need to show that any modular automorphism  $\tau$  restricts to a conjugation on  $\hat{G}_v$ .

We first prove that for each edge group  $G_e$ ,  $\tau$  coincides with a conjugation on  $\hat{G}_e$ . Indeed, since  $\hat{G}_e$  is elliptic, let  $G_v$  be a vertex group containing a conjugate of  $\hat{G}_e$ . If  $G_v$  is not of surface type, then this is clear since the restriction of  $\tau$  to  $G_v$  is a conjugation. If  $G_v$  is of surface type, then  $\hat{G}_e$  is conjugate to an edge group of  $\Gamma$ , and the restriction of  $\tau$  to  $\hat{G}_e$  is a conjugation.

We now prove that for each non-surface type vertex group  $G_v$ ,  $\tau$  coincides with a conjugation in restriction to  $\hat{G}_v$ . Remember that  $\hat{G}_v$  is generated by  $G_v$  and by the groups  $\hat{G}_e$  for  $e$  incident on  $v$ . Moreover,  $\tau$  coincides with a conjugation  $i_g$  on  $G_v$ , and with some (maybe different) conjugation  $i_h$  on  $\hat{G}_e$ . But  $i_g^{-1} \circ i_h$  fixes  $G_e$ , so  $g^{-1}h$  commutes with  $G_e$ . Since  $G$  is commutative transitive,  $i_h$  coincides with  $i_g$  on  $\hat{G}_e$  so  $\tau$  coincides with  $i_g$  on  $\hat{G}_v$ .

Finally, if  $G_v$  is a surface vertex group, then  $\sigma(G_v)$  is non abelian as a limit of the non abelian groups  $\varphi_i \circ \tau_i(G_v)$  in  $\mathcal{G}(G_v)$ .  $\square$

In a simple graph of limit groups, the edge groups are asked to be maximal abelian in both of their adjacent vertex groups, and the morphism  $\varphi$  is asked to be one-to-one in restriction to *all* vertex groups. However, those properties need not be satisfied by the canonical splitting  $\Gamma_{\text{can}}$  and by the morphism  $\sigma$ . The goal of next proposition is to show how those assumptions can be dropped. The key result follows immediately.

**Definition 4.20 (General graph of limit groups).** *A group  $G$  is a general graph of limit groups over a limit group  $L$  if  $G$  is the fundamental group of a graph of groups  $\Gamma$  whose vertex groups are finitely generated and such that*

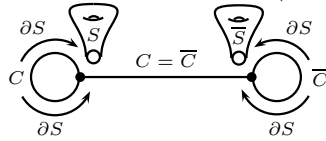
- $G$  is commutative-transitive;
- each edge group is a non-trivial abelian group;
- there is an epimorphism  $\varphi : G \rightarrow L$  such that
  - $\varphi$  is one to one in restriction to each edge group;
  - for each vertex  $v \in \Gamma$  of surface type,  $\varphi(G_v)$  is non-abelian;
  - for each non surface type vertex  $v$ ,  $\varphi$  is one-to-one in restriction to the elliptic abelian neighbourhood  $\tilde{G}_v$  of  $G_v$ .

This proposition gives a general statement that precises the statements in Definition 5.11 of [Sel01a].

**Proposition 4.21.** *Let  $G = \pi_1(\Gamma)$  be a general graph of limit groups over  $\varphi : G \rightarrow L$ .*

*Then  $G$  can be written as a simple graph of limit groups over the same morphism  $\varphi$ . In particular,  $G$  is a limit group.*

- Remark.*
1. If for some abelian vertex group  $G_v$ , we allow  $\varphi$  to be one-to-one only in restriction to the direct summand of the incident edge groups, then it is not true that there exists a sequence of Dehn twists  $\tau_i$  such that  $\varphi \circ \tau_i$  converges to  $\text{id}_G$  in  $\mathcal{G}(G)$ . As a matter of fact, such a Dehn twist restricts to a conjugation on each abelian vertex group. Sela’s proof misses this point.
  2. The statements in definition 5.11 and theorem 5.12 of [Sel01a] seem to be slightly incorrect. A simple counterexample is the following double :  $G = (C * S) *_{C=\bar{C}} (S * \bar{C})$  where  $S$  is the fundamental group of a punctured torus, and  $C$  is conjugate to the fundamental group of its boundary component (see figure below).



The fundamental group of this graph of groups is a double of a surface group with extended centralizer, and it is not commutative transitive (it contains a subgroup isomorphic to  $\mathbb{Z}^2 *_\mathbb{Z} \mathbb{Z}^2 \simeq F_2 \times \mathbb{Z}$ ).

To avoid stating technical conditions on the centralizers of edges, we include the hypothesis that  $G$  is commutative transitive in the result above. In view of the characterization of CSA graph of groups given in Corollary A.8 in appendix A, one could replace this hypothesis with the stronger assumption that  $\Gamma$  is acylindrical.

*Proof.* There are two main steps to prove this proposition. First, we cut surfaces occurring in  $\Gamma$  so that  $\varphi$  is one-to-one in restriction to the elliptic abelian neighbourhood of *all* vertex groups of the new graph of groups (Proposition 4.22). In second step, we *pull centralizers* so that edge groups become maximal abelian in neighbouring vertex groups (Proposition 4.26). The proposition follows.  $\square$



### 4.5.1 Step 1: cutting surfaces

**Proposition 4.22.** *Let  $G = \pi_1(\Gamma)$  be a general graph of limit groups over  $\varphi : G \rightarrow L$ . Then one can refine  $\Gamma$  into a graph of groups  $\Gamma_1$  such that*

- $\Gamma_1$  is a general graph of limit groups over  $L$ ;
- $\varphi$  is one-to-one in restriction to the elliptic abelian neighbourhood of each vertex group.

*Remark.* The proof follows Sela when all the fundamental groups of boundary components of surface type vertices are maximal abelian in  $G$ . The general case needs the additional easy Lemma 4.24.

The proof is based on the following elementary lemma of Sela.

**Lemma 4.23** ([Sel01a, Lemma 5.13]). *Let  $S$  be the fundamental group of a surface  $\Sigma$  (maybe with boundary) with Euler characteristic at most  $-1$ . Let  $\varphi : S \rightarrow L$  be a morphism to a limit group  $L$  with non abelian image, and which is one-to-one in restriction to the fundamental groups of its boundary components.*

*Then there exists a family of disjoint simple closed curves  $c_1, \dots, c_p$  of  $\Sigma$ , such that  $\varphi(c_i)$  is non-trivial for all  $i$ , all the connected components of  $\Sigma \setminus (c_1 \cup \dots \cup c_p)$  is either a pair of pants or a punctured Möbius band, and  $\varphi$  is one to one in restriction to the fundamental group of each of these components.*

*Remark.* Note that the fundamental group of a surface of Euler characteristic  $-1$  with non-empty boundary is a free group of rank 2. Since its image under  $\varphi$  is a non-abelian limit group,  $\varphi$  is one-to-one in restriction to this fundamental group as soon as its image is non abelian (see point 4 in Prop. 3.1). The idea to prove the lemma is to find an essential simple closed curve whose image in  $L$  is non-trivial, and such that the images in  $L$  of the connected components of the complement are non-abelian. Then one iterates the procedures on connected components of the complement.

*Proof of the proposition.* Using Lemma 4.23, we refine the graph of groups  $\Gamma$  into a graph of groups  $\Gamma_1$  by splitting the surface type vertices occuring in  $\Gamma$  along the simple closed curves given by the lemma.

Call *new vertices* of  $\Gamma_1$  all the vertices coming from the subdivision of surface type vertices of  $\Gamma$ , and *old vertices* the other ones. We want to prove that for each (old or new) vertex group  $G_v$ ,  $\varphi$  is one-to-one in restriction to its elliptic abelian neighbourhood  $\hat{G}_v$ .

The elliptic abelian neighbourhood of an old vertex group in  $\Gamma_1$  is not larger than in  $\Gamma$  since elliptic elements in  $\Gamma_1$  are elliptic in  $\Gamma$ . Thus  $\varphi$  is still one-to-one in restriction to  $\hat{G}_v$  for each old vertex  $v$  of  $\Gamma_1$ .

According to Lemma 4.23, we also know that  $\varphi$  is one-to-one in restriction to each new vertex group  $G_v$  of  $\Gamma_1$ . This implies that for each edge  $e$  of  $\Gamma_1$ ,  $\varphi$  is one-to-one in restriction to  $\hat{G}_e$ :  $\hat{G}_e$  is elliptic, so it is conjugate into some vertex group of  $\Gamma_1$ .

