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# Skew category, Galois covering and smash product of a category over a ring

Claude Cibils, Eduardo N. Marcos\*

## Abstract

In this paper we consider categories over a commutative ring provided either with a free action or with a grading of a not necessarily finite group. We define the smash product category and the skew category and we show that these constructions agree with the usual ones for algebras. In case of the smash product for an infinite group our construction specialized for a ring agrees with M. Beattie's construction of a ring with local units in [1]. We recover in a categorical generalized setting the duality theorems of M. Cohen and S. Montgomery in [5], and we provide a unification with the results on coverings of quivers and relations by E. Green in [8], see also [9]. We obtain a confirmation in a quiver and relations free categorical setting that both constructions are mutual inverses, namely the quotient of a free action category and the smash product of a graded category. Finally we describe functorial relations between the representation theories of a category and of a Galois cover of it.

## 1 Introduction

Throughout this paper we will consider small categories  $\mathcal{C}$  over a commutative ring  $k$ , which means that the objects  $\mathcal{C}_0$  forms a set, the morphism set  ${}_y\mathcal{C}_x$  from an object  $x$  to an object  $y$  is a  $k$ -module and the composition of morphisms is  $k$ -bilinear.

Inspired by the work of [5, 8, 9, 6, 12, 10, 4] we consider in addition  $G$ -categories over  $k$  and  $G$ -graded categories where  $G$  is a group. A  $G$ -category over  $k$  is provided by a category  $\mathcal{C}$  with firstly a set action of  $G$  on the objects and secondly  $k$ -module maps  $s : {}_y\mathcal{C}_x \rightarrow {}_{sy}\mathcal{C}_{sx}$  for each  $s \in G$  and for each couple of objects  $x$  and  $y$ , verifying  $s(gf) = (sg)(sf)$  in case  $g$  and  $f$  are morphisms which can be composed in the category. If the action of  $G$  is free on  $\mathcal{C}_0$  we say that  $\mathcal{C}$  is a free  $G$ -category.

A  $G$ -graded category  $\mathcal{C}$  over  $k$  has a direct sum decomposition of  $k$ -modules  ${}_y\mathcal{C}_x = \bigoplus_{s \in G} {}_y\mathcal{C}_x^s$  for each couple of objects, verifying  ${}_z\mathcal{C}_y^t {}_y\mathcal{C}_x^s \subset {}_z\mathcal{C}_x^{ts}$  and  ${}_x1_x \in {}_x\mathcal{C}_x^1$ .

In the first section we recall the definition of the quotient category  $\mathcal{C}/G$  which makes sense only if  $\mathcal{C}$  is a free  $G$ -category. It has been introduced and considered

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in [13, 2] as well as in subsequent work by several authors and recently in [4]. Note that  $\mathcal{C}$  is a Galois covering of the quotient  $\mathcal{C}/G$ . Next we consider the skew-category  $\mathcal{C}[G]$  which is an analogue of the skew-ring construction. We prove a coherence result between our approach and the ring theoretical construction in case  $G$  is finite and the category has a finite number of objects. The main result in this section is that  $\mathcal{C}/G$  and  $\mathcal{C}[G]$  are equivalent. This property unifies and generalizes previous results obtained for rings by M. Cohen and S. Montgomery [5] and for quivers and relations in [8, 9, 14].

In the second section we consider a  $G$ -graded category  $\mathcal{B}$  over  $k$  and we define the smash product category  $\mathcal{B}\#G$ . In case  $G$  is finite, we obtain again a coherence result with the  $k$ -algebra case used in [5]. The categorical approach for the smash product that we introduce here is equivalent to the ring without identity construction obtained by M. Beattie in [1]. Therefore the same remarks from the introduction of [1] are in force: the rings considered in [11, 14] differ from the present approach. Note that in the context of quivers and relations - that is categories given through a presentation - the categorical smash construction has been considered in [8, 9]. The first result in this section relates the categorical smash product with quotients: let  $\mathcal{B}$  be a  $G$ -graded category over  $k$ , then  $\mathcal{B}\#G$  is a free  $G$ -category and there is a canonical isomorphism  $(\mathcal{B}\#G)/G \simeq \mathcal{B}$ .

The second result of this section concerns a free  $G$ -category  $\mathcal{C}$  over  $k$ : the quotient category  $\mathcal{C}/G$  is graded and the smash product  $(\mathcal{C}/G)\#G$  is isomorphic to  $\mathcal{C}$ .

In other words we provide in this paper a confirmation in a uniform categorical setting that gradings of a category over  $k$  are in one-to-one correspondence with Galois coverings, after the work of E. Green [8]. Moreover the skew-category of the smash product is equivalent to the starting category, which is a general statement of Cohen–Montgomery duality for a non necessarily finite group and a category with possibly infinitely many objects.