There remains to prove that for each new vertex group  $G_v$  of  $\Gamma_1$ ,  $\varphi$  is one-to-one in restriction to its elliptic abelian neighbourhood  $\hat{G}_v$ .

We remark that at least one of the edges  $e$  incident on  $v$  corresponds to a subdivision curve (otherwise  $v$  would be an old vertex). Therefore, for this edge  $e$ , one has  $\hat{G}_e = G_e$ . Since  $G_v$  is the fundamental group of a pair of pants or of a punctured Möbius band, next lemma concludes.  $\square$

**Lemma 4.24 (Embedding of abelian neighbourhood of small surfaces).** *Let  $\Sigma$  be a pair of pants or a punctured Möbius band. Let  $G$  be a group containing  $S = \pi_1(\Sigma)$ . Denote by  $b_1, b_2, b_3$  (or by  $b_1, b_2$  in the case of a punctured Möbius band) some generators of the fundamental groups of the boundary components of  $\Sigma$ . For each index  $i$ , consider an abelian group  $B_i \subset G$  containing  $b_i$  such that for at least one  $i$ , one has  $B_i = \langle b_i \rangle$ . Let  $\hat{S}$  be the subgroup of  $G$  generated by  $S$  and the abelian groups  $B_i$ .*

*Let  $\varphi : \hat{S} \rightarrow L$  be a morphism to a limit group  $L$  which is one-to-one in restriction to each group  $B_i$ . and such that  $\varphi(\hat{S})$  is non-abelian.*

*Then  $\varphi$  is one-to-one in restriction to  $\hat{S}$ .*

*Proof.* We first consider the case of a pair of pants, so that  $S$  has a presentation of the form  $\langle b_1, b_2, b_3 \mid b_1 b_2 b_3 = 1 \rangle$  where each  $b_i$  is a generator of the fundamental group of a boundary components of  $\Sigma$ . Assume for instance that  $B_3 = \langle b_3 \rangle$ . Then  $\hat{S}$  is generated by  $B_1$  and  $B_2$  and the following claim concludes.

In the case of a a punctured Möbius band, one has a presentation of the form  $S = \langle a, b_1, b_2 \mid a^2 b_1 b_2 = 1 \rangle$ . If  $B_2 = \langle b_2 \rangle$ , then  $\hat{S}$  is generated by  $B_1$  and  $\langle c \rangle$ , and the following claim also concludes.  $\square$

**Claim 4.25.** *Let  $A, B$  be two abelian groups, and  $L$  be a limit group. If a morphism  $\varphi : A * B \rightarrow L$  has non-abelian image and is one-to-one in restriction to  $A$  and  $B$ , then  $\varphi$  is one-to-one on  $A * B$ .*

*Proof of the claim.* Consider  $a_1 b_1 \dots a_n b_n$  a reduced word in  $A * B$ . Let  $\rho : L \rightarrow F$  be a morphism into a free group such that  $\rho \circ \varphi(a_i)$  and  $\rho \circ \varphi(b_i)$  are non-trivial and do not commute. Since  $\rho \circ \varphi(A)$  and  $\rho \circ \varphi(B)$  are abelian in  $F$ , there exist  $\alpha, \beta \in F$  such that the elements  $\rho \circ \varphi(a_i)$  are powers of  $\alpha$  and  $\rho \circ \varphi(b_i)$  are powers of  $\beta$ . Thus  $\alpha$  and  $\beta$  do not commute, and they freely generate a free group. The image under  $\rho \circ \varphi$  of the word  $a_1 b_1 \dots a_n b_n$  is a reduced word in  $\langle \alpha, \beta \rangle$  and is thus non-trivial.  $\square$

#### 4.5.2 Step 2: pulling centralizers

**Proposition 4.26 (Pulling centralizers).** *Consider  $G = \pi_1(\Gamma)$  be a splitting of a commutative transitive group with abelian edge groups.*

*Then there exists a splitting  $G = \pi_1(\Gamma')$  with the same underlying graph as  $\Gamma$  such that*

- *each edge group of  $\Gamma'$  is maximal abelian in the neighbouring vertex groups*
- *each edge group  $G'_e$  of  $\Gamma'$  is the elliptic abelian neighbourhood  $\hat{G}_e$  in  $\Gamma$  of the corresponding edge group  $G_e$*
- *each vertex group  $G'_v$  of  $\Gamma'$  is the elliptic abelian neighbourhood  $\hat{G}_v$  in  $\Gamma$  of the corresponding edge group  $G_v$*

The graph of groups  $\Gamma'$  will be obtained from  $\Gamma$  by a finite sequence of the following operation:

**Definition 4.27 (Pulling centralizers across an edge).** *Consider a graph of groups  $\Gamma$  with abelian edge group, and an oriented edge  $e \in E(\Gamma)$  with  $u = o(e)$  and  $v = t(e)$ . Consider the graph of groups  $\hat{\Gamma}$  with same underlying graph and fundamental group as  $\Gamma$  and obtained from  $\Gamma$  by the following operation: replace  $G_e$  by  $\hat{G}_e$ , and replace  $G_v$  by  $\hat{G}_v$ . The edge morphisms are the natural ones.*

*The graph of groups  $\hat{\Gamma}$  is said to be obtained from  $\Gamma$  by pulling centralizers across  $e$ .*

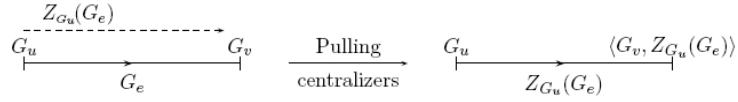


Figure 6: Pulling centralizers across an edge

*Proof of the Proposition.* We say that  $e$  is *full* if  $G_e = \hat{G}_e$ . If all edges of  $\Gamma$  are full, then one can take  $\hat{\Gamma} = \Gamma$  so we argue by induction on the number of edges which are not full.

Let  $T$  be the Bass-Serre tree of  $\Gamma$ . We already know that for any edge  $e$ ,  $\hat{G}_e$  fixes a point in  $T$ . Thus, if an edge  $e$  is not full, let  $v_0$  be the vertex fixed by  $\hat{G}_e$  closest to  $e$ . Then  $G_e$  fixes the arc joining  $e$  to  $v_0$ , and the edge  $e_0$  of this arc incident to  $v_0$  satisfies  $G_{e_0} \subsetneq \hat{G}_{e_0} = \hat{G}_e$ , and  $\hat{G}_{e_0} \subset G_{v_0}$ .

Denote by  $\Gamma'$  the graph of groups obtained by pulling the centralizer  $\hat{G}_{e_0}$  of  $G_{e_0}$  across  $e_0$  in  $\Gamma$ . Clearly, each edge group  $G'_e$  (resp. vertex group  $G'_v$ ) in  $\Gamma'$  is contained in the elliptic abelian neighbourhood  $\hat{G}_e$  (resp.  $\hat{G}_v$ ) of the corresponding group in  $\Gamma$ .

To conclude we just need to check that  $\Gamma'$  has strictly fewer non-full edges than  $\Gamma$ . Note that pulling centralizers increases the set of elliptic elements, so one may imagine that  $e_0$  might not become full or that some other edge  $e$  which used to be full becomes not full after the operation. We prove that this does not occur by proving that for all edge  $e$  of  $\Gamma$ , the set of elliptic elements in  $Z(G_e)$  does not increase when pulling centralizers. Thus, the following claim will conclude the proof.  $\square$

**Claim.** *Consider two non-trivial commuting elements  $h, g \in G$  such that  $h$  is hyperbolic and  $g$  is elliptic in  $T$ . Then  $h$  is still hyperbolic in the Bass-Serre tree  $T'$  of  $\Gamma'$ .*

*Proof.* The operation of pulling centralizers might be seen at the level of Bass-Serre tree as follows:  $T'$  is the quotient of  $T$  under the smallest equivariant equivalence relation  $\sim$  such that  $e_0 \sim g_0 \cdot e_0$  for all  $g_0 \in \hat{G}_{e_0}$ . More precisely, two edges  $\varepsilon_1, \varepsilon_2$  are folded together if and only if there exists  $k \in G$  such that  $k \cdot \varepsilon_1 = e_0$  and  $k \cdot \varepsilon_2 = g_0 \cdot e_0$  for some  $g_0 \in \hat{G}_{e_0}$ .

Now consider two non-trivial commuting elements  $h, g \in G$  such that  $h$  is hyperbolic and  $g$  is elliptic in  $T$ . If  $h$  is elliptic in  $T'$ , then there are two distinct edges  $\varepsilon_1, \varepsilon_2$  in the axis of  $h$  which are folded together. Up to conjugating  $h$ , we can assume that  $\varepsilon_1 = e_0$  and  $\varepsilon_2 = g_0 \cdot e_0 = g_0 \cdot \varepsilon_1$  for some  $g_0 \in \hat{G}_{e_0}$ . On the other hand, since  $[g, h] = 1$ ,  $g$  fixes pointwise the axis of  $h$  in  $T$ , so  $g \in G_{e_0}$ . By commutative transitivity, the element  $g_0 \in \hat{G}_{e_0}$  also commutes with  $h$ . Since  $g_0$  is elliptic,  $g_0$  fixes the axis of  $h$ , and thus fixes  $\varepsilon_1$  and  $\varepsilon_2$ . This contradicts the fact that  $g_0 \cdot \varepsilon_1 = \varepsilon_2$ .  $\square$

This terminates the proof of Proposition 4.21 showing that a general graph of limit groups over  $L$  can be turned into a simple graph of limit groups. Thus, the key result and the two characterizations of limit groups follow.

## 4.6 Fully residually free towers

As a corollary of the fact that general graph of limit groups over limit groups are limit groups, one way to construct limit groups will be by *gluing retracting surfaces* (Proposition 4.28). Together with the extension of centralizers, this construction is the building block for Sela's *fully residually free towers* (Definition 4.29). In topological terms, consider a space  $X$  obtained by gluing a surface  $\Sigma$  onto a space  $L$  whose fundamental group is a limit

group by attaching the boundary components of  $\Sigma$  to non-trivial loops of  $L$ . If there is a retraction of  $X$  onto  $L$  which sends  $\Sigma$  to a subspace of  $L$  with non-abelian fundamental group, then  $\pi_1(X)$  is a limit group.

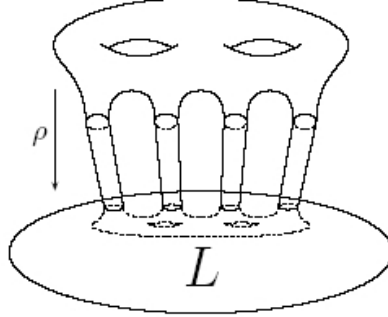


Figure 7: Gluing retracting surfaces

**Proposition 4.28 (Gluing retracting surfaces).** *Let  $L$  be a limit group, and  $\Sigma$  be a surface with boundary with Euler characteristic at most  $-2$  or a punctured torus or a punctured Klein bottle. Consider a morphism  $\rho : \pi_1(\Sigma) \rightarrow L$  with non-abelian image which is one-to-one in restriction to the fundamental groups  $C_1, \dots, C_n$  of the boundary components of  $\Sigma$ .*

*Consider the graph of groups  $\Gamma$  with two vertex groups  $L$  and  $\pi_1(\Sigma)$ , and  $n$  edge groups  $C_1, \dots, C_n$ , the two edge morphisms being the identity and the restriction of  $\rho$ .*

*Then  $\pi_1(\Gamma)$  is a limit group.*

*Remark.* In fully residually free towers, the groups  $\rho(C_i)$  will be asked to be maximal cyclic in  $L$  (see definition 4.29).

*Proof.* Consider a retraction  $\varphi : \pi_1(\Gamma) \rightarrow L$  such that  $\varphi$  restricts to the identity on  $L$ , and coincides with  $\rho$  on  $\pi_1(\Sigma)$ . Clearly, the centralizer of each edge group is contained in the vertex group  $L$ . In particular, the elliptic abelian neighbourhood  $\hat{L}$  of  $L$  coincides with  $L$ . Thus, to show that Proposition 4.21 applies, we only need to check that  $\pi_1(\Gamma)$  is commutative transitive. This is for instance a consequence of corollary A.8 given in appendix A since  $\Gamma$  is clearly 2-acylindrical since the fundamental group of a boundary component of  $\Sigma$  is malnormal in  $\pi_1(\Sigma)$ .  $\square$

We are now ready to give Sela's definition of fully residually free towers.

**Definition 4.29 (Fully residually free towers [Sel01a, Def. 6.1]).** *A finitely generated group is a fully residually free tower if it belongs to the smallest class of groups  $\mathcal{T}$  containing all the finitely generated free groups and surface groups and stable under free products, free extension of centralizers, and gluing of retracting surfaces on maximal cyclic subgroups (see Proposition 3.4 and Corollary 4.28).*

**Theorem 4.30 ([Sel01a]).** *Fully residually free towers are fully residually free.*

If we never extend centralizers, this construction gives only hyperbolic limit groups. Sela announces in his sequence of six papers the following answer to Tarski's problem. We state it as a conjecture since the referring process is not yet completed.

**Conjecture ([Sel01b, Th.7]).** *A finitely generated group is elementary equivalent to a non abelian free group if and only if it is a non elementary hyperbolic fully residually free tower.*

## 5 Some logic

The goal of this section is to give an intuitive feeling of basic logical notions, in order to state the following result: a finitely generated group is a limit group if and only if it has the same *universal theory* as free groups. This is by no means a substitute to a real introduction to model theory. For more precise information, see for instance [CK90, Hod97]. As a motivation, we first present a short introduction to the *elementary theory* of a group, and the Tarski problem.

### 5.1 The Tarski Problem

Z. Sela and Kharlampovich-Myasnikov have recently announced a positive solution to this problem ([Sel01a], [KM98a], [KM98b]). It is in the solution to this problem that Z. Sela introduced limit groups.

**Problem (Tarski, 1945).** *Do finitely generated non-abelian free groups have the same elementary theory?*

The language of groups uses the following symbols:

1. The binary function group multiplication “.”, the unary function inverse “ $^{-1}$ ”, the constant “1”, and the equality relation “=”,
2. variables  $x_1, \dots, x_n, \dots$  (which will have to be interpreted as individual group elements),
3. logical connectives “ $\wedge$ ” (meaning *and*), “ $\vee$ ” (meaning *or*), “ $\neg$ ” (meaning *not*), the quantifiers  $\forall$  (*for all*) and  $\exists$  (*there exists*), and parentheses: “(” “)”.

In the language of groups, the *terms* (used to be interpreted as elements of the group) are words in the variables, their inverses, and the identity element. For instance,  $((x_1.1).(x_2^{-1}))$  will be interpreted as an element of the group. Of course, since we will be working in a group, we may rather rewrite this term as  $x_1x_2^{-1}$ , dropping parentheses for convenience. To make a *formula* (interpreted as true or false) from terms, one can first compare two terms using “=”. For instance,  $x_1x_2^{-1} = x_3$  is a formula (called an *atomic* formula). Then, one can use logical connectors and quantifiers to make a new formula from other formulae. For instance,  $\forall x_1((x_1x_3 = 1) \wedge (\exists x_2 x_1x_2 = x_3))$  is a formula. Note that this formula has a free variable  $x_3$ : when interpreted, the fact that it is true or false will depend on  $x_3$ . A formula with no free variable is called a *sentence*, like for instance  $\forall x_3\forall x_1((x_1x_3 = 1) \vee (\exists x_2 x_1x_2 = x_3))$ . Given a group  $G$  and a sentence  $\sigma$  (with no free variable), we will say that  $G$  *satisfies*  $\sigma$  if  $\sigma$  is true if we interpret  $\sigma$  in  $G$  (in the usual sense). We denote by  $G \models \sigma$  the fact that  $G$  satisfies  $\sigma$ . For instance,  $G \models \forall x_1, x_2[x_1, x_2] = 1$  if and only if  $G$  is abelian.

Note that the following statement  $\forall x_1\exists k \in \mathbb{N} x_1^k = 1$  is not allowed, because it quantifies over an integer, and not a group element; similarly, one cannot quantify over subsets, subgroups or morphisms. In fact the quantifier does not mention to which group the variables belong, it is the interpretation which specifies the group. The *elementary theory* of  $G$ , denoted by  $\text{Elem}(G)$  is the set of sentences which are satisfied by  $G$ .

The Tarski problem is a special aspect of the more general problem to know which properties of a group can be read from its elementary theory. Two groups are said *elementarily equivalent* if they have the same elementary theory. For instance, the fact of being abelian, can be expressed in one sentence, and can therefore be read from the elementary theory of a group. There is no sentence saying that a group is torsion free. The sentence  $\forall x(x \neq 1 \rightarrow x.x \neq 1)$  says that a group has no 2-torsion. Similarly, for any integer  $k$ , there is a sentence saying that a group has no  $k$ -torsion. Thus, the property of being torsion free can be read in an infinite set of sentences, so a group elementarily equivalent to a torsion-free group is torsion-free.