In order to make precise the link with [5] we specialize our results in case to the case of a finite group  $G$ . Let  $A$  be a  $G$ -graded  $k$ -algebra, which we consider as a single object category  $\mathcal{C}_A$  with endomorphism ring  $A$ . The smash product algebra  $A\#G$  has a free  $G$ -action and the corresponding skew group algebra  $(A\#G)[G]$  is proved in [5] to be a matrix algebra over  $A$  of size  $|G|$ . Now the smash product category  $\mathcal{C}_A\#G$  we consider in this paper has  $G$  as set of objects. Since we assume here that  $G$  is finite, we can consider the  $k$ -algebra of the category  $a(\mathcal{C}_A\#G)$ , namely the direct sum of all the  $k$ -modules morphisms of  $\mathcal{C}_A\#G$  equipped with the usual matrix product. We show that this algebra is precisely  $A\#G$ .

Our results in this specific case provide the existence of a single object full subcategory of  $\mathcal{C}_A\#G[G]$  which is equivalent to the ambient one and isomorphic to  $\mathcal{C}_A$ . Since  $\mathcal{C}_A\#G$  has precisely  $|G|$  objects we infer Cohen–Montgomery result quoted above.

In another direction note that Hochschild–Mitchell cohomology of equivalent  $k$ -categories remain isomorphic through the corresponding change of bimodules. In [4] a Cartan–Leray spectral sequence is obtained relating the Hochschild–Mitchell cohomology of the categories of a Galois covering  $\mathcal{C} \rightarrow \mathcal{C}/G$ . Our results show that this Cartan–Leray spectral sequence translates into a spectral sequence relating

Hochschild–Mitchel cohomology of  $\mathcal{C}$  and of  $\mathcal{C}[G]$ , since  $\mathcal{C}[G]$  is equivalent to  $\mathcal{C}/G$ .

Finally in the last section we consider left modules over a category  $\mathcal{C}$ , namely contravariant functors from  $\mathcal{C}$  to the category of  $k$ -modules. Note that if a category  $\mathcal{C}$  over  $k$  is presented by a quiver with relations, left  $\mathcal{C}$ -modules are precisely representations of the quiver subjected to the relations. In case of a Galois covering of categories  $\mathcal{C} \rightarrow \mathcal{B}$ , modules over the base category  $\mathcal{B}$  coincide with the fixed modules over the covering category under the action of the group, while all the modules over the covering category coincides with graded ones over the base category.

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## 2 Galois coverings and the skew-category

Let  $\mathcal{C}$  be a free  $G$ -category over a commutative ring  $k$ . Recall that the  $G$ -set of objects  $\mathcal{C}_0$  is free, namely if  $sx = x$  for an object  $x$ , then  $s = 1$ .

**DEFINITION 2.1** *Let  $\mathcal{C}$  be a free  $G$ -category over  $k$ . The quotient category  $\mathcal{C}/G$  has set of objects the orbits of  $\mathcal{C}_0$ . The  $k$ -module of morphisms of  $\mathcal{C}/G$  from the orbit  $\alpha$  to the orbit  $\beta$  is*

$$\beta(\mathcal{C}/G)_\alpha = \left( \bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x \right) / G.$$

*Note that  $\bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x$  is a  $kG$ -module and  $(\bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x) / G$  is the largest  $k$ -module quotient with trivial  $G$ -action: if  $X$  is a  $kG$ -module  $X/G = X / (\ker \epsilon)X$  where  $\epsilon : kG \rightarrow k$  is the augmentation map given by  $\epsilon(s) = 1$ , for all  $s \in G$ .*

*Composition in  $\mathcal{C}/G$  is deduced from composition in  $\mathcal{C}$  and is well defined precisely because the action of  $G$  on the objects is free.*

**LEMMA 2.2** *Let  $\mathcal{C}$  be a free  $G$ -category over  $k$ , let  $\alpha$  and  $\beta$  be orbits of objects and let  $x_\alpha \in \alpha$  and  $x_\beta \in \beta$  be representatives. Then*

$$\beta(\mathcal{C}/G)_\alpha = \bigoplus_{s \in G} {}_{x_\beta}\mathcal{C}_{sx_\alpha}$$

*Proof.* There is a normalization map

$$\bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x \xrightarrow{\phi} \bigoplus_{s \in G} {}_{x_\beta}\mathcal{C}_{sx_\alpha}$$

given by  $\phi({}_y f_x) = s f$  where  $s$  is the unique group element such that  $sy = x_\beta$ . Note that  $\phi(tf) = \phi(f)$  hence  $\phi : \beta(\mathcal{C}/G)_\alpha \rightarrow \bigoplus_{s \in G} {}_{x_\beta}\mathcal{C}_{sx_\alpha}$  is well defined. Conversely, consider

$$\psi : \bigoplus_{s \in G} {}_{x_\beta}\mathcal{C}_{sx_\alpha} \hookrightarrow \bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x \rightarrow \beta(\mathcal{C}/G)_\alpha$$

which is an inverse for  $\phi$ . □

**DEFINITION 2.3** Let  $\mathcal{C}$  be a  $G$ -category over  $k$ . The skew-category  $\mathcal{C}[G]$  has set of vertices  $(\mathcal{C}[G])_0 = \mathcal{C}_0$ , and morphisms  ${}_y(\mathcal{C}[G])_x = \bigoplus_{s \in G} {}_y\mathcal{C}_{sx}$ .