*Example.*  $\mathbb{Z}$  and  $\mathbb{Z}^2$  don't have the same elementary theory. Indeed one can encode in a sentence the fact there are at most 2 elements modulo the doubles: the sentence

$$\forall x_1, x_2, x_3 \exists x_4 (x_1 = x_2 + 2x_4) \vee (x_1 = x_3 + 2x_4) \vee (x_2 = x_3 + 2x_4)$$

holds in  $\mathbb{Z}$  and not in  $\mathbb{Z}^2$  since  $\mathbb{Z}/2\mathbb{Z} \leq 2$  has only two elements, and  $\mathbb{Z}^2/2\mathbb{Z}^2$  has four elements.

## 5.2 Universal theory

A *universal formula* is a formula which can be written  $\forall x_1 \dots \forall x_p \varphi(x_1, \dots, x_p)$  for some quantifier free formula  $\varphi(x_1, \dots, x_p)$ . If it has no free variables, a universal formula is called a *universal sentence*. The *universal theory* of a group is the set of universal sentences satisfied by  $G$ . Similarly, one can define *existential* formulae and sentences, and the existential theory of a group. Note that two groups which have the same universal theory also have the same existential theory since the negation of a universal sentence is equivalent to an existential statement.

Note that if  $H < G$ , a quantifier free formula which is satisfied for every tuple of elements of  $G$  is also satisfied for every tuple of elements of  $H$ , hence  $\text{Univ}(G) \subset \text{Univ}(H)$ . Similarly, if  $H < G$ ,  $\text{Exist}(H) \subset \text{Exist}(G)$ . As a corollary, for  $n \geq 3$ , since  $F_n$  contains  $F_2$  and  $F_2$  contains  $F_n$ , one gets  $\text{Univ}(F_2) = \text{Univ}(F_n)$ .

**Theorem 5.1** ([GS93, Chi95, Rem89]). *A finitely generated group  $G$  has the same universal theory as a non-abelian free group (resp. an infinite cyclic group) if and only if  $G$  is a non-abelian limit group (resp. an abelian limit group, i. e. a free abelian group).*

This theorem was stated by Remeslennikov in the context of non-standard free group (see section 6). For instance, this theorem means that a non-exceptional surface group has the same universal theory as  $F_2$ . It also means that  $\mathbb{Z}$  and  $\mathbb{Z}^2$  have the same universal theory.

*Proof.* Let's first prove that a non-abelian limit group  $\Gamma$  has the same universal theory as  $F_2$ . Since a non-abelian limit group contains  $F_2$  (prop. 3.1), one has  $\text{Univ}(\Gamma) \subset \text{Univ}(F_2)$ . There remains to check that  $\text{Univ}(\Gamma) \supset \text{Univ}(F_2)$ . This follows from the following proposition:

**Proposition 5.2.** *Let  $\sigma$  be a universal sentence. Then the property  $G \models \sigma$  is closed in  $\mathcal{G}_n$ . Equivalently, if a sequence of marked groups  $(G_i, S_i)$  converge to a marked group  $(G, S)$ , then  $\text{Univ}(G) \supset \limsup \text{Univ}(G_i)$ .*

*Proof of the proposition.* We will prove that for any existential sentence  $\sigma$ , the property  $G \models \sigma$  is open.

Consider the sentence  $\exists x_1, \dots, x_p \varphi(x_1, \dots, x_p)$  where  $\varphi(x_1, \dots, x_p)$  is quantifier free. Using distributivity of  $\wedge$  with respect to  $\vee$ , one easily checks that  $\varphi(x_1, \dots, x_p)$  is equivalent to a formula  $\Sigma_1(x_1, \dots, x_p) \vee \dots \vee \Sigma_q(x_1, \dots, x_p)$ , where each  $\Sigma_i$  is a system of equations or inequations in the following sense: a set of equations or inequations of the form  $w(x_1, \dots, x_p) = 1$  or  $w(x_1, \dots, x_p) \neq 1$  separated by the symbol “ $\wedge$ ” (*and*) where  $w(x_1, \dots, x_p)$  is a word on  $x_1^{\pm 1}, \dots, x_p^{\pm 1}$ .

Consider a marked group  $(G, S) \in \mathcal{G}_n$  with  $S = (s_1, \dots, s_n)$ , and consider an existential sentence  $\sigma$  of the form

$$\exists x_1, \dots, x_p \Sigma_1(x_1, \dots, x_p) \vee \dots \vee \Sigma_q(x_1, \dots, x_p)$$

such that  $G \models \sigma$ . So consider  $a_1, \dots, a_p \in G$  and  $i \in \{1, \dots, q\}$  such that  $\Sigma_i(a_1, \dots, a_p)$  holds. Consider  $R$  large enough so that the ball of radius  $R$  in  $(G, S)$  contains  $\{a_1, \dots, a_p\}$  and so that for each word  $w$  occurring in  $\Sigma_i$ , the corresponding word on  $\{a_1, \dots, a_p\}$  can be read in this ball (for instance, one can take  $R$  to be the maximal length of the words times the maximal length of the  $a_i$ 's in  $(G, S)$ ). Now assume that a marked group  $(H, S')$  has the same ball of radius  $R$  as  $(G, S)$ . Clearly, this implies that the corresponding elements  $a'_1, \dots, a'_p$  in the ball of  $(H, S')$  satisfy  $\Sigma_i$ , so that  $H \models \sigma$ .  $\square$

We now prove that if  $G$  has the same existential theory as  $F_2$ , then it is a non-abelian limit group. It is clearly non-abelian since the property of being non-abelian expresses as an existential sentence. Let  $S = (s_1, \dots, s_n)$  be a finite generating family of  $G$ , let  $R > 0$ , and let  $B$  be the ball of radius  $R$  of  $(G, S)$ . We aim to find a generating set of a free group having the same ball. For this purpose, we are going to encode the ball in a system of equations and inequations. Let  $w_1, \dots, w_p$  be an enumeration of all the words on  $x_1^{\pm 1}, \dots, x_n^{\pm 1}$  of length at most  $R$ . For each of these words  $w_i$ , consider  $g_i = w_i(s_1, \dots, s_n) \in B$ . We consider the following system  $\Sigma(x_1, \dots, x_p)$  of equations and inequations: for each  $i, j \in \{1, \dots, p\}$ , we add to  $\Sigma$  the equation  $w_i = w_j$  or the inequation  $w_i \neq w_j$  according to the fact that  $g_i = g_j$  or  $g_i \neq g_j$ . Of course,  $\Sigma(s_1, \dots, s_p)$  holds. Thus

$$G \models \exists x_1, \dots, x_p \Sigma(x_1, \dots, x_p).$$

Since  $F_2$  has the same existential theory, let  $s'_1, \dots, s'_p \in F_2$  such that  $\Sigma(s'_1, \dots, s'_p)$  holds in  $G$ . Let  $F = \langle s'_1, \dots, s'_p \rangle < F_2$ . Thus  $F$  is a free group, and  $(F, (s'_1, \dots, s'_p))$  has the same ball of radius  $R$  as  $(G, S)$ .

Finally, the abelian part of the theorem states that all the finitely generated free abelian groups have the same universal theory. The same proofs as above work in the abelian context using the facts that  $\mathbb{Z}^p \supset \mathbb{Z}$  and that  $[\mathbb{Z}^p]_{\mathcal{G}_n} \subset \overline{[\mathbb{Z}]_{\mathcal{G}_n}}$ .  $\square$

Note that the second part of the proof actually shows the following more general statement:

**Proposition 5.3.** *If  $\text{Univ}(G) \supset \text{Univ}(H)$ , then for all generating family  $S$  of  $G$ ,  $(G, S)$  is a limit of marked subgroups of  $H$ .*

This kind of result is usually stated using ultra-products. The next section details the relation between convergence in  $\mathcal{G}_n$  and ultra-products.

## 6 A little non-standard analysis

### 6.1 definitions

**Definition 6.1 (ultrafilter).** An ultrafilter on  $\mathbb{N}$  is a finitely additive measure of total mass 1 (a mean) defined on all subsets of  $\mathbb{N}$ , and with values in  $\{0, 1\}$ . In other words, it is a map  $\omega : \mathcal{P}(\mathbb{N}) \rightarrow \{0, 1\}$  such that for all subsets  $A, B$  such that  $A \cap B = \emptyset$ ,  $\omega(A \cup B) = \omega(A) + \omega(B)$ ,  $\omega(\mathbb{N}) = 1$ .