Composition of morphisms is provided by the composition of  $\mathcal{C}$  after adjustment, more precisely let  $g \in {}_z\mathcal{C}_{ty}$  and  $f \in {}_y\mathcal{C}_{sx}$ . Then  $gf = g \circ (tf)$ .

We show now a coherence result with the usual skew-algebra construction.

**PROPOSITION 2.4** Let  $G$  be a finite group and let  $\mathcal{C}$  be a free  $G$ -category over  $k$ , with a finite number of objects. Let  $a(\mathcal{C})$  be the  $k$ -algebra associated to  $\mathcal{C}$ , namely  $a(\mathcal{C}) = \bigoplus_{x, y \in \mathcal{C}_0} ({}_y\mathcal{C}_x)$  provided with the matrix product induced by the composition of morphisms.

Then  $G$  acts on  $a(\mathcal{C})$  by algebra automorphisms and  $a(\mathcal{C}[G])$  is the skew-algebra  $a(\mathcal{C})[G]$ .

*Proof.* Recall that if  $A$  is a  $k$ -algebra with  $G$  acting by automorphisms on it, the skew group algebra  $A[G]$  is the  $k$ -module  $A \otimes kG$  with the twisted product provided by  $sa = s(a)s$  for all  $s \in G$  and  $a \in A$ . As it is well known and easy to check the twisting (see for instance [3]) conditions are satisfied and the skew group algebra is a  $k$ -algebra which has  $A$  and  $kG$  as subalgebras.

The isomorphism of algebras  $\psi : a(\mathcal{C}[G]) \rightarrow a(\mathcal{C})[G]$  is defined as follows. Let  $f \in {}_y\mathcal{C}_{sx}$  be an elementary morphism of  ${}_y\mathcal{C}[G]_x$ . Then  $\psi(f) = f \otimes s$ .

We have that

$$\psi(gf) = \psi(g \circ (tf)) = g(tf) \otimes ts,$$

while

$$\psi(g)\psi(f) = (g \otimes t)(f \otimes s) = g(tf) \otimes ts.$$

Moreover

$$\psi(1) = \psi\left(\sum_x 1_x\right) = \sum_x \psi(1_x) = \sum_x (1_x \otimes 1_G) = \left(\sum_x 1_x\right) \otimes 1_G = 1 \otimes 1_G.$$

□

The following result shows a fact that we need to be true in  $\mathcal{C}[G]$ . Together with the coherence property above it justifies the definition we gave for the skew-category  $\mathcal{C}[G]$ .

**LEMMA 2.5** Let  $\mathcal{C}$  be a  $G$ -category over  $k$ . Objects in the same  $G$  orbit are isomorphic in  $\mathcal{C}[G]$ .

*Proof.* If  $sx = y$  then  ${}_y1_y \in {}_y\mathcal{C}[G]_x$  and  ${}_x1_x \in {}_x\mathcal{C}[G]_y$ . These morphisms are mutual inverses. □

**PROPOSITION 2.6** Let  $\mathcal{C}$  be a  $G$ -category over  $k$  and let  $\mathcal{C}[G]$  be the skew-category. Choose an element  $x_\alpha$  in each orbit  $\alpha$  and let  $\underline{\mathcal{C}[G]}$  be the full subcategory of  $\mathcal{C}[G]$  given by these objects. Then the categories  $\mathcal{C}[G]$  and  $\underline{\mathcal{C}[G]}$  are equivalent.

Proof. Consider the functor  $F : \mathcal{C}[G] \rightarrow \underline{\mathcal{C}}[G]$  defined on objects by  $F(x) = x_\alpha$  if the orbit of  $x$  is  $\alpha$ . Since  $x$  and  $x_\alpha$  are canonically isomorphic - see the Lemma above - a morphism  $f : x \rightarrow y$  provides a unique morphism  $x_\alpha \rightarrow x_\beta$  where  $\beta$  is the orbit of  $y$ . This construction is functorial, moreover  $F$  and the natural embedding are inverse equivalences.  $\square$

**PROPOSITION 2.7** *Let  $\mathcal{C}$  be a free  $G$ -action category over  $k$ . Then the categories  $\mathcal{C}/G$  and  $\underline{\mathcal{C}}[G]$  are isomorphic.*

Proof. This is a consequence of Lemma 2.2.  $\square$

We have proved the following

**THEOREM 2.8** *Let  $\mathcal{C}$  be a free  $G$ -category over  $k$ . The quotient category  $\mathcal{C}/G$  and the skew category  $\underline{\mathcal{C}}[G]$  are equivalent.*

**Remark 2.9** *This Theorem suggests that the categories  $\mathcal{C}[G]$  or  $\underline{\mathcal{C}}[G]$  can be considered as a substitute for the quotient category in case the action of  $G$  on  $\mathcal{C}$  is not free.*

### 3 Smash Product Category

Let  $\mathcal{B}$  be a  $G$ -graded category over  $k$  where  $G$  is a group: each  $k$ -module of morphisms  ${}_y\mathcal{B}_x$  is a direct sum of sub  $k$ -modules  ${}_y\mathcal{B}_x^s$  where  $s \in G$ , and  ${}_z\mathcal{B}_y^t {}_y\mathcal{B}_x^s \subset {}_z\mathcal{B}_x^{ts}$ . Moreover  ${}_x1_x \in {}_x\mathcal{B}_x^1$ .