An ultrafilter is non-principal if it is not a Dirac mass, i. e. if finite sets have mass 0.

We will say that a property  $P(k)$  depending on  $k \in \mathbb{N}$  is true  $\omega$ -almost everywhere if  $\omega(\{k \in \mathbb{N} | P(k)\}) = 1$ . Note that a property which is not true almost everywhere is false almost everywhere. Given an ultrafilter  $\omega$  (which will usually be supposed to be non-principal), and a family of groups  $(G_k)_{k \in \mathbb{N}}$ , there is a natural equivalence relation  $\sim_\omega$  on  $\prod_{k \in \mathbb{N}} G_k$  defined by equality  $\omega$ -almost everywhere. When there is no risk of confusion, we may drop the reference to the ultrafilter  $\omega$ .

**Definition 6.2 (ultraproduct, ultrapower).** The ultraproduct with respect to  $\omega$  of a sequence of groups  $G_k$  is the group

$$\left( \prod_{k \in \mathbb{N}} G_k \right) / \sim_\omega.$$

When starting with a constant sequence  $G_k = G$ , the ultraproduct is called an ultrapower, and it is often denoted by  ${}^*G$  (though depending on the ultrafilter  $\omega$ ).

The main interest of ultraproducts and ultrapowers is Łos Theorem, which claims that ultrapowers of a group  $G$  have the same elementary theory as  $G$  (see for instance [BS69]).

**Theorem 6.3 (Łos).** Let  $G$  be a group, and  ${}^*G$  an ultrapower of  $G$ . Then  $G$  and  ${}^*G$  have the same elementary theory.

More generally, for every formula  $\varphi(x_1, \dots, x_n)$ ,  ${}^*G \models_{(x_1=(a_{1,k})_{k \in \mathbb{N}}, \dots, x_n=(a_{n,k})_{k \in \mathbb{N}})}$  if and only if for almost every  $k \in \mathbb{N}$ ,  $G \models_{(x_1=a_{1,k}, \dots, x_n=a_{n,k})}$ .

### 6.2 Ultraproducts and the topology on the set of marked groups

The link between ultraproducts and convergence of groups in  $\mathcal{G}_n$  is contained in the following lemma.

**Proposition 6.4.** 1. Consider a sequence of marked groups  $(G_k, S_k) \in \mathcal{G}_n$  which accumulates on  $(G, S) \in \mathcal{G}_n$ . Then there exists some non-principal ultrafilter  $\omega$  such that  $G$  embeds in the ultraproduct  $\prod_{k \in \mathbb{N}} G_k / \sim_\omega$ .

2. If the sequence above is convergent, then  $(G, S)$  embeds in any ultraproduct  $\prod_{k \in \mathbb{N}} G_k / \sim_\omega$  (assuming only that  $\omega$  non-principal).

3. Consider a finitely generated subgroup  $(G, S)$  of an ultraproduct  $\prod_{k \in \mathbb{N}} G_k / \sim_\omega$  (where  $\omega$  is non-principal). Then there exists a subsequence  $(i_k)_{k \in \mathbb{N}}$ , and marked subgroups  $(H_{i_k}, S_{i_k}) < G_{i_k}$ , such that  $(H_{i_k}, S_{i_k})$  converge to  $(G, S)$ .



*Proof of the lemma.* 1. Denote  $S_k = (s_1^{(k)}, \dots, s_n^{(k)})$ . Assume that there is a subsequence  $(G_{i_k}, S_{i_k})$  converging to  $(G, S)$ . Consider an ultrafilter  $\omega$  such that the subsequence  $\{i_k | k \in \mathbb{N}\}$  has full  $\omega$ -measure (the existence of  $\omega$ , which uses the axiom of choice, is proved in [Bou71, p.39]). Let  $U = \prod_{k \in \mathbb{N}} G_k / \sim_\omega$  be the corresponding ultraproduct, and let  $\overline{S} = (\overline{s_1}, \dots, \overline{s_n})$  the family of elements of  $U$  defined by  $\overline{s_p} = (s_p^{(k)})_{k \in \mathbb{N}} \in U$  for  $p \in \{1, \dots, n\}$ . Let  $\overline{G}$  be the subgroup of  $U$  generated by  $\overline{S}$ . We prove that  $(G, S)$  is isomorphic to  $(\overline{G}, \overline{S})$  as a marked group. For any word  $w$  on the generators  $s_1, \dots, s_n$  and their inverses, if  $w$  is trivial in  $(G, S)$  then it is trivial in  $(G_{i_k}, S_{i_k})$  for all but finitely many  $k$ 's, and hence for  $\omega$ -almost every  $k \in \mathbb{N}$ , thus  $w$  is trivial in  $(\overline{G}, \overline{S})$ . If  $w$  is non-trivial in  $(G, S)$  then it is non-trivial in  $(G_{i_k}, S_{i_k})$  for all but finitely many  $k$ 's, and hence for  $\omega$ -almost every  $k \in \mathbb{N}$ , thus  $w$  is non-trivial in  $(\overline{G}, \overline{S})$ . This proves that  $(G, S) \simeq (\overline{G}, \overline{S})$ .

2. If the sequence above is convergent, then taking  $i_k = k$  in the argument above, allows to choose any non-principal ultrafilter.

3. Let  $S = (s_1, \dots, s_n)$ , where each  $s_p$  is an element of  $U$ . So write each generator  $s_p$  as a sequence  $(s_p^{(k)})_{k \in \mathbb{N}}$ . Let  $S_k = (s_1^{(k)}, \dots, s_n^{(k)})$ , and let  $H_k$  be the subgroup of  $G_k$  generated by  $S_k$ . We prove that  $(H_k, S_k)$  accumulates on  $(G, S)$ . Consider a ball of radius  $R$  in  $(G, S)$  and consider a word  $w$  on  $s_1, \dots, s_n$  of length at most  $R$ . The word  $w$  is trivial in  $(G, S)$  if and only if it is trivial in  $(H_k, S_k)$  for  $\omega$ -almost every  $k$ . Thus the set of indices  $k$  such that the ball of radius  $R$  of  $(G_k, S_k)$  coincides with the ball of radius  $R$  of  $(G, S)$  has full measure as an intersection of finitely many full measure subsets. This set of indices is therefore infinite since  $\omega$  is non-principal, thus  $(G_k, S_k)$  accumulates on  $(G, S)$ .  $\square$

The following corollary is immediate:

**Corollary 6.5.** *A group is a limit group if and only if it is a finitely generated subgroup of an ultraproduct of free groups, and any ultraproduct of free groups contains all the limit groups.*

### 6.3 Application to residual freeness

We review here the following result of Remeslennikov which proves that limit groups are residually free ([Rem89], see also [Chi01, lem.5.5.7]).

**Proposition 6.6 (Remeslennikov).** *A finitely generated subgroup of an ultrapower  ${}^*F_2$  is fully residually free.*

Using corollary 6.5, this result gives an elementary proof (without use finite presentation of limit groups) that limit groups are fully residually free.

*Proof.* Fix an ultrafilter  $\omega$ , the corresponding ultrapower  ${}^*F_2$  of  $F_2$ , and  $G < {}^*F_2$  a finitely generated subgroup. It is well known that for any odd prime  $p$ , the kernel of  $\varphi : SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/p\mathbb{Z})$  is a non-abelian free group. Thus  $F_2$  embeds in  $\ker \varphi \subset SL_2(\mathbb{Z})$ . Therefore,  ${}^*F_2$  embeds in the kernel of the natural morphism  ${}^*\varphi : SL_2({}^*\mathbb{Z}) \rightarrow SL_2({}^*(\mathbb{Z}/p\mathbb{Z}))$  where  ${}^*\mathbb{Z}$  is the ring obtained by taking the  $\omega$ -ultrapower of the ring  $\mathbb{Z}$ . In particular  ${}^*F_2$  embeds in  $SL_2({}^*\mathbb{Z})$ .  $G$  being finitely generated,  $G$  embeds in  $SL_2(R)$  for a ring  $R$  which is finitely generated subring of  ${}^*\mathbb{Z}$ .