The smash product category  $\mathcal{B}\#G$  is defined below. We will show a coherence result in case  $\mathcal{B}$  is a single object category given by a  $G$ -graded  $k$ -algebra. Then we will prove that a Galois covering of categories is isomorphic to a smash product of categories.

**DEFINITION 3.1** *The smash product category  $\mathcal{B}\#G$  has object set  $\mathcal{B}_0 \times G$ . Let  $(x, s)$  and  $(y, t)$  be objects. The  $k$ -module of morphisms is defined as follows:*

$${}_{(y,t)}(\mathcal{B}\#G)_{(x,s)} = {}_y\mathcal{B}_x^{t^{-1}s}$$

*Note that  ${}_{(y,t)}(\mathcal{B}\#G)_{(x,s)}$  and  ${}_{(y,ut)}(\mathcal{B}\#G)_{(x,us)}$  are  $k$ -modules morphism from different objects which coincide as  $k$ -modules.*

*In order to define the composition map*

$${}_{(z,u)}(\mathcal{B}\#G)_{(y,t)} \otimes_k {}_{(y,t)}(\mathcal{B}\#G)_{(x,s)} \longrightarrow {}_{(z,u)}(\mathcal{B}\#G)_{(x,s)}$$

*note that the left hand side is*

$${}_z\mathcal{B}_y^{u^{-1}t} \otimes_k {}_y\mathcal{B}_x^{t^{-1}s}$$

*while the right hand side is  ${}_z\mathcal{B}_x^{u^{-1}s}$ . The graded composition of  $\mathcal{B}$  provides the required map.*

**PROPOSITION 3.2** *Let  $\mathcal{B}$  be a  $G$ -graded category over  $k$ . The category  $\mathcal{B}\#G$  is a free  $G$ -category and  $(\mathcal{B}\#G)/G = \mathcal{B}$ .*

Proof. On objects we define  $u(x, s) = (x, us)$ .

Let  $f$  be a morphism in  ${}_{(y,t)}(\mathcal{B}\#G)_{(x,s)} = {}_y\mathcal{B}_x^{t^{-1}s}$ , we define

$$uf = f \in {}_{(y,ut)}(\mathcal{B}\#G)_{(x,us)}.$$

In other words the action is obtained by translation. Clearly this is an action and moreover the action on objects is free. Note that orbits of objects are in one to one correspondence with objects of  $\mathcal{B}$ , by retaining the first component of each couple.

This shows that  $[(\mathcal{B}\#G)/G]_0 \cong \mathcal{B}_0$ . Moreover recall that by definition we have

$${}_y[(\mathcal{B}\#G)/G]_x = \left[ \bigoplus_{s \in G, t \in G} {}_{(y,t)}(\mathcal{B}\#G)_{(x,s)} \right] / G.$$

The action of  $G$  on the numerator transports  ${}_{(y,1)}(\mathcal{B}\#G)_{(x,s)}$  to  ${}_{(y,u)}(\mathcal{B}\#G)_{(x,us)}$  consequently those  $k$ -modules are identified in the quotient. This remark shows the following:

$$\left[ \bigoplus_{s \in G, t \in G} {}_{(y,t)}(\mathcal{B}\#G)_{(x,s)} \right] / G = \bigoplus_{s \in G} {}_{(y,1)}(\mathcal{B}\#G)_{(x,s)} = \bigoplus_{s \in G} {}_y\mathcal{B}_x^s.$$

□

The following is a natural consequence of the former proposition.

**COROLLARY 3.3** *Let  $G$  be a group and  $\mathcal{B}$  be a  $G$ -graded category over  $k$ . Then the categories  $(\mathcal{B}\#G)[G]$  and  $\mathcal{B}$  are equivalent.*

Proof. We know that  $\mathcal{B}\#G$  is a Galois covering of  $\mathcal{B}$ . The result of the previous section shows that  $\mathcal{B}[G]$  is equivalent to  $\mathcal{B}/G$ , namely  $(\mathcal{B}\#G)[G]$  is equivalent to  $(\mathcal{B}\#G)/G = \mathcal{B}$ . □

**Remark 3.4** *In case a ring  $A$  is graded by a finite group the coherence result below shows that the preceding Corollary is Cohen-Montgomery duality theorem, (see [5] and [7]).*

**Remark 3.5** *In case of an infinite group grading of a  $k$ -algebra  $A$  the preceding Corollary shows that the category  $(\mathcal{C}_A\#G)[G]$  is equivalent to  $\mathcal{C}_A$ , where  $\mathcal{C}_A$  is the single object category over  $k$  determined by  $A$ .*

In order to prove the coherence property of the categorical smash product with the smash product of a graded  $G$ -algebra we first recall its definition. Note that the usual smash product of a graded algebra requires  $G$  to be finite, see for instance [5]. As quoted in the Introduction, the categorical smash product we consider in this paper is an alternative approach to the ring without identity defined by M. Beattie in [1].