**Lemma 6.7 (Remeslennikov).** *Consider a finitely generated subring  $R$  of  ${}^*\mathbb{Z}$ . Then  $R$ , as a ring, is fully residually  $\mathbb{Z}$ , i. e. for  $a_1, \dots, a_k \in R \setminus \{0\}$ , there exists a ring morphism  $\rho : R \rightarrow \mathbb{Z}$  such that  $\rho(a_i) \neq 0$  for all  $i = 1, \dots, k$ .*

Let's conclude the proof of the proposition using the lemma. Consider finitely many elements  $g_1, \dots, g_k \in G \setminus \{1\} \subset SL_2(R)$ , and let  $a_1, \dots, a_k$  be the set of non-zero coefficients of the matrices  $g_j - \text{Id}$  ( $j = 1, \dots, k$ ) (there is at least one non-zero coefficient for each  $g_j$  since  $g_j \neq \text{Id}$ ). Consider a morphism  $\rho : R \rightarrow \mathbb{Z}$  given by the lemma. The induced morphism  $\psi : SL_2(R) \rightarrow SL_2(\mathbb{Z})$  maps the elements  $g_j$  to non-trivial elements, and since  $G \subset \ker {}^*\varphi$ ,  $\psi(G) \subset \ker \varphi$  which is free. Thus  $\psi(G)$  is free and  $G$  is fully residually free.  $\square$

*Proof of the lemma.* Let  $t_1, \dots, t_n \in {}^*\mathbb{Z}$  such that  $R = \mathbb{Z}[t_1, \dots, t_n]$ . Consider the corresponding exact sequence  $J \hookrightarrow \mathbb{Z}[T_1, \dots, T_n] \twoheadrightarrow R$ , where  $\mathbb{Z}[T_1, \dots, T_n]$  is the ring of polynomials with  $n$  commuting indeterminates. Since  $\mathbb{Z}[T_1, \dots, T_n]$  is Noetherian, the ideal  $J$  is generated by finitely many polynomials  $f_1, \dots, f_q$ . Let  $a_1, \dots, a_k \in R \setminus \{0\}$ , and let  $g_1, \dots, g_k$  some preimages in  $\mathbb{Z}[T_1, \dots, T_n] \setminus J$ . Note that  $(t_1, \dots, t_n)$  is a solution of the system of equations and inequations

$$\begin{cases} f_i(x_1, \dots, x_n) = 0 & (i = 1, \dots, q) \\ g_j(x_1, \dots, x_n) \neq 0 & (j = 1, \dots, k) \end{cases}$$

Now one can invoke Los theorem, or just remember that each  $t_i$  is a sequence of integers modulo the ultrafilter  $\omega$  to check that almost all the components of  $t_i$  provide a solution  $(x_1, \dots, x_n)$  to this system in  $\mathbb{Z}$ . The morphism  $\mathbb{Z}[T_1, \dots, T_n] \rightarrow \mathbb{Z}$  sending  $T_i$  to  $x_i$  induces the desired morphism  $\rho : R \rightarrow \mathbb{Z}$ .  $\square$

## 6.4 Maximal limit quotients

The result of this section is lemma 1.1 of [Raz84] and lemma 3 of [KM98b]. It also appears in [BMR99]. We give a short proof inspired by [Cha]. It is a clever way to get the existence of maximal limit quotients without using the finite presentation of limit groups. It just uses the fact that limit groups are residually free.

*Remark.* One could avoid using the fact that limit groups are residually free by replacing the mentions of *residually free* groups by groups that are *residually a limit group*, and by replacing the field  $\mathbb{C}$  by an ultrapower  ${}^*\mathbb{C}$  to ensure that limit groups embed in  $SL_2({}^*\mathbb{C})$  in lemma 6.8.

**Proposition 6.8.** *Consider a sequence of quotients of finitely generated groups*

$$G_1 \twoheadrightarrow G_2 \twoheadrightarrow \dots \twoheadrightarrow G_k \twoheadrightarrow \dots$$

*If every group  $G_i$  is residually free, then all but finitely many epimorphisms are isomorphisms.*

*Proof.* Take  $S_1 = (s_1, \dots, s_n)$  a finite generating family of  $G_1$ , and let  $S_k = (s_1^{(k)}, \dots, s_n^{(k)})$  its image in  $G_k$  under the quotient map. Let  $V_k \subset SL_2(\mathbb{C})^n$  be the variety of representations of  $(G_k, S_k)$  in  $SL_2(\mathbb{C})$ , i. e.

$$V_k = \left\{ (M_1, \dots, M_n) \in SL_2(\mathbb{C})^n \mid \forall \text{ relation } r \text{ of } (G_k, S_k), r(M_1, \dots, M_n) = \text{Id} \in SL_2(\mathbb{C}) \right\}$$

Note that  $V_k$  is an affine algebraic variety in  $(\mathbb{C})^{4n}$ , and that  $V_1 \supset V_2 \supset \dots \supset V_k \dots$ . By noetherianity, for all but finitely many indices  $k$ , one has  $V_k = V_{k+1}$ . There remains

to check that if  $G_{k+1}$  is a strict quotient of  $G_k$ , then  $V_{k+1}$  is strictly contained in  $V_k$ . So consider a word  $r$  on  $S^{\pm 1}$  which is trivial in  $G_{k+1}$  but not in  $G_k$ .

Since  $G_k$  is residually free, there exists a morphism  $\varphi : G_k \rightarrow F_2$  such that  $\varphi(r) \neq 1$ . Since  $F_2$  embeds in  $SL_2(\mathbb{C})$ , there exists a representation  $\rho : G_k \rightarrow SL_2(\mathbb{C})$  such that  $\rho(r) \neq 1$ . This representation provides a point in  $V_k \setminus V_{k+1}$ .  $\square$

Remember that any group  $G$  has a largest residually free quotient  $RF(G)$ :  $RF(G)$  is the quotient of  $G$  by the intersection of the kernels of all morphisms from  $G$  to free groups. The following corollary says that a residually free group is presented by finitely many relations plus all the relations necessary to make it residually free.

**Corollary 6.9 ([Cha]).** *If  $(G, S)$  is residually free, then there exists finitely  $S$ -words  $r_1, \dots, r_p$  such that  $G = RF(H)$  where  $H$  is the group presented by  $\langle S; r_1, \dots, r_p \rangle$ .*

*Proof.* Enumerate the relations  $r_i$  of  $(G, S)$ , and take  $G_k = RF(\langle S; r_1, \dots, r_k \rangle)$ . The previous lemma says that for  $k$  large enough,  $(G, S) = (G_k, S)$ .  $\square$

**Corollary 6.10.** *Given a residually free marked group  $(G, S)$ , there is a neighbourhood  $V_{(G,S)}$  of  $(G, S)$  such that every residually free group in  $V_{(G,S)}$  is a quotient of  $(G, S)$ .*

*Proof.* Take  $r_1, \dots, r_p$  relations of  $(G, S)$  as in the corollary above. One can take  $V_{(G,S)}$  to be the set of marked groups  $(G', S')$  such that the relations  $r_1, \dots, r_p$  hold in  $(G', S')$ .  $\square$

We can now give an elementary proof (without using finite presentation of limit groups) of Proposition 3.12:

**Corollary 6.11.** *Let  $G$  be a finitely generated group. Then there exist finitely many quotients  $\Gamma_1, \dots, \Gamma_k$  of  $G$ , such that each  $\Gamma_i$  is a limit group and such that any morphism from  $G$  to a free group factors through one  $\Gamma_i$ .*

*Proof.* Consider a marking  $(G, S)$  of  $G$ , and consider the set  $K \subset \mathcal{G}_n$  of marked quotients of  $(G, S)$  which are limit groups. This is clearly a compact subset of  $\mathcal{G}_n$ , and each of its points is residually free. Now cover  $K$  by finitely many  $V_{(\Gamma_i, S_i)}$  as in the previous corollary to get the result.  $\square$

*Remark.* The proof given in [Cha] is more logical in nature, and it is quite appealing. It relies on the following ideas. Let  $r_1, \dots, r_p$  be some relations of  $(G, S)$ . Say that an  $S$ -word  $r'$  is *deducible from  $r_1, \dots, r_p$  modulo  $\text{Univ}(F_2)$*  if the statement

$$\forall s_1, \dots, s_n, \begin{cases} r_1 = 1 \\ \vdots \\ r_p = 1 \end{cases} \rightarrow r' = 1$$

holds in  $F_2$ . Say that a marked group  $(G, S)$  is *closed under deduction mod  $\text{Univ}(F_2)$*  if for all relations  $r_1, \dots, r_p$  of  $(G, S)$ , any word  $r'$  which is deducible from  $r_1, \dots, r_p$  modulo  $\text{Univ}(F_2)$  is a relation of  $(G, S)$ . One can easily show that a group is closed under deduction mod  $\text{Univ}(F_2)$  if and only if it is residually a limit group (and hence if and only if it is residually free). Then the statement of lemma 6.9 says that a group  $(G, S)$  which is closed under deduction mod  $\text{Univ}(F_2)$  is finitely presented mod  $\text{Univ}(F_2)$ .