**DEFINITION 3.6** Let  $A$  be a  $G$ -graded  $k$ -algebra where  $G$  is a finite group. The smash product  $A\#k^G$  is the twisted (see for instance [3]) tensor product of algebras  $A \otimes k^G$  where the twist map  $\tau : k^G \otimes A \rightarrow A \otimes k^G$  is given as follows: let  $\delta_s$  be the Dirac mass on an element  $s \in G$ , and let  $f_t$  be an element of degree  $t$  in  $A$ . Then

$$\tau(\delta_t \otimes f_s) = f_s \otimes \delta_{s^{-1}t}.$$

The verifications of the twisting properties of  $\tau$  insuring associativity are well known, we provide them for the convenience of the reader and in order to avoid any missprint for  $\tau$ . We omit the tensor product symbol as well as the map  $\tau$ , in other words we consider  $\delta_t f_s = f_t \delta_{s^{-1}t}$  as the product in  $A \otimes k^G$ .

First observe that

$$(\delta_u \delta_t) f_s = 0 \text{ if } u \neq t \text{ while } (\delta_u \delta_t) f_s = f_s \delta_{s^{-1}t} \text{ if } u = t$$

and

$$\delta_u(\delta_t f_s) = \delta_u f_s \delta_{s^{-1}t} = f_s \delta_{s^{-1}u} \delta_{s^{-1}t} = 0 \text{ if } u \neq t \text{ while } \delta_u(\delta_t f_s) = f_s \delta_{s^{-1}t} \text{ if } u = t.$$

Second

$$\begin{aligned} \delta_u(g_t f_s) &= g_t f_s \delta_{(ts)^{-1}u} \text{ since } g_t f_s \in A_{ts}, \\ (\delta_u g_t) f_s &= g_t \delta_{t^{-1}u} f_s = g_t f_s \delta_{s^{-1}t^{-1}u}. \end{aligned}$$

Moreover

$$1_{k^G} f_s = \sum_{u \in G} \delta_u f_s = \sum_{s \in G} f_s \delta_{s^{-1}u} = f_s 1_{k^G}.$$

Recall that in case  $\mathcal{B}$  is a finite object set category over  $k$ , the  $k$ -algebra of morphisms of  $\mathcal{B}$  is denoted  $a(\mathcal{B})$ , it is the direct sum of the morphisms  $k$ -modules of  $\mathcal{B}$  equipped with the matrix product.

**PROPOSITION 3.7** Let  $G$  be a finite group and let  $A$  be a  $G$ -graded  $k$ -algebra. Let  $\mathcal{B}_A$  be the single object category over  $k$  with endomorphism  $k$ -algebra  $A$ . Then the category  $\mathcal{B}_A$  is  $G$ -graded and  $a(\mathcal{B}_A\#G) = A\#G$ .

*Proof.* We define a map  $\phi : a(\mathcal{B}_A\#G) \rightarrow A\#G$  as follows. Consider an elementary matrix  ${}_t F_s \in a(\mathcal{B}_A\#G)$  with all entries zero except perhaps the entry  $(x_0, t) - (x_0, s)$  with value an element  $f_{t^{-1}s} \in {}_{(x_0, t)}(\mathcal{B}_A\#G)_{(x_0, s)}$ . Observe that

$${}_{(x_0, t)}(\mathcal{B}_A\#G)_{(x_0, s)} = {}_{x_0}(\mathcal{B}_A)_{x_0} {}^{t^{-1}s} = A_{t^{-1}s}.$$

We put

$$\phi({}_t F_s) = f_{t^{-1}s} \delta_{s^{-1}}.$$

Note first that the unit element in  $a(\mathcal{B}_A\#G)$  is  $1 = \sum_{s \in G} {}_s 1_s$  where  ${}_s 1_s$  is the elementary matrix with nonzero entry only on the entry  $(x_0, s) - (x_0, s)$  with value  $1 \in A_{s^{-1}s} = A_1$ . Then

$$\phi\left(\sum_{s \in G} {}_s 1_s\right) = \sum_{s \in G} 1_1 \delta_{s^{-1}} = 1_{A\#G}.$$



Let  ${}_vG_u$  and  ${}_tF_s$  be elementary matrices. Then  ${}_vG_u {}_tF_s = 0$  if  $u \neq t$  and  ${}_vG_u {}_uF_s$  is the elementary matrix  ${}_v(GF)_s$  with zero entries except a possible nonzero entry  $(x_0, v) - (x_0, s)$  with value  $g_{v^{-1}t}f_{t^{-1}s}$ . We have that

$$\phi({}_vG_u)\phi({}_tF_s) = g_{v^{-1}u} \delta_{u^{-1}} f_{t^{-1}s} \delta_{s^{-1}} = g_{v^{-1}u} f_{t^{-1}s} \delta_{s^{-1}tu^{-1}} \delta_{s^{-1}}.$$

Note that if  $u \neq t$  we have  $\phi({}_vG_u)\phi({}_tF_s) = 0$  while if  $u = t$

$$\phi({}_vG_u)\phi({}_tF_s) = g_{v^{-1}u} f_{t^{-1}s} \delta_{s^{-1}} = \phi({}_vG_t {}_tF_s).$$

Clearly the morphism we have defined is an injective and surjective  $k$ -module map, since the image of the elementary matrix  ${}_tF_s$  belongs to  $A_{t^{-1}s} \otimes \delta_{s^{-1}}$ .  $\square$

The following result can be linked with Proposition 3.2, in order to obtain that the categorical smash product and the categorical quotient are inverse constructions between graded categories and free action categories.

**THEOREM 3.8** *Let  $G$  be a group and let  $\mathcal{C}$  be a free  $G$ -category over  $k$ . Then  $\mathcal{C}/G$  is  $G$ -graded and the category  $(\mathcal{C}/G) \# G$  is isomorphic to  $\mathcal{C}$*

Proof. Let  $\{x_\alpha \mid x_\alpha \in \alpha\}$  be a choice of representatives in each  $G$ -orbit of objects of  $\mathcal{C}$ . Recall that  ${}_\beta(\mathcal{C}/G)_\alpha = [\oplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x] / G$ . Let  ${}_y f_x \in {}_y\mathcal{C}_x$ . In order to define the degree  $\deg[{}_y f_x]$  of the class  $[{}_y f_x]$  we first normalize  ${}_y f_x$  in order to obtain as source object  $x_\alpha$ , then we retain the gap between the target object obtained and  $x_\beta$ . More precisely there is a unique  $s \in G$  such that  $sx = x_\alpha$ , then there is a unique element of  $G$  that we denote  $(\deg[f])^{-1}$  such that  $(\deg[f])^{-1}x_\beta = sy$ .

First we note that  $\deg[f]$  is well defined: let  $u \in G$ , then  $uf \in {}_{uy}\mathcal{C}_{ux}$ . The normalization uses  $su^{-1}$  and the target object is again  $sy$ .

Second we define

$${}_\beta(\mathcal{C}/G)_\alpha^s = \left\{ \sum [f] \mid f \in {}_y\mathcal{C}_x \text{ where } x \in \alpha, y \in \beta \text{ and } \deg[f] = s \right\}.$$

$$\text{Then } {}_\beta(\mathcal{C}/G)_\alpha = \bigoplus_{s \in G} {}_\beta(\mathcal{C}/G)_\alpha^s.$$

Finally we check that  $\deg(gf) = \deg(g)\deg(f)$ . Indeed consider  $(\deg g)^{-1}x_\gamma g_{x_\beta}$  and  $(\deg f)^{-1}x_\beta f_{x_\alpha}$ . In order to compose those morphisms we first have to adjust  $g$  in order that its source object coincide with the target object of  $f$ . The resulting composition is

$$(\deg f)^{-1}(\deg g)^{-1}x_\gamma \left[ (\deg f^{-1}) gf \right]_{x_\alpha}.$$

Hence  $(\deg(gf))^{-1} = (\deg f)^{-1}(\deg g)^{-1}$  and  $\deg(gf) = (\deg g)(\deg f)$ .

Using the graduation we have just described we define a functor

$$F : (\mathcal{C}/G) \# G \rightarrow \mathcal{C}$$

as follows. Let  $(\alpha, s)$  be an object of the smash category. Then  $F(\alpha, s) = sx_\alpha$ .

Recall that by definition

$$({}_{\beta,t}(\mathcal{C}/G))_{(\alpha,s)} = {}_\beta(\mathcal{C}/G)_\alpha^{t^{-1}s}.$$

Let  $s^{-1}tx_\beta[f]_{x_\alpha} \in {}_{(\beta,t)}(\mathcal{C}/G)\#G_{(\alpha,s)}$  where  $f \in {}_{s^{-1}tx_\beta}(\mathcal{C})_{x_\alpha}$ . We define

$$F(s^{-1}tx_\beta[f]_{x_\alpha}) = sf \in {}_{tx_\beta}\mathcal{C}_{sx_\alpha}.$$

We consider now

$$t^{-1}ux_\gamma[g]_{x_\beta} \in {}_{(\gamma,u)}(\mathcal{C}\#G)_{(\beta,t)}.$$

In order to compose  $[g]$  and  $[f]$  in  $(\mathcal{C}/G)\#G$  we have to adjust  $g$  using the  $G$  action, obtaining the right object for its source namely the target of  $f$ .