## A Reading property CSA from a graph of groups.

This appendix explains how to read CSA property (Definition 2.8) on a graph of groups  $\Gamma$  with abelian edge groups.

Let us start with two basic cases. In an amalgamated product  $G = A *_C B$  with abelian edge group  $C$ , if  $C$  is not maximal in  $A$  nor in  $B$ , then  $G = A *_C B$  is not commutative transitive. Thus for example  $\mathbb{Z} * \mathbb{Z} *_\mathbb{Z} \mathbb{Z} *_\mathbb{Z} \mathbb{Z} \simeq \mathbb{Z}^2 *_\mathbb{Z} \mathbb{Z}^2$  (with obvious embeddings) is not a limit group, since it is not commutative transitive. Similarly, the amalgam  $\mathbb{Z}^2 *_\mathbb{Z} \mathbb{Z} *_\mathbb{Z} \mathbb{Z}^2$  is not commutative transitive and is therefore not a limit group.

The second basic case concerns the HNN extension  $\langle A, t | tat^{-1} = \varphi(a) \rangle$  for an injective endomorphism  $\varphi : A \rightarrow A$ . This group may be commutative transitive for non trivial  $\varphi$  (for example the Baumslag-Solitar groups are commutative transitive), but such an extension is CSA if and only if  $\varphi$  is the identity. We are going to prove that those two basic phenomena are the only obstructions for getting a CSA group (see Cor. A.7).

Let  $\Gamma$  be a graph of group and let  $T$  be the Bass-Serre tree of  $\Gamma$ . Consider the equivalence relation on the set of non-oriented edges of  $T$  generated by  $e \sim e'$  if  $e$  and  $e'$  have a common endpoint and their stabilizers commute. We call a *cylinder* the subtree of  $T$  defined by an equivalence class.

Before stating a CSA criterion for a graph of groups, we first give the following necessary condition.

**Lemma A.1.** *Let  $\Gamma$  be a graph of groups with non-trivial abelian edge groups, and let  $T$  be its Bass-Serre tree.*

*If  $\pi_1(\Gamma)$  is CSA, then the vertex groups of  $\Gamma$  are CSA and the global stabilizer of every cylinder in  $T$  is abelian.*

*Proof.* Since CSA property is stable under taking subgroups, if  $G = \pi_1(\Gamma)$  is CSA then vertex groups are CSA. Consider a cylinder  $C$  in  $T$ . By commutative transitivity of  $G$ , the stabilizers of all the edges of  $C$  commute. Since edge groups are non-trivial, there exist a unique maximal abelian subgroup  $A$  of  $G$  containing these edge stabilizers. Let  $g \in G$  such that  $g.C = C$ . Then  $gAg^{-1} = A$ , so  $g \in A$  because  $G$  is CSA.  $\square$

Before proving the converse of this lemma, we first introduce the *tree of cylinders* of  $\Gamma$ , and give some properties of this tree. A similar construction is used in [GL] to get splittings invariant under automorphisms.

A trivial but fundamental property of cylinders is that two distinct cylinders intersect in at most one vertex (they have no edge in common). Moreover, the set of cylinders is  $G$ -invariant. Therefore, there is a natural bipartite  $G$ -tree  $T_C$  called *the tree of cylinders* defined as follows:  $V(T_C) = V_1(T_C) \sqcup V_0(T_C)$  where  $V_1(T_C)$  is the set of cylinders of  $T$ , and  $V_0(T_C)$  is the set of vertices of  $T$  belonging to at least two distinct cylinders, and there is an edge  $\varepsilon = (x, C)$  between  $x \in V_0(T_C)$  and  $C \in V_1(T_C)$  if  $x \in C$ . The fact that this graph is a tree is straightforward (see the notion of transverse coverings in [Gui03]).

**Definition A.2.** *An action of a group  $G$  on a tree is  $k$ -acylindrical if for all  $g \in G \setminus \{1\}$ , the set of fix points of  $g$  has diameter at most  $k$ . Similarly, a graph of groups is  $k$ -acylindrical if the action of  $\pi_1(\Gamma)$  on its Bass-Serre tree is  $k$ -acylindrical. We say that  $\Gamma$  is acylindrical if it is  $k$ -acylindrical for some  $k$ .*

We gather a few simple facts about the tree of cylinders.

**Lemma A.3.** *Let  $\Gamma$  be a graph of groups with CSA vertex groups and abelian edge groups. Suppose that the global stabilizers of cylinders of  $\Gamma$  are abelian.*

*Then:*

1. *the stabilizer of each vertex  $x \in V_0(T_{\mathcal{C}})$  is CSA*
2. *the stabilizer of each vertex  $C \in V_1(T_{\mathcal{C}})$  is abelian*
3. *The stabilizer of any cylinder intersect a vertex group in maximal abelian subgroup of this vertex group, that is for all edge  $\varepsilon \in E(T_{\mathcal{C}})$  incident on  $x \in V_0(T_{\mathcal{C}})$ ,  $G_{\varepsilon}$  is maximal abelian in  $G_x$*
4. *if  $\varepsilon, \varepsilon' \in E(T_{\mathcal{C}})$  are such that  $G_{\varepsilon} \cap G_{\varepsilon'} \neq \{1\}$  then  $\varepsilon$  and  $\varepsilon'$  have a common endpoint in  $V_1(T_{\mathcal{C}})$  (in particular,  $T_{\mathcal{C}}$  is 2-acylindrical)*
5. *for every abelian subgroup  $A \subset G$ , either  $A$  fixes a point in  $T_{\mathcal{C}}$ , or  $A$  is a cyclic group acting freely on  $T_{\mathcal{C}}$ .*

*Proof.* The first two claims result from the definitions. For claim 3, consider an edge  $\varepsilon = (x, C) \in E(T_{\mathcal{C}})$ . Assume that  $g \in G_x$  commutes with  $G_{\varepsilon}$ . Consider an edge  $e \in C$  incident on  $x$ . Since  $G_e \subset G_{\varepsilon}$ ,  $g$  commutes with  $G_e$ , so  $G_e = G_{g.e}$  and  $g.e \in C$ . This implies that  $g.C = C$ , and that  $g$  fixes the edge  $\varepsilon = (x, C)$ .

For claim 4, assume that  $\varepsilon = (x, C), \varepsilon' = (x', C') \in E(T_{\mathcal{C}})$  are such that  $G_{\varepsilon} \cap G_{\varepsilon'} \neq \{1\}$ . We want to prove that  $C = C'$ . If  $x = x'$ , claim 3 states that  $G_{\varepsilon}$  and  $G_{\varepsilon'}$  are maximal abelian in  $G_x$ . Since  $G_{\varepsilon} \cap G_{\varepsilon'} \neq \{1\}$ ,  $G_{\varepsilon} = G_{\varepsilon'}$ . Let  $e$  and  $e'$  be any edges of  $C$  and  $C'$  adjacent to  $x$ . We have  $G_e \subset G_{\varepsilon}$  and  $G_{e'} \subset G_{\varepsilon'}$ , thus  $G_e$  and  $G_{e'}$  commute. This proves  $C = C'$ . If  $x \neq x'$ , then any non trivial element  $h \in G_{\varepsilon} \cap G_{\varepsilon'}$  fixes  $[x, x']$ . Thus  $[x, x']$  is contained in a cylinder  $C''$ . Let  $\varepsilon'' = (x, C'')$ . Then  $h \in G_{\varepsilon} \cap G_{\varepsilon''}$ , and the previous case shows that  $C = C''$ . Similarly,  $C' = C''$ .

For claim 5, consider an abelian group  $A$ . Suppose  $A$  contains a hyperbolic element  $h$ . For any element  $g \in A \setminus \{1\}$ , let  $\text{Fix}(g)$  be its set of fix points in  $T_{\mathcal{C}}$ . Since  $\text{Fix}(g)$  is  $h$ -invariant and bounded by 2-acylindricity, one has  $\text{Fix}(g) = \emptyset$ . This means that the action of  $A$  is free, so  $A$  is a cyclic group. Suppose  $A$  contains an elliptic element  $h \neq 1$ . Let  $F$  be the set of fix points of  $h$  in  $T_{\mathcal{C}}$ . Acylindricity shows that  $F$  is bounded. Since it is  $A$ -invariant,  $A$  fixes a point in  $F$ .  $\square$

**Proposition A.4 (a CSA criterion).** *Consider a graph of groups  $\Gamma$  with torsion-free vertex groups and non-trivial abelian edge groups.*

*Then  $\pi_1(\Gamma)$  is CSA if and only if vertex groups of  $\Gamma$  are CSA and the global stabilizer of every cylinder in the Bass-Serre tree of  $\Gamma$  is abelian.*

*Remark.* One could replace the assumption that vertex groups are torsion free by the assumption that  $\pi_1(\Gamma)$  contains no infinite dihedral subgroup acting faithfully on  $T$ . This assumption is more natural since the infinite dihedral group is not CSA.