Then  $[g] \circ [f] = [s^{-1}tg][f]$ . So

$$F(s^{-1}ux_\alpha[s^{-1}tg][f]_{x_\alpha}) = s((s^{-1}tg)(f)) = (tg)(sf) = F(g)F(f)$$

It is not hard to see that  $F$  is a bijection on objects and an isomorphism of  $k$ -modules on morphisms.  $\square$

## 4 Modules

Let us recall the following definition:

**DEFINITION 4.1** *Let  $\mathcal{C}$  be a category over a commutative ring  $k$ . A left  $\mathcal{C}$ -module  $\mathcal{M}$  is a collection of  $k$ -modules  $\{{}_x\mathcal{M}\}_{x \in \mathcal{C}_0}$  provided with a left action of the  $k$ -modules morphisms of  $\mathcal{C}$ , given by  $k$ -module maps  ${}_y\mathcal{C}_x \otimes_k {}_x\mathcal{M} \rightarrow {}_y\mathcal{M}$  where the image of  ${}_y f_x \otimes {}_x m$  is denoted  ${}_y f_x {}_x m$ , verifying the usual axioms*

1.  ${}_z f_y ({}_y f_x {}_x m) = ({}_z f_y {}_y f_x) {}_x m$
2.  ${}_x 1_x {}_x m = {}_x m$

In other words  $\mathcal{M}$  is a contravariant functor from  $\mathcal{C}$  to the category of  $k$ -modules. We denote by  $\mathcal{C} - \text{Mod}$  the category of left  $\mathcal{C}$ -modules. As it is well known and easy to establish the agreement property holds, namely if  $\mathcal{C}$  is a finite number objects category and  $a(\mathcal{C})$  is the corresponding algebra of morphisms,  $\mathcal{C}$ -modules and usual  $a(\mathcal{C})$ -modules coincide. In particular if  $A$  is a  $k$ -algebra and  $\mathcal{C}_A$  is the single object category with endomorphism ring  $A$ , then  $\mathcal{C}_A$ -modules and  $A$ -modules are the same.

Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a functor between categories over  $k$ . Note that functors do not correspond really to algebra maps. In case  $\mathcal{C}$  and  $\mathcal{B}$  have finite number of objects, the corresponding map  $a(\mathcal{C}) \rightarrow a(\mathcal{B})$  do not send, in general,  $1_{a(\mathcal{C})}$  to  $1_{a(\mathcal{B})}$ . However the map is additive and multiplicative.

**DEFINITION 4.2** *Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a functor of  $k$ -categories, and let  $\mathcal{N}$  be a  $\mathcal{B}$ -module. We denote by  $F^*\mathcal{N}$  the module  $\mathcal{N} \circ F$  and by  $F^*$  the corresponding functor  $F^* : \mathcal{B} - \text{Mod} \rightarrow \mathcal{C} - \text{Mod}$ .*

In other words  $F^*$  is the restriction functor and  $F^*\mathcal{N}$  is the  $\mathcal{C}$ -module given by  ${}_x(F^*\mathcal{N}) = {}_{F(x)}\mathcal{N}$  and

$$({}_y c_x) \cdot ({}_{F(x)} n) = ({}_{F(y)} F(c)_{F(x)}) ({}_{F(x)} n) \in {}_y(F^*\mathcal{N}).$$

Let  $f$  be a natural transformation between modules, then  $F^*(f) = f$ . Note that  $F^*$  is faithful. In case  $F$  is a full functor the functor  $F^*$  is also full.

In case of a  $G$ -category  $\mathcal{C}$  over  $k$  the group  $G$  acts on the category  $\mathcal{C} - \text{Mod}$  in the following way: let  $\mathcal{M}$  be a  $\mathcal{C}$ -module and  $s \in G$ , then  ${}^s\mathcal{M}$  is given by

$$x({}^s\mathcal{M}) = {}_{s^{-1}x}\mathcal{M}$$

and

$$({}_y c_x)({}_{s^{-1}x}m) = (s^{-1}c)m \in {}_{s^{-1}y}\mathcal{M} = {}_y({}^s\mathcal{M}).$$

**THEOREM 4.3** *Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a Galois covering where  $\mathcal{C}$  is a free  $G$ -category over  $k$  and  $\mathcal{B} = \mathcal{C}/G$ . Then  $\mathcal{B} - \text{Mod}$  and  $(\mathcal{C} - \text{Mod})^G$  are isomorphic categories.*

**Remark 4.4**  $(\mathcal{C} - \text{Mod})^G$  is a full subcategory of  $\mathcal{C} - \text{Mod}$  provided by fixed objects under the action of  $G$ , namely modules  $\mathcal{M}$  such that  ${}^s\mathcal{M} = \mathcal{M}$  for all  $s \in G$ .