*Proof.* Lemma A.1 shows one part of the equivalence. We now assume that vertex stabilizers are CSA and that the stabilizer of each cylinder is abelian, and we have to prove that  $G$  is CSA. Let  $A$  be a maximal abelian subgroup of  $G$  and assume that  $gAg^{-1} \cap A \neq \{1\}$ . Suppose first that  $A$  acts freely on  $T_{\mathcal{C}}$ . Denote by  $l$  its axis and let  $G_l$  be the global stabilizer of  $l$ . Clearly,  $g \in G_l$  so we are reduced to prove that  $G_l$  is abelian, since it will follow that  $A = G_l \ni g$ . Because of the acylindricity of  $T_{\mathcal{C}}$ , no element of  $G_l$  fixes  $l$  so  $G_l$  acts faithfully by isometries on  $l$ , and  $G_l$  is either cyclic or dihedral. But since vertex groups are torsion free (this is the only place where we use this assumption),  $G_l$  cannot be dihedral.

If  $A$  fixes a point in  $T_C$ , let  $F$  be its set of fix points in  $T_C$ . If  $g$  fixes a point in  $F$ , then we are done since  $g$  and  $A$  both belong to a vertex stabilizer of  $T_C$  which is known to be CSA. Let  $h \in A \cap gAg^{-1} \setminus \{1\}$ . Since  $h$  fixes pointwise  $F \cup g.F$ , and thus its convex hull,  $F \cup g.F$  is contained in the 1-neighbourhood of a vertex  $v_1 \in V_1(T_C)$  (claim 4 of the fact). Since  $v_1$  is the only vertex of this neighbourhood lying in  $V_1(T_C)$  ( $T_C$  is bipartite), if  $v_1 \in F$ , then  $v_1$  is fixed by both  $g$  and  $A$ , and this case was already settled. Thus one can assume that  $v_1 \notin F$  so  $F$  consists in a single vertex  $v_0 \in V_0(T_C)$ . One can also assume that  $g.v_0 \neq v_0$ . Note that in this case,  $v_1$  is the midpoint of  $[v_0, g.v_0]$ . Let  $\varepsilon$  be the edge joining  $v_1$  to  $v_0$ . One has  $h \in G_\varepsilon \subset G_{v_0}$ , and  $h \in A \subset G_{v_0}$ . Since  $G_\varepsilon$  is maximal abelian in  $G_{v_0}$  (claim 5 in the fact), one has  $G_\varepsilon = A$ . Thus  $A$  fixes  $\varepsilon$ , so  $A$  fixes  $v_1$ , so  $v_1 \in F$  which has been excluded.  $\square$

We now translate our criterion into a more down-to-earth property (compare [Sel01a, Definition 5.11])

**Definition A.5 (Cylinders).** Let  $\Gamma$  be a graph of groups with abelian edge groups and CSA vertex groups. Denote by  $Cyl(\Gamma)$  the following (non-connected) graph of groups.

Edges of  $Cyl(\Gamma)$  are the edges of  $\Gamma$ , and they hold the same edge groups. We define the vertices of  $Cyl(\Gamma)$  by describing when two oriented edges have the same terminal vertex:  $e$  and  $e'$  have the same terminal vertex in  $Cyl(\Gamma)$  if they have the same terminal vertex  $v$  in  $\Gamma$  and if there exists  $g \in G_v$  such that  $i_e(G_e)$  and  $g.i_{e'}(G_{e'})g^{-1}$  commute. The corresponding vertex group is the maximal abelian group containing  $i_e(G_e)$  (which is well defined up to conjugacy) and the edge morphism is  $i_e : G_e \rightarrow i_e(G_e)$ , which is well defined since any conjugation preserving the maximal abelian group  $A \subset G_v$  containing  $G_e$  fixes  $A$  because  $G_v$  is CSA.

The connected components of  $Cyl(\Gamma)$  correspond to the orbits of cylinders in  $T_C$  in the following sense:

**Lemma A.6.** Let  $C$  be a cylinder of  $T$ ,  $G_C$  its global stabilizer. Consider the graph of groups  $\Lambda = C/G_C$ . Then  $\Lambda$  corresponds to a connected component of  $Cyl(\Gamma)$ . The fundamental group of a connected component of  $Cyl(\Gamma)$  is conjugate to the stabilizer of the corresponding cylinder in  $T_C$ . In particular they are maximal abelian subgroups of  $G$ .

*Proof.* The proof is straightforward and left as an exercise.  $\square$

**Corollary A.7.** Consider a graph of groups  $\Gamma$  with torsion-free vertex groups and non-trivial abelian edge groups.

Then  $\pi_1(\Gamma)$  is CSA if and only vertex groups of  $\Gamma$  are CSA and each connected component  $\Lambda$  of  $Cyl(\Gamma)$  is of one of the following form:

- either  $\Lambda$  is a trivial splitting: there is a vertex  $v_0 \in \Lambda$  such that the injection  $G_{v_0} \subset \pi_1(\Lambda)$  is actually an isomorphism. This translates into the fact that  $\Lambda$  is a tree of groups and that for all vertex  $v \neq v_0$  and for all edge  $e$  with  $t(e) = v$  and separating  $v$  from  $v_0$ , the edge morphism  $i_e : G_e \rightarrow G_{t(e)}$  is an isomorphism.
- or  $\Lambda$  has the homotopy type of a circle  $\mathcal{C} \subset \Lambda$  (as a simple graph), the edge morphisms of edges of  $\mathcal{C}$  are isomorphisms, the composition of all the edge morphisms around  $\mathcal{C}$  is the identity, and the injection  $\pi_1(\mathcal{C}) \subset \pi_1(\Lambda)$  is actually an isomorphism. This last property translates into the fact that for all edge  $v \notin \mathcal{C}$  and for all edge with  $t(e) = v$  and separating  $v$  from  $\mathcal{C}$ , the edge morphism  $i_e : G_e \rightarrow G_{t(e)}$  is an isomorphism.

*Proof.* It is clear that each of the cases implies that  $\pi_1(\Lambda)$  is abelian since vertex groups of  $\Lambda$  are abelian.

Conversely, if  $\pi_1(\Lambda)$  is abelian, then the fundamental group of the graph underlying  $\Lambda$  has to be abelian, so  $\Lambda$  has the homotopy type of a point or of a circle.

In the first case,  $\pi_1(\Lambda)$  is generated by finitely many vertex stabilizers. But it is an easy exercise to check that two commuting vertex stabilizers must fix a common vertex, so whole group  $\pi_1(\Lambda)$  must fix a vertex.

If  $\Lambda$  has the homotopy type of a circle  $\mathcal{C} \subset \Lambda$ , then the action of  $\pi_1(\Lambda)$  on its Bass-Serre tree is non-trivial, but since  $\pi_1(\Lambda)$  is abelian, its minimal invariant subtree is a line  $l$ , and  $\pi_1(\Lambda)$  acts by translations on  $l$ . Moreover,  $\mathcal{C} = l/\pi_1(\Lambda)$  is such that the injection  $\pi_1(\mathcal{C}) \subset \pi_1(\Lambda)$  is an isomorphism, and the edge morphisms of edges of  $\mathcal{C}$  are isomorphisms, and that the composition of all the edge morphisms around  $\mathcal{C}$  is the identity (as it is induced by a conjugation in the abelian group  $\pi_1(\Lambda)$ ).  $\square$

**Corollary A.8.** *Consider a graph of groups  $\Gamma$  with CSA vertex groups and abelian edge groups. If every edge group is maximal abelian in the two neighbouring vertex groups, and if the Bass-Serre tree of  $\Gamma$  is acylindrical, then  $\pi_1(\Gamma)$  is CSA.*

*Proof.* Acylindricity of  $\Gamma$  implies that each component  $\Lambda$  of  $Cyl(\Gamma)$  is a tree of groups. Moreover, every edge morphism in  $\Lambda$  is an isomorphism, therefore the splitting corresponding to  $\Lambda$  is trivial. Thus the first condition of the previous corollary holds.  $\square$

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