Proof. The image of the functor  $F^*$  is precisely  $(\mathcal{C} - \text{Mod})^G$ . Indeed

$$x({}^s(F^*\mathcal{N})) = {}_{s^{-1}x}(F^*\mathcal{N}) = {}_{F(s^{-1}x)}\mathcal{N} = {}_{F(x)}\mathcal{N}.$$

Note that by definition the functor  $F$  of a Galois covering is full. □

Next we want to prove that the entire category  $\mathcal{C} - \text{Mod}$  is isomorphic to the category of graded modules over  $\mathcal{B}$ . Recall that in case of a Galois covering  $F : \mathcal{C} \longrightarrow \mathcal{B}$  (where  $\mathcal{C}$  is a free  $G$ -category over  $k$  and  $\mathcal{B} = \mathcal{C}/G$ ) we have shown that  $\mathcal{B}$  is  $G$ -graded. We also have shown that  $\mathcal{C} = \mathcal{B}\#G$  and we will use this description.

First recall that in case  $\mathcal{B}$  is graded, a graded  $\mathcal{B}$ -module is a  $\mathcal{B}$ -module  $\mathcal{N}$  provided with a direct sum decomposition at each  $k$ -module, namely

$${}_x\mathcal{N} = \bigoplus_{s \in G} {}_x\mathcal{N}^s \text{ such that } {}_y\mathcal{B}_x^t {}_x\mathcal{N}^s \subset {}_y\mathcal{N}^{ts}.$$

We denote by  $\mathcal{B} - \text{Mod}_G$  the category of graded  $\mathcal{B}$ -modules.

**THEOREM 4.5** *Let  $\mathcal{B}$  be a  $G$ -graded category over  $k$  and let  $\mathcal{C} = \mathcal{B}\#G$  be its Galois covering. Then  $\mathcal{B} - \text{Mod}_G$  and  $\mathcal{C} - \text{Mod}$  are isomorphic categories.*

Proof. Let  $\mathcal{N}$  be a graded  $\mathcal{B}$ -module. We define the  $\mathcal{C}$  module  $C(\mathcal{N})$  as follows:

$$({}_{x,s})C(\mathcal{N}) = {}_x\mathcal{N}^{s^{-1}},$$

and the action maps

$$({}_{y,t})(\mathcal{B}\#G)({}_{x,s}) \otimes_k ({}_{x,s})C(\mathcal{N}) \longrightarrow ({}_{y,t})C(\mathcal{N})$$

are given by the maps

$${}_y\mathcal{B}_x^{t^{-1}s} \otimes_k {}_x\mathcal{N}^{s^{-1}} \longrightarrow {}_y\mathcal{N}^{t^{-1}}.$$

Clearly this is an action of  $\mathcal{C}$  on  $C(\mathcal{N})$  and a map between graded  $\mathcal{B}$ -modules gives in a functorial way a map between the corresponding  $\mathcal{C}$ -modules.

Conversely, let  $\mathcal{M}$  be a  $\mathcal{C}$ -module. Let  $B(\mathcal{M})$  be the  $\mathcal{B}$ -module defined by

$${}_x B(\mathcal{M}) = \bigoplus_{s \in G} (x, s) \mathcal{M}.$$

The graduation is given by

$${}_x B(\mathcal{M})^s = (x, s^{-1}) \mathcal{M}.$$

The action is graded by construction. In order to give the maps

$${}_y \mathcal{B}_x^t \otimes_k {}_x B(\mathcal{M})^s \longrightarrow {}_y B(\mathcal{M})^{ts}$$

recall that

$${}_{(y, t)} (B \# G)_{(x, s)} = {}_y \mathcal{B}_x^{t^{-1} s}.$$

Then

$${}_y \mathcal{B}_x^t = {}_{(y, (ts)^{-1})} (B \# G)_{(x, s^{-1})}.$$

The required maps

$${}_{(y, (ts)^{-1})} (B \# G)_{(x, s^{-1})} \otimes_k (x, s^{-1}) \mathcal{M} \longrightarrow {}_{(y, (ts)^{-1})} \mathcal{M}$$

are obtained through the given action of  $B \# G$  on  $\mathcal{M}$ .

Clearly the two constructions provide functors which are inverse to each other.

□

The following result is now clear:

**PROPOSITION 4.6** *Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a Galois covering of categories over  $k$  with Galois group  $G$ . Let  $\mathcal{N}$  be a  $\mathcal{B}$ -module. Then*

$$BF^*(\mathcal{N}) = \bigoplus_{s \in G} \mathcal{N}^s \text{ where } \mathcal{N}^s = \mathcal{N} \text{ for each } s \in G.$$

*The graded action on  $\bigoplus_{s \in G} \mathcal{N}^s$  is given by  ${}_y \mathcal{B}_x^t \otimes_k {}_x \mathcal{N}^s \rightarrow {}_y \mathcal{N}^t$  using the  $\mathcal{B}$ -action on  $\mathcal{N}$ . Ignoring the grading, the  $\mathcal{B}$ -module  $BF^*(\mathcal{N})$  is a direct sum of copies of  $\mathcal{N}$  indexed by  $G$ .*

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